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* It is very hard to describe and convey the W.A (prob) you are using to others.

$\Rightarrow$ Mathematicians thus give names to some widely used W.As.

For example: ① A R.V X is of geometric distribution

② X is a "geometric" R.V.

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1. Bernoulli distribution /

Bernoulli R.V.

has 1 parameter p

$S = \{0, 1\}$.

$p_0 = 1 - p$, $p_1 = p$.

For example, ① a fair coin is a Bernoulli R.V w. $p = \frac{1}{2}$

② a bent coin is a Bernoulli R.V w. p

③ Winning a lotter is Bernoulli R.V w. $p = \frac{1}{150M}$

④ # of touchdowns of Purdue football team is (modeled as) a Bernoulli w. $p = 0.7$

2. Binomial R.V with 2 parameters n and p

$S = \{0, 1, \ldots, n\}$

$p_k = \binom{n}{k} p^k (1-p)^{n-k}$

Q: The score of a baseball team is a binomial R.V with para p, n

$E(X) = ?$ $Var(X) = ?$

Ans:

$$E(X) = \sum_{k=0}^{n} p_k \times k$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \times k$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\,(k!)} \times p^k (1-p)^{n-k} \times k$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \times \frac{(n-1)!}{(n-k)!\,(k-1)!} p \times p^{k-1} (1-p)^{n-k}$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad 0! = 1.}$$

Note Title 1/31/2011

Let $k' = k-1$, $n' = n-1$

$$= \sum_{k'=0}^{n'} n \cdot \frac{n'!}{(n'-k')!\,k'} p \cdot p^{k'} \cdot (1-p)^{n'-k'}$$

$$= np \left(\underbrace{\sum_{k'=0}^{n'} \frac{n'!}{(n'-k')!\,k'!} p^{k'} (1-p)^{n'-k'}}_{=1} \right)$$

→ It is the total prob of a binomial R.V w. para. p, n'.

$Var(X) = np(1-p)$ ← will come back later.

* Usually (but not necessarily), a binomial distri models the scenario of tossing a bent coin n times & count the # of heads.

Q: The score of a baseball team is a binomial R.V with para p, n

$E(X) = ?$ $Var(X) = ?$

3. Geometric Random Variable with 1 parameter $0<p<1$

$S_X = \{0, 1, \ldots\}$ all non-negative integers

$p_k = p(1-p)^k$

Example: $p = \frac{1}{4}$. Geometric R.V has

$p_0 = \frac{1}{4}(1-\frac{1}{4})^0$

$p_1 = \frac{1}{4}(1-\frac{1}{4})^1$ $\ldots$

* The geometric R.V. generally models the number of trials before the first head when flipping an unfair coin with head prob p.

Q The number of cars in a parking lot is a geometric R.V w. para p.

$E(X) = ?$

Ans: $E(X) = \sum_{k=0}^{\infty} p_k \times k$

$$= \sum_{k=0}^{\infty} p(1-p)^k \times k$$

$$= \sum_{k=1}^{\infty} p(1-p)^k \times k$$

$$= \frac{\text{the first term}}{(1-\text{ratio})^2}$$

$$= \frac{p(1-p)}{(1-(1-p))^2} = \frac{1-p}{p}$$

$Var(X) = \frac{1-p}{p^2}$

*

④ Poisson Random Var with para $\alpha > 0$

063 $S = \{0, 1, 2, \dots\}$: all non-negative integers

$p_k = \frac{\alpha^k}{k!} e^{-\alpha}$

$\boxed{\text{We use the convention } 0! = 1.}$

Ex: if $\alpha = 0.5$

$p_0 = \frac{0.5^0}{0!} e^{-0.5} = \frac{1}{1} e^{-0.5} = e^{-0.5}$

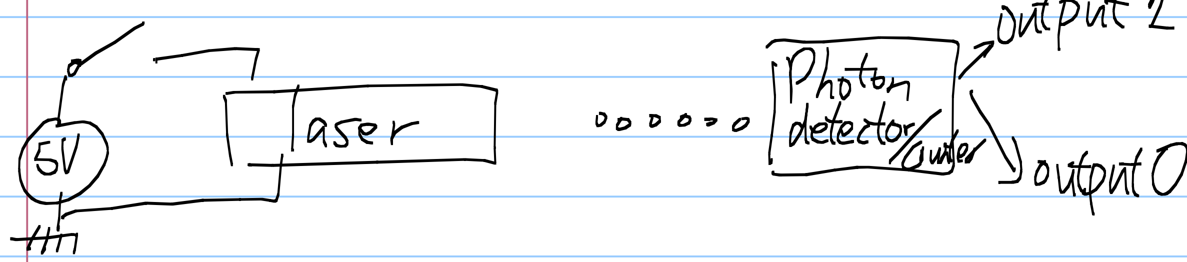
$p_1 = \frac{0.5^1}{1!} e^{-0.5} = \frac{0.5}{1} e^{-0.5} = 0.5 e^{-0.5}$

$p_2 = \frac{0.5^2}{2!} e^{-0.5} = \frac{0.5^2}{2} e^{-0.5} = \frac{1}{8} e^{-0.5}$

* The sample space of a Poisson R.V is exactly the same as that of a geometric R.V.

* Poisson random variable is used to model the experiment that ① Consider a fixed time interval, ② "A customer may show up or not" has the same prob for any time instant. ③ Knowing the average number of customers within this interval/duration is α ④ The actual number of customers is a

* Poisson is quite common, especially in physics.



A laser that can be turned on (5V) or off (0V)

Once it is on, in average 1000 photons/msec.

A photon detector count the number of photons X in 0.1 msec. And output 0 or 1 depending on X.

If there is no "ambient noise"

$$\text{Output} = \begin{cases} 1 & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}$$

(but in reality, we choose $\boxed{X > 30}$ instead.

What is the prob that the output

is 0 even if the laser is ON.

Ans: X is a Poisson R.V.

with α = the average # of photons in 0.1 msec

$= 1000 \times 0.1 = 100$

Then the answer is simple

$$P(X \leq 30) = \sum_{k=0}^{30} \frac{100^k}{k!} \times e^{-100}$$

$$\approx 2 \times 10^{-16}$$

Q: Suppose we reduce the interval to 0.01 msec

Y is the # of photons in 0.01 msec.

$$\text{Output} = \begin{cases} 1 & \text{if } Y \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

$P(\text{Output} = 0 \text{ given the laser is ON}) = ?$

Ans: $\alpha = 1000 \times 0.01 = 10$

$$P(Y \leq 3) = \sum_{k=0}^{3} \frac{10^k}{k!} e^{-10} = 1\%$$

Namely: sending at rate $\frac{1}{0.1\text{ msec}} = 10k$. the error prob $\approx 2 \times 10^{-16}$

if we increase the rate to 100k.