${\bf Homework~1}$ ECE695C Inference Method for Codes on Graphs

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1. Find the maximum a posteriori probability (MAP) detector as a function of y.

To fine the MAP detector of $\hat{X}_{MAP}(y) = \arg\max_{x} p(x|y)$, the MAP decision rule is required as below:

$$\frac{p(x=0|y)}{p(x=1|y)} \stackrel{\geq}{\geq} 1$$
(Likelihood-ratio)
$$\frac{p(y|x=0)}{p(y|x=1)} \stackrel{\geq}{\geq} \frac{p(x=1)}{p(x=0)}$$

$$\frac{\exp\{-|y-1|\}}{\exp\{-|y+1|\}} \stackrel{\geq}{\geq} 2$$

$$-|y-1|+|y+1| \stackrel{\geq}{\geq} \ln 2$$

Taking the prior probability, we have the Bayes rule as

i) for
$$y < -1$$

 $\hat{X}_{MAP}(y) = 1$,

$$\begin{split} \text{ii) for } -1 &\leq y < 1 \\ \hat{X}_{\text{MAP}}(y) &= \left\{ \begin{array}{ll} 0 & \text{if } y \geq \frac{\ln 2}{2} \\ 1 & \text{if } y < \frac{\ln 2}{2} \end{array} \right., \quad \hat{X}_{\text{MAP}}(y) = 0. \end{split}$$

In short, an equivalent Bayes test for this case is given by

i) for
$$y < \frac{\ln 2}{2}$$

 $\hat{X}_{MAP}(y) = 1$,

ii) for
$$\frac{\ln 2}{2} \le y$$

 $\hat{X}_{\text{MAP}}(y) = 0$.

2. Find the maximum likelihood (ML) detector as a function of y.

To fine the ML detector of $\hat{X}_{\text{ML}}(y) = \arg\max_{x} p(y|x)$, the likelihood-ratio is thus given by

$$\frac{p(y|x=0)}{p(y|x=1)} \stackrel{\geq}{\geq} 1$$

$$\frac{\exp\{-|y-1|\}}{\exp\{-|y+1|\}} \stackrel{\geq}{\geq} 1$$

$$-|y-1|+|y+1| \stackrel{\geq}{\geq} 0.$$

The boundary points where the sign of y change are ± 1 , then we have the following:

i) for
$$y < -1$$

 $\hat{X}_{ML}(y) = 1$,

$$\begin{split} &\text{ii) for } -1 \leq y < 1 \\ &\hat{X}_{\text{ML}}(y) = \left\{ \begin{array}{ll} 0 & \text{if } y \geq 0 \\ 1 & \text{if } y < 0 \end{array} \right., \end{aligned}$$

iii) for
$$1 \leq y$$
 . $\hat{X}_{\mathrm{ML}}(y) = 0$.

Simply, a received signal y is interpreted as 1 when y < 0 and vice versa as

i) for
$$y < 0$$

 $\hat{X}_{ML}(y) = 1$,

ii) for
$$0 < y$$

 $\hat{X}_{ML}(y) = 0$,

iii) for
$$y = 0$$

 $\hat{x}_{\text{ML}}(\bar{Y}) = 0 \text{ or } 1.$

3. Are these two detectors the same?

Not the same. The threshold to detect a received symbols is different: $\tau = \ln 2$ for MAP and $\tau = 0$ for ML.

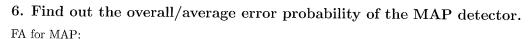
4. Find out the mis-detection probability of the MAP detector.

MD for MAP:

$$p(f(Y) = 0|X = 1) = \int_{\frac{\ln 2}{2}}^{\infty} \frac{1}{2} e^{-|y+1|} dy$$
$$= \frac{1}{2} e^{-(\frac{\ln 2}{2} + 1)}$$

5. Find out the false-alarm probability of the following naive detector: FA for NAIVE:

$$p(f(Y) = 1|X = 0) = \int_{-\infty}^{\frac{1}{4}} \frac{1}{2} e^{-|y-1|} dy$$
$$= \frac{1}{2} e^{(\frac{1}{4} - 1)} = \frac{1}{2} e^{-\frac{3}{4}}$$



$$p(f(Y) = 1|X = 0) = \int_{-\infty}^{\frac{\ln 2}{2}} \frac{1}{2} e^{-|y-1|} dy$$
$$= \frac{1}{2} e^{(\frac{\ln 2}{2} - 1)}$$

Average error probability for MAP:

$$\begin{split} p(f(Y) \neq X) &= p(f(Y) = 1 | X = 0) p(X = 0) + p(f(Y) = 0 | X = 1) p(X = 1) \\ &= \frac{1}{2} e^{\left(\frac{\ln 2}{2} - 1\right)} \times \frac{1}{3} + \frac{1}{2} e^{-\left(\frac{\ln 2}{2} + 1\right)} \times \frac{2}{3} \\ &= \frac{1}{6} e^{\left(\frac{\ln 2}{2} - 1\right)} + \frac{1}{3} e^{-\left(\frac{\ln 2}{2} + 1\right)} \end{split}$$

7. Find out the overall/average error probability of the ML detector. FA for ML:

$$p(f(Y) = 1|X = 0) = \int_{-\infty}^{0} \frac{1}{2} e^{-|y-1|} dy$$
$$= \frac{1}{2} e^{-1}$$

MD for ML:

$$p(f(Y) = 0|X = 1) = \int_0^\infty \frac{1}{2} e^{-|y+1|} dy$$
$$= \frac{1}{2} e^{-1}$$

Average error probability for ML:

$$p(f(Y) \neq X) = p(f(Y) = 1|X = 0)p(X = 0) + p(f(Y) = 0|X = 1)p(X = 1)$$

$$= \frac{1}{2}e^{-1} \times \frac{1}{3} + \frac{1}{2}e^{-1} \times \frac{2}{3}$$

$$= \frac{1}{2}e^{-1}$$

1. Find the maximum a posteriori probability (MAP) detector as a function of y.

To fine the MAP detector of $\hat{X}_{MAP}(y) = \arg\max_{x} p(x|y)$, the MAP decision rule is required as below:

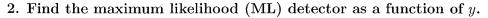
$$\frac{p(x=0|y)}{p(x=1|y)} \ \stackrel{\geq}{\geq} \ 1$$

$$\frac{p(y|x=0)}{p(y|x=1)} \ \stackrel{\geq}{\geq} \ \frac{p(x=1)}{p(x=0)} = 2.$$

Taking the prior probability, we have the Bayes rule as



ii) Otherwise
$$\hat{X}_{MAP}(y) = 1$$
.



To fine the ML detector of $\hat{X}_{\text{ML}}(y) = \arg\max_{x} p(y|x)$, the likelihood-ratio is thus given by

$$\frac{p(y|x=0)}{p(y|x=1)} \stackrel{\geq}{=} 1$$

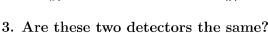
$$\frac{2(y+1)}{3} \stackrel{\geq}{=} 1, \quad \text{if } 0 \le y \le 1.$$

i) for
$$0 \le y \le 1$$
 ii) Otherwise
$$\hat{X}_{\text{MAP}}(y) = \left\{ \begin{array}{ll} 0 & \text{if } y \ge \frac{1}{2} \\ 1 & \text{if } y < \frac{1}{2} \end{array} \right. \qquad \hat{X}_{\text{MAP}}(y) = 1.$$

$$\hat{X}_{\mathrm{MAP}}(y) = 1.$$

i) for
$$\frac{1}{2} \le y \le 1$$

 $\hat{X}_{MAP}(y) = 0$.



Not the same. The MAP detector always says $\hat{X} = 1$; however the ML detector has a threshold, $\tau = 0$.

4. Find out the mis-detection probability of the MAP detector.

MD for MAP:

$$p(f(Y) = 0|X = 1) = \int_{y \notin [0,1]} p_1(y)dy = 0.$$

5. Find out the false-alarm probability of the following naive detector: FA for NAIVE:

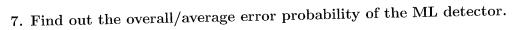
$$p(f(Y) = 1|X = 0) = \int_0^{\frac{1}{4}} p_0(y)dy$$
$$= \int_0^{\frac{1}{4}} \frac{2}{3}(y+1)dy = \frac{3}{16}$$

6. Find out the overall/average error probability of the MAP detector. FA for MAP:

$$p(f(Y) = 1|X = 0) = \int_0^1 p_0(y)dy$$
$$= \int_0^1 \frac{2}{3}(y+1)dy = 1$$

Average error probability:

$$p(f(Y) \neq X) = p(f(Y) = 1 | X = 0)p(X = 0) + p(f(Y) = 0 | X = 1)p(X = 1)$$
$$= 1 \times \frac{1}{3} + 0 \times \frac{2}{3} = \frac{1}{3}$$



FA for ML:

$$p(f(Y) = 1|X = 0) = \int_0^{\frac{1}{2}} p_0(y)dy$$
$$= \int_0^{\frac{1}{2}} \frac{2}{3}(y+1)dy = \frac{5}{12}$$

MD for ML:

$$p(f(Y) = 0|X = 1) = \int_{\frac{1}{2}}^{1} p_1(y)dy$$
$$= \int_{\frac{1}{2}}^{1} 1dy = \frac{1}{2}$$

Average error probability for ML:

$$p(f(Y) \neq X) = p(f(Y) = 1|X = 0)p(X = 0) + p(f(Y) = 0|X = 1)p(X = 1)$$

$$= \frac{5}{12} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3}$$

$$= \frac{17}{36}$$



1. Model this CDMA system as a hypothesis testing problem.

Consider the following two hypotheses concerning a real-valued observation $\mathbf{y} := [Y_1, \dots, Y_{10}]^T$:

$$\mathcal{H}_0: \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_0, \sigma^2 \mathbf{I})$$
 versus $\mathcal{H}_1: \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_1, \sigma^2 \mathbf{I})$

where $\mu_0 = [-1 \ 1 \cdots - 1 \ 1]^T$, $\mu_1 = [1 \ 1 \cdots \ 1]^T$.

2. Find the MAP detector and express the MAP detector in the form of log-likelihoodratio test.

$$\hat{x}_{\text{MAP}}(\bar{Y}) = \arg\max_{x} p(x|\bar{Y})$$

$$\frac{p(\operatorname{bit} = 0|\mathbf{y})}{p(\operatorname{bit} = 1|\mathbf{y})} \stackrel{\geq}{\geq} 1$$

$$\frac{p(\mathbf{y}|\operatorname{bit} = 0)}{p(\mathbf{y}|\operatorname{bit} = 1)} \stackrel{\geq}{\geq} \frac{p(\operatorname{bit} = 1)}{p(\operatorname{bit} = 0)} = 1$$

$$\frac{\exp\left\{-\frac{1}{2\sigma^2}(\mathbf{y} - \boldsymbol{\mu}_0)^T(\mathbf{y} - \boldsymbol{\mu}_0)\right\}}{\exp\left\{-\frac{1}{2\sigma^2}(\mathbf{y} - \boldsymbol{\mu}_1)^T(\mathbf{y} - \boldsymbol{\mu}_1)\right\}} \stackrel{\geq}{\geq} 1$$

$$-\frac{1}{2\sigma^2}\left\{(\mathbf{y} - \boldsymbol{\mu}_0)^T(\mathbf{y} - \boldsymbol{\mu}_0) - (\mathbf{y} - \boldsymbol{\mu}_1)^T(\mathbf{y} - \boldsymbol{\mu}_1)\right\} \stackrel{\geq}{\geq} 0$$

$$\underbrace{(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T}_{=:\boldsymbol{\mu}^T} \mathbf{y} \stackrel{\leq}{\leq} \frac{\boldsymbol{\mu}_1^T \boldsymbol{\mu}_1 - \boldsymbol{\mu}_0^T \boldsymbol{\mu}_0}{2}$$

$$\boldsymbol{\mu}^T \mathbf{y} \stackrel{\leq}{\leq} 0$$

$$\bar{Y} := Y_1 + Y_3 + Y_5 + Y_7 + Y_9 \stackrel{\leq}{\leq} 0$$

i) for
$$\bar{Y} < 0$$

 $\hat{\text{bit}}_{MAP}(\bar{Y}) = 0$,

ii) for
$$0 < \bar{Y}$$

 $\hat{\text{bit}}_{MAP}(\bar{Y}) = 1$,

iii) for
$$\bar{Y} = 0$$

 $\hat{\text{bit}}_{\text{MAP}}(\bar{Y}) = 0 \text{ or } 1.$



3. Find the average error probability.

Since \bar{Y} has probability density function (pdf) $\mathcal{N}(-5, 5\sigma^2)$ when \mathcal{H}_0 is true, and $\mathcal{N}(5, 5\sigma^2)$ when \mathcal{H}_1 is true, then

FA for MAP:

$$p(f(\bar{Y}) = 1|\text{bit} = 0) = \int_0^\infty p_0(\bar{Y})d\bar{y}$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right)$$



where $Q(x):=\frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\frac{u^2}{2}\right)du.$ MD for MAP:

$$p(f(\bar{Y}) = 0|X = 1) = \int_{-\infty}^{0} p_1(\bar{Y})d\bar{y}$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right)$$

Average error probability for MAP:

$$p(f(\bar{Y}) \neq X) = p(f(\bar{Y}) = 1|X = 0)p(X = 0) + p(f(\bar{Y}) = 0|X = 1)p(X = 1)$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2} + Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2}$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right)$$



4. Use the Chernoff bound to derive/approximate the average error probability for general N-bit signature sequence.

When
$$\bar{Y} := Y_1 + Y_3 + Y_5 + Y_7 + Y_9$$
, FA:

$$\begin{aligned} p_0 \left(\bar{Y} > 0 \right) &\lessapprox \min_{s \geq 0} \frac{\mathbb{E} \left\{ e^{s\bar{Y}} \right\}}{e^{s \cdot 0}} \\ &= \left(\min_{s \geq 0} \mathbb{E} \left\{ e^{sY} \right\} \right)^5 \\ &= e^{-\frac{5}{2\sigma^2}}, \end{aligned}$$

where

$$\begin{split} \mathbb{E}\left\{e^{sY}\right\} &= e^{-s + \frac{\sigma^2}{2}s^2} \\ \frac{\partial \mathbb{E}\left\{e^{sY}\right\}}{\partial s} &= (-1 + \sigma^2 s)e^{-s + \frac{\sigma^2}{2}s^2} = 0 \ \Rightarrow \ s^* = \frac{1}{\sigma^2}. \end{split}$$

MD:

$$p_1(\bar{Y} < 0) = p_1(-\bar{Y} > 0) \lessapprox \min_{s \ge 0} \frac{\mathbb{E}\left\{e^{-s\bar{Y}}\right\}}{e^{s \cdot 0}}$$
$$= \left(\min_{s \ge 0} \mathbb{E}\left\{e^{-sY}\right\}\right)^5$$
$$= e^{-\frac{5}{2\sigma^2}},$$



where

$$\mathbb{E}\left\{e^{-sY}\right\} = e^{-s + \frac{\sigma^2}{2}s^2}$$

$$\frac{\partial \mathbb{E}\left\{e^{sY}\right\}}{\partial s} = (-1 + \sigma^2 s)e^{-s + \frac{\sigma^2}{2}s^2} = 0 \implies s^* = \frac{1}{\sigma^2}.$$

Average error probability:

$$p(f(\bar{Y}) \neq X) = p(f(\bar{Y}) = 1|X = 0)p(X = 0) + p(f(\bar{Y}) = 0|X = 1)p(X = 1)$$
$$= e^{-\frac{5}{2\sigma^2}} \times \frac{1}{2} + e^{-\frac{5}{2\sigma^2}} \times \frac{1}{2}$$
$$= e^{-\frac{5}{2\sigma^2}}$$

5. [Optional for those who has learned CDMA/digital communication before.] Discuss its relationship to the matched filter in digital communication. Is the matched filter optimal?

Matched filter

$$\begin{split} \mathbf{u}_0 &:= [-1 \ 1 - 1 \cdots 1]; \\ \mathbf{u}_1 &:= [\ 1 \ 1 \ 1 \cdots 1]; \\ \mathbf{\bar{u}} &:= \mathbf{u}_0 - \mathbf{u}_1 = [-2 \ 0 - 2 \cdots 0]; \end{split}$$

Front-end processing

$$\bar{Y} := \sum_{i=1}^{N} \bar{\mathbf{u}}(i) Y_i$$

Hypothesis testing

$$H_0: \bar{Y} \sim \mathcal{N}(10, 20\sigma^2)$$
 versus $H_1: \bar{Y} \sim \mathcal{N}(-10, 20\sigma^2)$

FA:

$$p(f(Y) = 1|X = 0) = \int_{-\infty}^{0} p_0(y)dy$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right)$$

MD:

$$p(f(Y) = 0|X = 1) = \int_0^\infty p_1(y)dy$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right)$$

Average error probability:

$$p(f(Y) \neq X) = p(f(Y) = 1|X = 0)p(X = 0) + p(f(Y) = 0|X = 1)p(X = 1)$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2} + Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2}$$
$$= Q\left(\frac{\sqrt{5}}{\sigma}\right)$$

The matched filter is optimal because the performance of matched filter is the same to that of MAP in white noise.



1. Model this CDMA system as a hypothesis testing problem.

Consider the following two hypotheses concerning a real-valued observation $\mathbf{y} := [Y_1, Y_2]^T$:

$$\mathcal{H}_0: \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})$$
 versus $\mathcal{H}_1: \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$

where
$$\boldsymbol{\mu}_0 = [1, 0.5]^T$$
, $\boldsymbol{\mu}_1 = [-1, -0.5]^T$, and $\boldsymbol{\Sigma} = \sigma^2 \left[\begin{array}{cc} 1 & 0.5 \\ 0.5 & 1 \end{array} \right]$.

2. Find the MAP detector and express the MAP detector in the form of log-likelihoodratio test.

 $\hat{x}_{\text{MAP}}(\mathbf{y}) = \arg\max_{x} p(x|\mathbf{y})$

$$\frac{p(\text{bit} = 0|\mathbf{y})}{p(\text{bit} = 1|\mathbf{y})} \stackrel{\geq}{\geq} 1$$

$$\frac{p(\mathbf{y}|\text{bit} = 0)}{p(\mathbf{y}|\text{bit} = 1)} \stackrel{\geq}{\geq} \frac{p(\text{bit} = 1)}{p(\text{bit} = 0)} = 1$$

$$\frac{\exp\left\{-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu}_0)\right\}}{\exp\left\{-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_1)^T \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu}_1)\right\}} \stackrel{\geq}{\geq} 1$$

$$-\frac{1}{2}\left\{(\mathbf{y} - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu}_0) - (\mathbf{y} - \boldsymbol{\mu}_1)^T \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu}_1)\right\} \stackrel{\geq}{\geq} 0$$

$$\underbrace{(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T}_{=:\boldsymbol{\mu}^T} \boldsymbol{\Sigma}^{-1} \mathbf{y} \stackrel{\leq}{\leq} \frac{\boldsymbol{\mu}_1^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_1 - \boldsymbol{\mu}_0^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_0}{2}$$

$$\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{y} \stackrel{\leq}{\leq} 0$$

$$-\frac{2}{\sigma^2} Y_1 \stackrel{\leq}{\leq} 0$$

$$Y_1 \stackrel{\geq}{\geq} 0$$

where $\mu^T \Sigma^{-1} = \frac{1}{\sigma^2} [-2 \ 0].$

i) for
$$Y_1 < 0$$

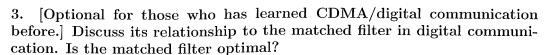
 $\hat{\text{bit}}_{MAP}(Y_1) = 1$,

ii) for
$$0 < Y_1$$

bît_{MAP} $(Y_1) = 0$,

iii) for
$$Y_1 = 0$$

 $\hat{\text{bit}}_{\text{MAP}}(Y_1) = 0 \text{ or } 1.$



Matrix decomposition:

$$\begin{split} \boldsymbol{\Sigma} &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{2} & 0 \\ 0 & \frac{3\sigma^2}{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\ \boldsymbol{\Sigma}^{-1} &= \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^T = \mathbf{D}^T\mathbf{D} \end{split}$$

where $\mathbf{D} = \sqrt{\mathbf{\Lambda}^{-1}} \mathbf{U}^T$. Front-end processing:

$$\tilde{\mathbf{y}} := \mathbf{D}\mathbf{y}$$
 (whitening noise)
 $\tilde{\boldsymbol{\mu}} := \mathbf{D}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1) = \mathbf{D}\boldsymbol{\mu}.$

The test statistic $T(\mathbf{y}) = \tilde{\boldsymbol{\mu}}^T \tilde{\mathbf{y}}$ has pdf $\mathcal{N}\left(\boldsymbol{\mu}^T \mathbf{C}^{-1} \boldsymbol{\mu}_0, \boldsymbol{\mu}^T \mathbf{C}^{-1} \boldsymbol{\mu}\right)$ under \mathcal{H}_0 , and pdf $\mathcal{N}\left(\boldsymbol{\mu}^T \mathbf{C}^{-1} \boldsymbol{\mu}_1, \boldsymbol{\mu}^T \mathbf{C}^{-1} \boldsymbol{\mu}\right)$ under \mathcal{H}_1 , then we see that in correlated noise, the shape of the signal can affect the performance:

$$\mathcal{H}_0: T(\mathbf{y}) \sim \mathcal{N}\left(\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right) \quad \text{versus} \quad \mathcal{H}_1: T(\mathbf{y}) \sim \mathcal{N}\left(-\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right).$$

FA for the Matched filter:

$$p(f(Y) = 1|X = 0) = \int_{-\infty}^{0} p_0(y)dy$$
$$= Q\left(\frac{1}{\sigma}\right)$$

MD for the Matched filter:

$$p(f(Y) = 0|X = 1) = \int_0^\infty p_1(y)dy$$
$$= Q\left(\frac{1}{\sigma}\right)$$

Average error probability for the Matched filter:

$$p(f(Y) \neq X) = p(f(Y) = 1|X = 0)p(X = 0) + p(f(Y) = 0|X = 1)p(X = 1)$$
$$= Q\left(\frac{1}{\sigma}\right) \times \frac{1}{2} + Q\left(\frac{1}{\sigma}\right) \times \frac{1}{2}$$
$$= Q\left(\frac{1}{\sigma}\right)$$

The matched filter is optimal because the performance of matched filter is the same to that of MAP in color noise.



FA:

$$p_{0}\left(\sum_{i=1}^{n} Y_{i} < \frac{n}{2}\right) = p_{0}\left(-\sum_{i=1}^{n} Y_{i} > -\frac{n}{2}\right) \lessapprox \min_{s \ge 0} \frac{\mathbb{E}\left\{e^{-s\sum_{i=1}^{n} Y_{i}}\right\}}{e^{-s\frac{n}{2}}}$$

$$= \left(\min_{s \ge 0} \underbrace{\frac{\mathbb{E}\left\{e^{-sY}\right\}}{e^{-\frac{s}{2}}}}_{=:f(s)}\right)^{n}$$

$$= 2^{n} (p(1-p))^{\frac{n}{2}},$$

where

$$f(s) = \frac{(1-p) + pe^{-s}}{e^{-\frac{s}{2}}}$$

$$\frac{\partial f(s)}{\partial s} = \frac{1-p}{2}e^{\frac{s}{2}} - \frac{p}{2}e^{-\frac{s}{2}} = 0 \implies s^* = \ln\frac{p}{1-p}.$$

MD:

$$p_1\left(\sum_{i=1}^n Y_i > \frac{n}{2}\right) \lessapprox \min_{s \ge 0} \frac{\mathbb{E}\left\{e^{s\sum_{i=1}^n Y_i}\right\}}{e^{s\frac{n}{2}}}$$

$$= \left(\min_{s \ge 0} \underbrace{\frac{\mathbb{E}\left\{e^{sY}\right\}}{e^{\frac{s}{2}}}}_{=:f(s)}\right)^n$$

$$= 2^n (p(1-p))^{\frac{n}{2}},$$

where

$$f(s) = \frac{(1-p)e^s + p}{e^{\frac{s}{2}}}$$

$$\frac{\partial f(s)}{\partial s} = \frac{1-p}{2}e^{\frac{s}{2}} - \frac{p}{2}e^{-\frac{s}{2}} = 0 \implies s^* = \ln\frac{p}{1-p}.$$

Average error probability:

of probability:

$$p(f(Y) \neq X) = p(f(Y) = 1 | X = 0) p(X = 0) + p(f(Y) = 0 | X = 1) p(X = 1)$$

$$= 2^{n} (p(1-p))^{\frac{n}{2}} \times \frac{1}{2} + 2^{n} (p(1-p))^{\frac{n}{2}} \times \frac{1}{2}$$

$$= 2^{n} (p(1-p))^{\frac{n}{2}}$$