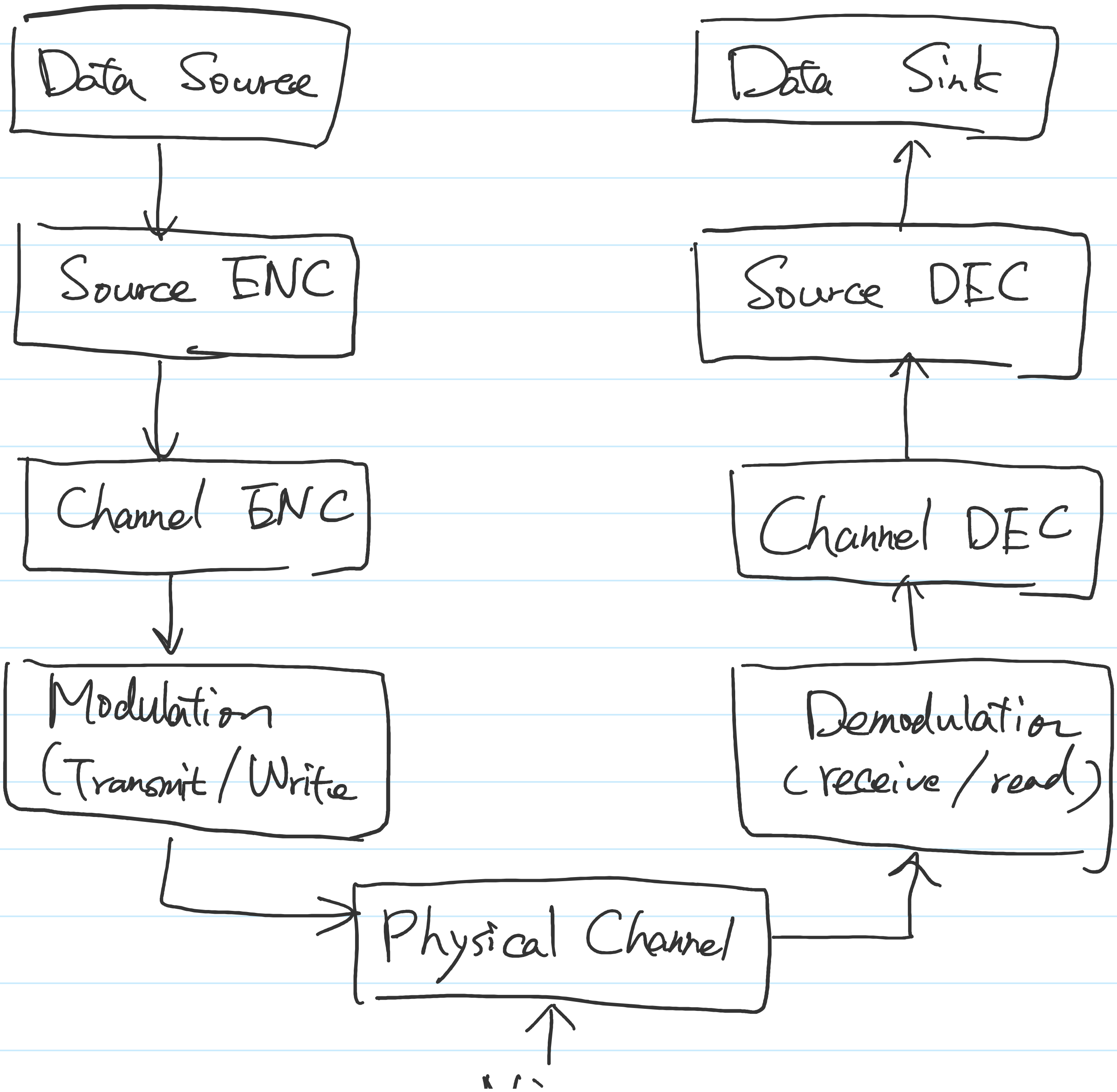
L01

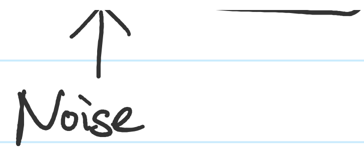
Monday, August 15, 2022 1:30 PM

(A significant part of the lecture notes was developed by Prof. David J. Love, who taught ECE639 many of the previous offerings.)

Digital Comm. Review and Motivation

Block Diagram





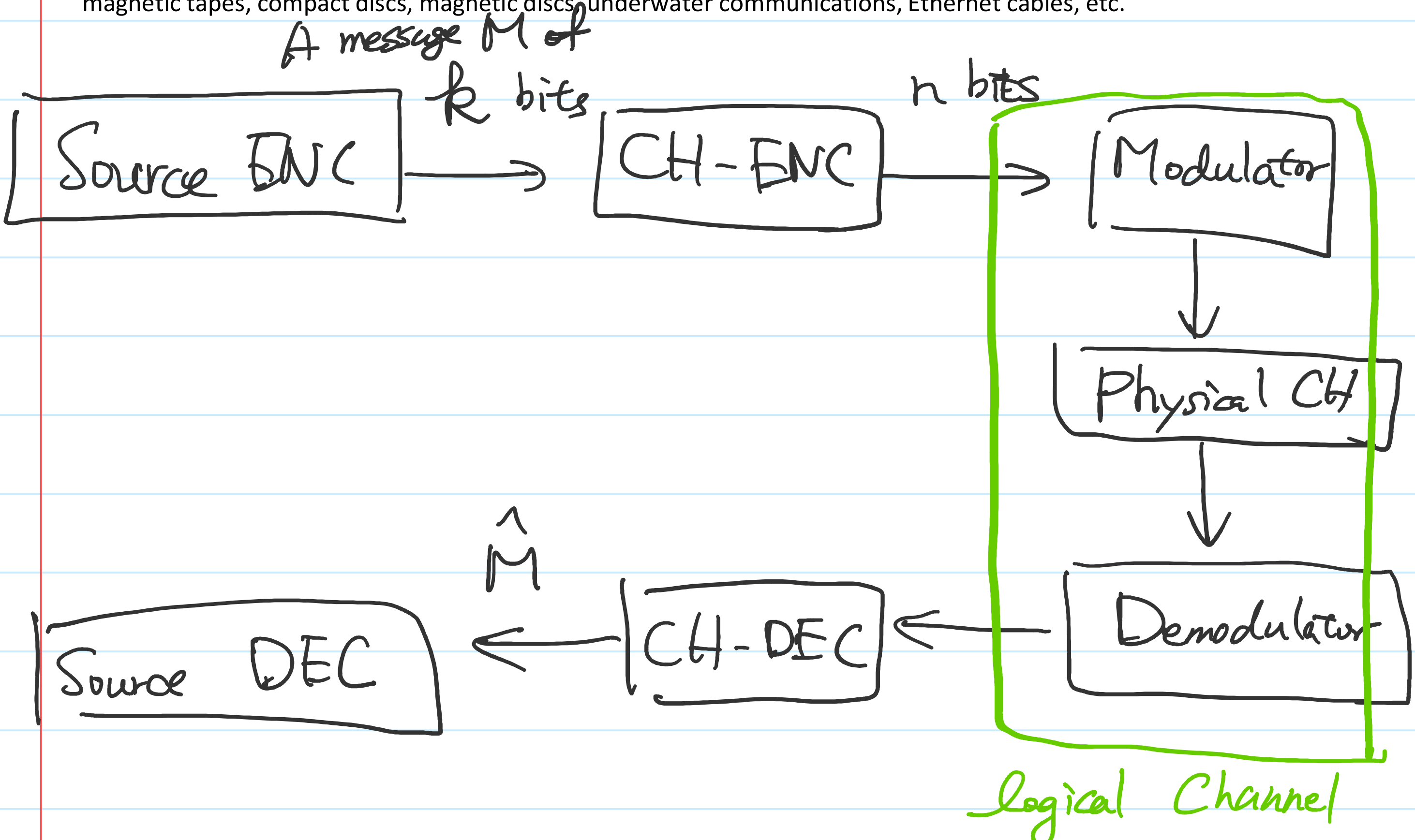
Data Source: Person or machine generating a continuous or discrete signal

Source Encoder: Transforms signal into a sequence of binary digits (bits) and removes uncontrolled redundancy

Channel Encoder: Formats the information to be transmitted so as to increase immunity to noise. This is accomplished by inserting controlled redundancy into data (symbol) stream:

Modulator: Transforms each output symbol of the channel encoder into a waveform of duration T seconds, that is suitable for transmission or recording.

Physical Channel: waveform is corrupted (with noise).
Examples: telephone lines, mobile cellular phones, wireless LAN, telemetry, semiconductor memories, magnetic tapes, compact discs, magnetic discs, underwater communications, Ethernet cables, etc.



Shannon's Channel Coding Theorem:
Every "logical channel" is associated with a channel capacity C. Furthermore, there exist channel codes for every rate k/n < C, such that the k-bit information can be transmitted over n usages of the noisy logical channel, and the channel decoder can successfully "decode" the k-bit message with arbitrarily small error probability.

Brief history:
Shannon's paper was published in 1948 in the Bell System Technical Journal and used the so-called random coding proof techniques. Also at Bell Labs, Richard Hamming was working on design "practical" ways of constructing codes to detect and/or correct channel errors. Hamming published his first results in 1950.

Questions:

* How to formalize the concept of "logical Channel?

* How to design "good" channel code to approach the predicted performance limit C?

Review of prob notation
(Assuming discrete random variables)

Capital $X\ Y\ Z$: Random Variables.

Small x, y, z: deterministic values

$P_X(x) = \text{Prob}(X = x)$ is a number
(i.e. a function w.r.t. x)

$P_X(\cdot): S_X \longmapsto [0,1]$ is a function

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$P_{Y|X}(y|x) = \text{Prob}(Y=y \mid X=x)$ is a number

$P_{Y|X}(\cdot|\cdot): S_Y \times S_X \longmapsto [0,1]$

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Inference Problem:

$$X \sim P_X \longrightarrow \boxed{P_{Y|X}} \xrightarrow{Y} \boxed{\text{Inference}} \xrightarrow{\hat{X}}$$

Deciding X by observing Y

We know ① P_X, ② $P_{Y|X}$, ③ and observe $Y=y$

Example 1:

$$P_X(x) = \begin{cases} 1/2 & \text{if } x=1 \text{ or } x=-1 \\ 0 & \text{otherwise} \end{cases}$$

$Y = X + N$ where N is Gaussian with mean 0 and var. σ^2

$\Rightarrow P_{Y|X} \sim \text{Gsn}(x, \sigma^2)$

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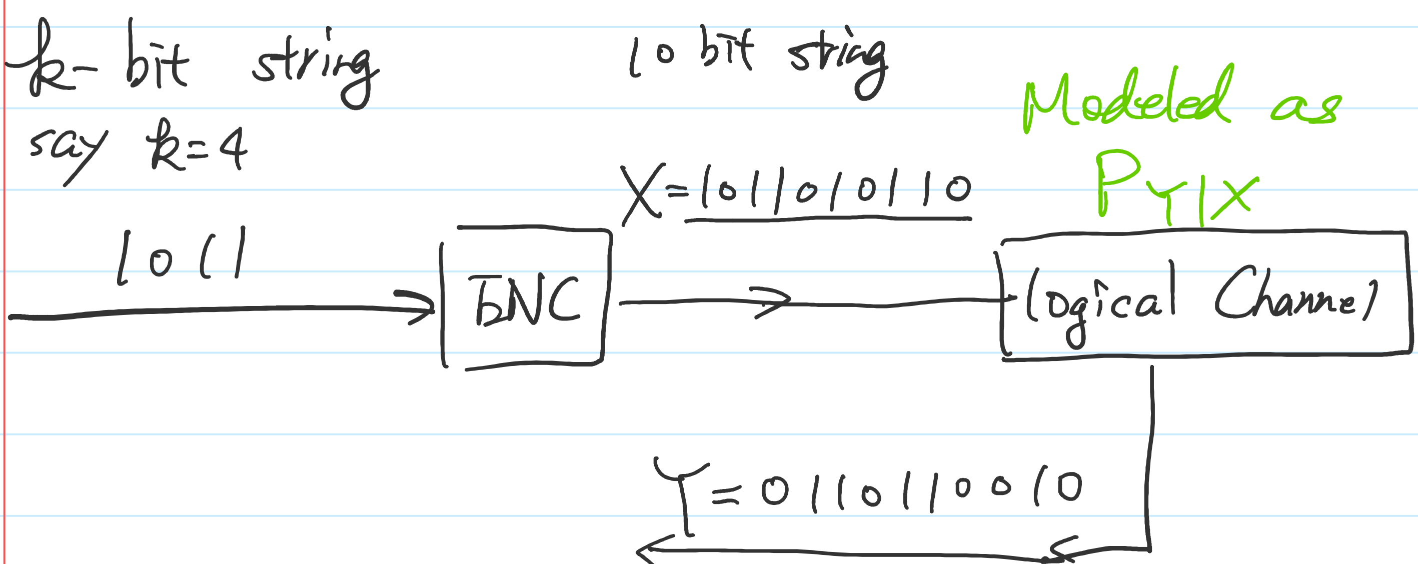
Q: Observing $Y = 2.7$, what is the $\hat{X}$ value?

Example 2:

Error Control Coding is an example of the inference problem:



Here $P_X(x) = \begin{cases} \frac{1}{16} & \text{if } x = x_i \text{ being one of the 16 possible outputs of } \boxed{\text{ENC}} \\ 0 & \text{otherwise} \end{cases}$

Notation: Each of the 16 potential outputs of ENC is called a <u>codeword</u>

Collectively the 16 codewords is called the

ENC is called a codeword

Collectively the 16 codewords is called the codebook.

We know ① P_X (from the codebook), ② $P_{Y|X}$ (from the logical ch.), ③

Question: Observing $Y=$
what is the best guess / inference of X.

It turns out that by casting it as an inference prob, we can explore the fundamental limit of CH coding.

A special case of the inference prob is the hypothesis testing problem

Two hypothesis

H_0: (null hypothesis) $Y \sim P_Y$

H_1: (alternative hypothesis) $Y \sim Q_Y$

$\mathrm{Prob}(H_0 \text{ is true}) = p_0$

$\mathrm{Prob}(H_1 \text{ is true}) = 1 - p_0$

Observe $Y = y$, Find $P(H_0 \text{ is true} \mid Y = y)$

We can convert it to an inference prob.

$X = 0$, or 1

$P_Y : P_{Y|X}(\cdot \mid 0) \qquad Q_Y : P_{Y|X}(\cdot \mid 1)$

$P(H_i \text{ is true}) = P(X = i)$

Optimal Solutions of HT

* Maximum A posteriori Probability (MAP) detector

First define the posterior probability

$$P_{X|Y}(x \mid y) = \mathrm{Prob}(X = x \mid Y = y)$$

The MAP detector finds

$$\hat{X}_{MAP}(y) = \arg\max_{x \in \{0,1\}} P_{X|Y}(x|y)$$

Q: How to compute $\hat{X}_{MAP}(y)$?

A1: Bayes Rule:

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$= \frac{P(X=x) \cdot P(Y=y \mid X=x)}{P(X=0) \cdot P(Y=y \mid X=0) + P(X=1) \cdot P(Y=y \mid X=1)}$$

repeat the above computation for $x=0$ and 1

then choose x^* that maximizes $P_{X|Y}(x|y)$

A2: We are NOT interested in the value of $P_{X|Y}(x|y)$. Instead, we are only interested in

$$P_{X|Y}(0|y) \gtreqless P_{X|Y}(1|y)$$

$$\Leftrightarrow \boxed{P_X(0) P_{Y|X}(y|0) \gtreqless P_X(1) P_{Y|X}(y|1)}$$

since they share the same denominator

A3:
$$\frac{P_X(0)\,P_{Y|X}(y|0)}{P_X(1)\,P_{Y|X}(y|1)} \underset{\text{output 1.}}{\overset{\text{output 0}}{\gtrless}} 1$$

A4:
$$\frac{P_{Y|X}(y|0)}{P_{Y|X}(y|1)} \gtrless \frac{P_X(1)}{P_X(0)}$$

A5:
$$\log\left(\frac{P_{Y|X}(y|0)}{P_{Y|X}(y|1)}\right) \gtrless \log\left(\frac{P_X(1)}{P_X(0)}\right)$$

$\boxed{\text{All 5 methods are used regularly.}}$

We can go a bit deeper

* Define the <u>likelihood</u> function as.

$$P_{Y|X}(y|\cdot)$$

Comparison: The sum of A posteriori Prob function is 1.

function is 1.
But the sum of the likelihood function is not 1.

* Define $L(y) = \frac{P_{Y|X}(y|0)}{P_{Y|X}(y|1)}$ as the likelihood ratio.

* A decision rule

$$L(y) \triangleq \frac{P_{Y|X}(y|0)}{P_{Y|X}(y|1)} \underset{\text{outputs } 1.}{\overset{\text{outputs } 0}{\gtrless}} \eta$$

is thus called a likelihood ratio test with threshold η.

* The $\hat{X}_{MAP}(y)$ in A4 is thus a likelihood ratio test with $\eta = \frac{P_X(1)}{P_X(0)}$

$$\hat{X}_{MAP}(y) = \begin{cases} 0 & \text{if } L(y) > \eta = \frac{P_X(1)}{P_X(0)} \\ 0 \text{ or } 1 & \text{if } L(y) = \eta \\ 1 & \text{if } L(y) < \eta \end{cases}$$

* Another equivalent form:

Define log likelihood ratio (LLR)

as $LLR(y) = \log(L(y))$

$$= \log\left(\frac{P_{Y|X}(y|0)}{P_{Y|X}(y|1)}\right)$$

$$\hat{X}_{MAP}(y) = \begin{cases} 0 & \text{if } LLR(y) > \log\left(\frac{P_X(0)}{P_X(1)}\right) \\ 0 \text{ or } 1 & \text{if } LLR(y) = \\ 1 & \text{if } LLR(y) < \end{cases}$$

Summary.

likelihood ratio test with threshold η

If $\eta = \frac{P_X(0)}{P_X(1)}$ then it becomes the maximum aposterior prob (MAP) detector

$\hat{X}_{MAP}(y)$ Optimal for HT

If $\eta = 1$, then it becomes a Maximum Likelihood (ML) detector

$\hat{X}_{ML}(y)$

not necessarily optimal

* $\hat{X}_{MAP}(y)$ is optimal for HT.

* $\hat{X}_{ML}(y)$ is not necessarily optimal for HT.

* When $P(X=0) = P(X=1) = \frac{1}{2}$

$$\hat{X}_{MAP}(\cdot) \equiv \hat{X}_{ML}(\cdot)$$

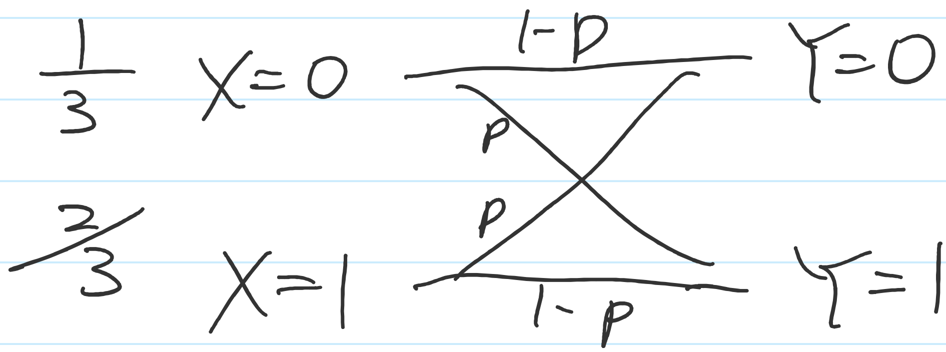
Multiple Hypothesis Testing (Multiple Testing)

$x = 0, 1, 2, \cdots, K-1$

$$\Rightarrow \hat{X}_{MAP}(y) = \underset{x \in \{0,1,\dots,k-1\}}{\operatorname{argmax}} P_{X|Y}(x|y)$$

$$\hat{X}_{ML}(y) = \underset{x \in \{0,1,\dots,k-1\}}{\operatorname{argmax}} P_{Y|X}(y|x)$$

Example: A binary symmetric channel (BSC) w. non-uniform prior distribution



Q: Find the $\hat{X}_{MAP}(y)$

Ans: We need to determine $\hat{X}_{MAP}(y)$ as a function of y

If $y=0$

$$P_{X|Y}(x|0) = \begin{cases} \dfrac{\frac{1}{3}(1-p)}{\frac{2}{3}p + \frac{1}{3}(1-p)} = \dfrac{1-p}{1+p} & \text{if } x=0 \end{cases}$$

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If $y=1$

$$P_{X|Y}(x|1) = \begin{cases} \dfrac{p}{2-p} & \text{if } x=0 \\ \dfrac{2-2p}{2-p} & \text{if } x=1 \end{cases}$$

Our goal: $\hat{X}_{MAP}(y) = \begin{cases} & \text{if } y=0 \\ & \text{if } y=1 \end{cases}$

We have three cases depending on the p value

Case 1: if $p < \frac{1}{3}$

$$\hat{X}_{MAP}(y) = \begin{cases} 0 & \text{if } y=0 \\ 1 & \text{if } y=1 \end{cases}$$

$$\qquad\quad 1 \qquad \text{if} \quad y=1$$

Case 2: If $\frac{1}{3} < p < \frac{2}{3}$

$$\hat{X}_{MAP}(y) = \begin{cases} 1 & \text{if} \quad y=0 \\ 1 & \text{if} \quad y=1 \end{cases}$$

Case 3: If $\frac{2}{3} < p < 1$

$$\hat{X}_{MAP}(y) = \begin{cases} 1 & \text{if} \quad y=0 \\ 0 & \text{if} \quad y=1 \end{cases}$$