ECE695C HW#2 Kuo, Wei-Cheng.

Question 1.

$P_{Y|X}(\cdot|0) \sim N(0, \sigma^2)$

$P_{Y|X}(\cdot|1) \sim N(0, 1+\sigma^2)$

$$D(P_0 \| P_1) = E_0\left(\log \frac{P_0}{P_1}\right)$$

$$= \int_{-\infty}^{\infty} P_0 \log \frac{P_0}{P_1}\, dx$$

It should be (1+\sigma^2)/(\sigma^2)

$$P_0 = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} ; \quad P_1 = \frac{1}{\sqrt{2\pi(1+\sigma^2)}} e^{-\frac{x^2}{2(1+\sigma^2)}}$$

$$\Rightarrow \frac{P_0}{P_1} = \sqrt{\frac{\sigma^2}{1+\sigma^2}} \exp\left(\frac{x^2}{2(1+\sigma^2)} - \frac{x^2}{2\sigma^2}\right) = \sqrt{\frac{\sigma^2}{1+\sigma^2}} \exp\left(\frac{x^2}{2}\left(\frac{\sigma^2 - 1 - \sigma^2}{\sigma^2(1+\sigma^2)}\right)\right) = \sqrt{\frac{\sigma^2}{1+\sigma^2}} \exp\left(-\frac{x^2}{2} \cdot \frac{1}{\sigma^2(1+\sigma^2)}\right)$$

$$\Rightarrow \ln \frac{P_0}{P_1} = \frac{1}{2} \ln\left(\frac{\sigma^2}{1+\sigma^2}\right) - \frac{x^2}{2\sigma^2(1+\sigma^2)}$$

$$\int_{-\infty}^{\infty} P_0 \left(\frac{1}{2}\ln\left(\frac{\sigma^2}{1+\sigma^2}\right) - \frac{x^2}{2\sigma^2(1+\sigma^2)}\right) dx$$

$$= \frac{1}{2}\ln\left(\frac{\sigma^2}{1+\sigma^2}\right) - \frac{1}{2\sigma^2(1+\sigma^2)} \underbrace{\int_{-\infty}^{\infty} x^2 \cdot P_0\, dx}_{\text{the variance of } P_0 = \sigma^2}$$

$$D(P_0 \| P_1) = \boxed{\frac{1}{2}\ln\left(\frac{\sigma^2}{1+\sigma^2}\right) - \frac{1}{2(1+\sigma^2)}}$$

40

Question 2

	X = 0	X = 1
Y = 0	5/36	5/18
Y = 1	5/18	5/36
Y = 2	1/48	1/16
Y = 3	1/16	1/48

1. For MAP, we circle the maximum prob. given Y.

Hence $p_e = \frac{5}{36} + \frac{5}{36} + \frac{1}{48} + \frac{1}{48} = \frac{\cancel{10}^{5}}{\underset{18}{\cancel{36}}} + \frac{\cancel{2}^{1}}{\underset{24}{\cancel{48}}} = \frac{20}{72} + \frac{3}{72} = \frac{23}{72}$.

2.

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$P_{X|Y}(X=0|y=0) = \frac{P_{XY}(X=0, y=0)}{P_Y(y=0)} = \frac{5/36}{5/36 + 5/18} = \frac{5}{5+10} = \frac{1}{3}.$$

$$P_{X|Y}(X=1|y=0) = \frac{5/18}{5/36 + 5/18} = \frac{2}{3}.$$

$$P_{X|Y}(X=0|y=1) = \frac{5/18}{5/18 + 5/36} = \frac{2}{3}$$

$$P_{X|Y}(X=1|y=1) = \frac{1}{3}.$$

$$P_{X|Y}(X=0|y=2) = \frac{1/48}{1/48 + 1/16} = \frac{1}{1+3} = \frac{1}{4}$$

$$P_{X|Y}(X=1|y=2) = \frac{3}{4}$$

$$P_{X|Y}(X=0|y=3) = \frac{3}{4}$$

$$P_{X|Y}(X=1|y=3) = \frac{1}{4}.$$

$$\Rightarrow f(0,0) = \sqrt{\frac{P_{X|Y}(1|0)}{P_{X|Y}(0|0)}} = \sqrt{\frac{2/3}{1/3}} = \sqrt{2}. \quad ; \quad f(1,0) = \sqrt{\frac{P_{X|Y}(0|0)}{P_{X|Y}(1|0)}} = \sqrt{\frac{1/3}{2/3}} = \sqrt{\frac{1}{2}}.$$

$$f(0,1) = \sqrt{\frac{P_{X|Y}(1|1)}{P_{X|Y}(0|1)}} = \sqrt{\frac{1/3}{2/3}} = \sqrt{\frac{1}{2}} \quad ; \quad f(1,1) = \sqrt{\frac{P_{X|Y}(0|1)}{P_{X|Y}(1|1)}} = \sqrt{\frac{2/3}{1/3}} = \sqrt{2}.$$

$$f(0,2) = \sqrt{\frac{P_{X|Y}(1|2)}{P_{X|Y}(0|2)}} = \sqrt{\frac{3/4}{1/4}} = \sqrt{3} \quad ; \quad f(\overset{1}{\cancel{0}},2) = \sqrt{\frac{P_{X|Y}(0|2)}{P_{X|Y}(1|2)}} = \sqrt{\frac{1/4}{3/4}} = \sqrt{\frac{1}{3}}.$$

$$f(0,3) = \sqrt{\frac{P_{X|Y}(1|3)}{P_{X|Y}(0|3)}} = \sqrt{\frac{1/4}{3/4}} = \sqrt{\frac{1}{3}} \quad ; \quad f(1,3) = \sqrt{\frac{P_{X|Y}(0|3)}{P_{X|Y}(1|3)}} = \sqrt{\frac{3/4}{1/4}} = \sqrt{3}.$$

$$b = E(f(x,y)) = \frac{5}{36}\cdot\sqrt{2} + \frac{5}{18}\cdot\sqrt{\frac{1}{2}} + \frac{5}{18}\cdot\frac{\sqrt{2}}{2} + \frac{5}{36}\cdot\sqrt{2} + \frac{1}{48}\cdot\sqrt{3} + \frac{1}{16}\cdot\frac{\sqrt{3}}{3} + \frac{1}{16}\cdot\frac{\sqrt{3}}{3} + \frac{1}{48}\cdot\sqrt{3}$$

$$= \frac{5}{\underset{9}{\cancel{36}}}\cdot\cancel{4}\cdot\sqrt{2} + \frac{1}{\underset{12}{\cancel{48}}}\cdot\cancel{4}\cdot\sqrt{3} = \frac{5}{9}\sqrt{2} + \frac{1}{12}\sqrt{3}.$$

3. We first calculate the Chernoff bound for single bit.

$$\hat{X}_{MAP}(y) = \begin{cases} 0 & \text{if } y = 1, 3 \\ 1 & \text{if } y = 2, 4. \end{cases}$$

Note that $p_X(0) = p_X(1) = \frac{1}{2}$. by summing up the column terms.

It is equivalent to

$$\log \frac{P_{Y|X}(y|X=0)}{P_{Y|X}(y|X=1)} \geq 0 \quad \text{then decide } \hat{X}_{MAP}(y) = 0.$$

$$\text{else decide } \hat{X}_{MAP}(y) = 1.$$

The error happens when

$$P\Big(\underbrace{\log \frac{P_{Y|X}(y|X=0)}{P_{Y|X}(y|X=1)}}_{\triangleq Z} \geq 0 \;\Big|\; X=1\Big) \leq \frac{E(e^{sZ})}{e^{sd}}\Big|_{d=0} = E(e^{sZ}).$$

Note Z is a function of Y.

$y=0$: $Z = \log_2 \frac{5/36 \times 2}{5/18 \times 2} = \log_e \frac{1}{2} = -\log_e 2$ (with prob. $\frac{5}{36} + \frac{5}{18} = \frac{15}{36} = \frac{5}{12}$), $P(Z|X=1) = 10/18$

$y=1$: $Z = \log_2 \frac{5/18 \times 2}{5/36 \times 2} = \log_e 2$ (with prob. $\frac{5}{18} + \frac{5}{36} = \frac{5}{12}$), $P(Z|X=1) = 5/18$

$y=2$: $Z = \log_2 \frac{1/48 \times 2}{1/16 \times 2} = \log_e \frac{1}{3} = -\log_e 3$ (with prob. $\frac{1}{48} + \frac{1}{16} = \frac{4}{48} = \frac{1}{12}$), $P(Z|X=1) = 1/8$

$y=3$: $Z = \log_2 \frac{1/16 \times 2}{1/48 \times 2} = \log_e 3$ (with prob $\frac{1}{12}$). $P(Z|X=1) = 1/24$

Hence $E(e^{sZ}) =$

$$\frac{5}{12} e^{-s\ln 2} + \frac{5}{12} e^{s\ln 2} + \frac{1}{12} e^{s\ln 3} + \frac{1}{12} e^{s\ln 3}$$

$$\Rightarrow E(e^{sZ}) = \left(\frac{10}{18}\cdot\left(\frac{1}{2}\right)^s + \frac{5}{18}\cdot 2^s + \frac{1}{8}\left(\frac{1}{3}\right)^s + \frac{1}{24}\cdot 3^s\right) \quad (\text{for } s \geq 0)$$

$$\frac{d E(e^{sZ})}{ds} = \frac{10}{18}\left(\frac{1}{2}\right)^s \log\left(\frac{1}{2}\right) + \frac{5}{18} 2^s \log 2 + \frac{1}{8}\left(\frac{1}{3}\right)^s \log\left(\frac{1}{3}\right) + \frac{1}{24} 3^s \log 3 = 0.$$

$$\Rightarrow \frac{5}{18} \log 2 \left(2^s - 2\left(\frac{1}{2}\right)^s\right) + \frac{1}{24} \log 3 \left(3^s - 3\left(\frac{1}{3}\right)^s\right) = 0$$

$$\Rightarrow 2^s = 2\cdot\left(\frac{1}{2}\right)^s \Rightarrow 2^s = 2^{-s+1} \Rightarrow s = \frac{1}{2} \quad ; \quad 3^s = 3\left(\frac{1}{3}\right)^s \Rightarrow s = \frac{1}{2}.$$

Hence $E(e^{sZ})\Big|_{s=\frac{1}{2}} = \frac{10}{18}\cdot\frac{\sqrt{2}}{2} + \frac{5}{18}\sqrt{2} + \frac{1}{8}\cdot\frac{\sqrt{3}}{3} + \frac{1}{24}\sqrt{3} = \frac{5}{9}\sqrt{2} + \frac{1}{12}\sqrt{3}$

$\Rightarrow$ prob of error $\leq \frac{5}{9}\sqrt{2} + \frac{\sqrt{3}}{12} = b.$

Question 3.

For a random variable X with pmf $p_X(x)$, let $u(x) = \frac{1}{|\mathcal{X}|}$ be the uniform probability mass function over $\mathcal{X}$.

Then

$$D(p_X(x) \| u(x)) = \sum p_X(x) \log \frac{p_X(x)}{u(x)} = \log |\mathcal{X}| - H(X).$$

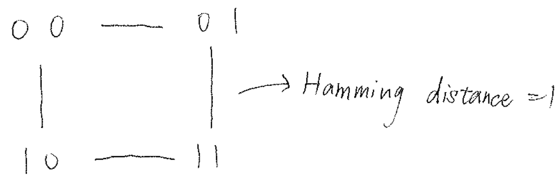
$$\Rightarrow \quad H(X) = \log |\mathcal{X}| - D(p_X(x) \| u(x))$$

Hence $H(X)$ is a concave function followed from the convexity of the divergence. ■

✓

Question 4

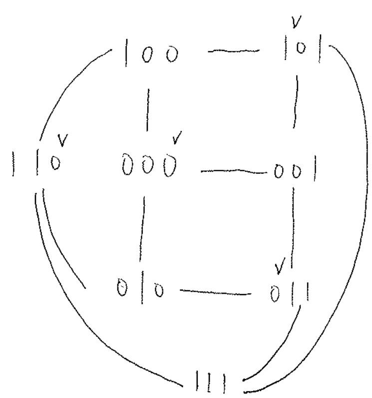
1. $n=2$, $\varepsilon = 1/2$.



We can choose $\{00, 11\}$.

$$\text{Rate} = \frac{\log_2 2}{2} = \frac{1}{2}$$

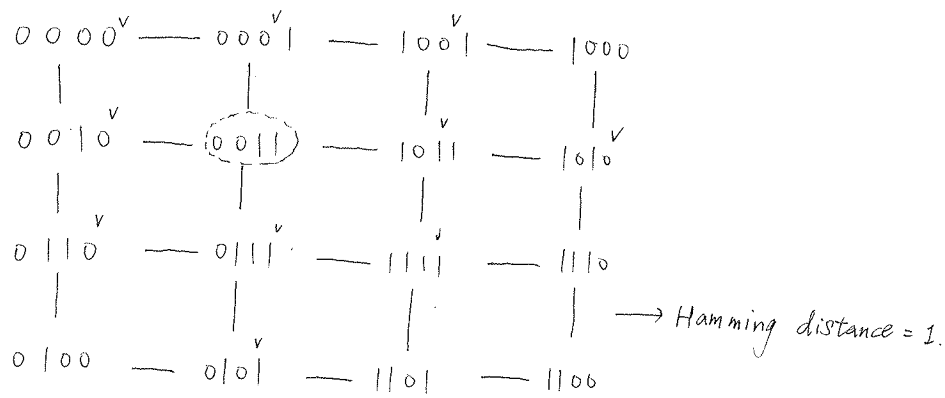
2. $n=3$, $\varepsilon = \frac{1}{3}$.



We can choose $\{000, 011, 101, 110\}$.

$$\text{Rate} = \frac{\log_2 4}{3} = \frac{2}{3}.$$

3. $n = 4$, $\varepsilon = \frac{1}{2}$ (possible erasure # = 2)



Note that this is a cyclic table.

We can see that after we first pick a codeword, it already occupies 11 slots. Any of the remaining slots will use up all the vacancy.

Choose = {0011, 1100}

$$\text{rate} = \frac{\log_2 2}{4} = \frac{1}{4}.$$