L04

Wednesday, August 31, 2022 1:01 PM

Example 2:

a bit string of n bits 01011 0...

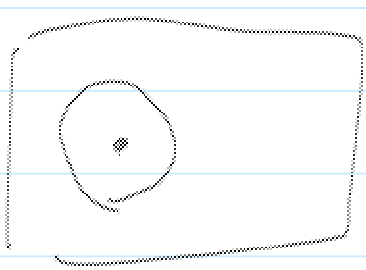
If noise-free, 2^n possibilities

$= \log_2(2^n) = n$ bits

A special noise model: At most $0.1n$ bits are flipped (corrupted)

Q: What is the achievable rate upper bound?

A: each choice occupies



$\sum_{k=0}^{0.1n} \binom{n}{k}$ different bit strings

$\Rightarrow$ the reduce info $= \frac{2^n}{\sum^{0.1n} \binom{n}{k}}$

$\Rightarrow$ we reduce info $=$

$$\sum_{k=0}^{0.1n} \binom{n}{k}$$

$$= \log_2 \left(\frac{2^n}{\sum \binom{n}{k}} \right) \text{ bits}$$

In terms of bits: $\log_2 \left(\dfrac{2^n}{\sum_{k=0}^{0.1n} \binom{n}{k}} \right)$ bits.

$$= n - \log_2 \left(\sum_{k=0}^{0.1n} \binom{n}{k} \right)$$

The normalized rate (upper bound) becomes

$$\frac{n - \log_2 \left(\sum_{k=0}^{0.1n} \binom{n}{k} \right)}{n} \triangleq \gamma(n)$$

$\gamma(n)$ is an increasing function

Q: $\lim_{n \to \infty} \gamma(n) = ?$

Or equivalently $\lim_{n \to \infty} \dfrac{\log_2 \left(\sum_{k=0}^{0.1n} \binom{n}{k} \right)}{n} = ?$

Ans:

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$$\frac{\log_2\left(\binom{n}{0.1n}\right)}{n} \le \frac{\log_2\left(\sum_{k=0}^{0.1n}\binom{n}{k}\right)}{n} \le \frac{\log_2\left(0.1n \cdot \binom{n}{0.1n}\right)}{n}$$

By Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\frac{n!}{(0.1n)!\,(0.9n)!} \approx \frac{\sqrt{2\pi n}\; n^n/e^n}{\dfrac{\sqrt{2\pi\, 0.1n}\,(0.1n)^{0.1n}\sqrt{2\pi\, 0.9n}\,(0.9n)^{0.9n}}{e^n}}$$

$$\approx \frac{1}{\sqrt{2\pi(0.1)(0.9)n}} \; \frac{n^n}{(0.1n)^{0.1n}(0.9n)^{0.9n}}$$

$$\frac{\log_2\binom{n}{0.1n}}{n} \approx \frac{n}{n}\log_2 n - \frac{(0.1n)}{n}\log_2(0.1n) - \left(\frac{0.9n}{n}\right)\log_2(0.9n)$$

$$= -0.1\log_2(0.1) - 0.9\log_2 0.9$$

$$= H(0.1)$$

$$\frac{\log\left(0.1n \cdot \binom{n}{0.1n}\right)}{n} \approx \frac{\log(0.1n)}{n} + H(0.1)$$

$$\Rightarrow \lim_{n\to\infty} r(n) = \underline{\underline{1 - H(0.1)}}$$

The Shannon capacity of a BSC with crossover prob 0.1.

The 2nd Application of $D(P_0 \| P_1)$

* Consider i.i.d channel

$$\xrightarrow{X_i} \boxed{P_{Y|X}} \xrightarrow{Y_i}$$

If the codeword satisfies that the marginal distribution of X_i is P_X then the achievable rate is

$$\leq \underline{I(X;Y)} \triangleq \underline{\underline{E_{XY}}}\left(\log \frac{P_{XY}(X,Y)}{P_X(X) \cdot P_Y(Y)}\right)$$

$I(X;Y) = E_{XY}\left(\log \frac{P_{XY}(X,Y)}{P_X(X)\cdot P_Y(Y)}\right)$

mutual information

$\hookrightarrow$ evaluated by P_{XY}

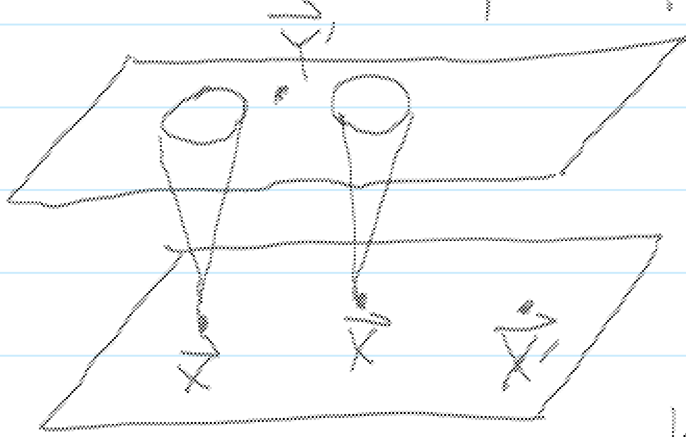
$= D(\underline{P_{XY}} \,\|\, \underline{P_X \times P_Y})$

look as if dependent ← actual indep

H_0: $(X_1, Y_1), \cdots, (X_n, Y_n)$ are i.i.d with joint distri P_{XY}

H_1: $(X_1, Y_1) \cdots (X_n, Y_n)$ are i.i.d with independent distri $P_X \times P_Y$.

* Consider the sphere packing bound.



$Y_1, \cdots, Y_N$

$X_1, \ldots, X_n$ decided by P_X

The larger the base $(x_1, \cdots, x_n)$

[illegible]

the more "cones" we can squeeze into.

Therefore, we need to fix $\boxed{P_X}$.

& discuss the # of cones one can fit.

$\rightarrow$ The stronger ~~the~~ signal power.

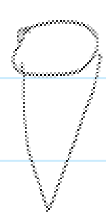
Question: Consider $\vec{X}' \sim (P_X)^n$

$\vec{Y}' \sim (P_Y)^n$

& $\vec{X}', \vec{Y}'$ are indep.

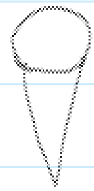
When will $\vec{X}', \vec{Y}'$ look like it is generated from $P_X, P_{Y|X}$? (or from P_{XY})

Ans: When $\vec{X}', \vec{Y}'$ fall into the

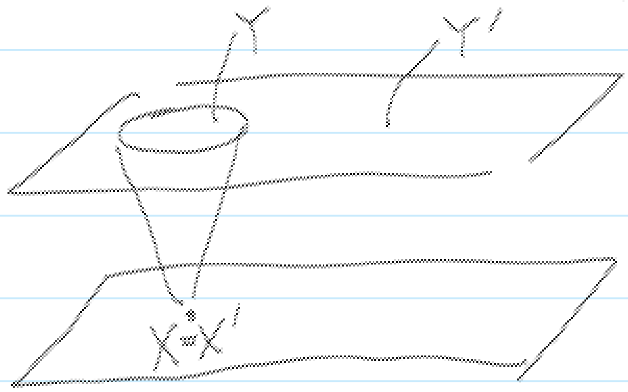


cone

$\Rightarrow$ [illegible] $-nD(P_0 \| P_1)$

$\Rightarrow$  is taking up $e^{-nD(P_0 \| P_1)}$

of the total (prob) volume of



where $P_0 : P_{XY}$ X, Y depend on each other

$P_1 : P_X \times P_Y$ are indep

As a result, we should be able to

pack $e^{nD(P_{XY} \| P_X \cdot P_Y)} = e^{nI(X;Y)}$

"Codewords" without significant overlap.

$$I(X;Y) = D(P_{XY} \| P_X \times P_Y)$$

$$= E_X \left[D\left(P_{Y|X}(\cdot|X) \| P_Y(\cdot) \right) \right]$$