☆ The prob of an event is the Total weight for all the outcomes in the event.

Ex: the prob of "X being a prime number" $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 0.5$

event

$\{2, 3, 5\}$

outcome

☆ Set / Event operations

① Empty set / Null event "ϕ"

I.e: No outcome in a null event.

$\{ \}$.

Q: What is the prob of a null event?

Ans: 0. ($\because$ count nothing)

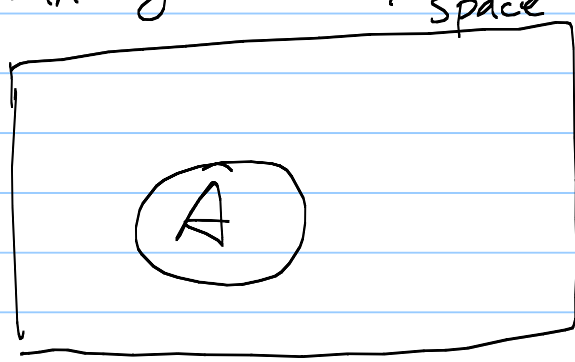
② Global set $S = \{$every thing$\}$

Q: What is the prob of a global event

Ans: 1 ($\because$ count everything)

☆ Venn's Diagram: A tool to help us visualize the set operations.

S: the global set / sample space



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③ Compliment event "c" e.g. $S = \{1, 2, 3, 7\}$

$\boxed{Ⓐ\ A^c}$ $A = \{1, 3\}$

$A^c = \{2, 7\}$ everything else

E.g. $S = \{x : x > 0\}$ $A = \{x : 1 < x < 3\}$, $A^c = \{x : 0 < x \leq 1 \text{ or } 3 \leq x\}$

④ Union "$\cup$"

E.g. $A = \{1, 2, 3\}$ $B = \{2, 3, 7\}_{2,2}$

$A \cup B = \{1, 2, 3, 7\}$, Not $\{1, 3, 3, 7\}$

E.g. $B = \{x : 2 \leq x \leq 5\}$.

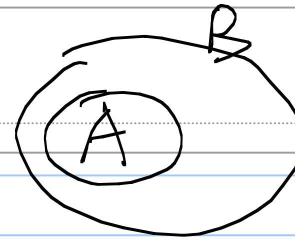
$A \cup B = \{x : 1 < x \leq 5\}$

⑤ Intersection "$\cap$"

$A \cap B = \{3, 2\}$ E.g. $A \cap B = \{x : 2 \leq x < 3\}$

Why are we interested in the set operations?

Ans: We are more interested in the Weights assigned to each set. Nonetheless, knowing how to include/exclude an outcome is essential before we can properly count the total weight assigned for an event.

★ $\boxed{A \text{ implies } B} \equiv$ 

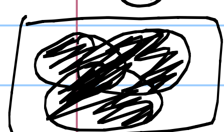
Ex: $A = \{\text{all multiples of } 4\}$. A implies B.
4, 8, 12
$B : \{\text{all multiples of } 2\}$. If X is a multiple of 4 then X must be a multiple of 2.
2, 4, 6, 8, ...

⑥ Commutativity

$A \cup B = B \cup A$ OR : In at least one

$A \cap B = B \cap A$ AND : In ALL.

⑦ Associativity


$A \cup (B \cup C) = (A \cup B) \cup C$

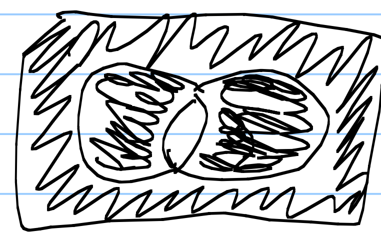

$A \cap (B \cap C) = (A \cap B) \cap C$

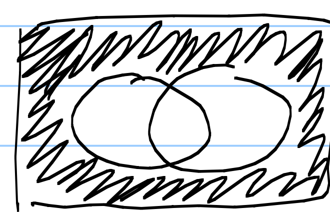
⑧ Distributivity


$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$


$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

⑨ Demorgan's Rule

$(A \cap B)^c = A^c \cup B^c$ 

$(A \cup B)^c = A^c \cap B^c$ 

Once we know how to include/exclude events/sets, we need to assign weights

A valid W.A satisfies the following 3 axioms

Axiom 1:

Each weight must be non-negative

$$P(A) \geq 0$$

Axiom 2: The total weight must be 1.

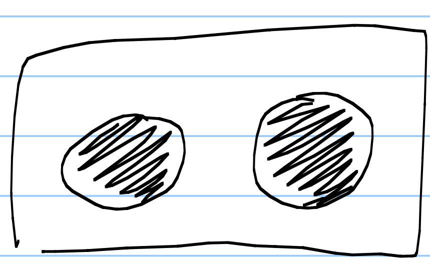
$$P(S) = 1.$$

Axiom 3: If <u>events</u> are disjoint, i.e, they can not happen simultaneously.

$$\| \; A \cap B = \phi$$

then the weights of either A or B happens must be the sum of individual weights

$$P(A \cup B) = P(A) + P(B)$$



Similarly: if $A \cap B = B \cap C = C \cap A = \phi$

Then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Axiom 3.1: If any two A_i, A_j are disjoint. then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The above "axioms" are very intuitive and can be taken as granted and used to show some non-intuitive results.

Corollary 1

$$P(A^c) = 1 - P(A)$$

Corollary 2

$$P(A) \leq 1$$

Corollary 3

$$P(\phi) = 0$$

Corollary 4

If $A_1 \cdots A_n$ are disjoint, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{k=1}^{n} P(A_k)$$

Corollary 5

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex: X is the outcome of a fair 6-faced die.

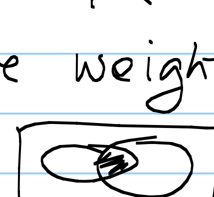
$$P(X \text{ is a prime or } X \geq 5)$$

$$= P(X = 2, 3, 5, 6) = \frac{4}{6}$$

$$= P(X = 2, 3, 5) + P(X = 5, 6) - P(X = 5)$$

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6}.$$

$\because$ The weight of $X=5$



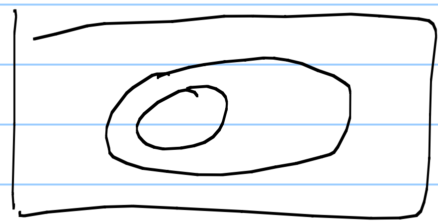
is double counted.

Corollary 7

If $A \subseteq B$, then $P(A) \leq P(B)$

A is a subset of B

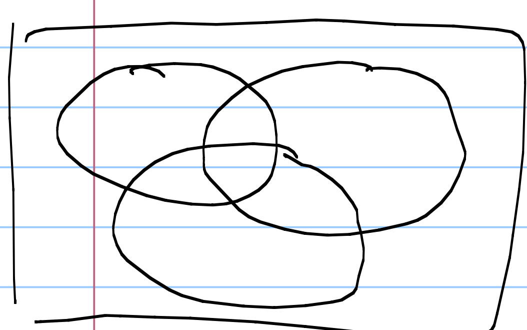
A implies B



Corollary 6

Question for the team: Explain the following "inclusion/exclusion" principle by the Venn Diagram

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$



Q: $P(A \cup B \cup C \cup D) = ?$

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① We start by "set operations": how to include/exclude the events.

② Then discuss properties of a valid W.A.

③ The next question is how to construct a valid W.A by ourselves.

Case 1: The sample space is discrete. (ex: A card game, a coin)

Step 1: Specify the non-negative weight for each outcome. & Step 2: Make sure the total sum is 1.

Ex: A coin has two outcomes $\{H, T\}$

We can assign $P(H) = \frac{1}{3}$ $P(T) = \frac{2}{3}$.

In many cases, we are interested in random experiments that have output being integers, then the weight assignment is described by

$$P_k = P(X = k)$$

This special type of experiments is called "discrete random variable" & the associated weight assignment is called "discrete distribution"

* random: we do not know what the outcome will be.
  Variable: The outcome is (usually) a number.
  Discrete: Values are integers
* The p_k used for describing a discrete distribution (W.A) is called the $\boxed{\text{prob mass function (pmf)}}$

Example: A fair die is a discrete random variable & its distribution is described by pmf

$$p_1 = p_2 = p_3 = \cdots = p_6 = \frac{1}{6}, \quad \text{all other } p_k = 0$$

If we let 0 denote tail, 1 for head, then the previous coin experiment is a discrete R.V. & its distribution is described by the following pmf.

$$p_0 = \frac{2}{3} \quad p_1 = \frac{1}{3} \quad \text{all other } p_k = 0$$

Example: A discrete R.V has sample space $S = \{0, 1, 2, \cdots, \infty\}$ and it pmf (W.A) is

$$p_k = \frac{1}{4}\left(1 - \frac{1}{4}\right)^k \quad \text{for } k = 0, \cdots, \infty$$

Q: Is this a valid W.A

Ans: Check ① $p_k \geq 0$ for all k ✓

$\boxed{\text{Yes}}$ ② $\sum_{k=0}^{\infty} p_k = \frac{0.25}{1-(1-0.25)} = 1$ ✓

* We define W.A first & then make prob statements.
* Be careful when we try to design a W.A to "retro-fit" some prob. statement.

Ex: $S = \{1, 2, 3\}$ ex: 1 : sunny

2 : rainy

3 : snowy

If someone says that

The prob $(X \neq 2)$ is $5/8$

$\text{prob}(X \neq 1) = \frac{1}{4}$

Q: Are these two statements consistent? (Equivalently, can we find a valid W.A satisfying the above two statements?

A: Suppose we can, then we will have the pmf p_1, p_2, p_3

Then we have

$\boxed{\text{Not valid}}$

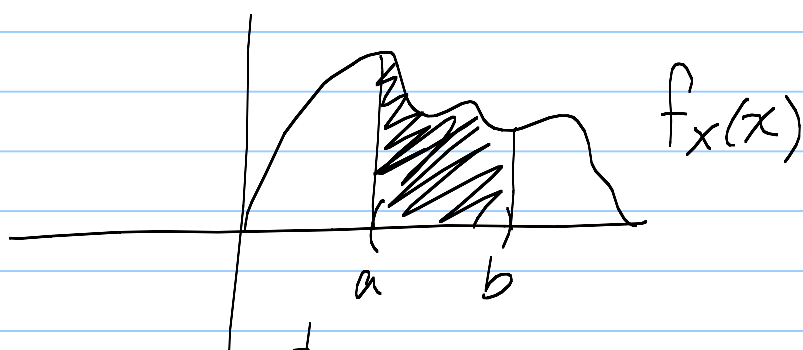
$$\begin{cases} p_1 + p_3 = 5/8 \\ p_2 + p_3 = \frac{1}{4} \\ p_1 + p_2 + p_3 = 1 \end{cases} \Longrightarrow \begin{aligned} &p_1 = \frac{3}{4},\ p_2 = \frac{3}{8} \\ &p_3 = -\frac{1}{8} \end{aligned}$$

Case 2: Suppose the sample space is continuous, and the output of a random experiment is the real number. Ex: the temperature, the time that the instructor enters the classroom.

We say this type of random experiment is a $\boxed{\text{continuous random variable}}$, its W.A is a continuous distribution

The W.A is described by <u>the area underneath a curve</u>.

Namely



$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

1 To make sure the sum is 1. we need

$\Rightarrow \int_{-\infty}^{\infty} f_X(x) = 1.$

2 Note that $f_X(x)$ stays above zero $\because$ all weights must be non-negative.

The curve $f_X(x)$ is termed the prob density function (pdf)

Ex: $f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Q: Is $f_X(x)$ a valid pdf (describing a valid W.A?

Ans: Check 1 $f_X(x) \geq 0$ for all x ✓

2 $\int_{-\infty}^{\infty} f_X(x)\,dx \stackrel{?}{=} 1$

$= \int_{0}^{\infty} 0.5\, e^{-0.5x}\, dx = 1.$

Since 1, 2 $\Rightarrow$ Yes. $f_X(x)$ is valid.

Q. Do we need to have $f_X(x) \leq 1$ for all x?

Ans: No. in this example $f_X(0) = 2.$

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Example: Today's temperature is uniformly/equally likely distributed between (5 F, 40F)

What is the prob that $P(T>32)$?

Ans: Step 1: Find the sample space

$S=(5,40)$ | $S=\mathbb{R}$ any real number

Step 2: Construct the W.A. Since it is a continuous random variable. We need to specify a curve $f_X(x)$.

$\because$ Uniformly/equally likely, the curve should

be flat | be flat over (5.40)

$f_X(x)=c$ | $f_X(x)=\begin{cases} c & \text{if } 5<x<40 \\ 0 & 0 \end{cases}$

To be a valid W.A. |

$$\int_5^{40} f_X(x)\,dx=1$$ | $$\int_{-\infty}^{\infty} f_X(x)\,dx=1.$$

$\Rightarrow 35c=1 \quad c=\frac{1}{35}$ | $=\int_5^{40} c\,dx=1$

$f_X(x)=\frac{1}{35}$ | $f_X(x)=\begin{cases} \frac{1}{35} & \text{if } 5<x<40 \\ 0 & \text{otherwise} \end{cases}$

Step 3: Count the prob of the desired event.