Midterm #1 of ECE 639, (CRN: 25576)
6–7pm, Wednesday, October 05, 2022, BHEE005.

1. Enter your student ID number, and signature in the space provided on this page.
2. This is a closed book exam.
3. The instructor will hand out loose sheets of paper for the rough work.
4. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [30%]

Consider the following scheme that sends a ternary message $M \in \{0,1,2\}$.

If $M=0$, we send a 1000-bit vector $\vec{X} = \overbrace{00\cdots0}^{1000}$;

if $M=1$, we send a 1000-bit vector $\vec{X} = \overbrace{11\cdots1}^{1000}$;

and if $M=2$, we send a 1000-bit vector $\vec{X} = \overbrace{0\cdots0}^{500}\overbrace{1\cdots1}^{500}$.

Suppose the binary vector $\vec{X}$ is sent through an i.i.d. Binary Symmetric Channel (BSC) with crossover probability $p < 0.5$. We denote the noisy observation by $\vec{Y}$ and the corresponding Maximum Likelihood decoder by $\hat{M}_{\text{ML}}(\vec{Y})$.

1. [30%] Please estimate/upper bound the probability $P(\hat{M}_{\text{ML}}(\vec{Y}) \neq M | M = 2)$ by the Chernoff inequality

   Hint 1: You may need to use the *union bound* in probability. That is, $P(A \cup B) \leq P(A) + P(B)$.

   Hint 2: $\hat{M}_{\text{ML}}(\vec{Y}) \neq 2$ is the event that EITHER the likelihood of $M = 0$ is larger than that of $M = 2$, OR the likelihood of $M = 1$ is larger than that of $M = 2$.

Question 2: [35%] Consider two sets of i.i.d. discrete random variables $\{X_i : i \geq 1\}$ and $\{Y_j : j \geq 1\}$, and we assume that $\{X_i\}$ and $\{Y_j\}$ are independent and their marginal distributions are $X_i \sim P_X$ and $Y_j \sim P_Y$, respectively. Also assume, without loss of generality, that $E(X_i) < 0$ and $E(Y_j) < 0$.

For any even integer $n \geq 1$, define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

$$\bar{Y}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j \tag{2}$$

$$\bar{Z}_n = \frac{1}{n}\left(\sum_{i=1}^{0.5n} X_i + \sum_{j=1}^{0.5n} Y_j\right) = 0.5\left(\frac{1}{0.5n}\sum_{i=1}^{0.5n}(X_i + Y_i)\right) \tag{3}$$

That is, $\bar{X}_n$ is the empirical average of X_i after n samples; $\bar{Y}_n$ is the empirical average of Y_j after n samples; and $\bar{Z}_n$ is the empirical average of X_i and Y_j after we sample $0.5n$ times from each of the i.i.d. random variables $\{X_i\}$ and $\{Y_j\}$.

Let c_X denote the asymptotic *decay rate* of the probability $P(\bar{X}_n > 1)$. That is, for any $\epsilon > 0$,

$$e^{-n\cdot(c_X+\epsilon)} < P(\bar{X}_n > 1) \leq e^{-n\cdot c_X}, \text{ for all sufficiently large } n. \tag{4}$$

Similarly, we can define the asymptotic decay rate of the probabilities $P(\bar{Y}_n > 1)$ and $P(\bar{Z}_n > 1)$, respectively. That is,

$$e^{-n\cdot(c_Y+\epsilon)} < P(\bar{Y}_n > 1) \leq e^{-n\cdot c_Y}, \text{ for all sufficiently large } n. \tag{5}$$

$$e^{-n\cdot(c_Z+\epsilon)} < P(\bar{Z}_n > 1) \leq e^{-n\cdot c_Z}, \text{ for all sufficiently large } n. \tag{6}$$

$$\tag{7}$$

Prove that $c_Z \leq \frac{c_X+c_Y}{2}$ for any possible marginal distributions P_X and P_Y.

Hint 1: The expressions e^{-c_X} and e^{-c_Y} can be obtained by standard Chernoff bound results.

Hint 2: You may want to upper bound $P(\bar{Z}_n > 1)$ by applying Chernoff bound based on a new random variable $W_i = X_i + Y_i$. If successful, this will give you the e^{-c_Z} value.

Hint 3: Comparing the expressions of e^{-c_X}, e^{-c_Y}, and e^{-c_Z} will give us the desired inequality.

Question 3: [15%]Consider any finite group G with the associated multiplication operator "$\cdot$", and H is a *sub-group* of G. Prove that for any three elements $x \in G$, $y \in G$, and $h_1 \in H$ satisfying $x = y \cdot h_1$, the following three statements are always true.

Statement 1: For any $h_2 \in H$, we can always find another $h_3 \in H$ such that $x \cdot h_2 = y \cdot h_3$.

Statement 2: For any $h_4 \in H$, we can always find another $h_5 \in H$ such that $x \cdot h_5 = y \cdot h_4$.

Statement 3: $\{x \cdot h : h \in H\} = \{y \cdot h : h \in H\}$.

Question 4: [20%] Consider the Galois Field $GF(3^3)$ generated by an irreducible polynomial $g(x) = x^3 + 2x + 2 \in \mathbb{F}_3[x]$. Under this finite field, what is the value of $12 \cdot 20$.

Hint: $12 = 110$ in base-3; and $20 = 202$ in base-3.