Independence

Two events are independent

① Physically not related

ex: the temperature today vs.
the lottery number.

② In this course, we use a different, <u>freq perspective</u> to say two events are independent.

Ex: Two virtual coins generated by a single computer/ iphone.

The outcomes are physically related, generated by the same program

Nonetheless, if we count the freq of the outcomes

* It is no different than two physical coins.

X \ Y	0	1
0	$\approx \frac{1}{4}$	$\approx \frac{1}{4}$
1	$\approx \frac{1}{4}$	$\approx \frac{1}{4}$

* then we say the two coins are independent (even though they are physically related)

A formal definition of independence is two events A & B are independent if

$$P(A) = P(A|B)$$

(or equivalently $P(A) \cdot P(B) = P(A \cap B)$)

Namely, conditioning on knowing whether B happens or not, does not change the freq of A happens.

Example: Are "the NY Stock Index" & the "weather of NYC" independent?

Suppose the historically data shows that

NYSE	Snow	Not snow
$\searrow$	$\frac{1}{75}$	$\frac{29}{75}$
$\nearrow$	$\frac{1}{100}$	$\frac{59}{100}$

$\Longrightarrow$ They are "dependent"

$$P(\nearrow) = \frac{1}{100} + \frac{59}{100} = 60\%$$

$$P(\nearrow \mid \text{Snow}) = \frac{\frac{1}{100}}{\frac{1}{75} + \frac{1}{100}} = \frac{3}{7} \neq 60\%$$

* A & B are $\boxed{\text{independent}}$ if $P(A) = P(A|B)$

(or equivalently $P(A) \cdot P(B) = P(A \cap B)$)

Another Example: Consider 1 fair coin X

& 1 unfair coin Y with

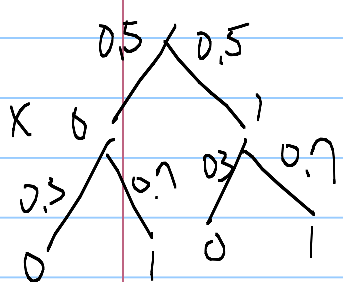
$$P(Y=0) = 0.3 \quad P(Y=1) = 0.7.$$

Suppose X & Y are <u>independent</u>

Q: Find the W.A.

Q: $P(Y=0 \mid X+Y \le 1)$

Ans: Table method



X \ Y	0	1
0	0.5×0.3	0.5×0.7
1	0.5×0.3	0.5×0.7
	0.3	0.7

$P(X=0 \mid Y=0)$

$= P(X=0) = 0.5$

The conditional prob is the same as the unconditional prob.

$$\frac{P(Y=0 \text{ and } X+Y \le 1)}{P(X+Y \le 1)}$$

$$= \frac{0.5\times0.3 + 0.5\times0.3}{0.5\times0.3 + 0.5\times0.7 + 0.5\times0.3}$$

$$= 0.4615$$

Part of

1045

Note Title 1/26/2011

Q38

Consider X is a discrete R.V with sample space $\{0, 1, 2, 3\}$ and weight assignment

$$\underset{p_0}{\frac{1}{8}},\ \underset{p_1}{\frac{3}{8}},\ \underset{p_2}{\frac{3}{8}},\ \underset{p_3}{\frac{1}{8}}$$

Consider another independent R.V Y that also has $S = \{0, 1, 2, 3\}$ and

W.A. $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$.

Suppose X and Y are independent.

Q What is the W.A when we consider jointly (X, Y)

X \ Y	0	1	2	3
0	$\frac{1}{8}\times\frac{1}{8}$	$\frac{1}{8}\times\frac{3}{8}$	$\frac{1}{8}\times\frac{3}{8}$	$\frac{1}{8}\times\frac{1}{8}$
1		$\frac{3}{8}\times\frac{3}{8}$		
2				
3				

Q: $P(X=Y)$?

Ans: $\frac{1}{8}\times\frac{1}{8}+\frac{3}{8}\times\frac{3}{8}+\frac{3}{8}\times\frac{3}{8}+\frac{1}{8}\times\frac{1}{8}$

$= \frac{20}{64} = \frac{5}{16}$ ※

Showing/proving 2 events A, B are indep

$\equiv$ Showing $P(A|B) = P(A)$

or $\frac{P(A \cap B)}{P(B)} = P(A)$

$\equiv P(A \cap B) = P(A) \cdot P(B)$

Showing/proving 3 events A, B, C are indep

$\equiv$ showing ① A, B are indep.

② B, C - - - -

③ C, A - - - -

☆ ④ $P(A \cap \quad \cap C) = P(A)P(B)P(C)$

Q39

2 independent fair coins X, Y,

& 1 magic coin

$$M = \begin{cases} 1 & \text{if } X \neq Y \\ 0 & \text{if } X = Y \end{cases}$$

Consider 3 events

$A: \{X=1\}$, $B: \{Y=1\}$, $C: \{M=1\}$

Q: What is the sample space?

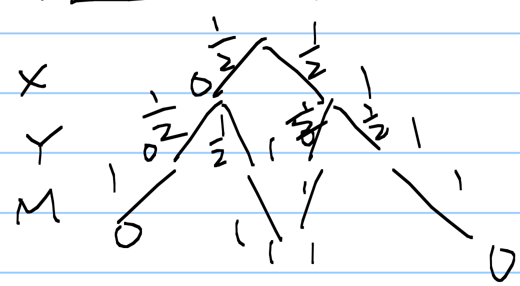
Ans: Since M depends on X and Y, the true randomness can be modeled as

X \ Y	0	1
0	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}$
1	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}$

or

M \ XY	00	01	10	11
0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
1	0	$\frac{1}{4}$	$\frac{1}{4}$	0

or



Q: $P(C) = P(M=1) = ?$

Ans: $= \underset{(0,1)}{\frac{1}{2}\times\frac{1}{2}} + \underset{(1,0)}{\frac{1}{2}\times\frac{1}{2}} = \frac{1}{2}$

Q: Are A, C independent?

Ans $P(A\cap C) = P(X=1, Y=0) = \frac{1}{4}$

$P(A) = \frac{1}{2}$ $P(C) = \frac{1}{2}$ $\Rightarrow$ $\boxed{\text{independent!}}$

Q: Are A, B, C independent?

Ans $P(A\cap B\cap C) = P(X=1, Y=1, M=1)$

$= 0$

$\neq P(A)\cdot P(B)\cdot P(C) = \left(\frac{1}{2}\right)^3$ $\boxed{\text{Not independent, dependent}}$

The hard drive example

Ans to Q1: $S = \{(0,0), (0,1), (1,0), (1,1)\}$

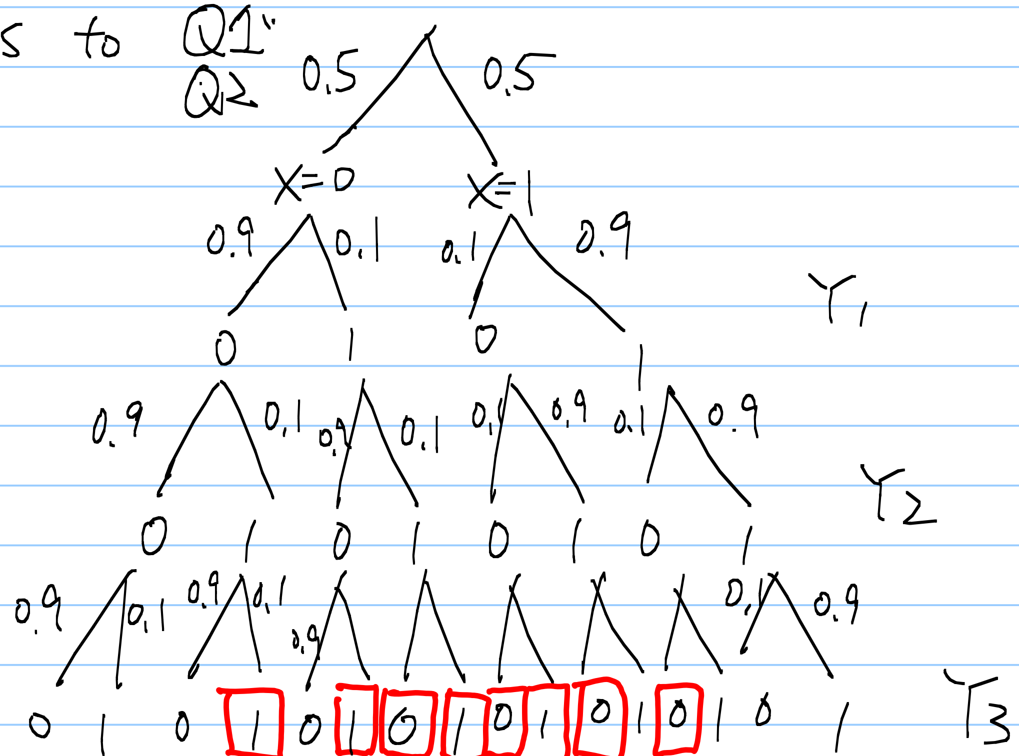
X \ Y	0	1	
0	0.5×0.9	0.5×0.1	0.5
1	0.5×0.1	0.5×0.9	0.5

Ans to Q2:

Ans to Q3: $0.5 \times 0.1 + 0.5 \times 0.1 = 0.1 = 10\%$

For repetition codes

Ans to Q1:
Q2



Ans to Q3: $P(X \neq Y)$

$= 0.5 \times 0.9 \times 0.1 \times 0.1 + 0.5 \times 0.1 \times 0.9 \times 0.1$

$+0.5\times0.1\times0.1\times0.9+0.5\times0.1\times0.1\times0.1$

$+0.5\times0.1\times0.1\times0.1+0.5\times0.1\times0.1\times0.9$

$+0.5\times0.1\times0.9\times0.1+0.5\times0.9\times0.1\times0.1$

$=2.8\%$

Table method

X \ Y	000	001	010	011	100	101	110	111
0	0.5×0.9^3	$0.5\times0.9^2\times0.1$	$0.5\times0.9^2\times0.1$	$0.5\times0.1^2\times0.9$	$0.5\times0.1\times0.9^2$	c	c	d
1	d	c	c	b	c	b	b	a

$a=0.5\times0.9^3$

$b=0.5\times0.9^2\times0.1$

$c=0.5\times0.9\times0.1^2$

$d=0.5\times0.1^3$

$\hat{Y}=0$

$X\neq\hat{Y}$

$p(X\neq\hat{Y})=c\cdot6+d\times2$

$=2.8\%$

* Basic prob concepts: W.A., Counting, Conditional prob, independence.

* We are now ready for some intermediate-level discussion, for which, we focus only on "Random Variables": Random experiments that have output being a <u>number</u>

* R.V is very useful as ① many experiments indeed output numbers. Ex: temperature, voltage,

② Moreover, in a digital world. more and more things are converted to numbers.

Ex: Black $\longrightarrow$ 0
white $\longrightarrow$ 255
light gray $\longrightarrow$ 200
dark gray $\longrightarrow$ 50

③ Easy manipulation. Suppose X, Y are Random variables, we can define a new R.V $Z = X^2 + Y^2$ and ask question like $P(Z<1)$

④ We can take weighted average

Ex: Flip a fair coin.

If the outcome is $\{H, T\}$. then the weighted average is meaningless.

(average of head ~~&~~ tail)

However, if we convert it to a R.V. $S = \{0, 1\}$ with weight assignment $\frac{1}{2}$ for $X=0$, $\frac{1}{2}$ for $X=1$. The weighted average becomes

$$\frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2} \#$$

For the following, we first consider "discrete R.V" (such that $S =$ integers)

* Discrete R.V:

① Sample space is discrete integers.

② The weight assignment is pmf.

$p_k = P(X = k)$ | Q: how to plot the pmf?

③ The "expectation / mean" of a discrete R.V. is the weighted average

$$E(X) \triangleq \sum_{k=-\infty}^{\infty} \underbrace{p_k}_{\text{weight}} \times \underbrace{k}_{\text{Value}}$$

Ex: If X is a fair dice,

$$E(X) = \sum_{k=1}^{6} \frac{1}{6} \times k = \frac{7}{2}$$

④ Expectation is simply weighted average.

Expectation of X^2 is

$$E(X^2) = \sum_{k=-\infty}^{\infty} \underbrace{p_k}_{\text{weight}} \cdot \underbrace{(k^2)}_{\text{value}}$$

Expectation of e^X is

$$E(e^X) = \sum_{k=-\infty}^{\infty} p_k \, (e^k)$$

Ex: X is a fair die

$$E(e^{-X}) = \sum_{k=1}^{6} \frac{1}{6} \times e^{-k} = \frac{\frac{1}{6}e^{-1}(1-e^{-6})}{1-e^{-1}}$$

$$E(X^3) = \sum_{k=1}^{6} \frac{1}{6} \times k^3 = \frac{441}{6}$$

Ex: Throw an unfair die with weight assignment

$$p_1 = \frac{2}{7}, \quad p_2 = \frac{1}{7} = p_3 = p_4 = p_5 = p_6$$

The casino gives you $f(X)$ dollars depending on the outcome of the die.

$$f(X) = \begin{cases} 1 & \text{if } 1 \le X \le 3 \\ X^2 & \text{if } 4 \le X \le 6 \end{cases}$$

Q: What is the expected return?

Ans: $E(f(X))$

$$= \sum_{k=1}^{6} p_k f(k)$$

$$= \frac{2}{7} \times 1 + \frac{1}{7} \times 1 + \frac{1}{7} \times 1 + \frac{1}{7} \times 4^2 + \frac{1}{7} \times 5^2 + \frac{1}{7} \times 6^2$$

$$= \frac{81}{7}$$

Important properties of expectation:

① Expectation is <u>linear</u>

Namely $E(af(X)) = aE(f(X))$

$E(f_1(X)+f_2(X)) = E(f_1(X)) + E(f_2(X))$

Pf: $E(af(X))$

$= \sum_{k=-\infty}^{\infty} p_k (af(k))$

$= a\sum_{k=-\infty}^{\infty} p_k (f(k)) = aE(f(X))$

$E(f_1(X)+f_2(X))$

$= \sum_{k=-\infty}^{\infty} p_k (f_1(k)+f_2(k))$

$= \sum_{k=-\infty}^{\infty} p_k (f_1(k)) + \sum_{k=-\infty}^{\infty} p_k (f_2(k))$

$= E(f_1(X)) + E(f_2(X))$

Namely: The (weighted) avg of a fair dice is $E(X) = 3.5 \Rightarrow$ The weighted average of $E(2X) = 2E(X) = 7$

Note Title 2/2/2011

③ Expectation of a constant is the constant itself. $E(2.75) = \sum_{k=-\infty}^{\infty} p_k (2.75)$

$= 2.75 \sum_{k=-\infty}^{\infty} p_k = 2.75$ | $\sum p_k = 1$ Total prob

④ Expectation may be infinite (or does not exist.

Example $p_k = \begin{cases} 0.5^k & \text{for all } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$

$$E(3^X) = \sum_{k=-\infty}^{\infty} p_k \cdot 3^k$$

$$= \sum_{k=1}^{\infty} (0.5 \cdot 3)^k \quad \text{diverges}$$

⑤ The "variance" of a discrete R.V is the weighted average of $(X-m)^2$ where m is the constant $E(X)$

Namely

$$\boxed{Var(X) \triangleq E((X-m)^2)}$$

Expected squared distance to the center "m"

Ex: X is a fair dice.

What is $Var(X)$

Ans: Step 1: Find the "center" m first

$$E(X) = \sum_{k=1}^{6} \frac{1}{6} \times k = \frac{7}{2} = m$$

Step 2:

$$Var(X) = E((X-m)^2) = \sum_{k=1}^{6} \frac{1}{6} \left(k - \frac{7}{2}\right)^2$$

$$= \frac{1}{6}\left[\left(1-\frac{7}{2}\right)^2 + \left(2-\frac{7}{2}\right)^2 + \left(3-\frac{7}{2}\right)^2 + \cdots + \left(6-\frac{7}{2}\right)^2\right]$$

$$= \frac{35}{12}$$

An alternative formula of

$$E((X-m)^2) = E(X^2 - 2mX + m^2)$$

$$= E(X^2) - E(2mX) + E(m^2)$$

$$= E(X^2) - 2m\underbrace{E(X)}_{m} + m^2$$

$$= E(X^2) - m^2$$

Step 1 Find $E(X^2)$

Step 2 $Var(X) = E(X^2) - m^2$

In many places, you see

$$Var(X) = E((X - E(X))^2) = E(X^2) - (E(X))^2$$

I prefer

$$Var(X) = E((X - m)^2) = E(X^2) - m^2$$

⑥ Standard deviation $\triangleq \sqrt{\text{Variance}}$

⑦ The n-th moment of X

$= E(X^n)$

⑧ The n-th <u>central moment</u> of X

$= E((X-m)^n)$

Example: 1. The mean $E(X)$ is the 1st moment of X.

2. The variance $Var(X)$ is the 2nd central moment of X.

3. The first central moment is $E(X-m) = E(X) - m = 0$ always zero.