**ECE 639, Homework #2 (CRN: 25576)**
**Due date: Wednesday 9/21/2022 during the lecture**

`https://engineering.purdue.edu/~chihw/22ECE639F/22F_ECE639.html`

*Question 6:* Consider the following inference problem:

$$P_X(0) = 1/3, P_X(1) = 2/3,$$
$$P_{Y|X}(\cdot|x) \sim Gsn(0, x + \sigma^2).$$

Find the divergence $D(P_0||P_1)$ between $P_0 = P_{Y|X}(\cdot|0)$ and $P_1 = P_{Y|X}(\cdot|1)$.

*Question 7:* Consider the following inference problem specified by the joint probability mass function of $X$ and $Y$:

| | $X = 0$ | $X = 1$ |
|---|---|---|
| $Y = 0$ | $\frac{5}{36}$ | $\frac{5}{18}$ |
| $Y = 1$ | $\frac{5}{18}$ | $\frac{5}{36}$ |
| $Y = 2$ | $\frac{1}{48}$ | $\frac{1}{16}$ |
| $Y = 3$ | $\frac{1}{16}$ | $\frac{1}{48}$ |

1. What is the corresponding $p_e$ error probability of a MAP detector?

2. If we define $b = E(\sqrt{\frac{P_{X|Y}((1-X)|Y)}{P_{X|Y}(X|Y)}})$, find out the $b$ value for this problem. Hint: you should first decide the function $f(x, y) = \sqrt{\frac{P_{X|Y}(1-x|y)}{P_{X|Y}(x|y)}}$, and then evaluate the $E(f(X, Y))$. In communications, this $b$ is termed the Bhattacharyya noise parameter for a given channel.

3. Consider the following hypothesis testing problem. $H_0$: A sequence of $n$ 1s $(111 \cdots, 1)$ is transmitted over the above binary-input/quaternary-output channel. $H_1$: A sequence of $n$ 0s $(000 \cdots, 0)$ is transmitted over he above binary-input/ternary-output channel and $Y_1$ to $Y_n$ are the corresponding output.

   Use the Chernoff bound to bound the average error probability of a MAP detector. Verify that the Chernoff bound is indeed the Bhattacharyya noise parameter.

4. [Optional] You might notice that $2p_e \leq b$. Prove that for any given table,

   | | $X = 0$ | $X = 1$ |
   |---|---|---|
   | $Y = 0$ | $p_{0,1}$ | $p_{1,1}$ |
   | $Y = 1$ | $p_{0,2}$ | $p_{1,2}$ |
   | $Y = 2$ | $p_{0,3}$ | $p_{1,3}$ |

   $2p_e \leq b$ always holds.

5. [Optional] You might notice that $b \leq 2\sqrt{p_e(1-p_e)}$. Prove that for any given table,

| | $X=0$ | $X=1$ |
|---|---|---|
| $Y=0$ | $p_{0,0}$ | $p_{1,0}$ |
| $Y=1$ | $p_{0,1}$ | $p_{1,1}$ |
| $Y=2$ | $p_{0,2}$ | $p_{1,2}$ |

$b \leq 2\sqrt{p_e(1-p_e)}$ always holds. Hint: Note that $f(p) = \sqrt{p(1-p)}$ is a concave function with respect to $p$.

*Question 8:* Show that entropy $H(X) = E_X\left\{\log(\frac{1}{P_X(X)})\right\}$ is a concave function. Hint: You can use the convexity of the divergence.

*Question 9:* Consider a special binary erasure channel as follows. Given a bit string of length $n$, at most $\epsilon n$ bits will be completely erased. For example: for $n=6$, $\epsilon=1/3$, if one sends 6 bits 010010, then at most 2 bits are erased. One possible received bit string is 01*0*0.

If we are allowed to use the channel exactly $n$ times, design three binary codes that can achieve *error-free* transmission for the following three difference scenarios, respectively.

1. $n=2$, $\epsilon=1/2$.
2. $n=3$, $\epsilon=1/3$.
3. $n=4$, $\epsilon=1/2$.

In your schemes, how many codewords can you pack into the space. The code rate of yours is then defined as

$$\frac{\log_2(\text{the number of codewords})}{n}. \tag{1}$$

Hint: the $X$ vector takes values in $\{0,1\}^n$, while the observation $Y$ vector takes values in $\{0,1,*\}^n$.