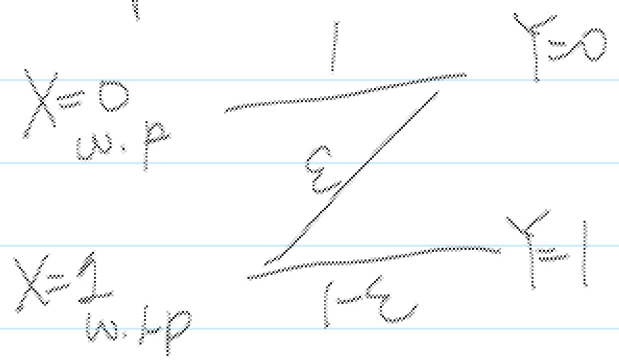
$$= E_X\left(D\left(P_{Y|X}(\cdot|X) \,\|\, P_Y(\cdot) \right) \right)$$

Example: The capacity of a Z channel



Step 1: Assume the prior distribution P_X

Step 2: Evaluate $I(X;Y)$

$$= E_{XY}\left(\log \frac{P_{XY}(X,Y)}{P_X(X)\cdot P_Y(Y)} \right)$$

$(x,y)=(0,0)$ $\qquad\qquad (1,0)$

$$I(X;Y) = p \log \frac{p}{p\cdot(p+\varepsilon(1-p))} + \varepsilon(1-p)\log\frac{(1-p)\varepsilon}{(1-p)(p+\varepsilon(1-p))}$$

$(1,1)$

$$+ (1-p)(1-\varepsilon)\log\frac{(1-p)(1-\varepsilon)}{(1-p)((1-p)(1-\varepsilon))}$$

$-$

Step 3: $\max_{p} I(X;Y)$

$$p^* = \frac{1-\varepsilon^{\frac{1}{1-\varepsilon}}}{1+(1-\varepsilon)\varepsilon^{\frac{\varepsilon}{1-\varepsilon}}}$$

Example: Consider a BPSK system over an i.i.d. channel.

I.e. $X_i = (-1)^{b_i} \cdot a$ $\quad b_i$ is the i-th bit.

$Y_i = X_i + N_i$

N_i: I.i.d. Gsn $\mathcal{N}(0,1)$

Q: Find out the optimal achivable rate of this channel.

Ans: Step 1: Design the marignal prob P_X

Say $P(X=a) = p$ $\quad P(X=-a) = (1-p)$

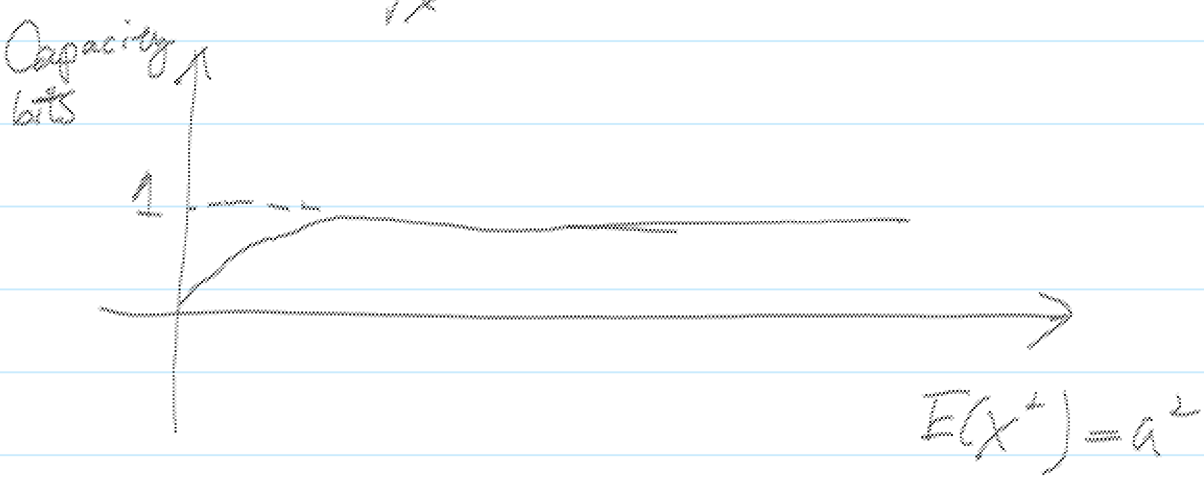
Step 2: $I(X;Y)$

$$= E_{XY}\left(\log \frac{P_{XY}(X,Y)}{P_X(X)P_Y(Y)}\right)$$

$$= \sum_{x=0,1}\int_{y=-\infty}^{\infty} P(X=x)\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2\times 1}} \cdot \log\left(\frac{P(X=x)\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-x)^2}{2}}}{P(X=x)\left(P(X=a)\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-a)^2}{2}} + P(X=-a)\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+a)^2}{2}}\right)}\right) dy$$

This can be evaluated numerically for any p values.

Step 3: $\max_{P_X} I(X;Y)$



The second example

Find out the channel capacity of an i.i.d Gussion channel $N_i \sim Gsn(0,1)$ with the "Power constraint γ". Here X_i does not need to be of antipodal values but only need to satisfy $E(X^2) \le \gamma$.

Ans: Step 1: Choose any P_X satisfying

$$E(X^2) \le \gamma$$

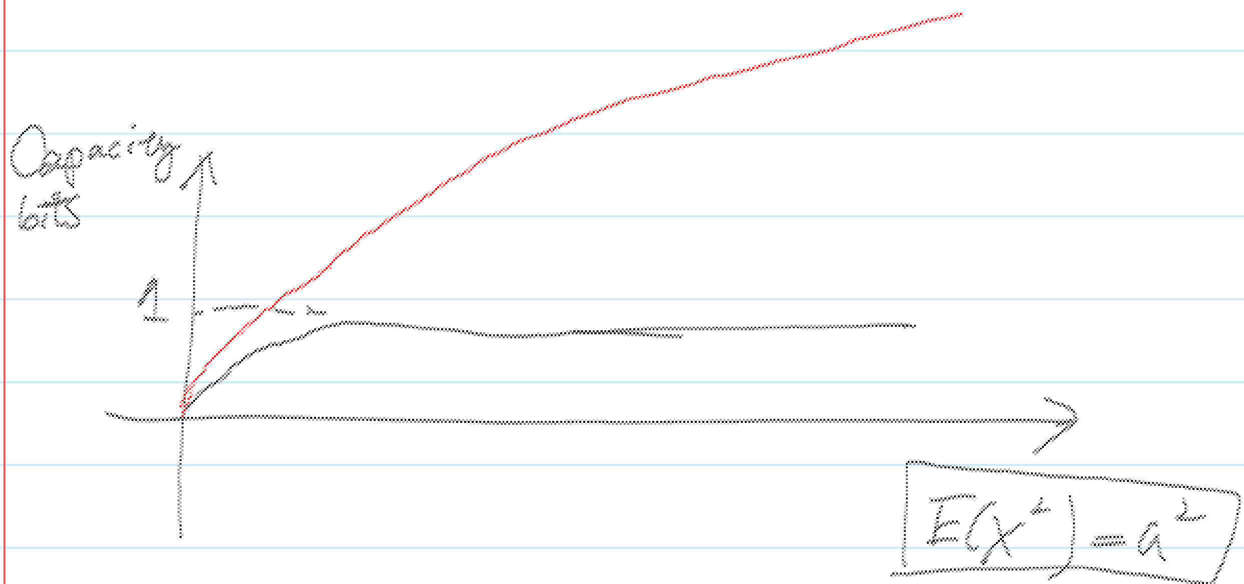
Step 2: Find $I(X;Y)$

Step 3: $\displaystyle \max_{P_X : E(X^2) \le \gamma} I(X;Y)$

With power constraint: $p^* \sim Gsn(0, \gamma)$

Capacity $= I_{P_X^*}(X;Y)$ (bits per symbol usage)

$$= \frac{1}{2}\log(1+\gamma) \leftarrow \text{(exercise)}$$



We can also impose different constraints on P_X, ex: $P(|X|>M)=0$

Question: How to achieve the "capacity" of BPSK?

How to achieve the capacity of arbitrary P_X?

Given any Signal to Noise Ratio

Given any Signal to Noise Ratio

$\Longrightarrow I(X;Y)$ is a function of SNR

and any code with good performance

must have rate $r < I(X;Y)$

* If we use logarithm with base 2,

then we have 2^{rn} distinct codewords in S_X^n

Alternatively,

given any code of rate r.

the good performance happens when

SNR satisfying $I(X;Y) > r$

* In error-control coding, we usually

study the alternative question by ploting

the error rate vs. SNR curve

Illustration:

SNR vs error-rates for

different codes of the same
rate