Homework 1
ECE695C Inference Method for Codes on Graphs

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Question 1

1. Find the maximum a posteriori probability (MAP) detector as a function of y.

To fine the MAP detector of $\hat{X}_{\mathrm{MAP}}(y) = \arg\max_x p(x|y)$, the MAP decision rule is required as below:

$$\begin{aligned} \frac{p(x=0|y)}{p(x=1|y)} &\gtreqless 1 \\ \text{(Likelihood-ratio)} \quad \frac{p(y|x=0)}{p(y|x=1)} &\gtreqless \frac{p(x=1)}{p(x=0)} \\ \frac{\exp\{-|y-1|\}}{\exp\{-|y+1|\}} &\gtreqless 2 \\ -|y-1|+|y+1| &\gtreqless \ln 2 \end{aligned}$$

Taking the prior probability, we have the Bayes rule as

i) for $y < -1$ $\quad \hat{X}_{\mathrm{MAP}}(y) = 1,$

ii) for $-1 \le y < 1$ $\quad \hat{X}_{\mathrm{MAP}}(y) = \begin{cases} 0 & \text{if } y \ge \frac{\ln 2}{2} \\ 1 & \text{if } y < \frac{\ln 2}{2} \end{cases},$

iii) for $1 \le y$ $\quad \hat{X}_{\mathrm{MAP}}(y) = 0.$

In short, an equivalent Bayes test for this case is given by

i) for $y < \frac{\ln 2}{2}$ $\quad \hat{X}_{\mathrm{MAP}}(y) = 1,$

ii) for $\frac{\ln 2}{2} \le y$ $\quad \hat{X}_{\mathrm{MAP}}(y) = 0.$

2. Find the maximum likelihood (ML) detector as a function of y.

To fine the ML detector of $\hat{X}_{\mathrm{ML}}(y) = \arg\max_x p(y|x)$, the likelihood-ratio is thus given by

$$\begin{aligned} \frac{p(y|x=0)}{p(y|x=1)} &\gtreqless 1 \\ \frac{\exp\{-|y-1|\}}{\exp\{-|y+1|\}} &\gtreqless 1 \\ -|y-1|+|y+1| &\gtreqless 0. \end{aligned}$$

The boundary points where the sign of y change are ± 1, then we have the following:

i) for $y < -1$ $\quad \hat{X}_{\mathrm{ML}}(y) = 1,$

ii) for $-1 \le y < 1$ $\quad \hat{X}_{\mathrm{ML}}(y) = \begin{cases} 0 & \text{if } y \ge 0 \\ 1 & \text{if } y < 0 \end{cases},$

iii) for $1 \le y$ $\quad \hat{X}_{\mathrm{ML}}(y) = 0.$

Simply, a received signal y is interpreted as 1 when $y < 0$ and vice versa as

i) for $y < 0$ $\quad \hat{X}_{\mathrm{ML}}(y) = 1,$

ii) for $0 < y$ $\quad \hat{X}_{\mathrm{ML}}(y) = 0,$

iii) for $y = 0$ $\quad \hat{x}_{\mathrm{ML}}(\bar{Y}) = 0$ or 1.

3. Are these two detectors the same?

Not the same. The threshold to detect a received symbols is different: $\tau = \ln 2$ for MAP and $\tau = 0$ for ML.

4. Find out the mis-detection probability of the MAP detector.

MD for MAP:

$$\begin{aligned} p(f(Y)=0|X=1) &= \int_{\frac{\ln 2}{2}}^{\infty} \frac{1}{2} e^{-|y+1|} dy \\ &= \frac{1}{2} e^{-(\frac{\ln 2}{2}+1)} \end{aligned}$$

5. Find out the false-alarm probability of the following naive detector:

FA for NAIVE:

$$\begin{aligned} p(f(Y)=1|X=0) &= \int_{-\infty}^{\frac{1}{4}} \frac{1}{2} e^{-|y-1|} dy \\ &= \frac{1}{2} e^{(\frac{1}{4}-1)} = \frac{1}{2} e^{-\frac{3}{4}} \end{aligned}$$

6. Find out the overall/average error probability of the MAP detector.

FA for MAP:

$$\begin{aligned} p(f(Y)=1|X=0) &= \int_{-\infty}^{\frac{ln2}{2}} \frac{1}{2} e^{-|y-1|} dy \\ &= \frac{1}{2} e^{(\frac{\ln 2}{2}-1)} \end{aligned}$$

Average error probability for MAP:

$$\begin{aligned} p(f(Y)\neq X) &= p(f(Y)=1|X=0)p(X=0) + p(f(Y)=0|X=1)p(X=1) \\ &= \frac{1}{2} e^{(\frac{\ln 2}{2}-1)} \times \frac{1}{3} + \frac{1}{2} e^{-(\frac{\ln 2}{2}+1)} \times \frac{2}{3} \\ &= \frac{1}{6} e^{(\frac{\ln 2}{2}-1)} + \frac{1}{3} e^{-(\frac{\ln 2}{2}+1)} \end{aligned}$$

7. Find out the overall/average error probability of the ML detector.

FA for ML:

$$\begin{aligned} p(f(Y)=1|X=0) &= \int_{-\infty}^{0} \frac{1}{2} e^{-|y-1|} dy \\ &= \frac{1}{2} e^{-1} \end{aligned}$$

MD for ML:

$$\begin{aligned} p(f(Y)=0|X=1) &= \int_{0}^{\infty} \frac{1}{2} e^{-|y+1|} dy \\ &= \frac{1}{2} e^{-1} \end{aligned}$$

Average error probability for ML:

$$\begin{aligned} p(f(Y)\neq X) &= p(f(Y)=1|X=0)p(X=0) + p(f(Y)=0|X=1)p(X=1) \\ &= \frac{1}{2} e^{-1} \times \frac{1}{3} + \frac{1}{2} e^{-1} \times \frac{2}{3} \\ &= \frac{1}{2} e^{-1} \end{aligned}$$

Question 2

1. Find the maximum a posteriori probability (MAP) detector as a function of y.

To fine the MAP detector of $\hat{X}_{\mathrm{MAP}}(y) = \arg\max_x p(x|y)$, the MAP decision rule is required as below:

$$\frac{p(x=0|y)}{p(x=1|y)} \gtreqless 1$$
$$\frac{p(y|x=0)}{p(y|x=1)} \gtreqless \frac{p(x=1)}{p(x=0)} = 2.$$

Taking the prior probability, we have the Bayes rule as

i) for $0 \leq y \leq 1$
$\hat{X}_{\mathrm{MAP}}(y) = 1,$

ii) Otherwise
$\hat{X}_{\mathrm{MAP}}(y) = 1.$

2. Find the maximum likelihood (ML) detector as a function of y.

To fine the ML detector of $\hat{X}_{\mathrm{ML}}(y) = \arg\max_x p(y|x)$, the likelihood-ratio is thus given by

$$\frac{p(y|x=0)}{p(y|x=1)} \gtreqless 1$$
$$\frac{2(y+1)}{3} \gtreqless 1, \qquad \text{if } 0 \leq y \leq 1.$$

i) for $0 \leq y \leq 1$
$$\hat{X}_{\mathrm{MAP}}(y) = \begin{cases} 0 & \text{if } y \geq \frac{1}{2} \\ 1 & \text{if } y < \frac{1}{2} \end{cases}.$$

ii) Otherwise
$\hat{X}_{\mathrm{MAP}}(y) = 1.$

i) for $\frac{1}{2} \leq y \leq 1$
$\hat{X}_{\mathrm{MAP}}(y) = 0.$

ii) Otherwise
$\hat{X}_{\mathrm{MAP}}(y) = 1.$

3. Are these two detectors the same?

Not the same. The MAP detector always says $\hat{X} = 1$; however the ML detector has a threshold, $\tau = 0$.

4. Find out the mis-detection probability of the MAP detector.

MD for MAP:

$$p(f(Y) = 0|X = 1) = \int_{y \notin [0,1]} p_1(y)dy = 0.$$

5. Find out the false-alarm probability of the following naive detector:

FA for NAIVE:

$$p(f(Y) = 1|X = 0) = \int_0^{\frac{1}{4}} p_0(y)dy$$
$$= \int_0^{\frac{1}{4}} \frac{2}{3}(y+1)dy = \frac{3}{16}$$

6. Find out the overall/average error probability of the MAP detector.

FA for MAP:

$$\begin{aligned} p(f(Y)=1|X=0) &= \int_0^1 p_0(y)dy \\ &= \int_0^1 \frac{2}{3}(y+1)dy = 1 \end{aligned}$$

Average error probability:

$$\begin{aligned} p(f(Y)\neq X) &= p(f(Y)=1|X=0)p(X=0) + p(f(Y)=0|X=1)p(X=1) \\ &= 1\times\frac{1}{3} + 0\times\frac{2}{3} = \frac{1}{3} \end{aligned}$$

7. Find out the overall/average error probability of the ML detector.

FA for ML:

$$\begin{aligned} p(f(Y)=1|X=0) &= \int_0^{\frac{1}{2}} p_0(y)dy \\ &= \int_0^{\frac{1}{2}} \frac{2}{3}(y+1)dy = \frac{5}{12} \end{aligned}$$

MD for ML:

$$\begin{aligned} p(f(Y)=0|X=1) &= \int_{\frac{1}{2}}^1 p_1(y)dy \\ &= \int_{\frac{1}{2}}^1 1dy = \frac{1}{2} \end{aligned}$$

Average error probability for ML:

$$\begin{aligned} p(f(Y)\neq X) &= p(f(Y)=1|X=0)p(X=0) + p(f(Y)=0|X=1)p(X=1) \\ &= \frac{5}{12}\times\frac{1}{3} + \frac{1}{2}\times\frac{2}{3} \\ &= \frac{17}{36} \end{aligned}$$

Question 3

1. Model this CDMA system as a hypothesis testing problem.

Consider the following two hypotheses concerning a real-valued observation $\mathbf{y} := [Y_1, \dots, Y_{10}]^T$:

$$\mathcal{H}_0 : \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_0, \sigma^2\mathbf{I}) \quad \text{versus} \quad \mathcal{H}_1 : \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_1, \sigma^2\mathbf{I})$$

where $\boldsymbol{\mu}_0 = [-1\ 1 \cdots -1\ 1]^T$, $\boldsymbol{\mu}_1 = [1\ 1 \cdots\ 1]^T$.

2. Find the MAP detector and express the MAP detector in the form of log-likelihoodratio test.

$\hat{x}_{\text{MAP}}(\bar{Y}) = \arg\max_x p(x|\bar{Y})$

$$\begin{aligned}
\frac{p(\text{bit}=0|\mathbf{y})}{p(\text{bit}=1|\mathbf{y})} &\gtreqless 1 \\
\frac{p(\mathbf{y}|\text{bit}=0)}{p(\mathbf{y}|\text{bit}=1)} &\gtreqless \frac{p(\text{bit}=1)}{p(\text{bit}=0)} = 1 \\
\frac{\exp\left\{-\frac{1}{2\sigma^2}(\mathbf{y}-\boldsymbol{\mu}_0)^T(\mathbf{y}-\boldsymbol{\mu}_0)\right\}}{\exp\left\{-\frac{1}{2\sigma^2}(\mathbf{y}-\boldsymbol{\mu}_1)^T(\mathbf{y}-\boldsymbol{\mu}_1)\right\}} &\gtreqless 1 \\
-\frac{1}{2\sigma^2}\left\{(\mathbf{y}-\boldsymbol{\mu}_0)^T(\mathbf{y}-\boldsymbol{\mu}_0) - (\mathbf{y}-\boldsymbol{\mu}_1)^T(\mathbf{y}-\boldsymbol{\mu}_1)\right\} &\gtreqless 0 \\
\underbrace{(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)^T}_{=:\boldsymbol{\mu}^T}\mathbf{y} &\lesseqgtr \frac{\boldsymbol{\mu}_1^T\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0^T\boldsymbol{\mu}_0}{2} \\
\boldsymbol{\mu}^T\mathbf{y} &\lesseqgtr 0 \\
\bar{Y} := Y_1 + Y_3 + Y_5 + Y_7 + Y_9 &\lesseqgtr 0
\end{aligned}$$

i) for $\bar{Y} < 0$
$\hat{\text{bit}}_{\text{MAP}}(\bar{Y}) = 0$,

ii) for $0 < \bar{Y}$
$\hat{\text{bit}}_{\text{MAP}}(\bar{Y}) = 1$,

iii) for $\bar{Y} = 0$
$\hat{\text{bit}}_{\text{MAP}}(\bar{Y}) = 0$ or 1.

3. Find the average error probability.

Since $\bar{Y}$ has probability density function (pdf) $\mathcal{N}(-5, 5\sigma^2)$ when $\mathcal{H}_0$ is true, and $\mathcal{N}(5, 5\sigma^2)$ when $\mathcal{H}_1$ is true, then
FA for MAP:

$$\begin{aligned}
p(f(\bar{Y}) = 1|\text{bit}=0) &= \int_0^\infty p_0(\bar{Y})d\bar{y} \\
&= Q\left(\frac{\sqrt{5}}{\sigma}\right)
\end{aligned}$$

where $Q(x) := \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\frac{u^2}{2}\right)du$.
MD for MAP:

$$\begin{aligned}
p(f(\bar{Y}) = 0|X=1) &= \int_{-\infty}^0 p_1(\bar{Y})d\bar{y} \\
&= Q\left(\frac{\sqrt{5}}{\sigma}\right)
\end{aligned}$$

Average error probability for MAP:

$$
\begin{aligned}
p(f(\bar{Y}) \neq X) &= p(f(\bar{Y}) = 1|X = 0)p(X = 0) + p(f(\bar{Y}) = 0|X = 1)p(X = 1) \\
&= Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2} + Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2} \\
&= Q\left(\frac{\sqrt{5}}{\sigma}\right)
\end{aligned}
$$

4. Use the Chernoff bound to derive/approximate the average error probability for general N-bit signature sequence.

When $\bar{Y} := Y_1 + Y_3 + Y_5 + Y_7 + Y_9$,
FA:

$$
\begin{aligned}
p_0\left(\bar{Y} > 0\right) &\lessapprox \min_{s \geq 0} \frac{\mathbb{E}\left\{e^{s\bar{Y}}\right\}}{e^{s \cdot 0}} \\
&= \left(\min_{s \geq 0} \mathbb{E}\left\{e^{sY}\right\}\right)^5 \\
&= e^{-\frac{5}{2\sigma^2}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E}\left\{e^{sY}\right\} &= e^{-s + \frac{\sigma^2}{2}s^2} \\
\frac{\partial \mathbb{E}\left\{e^{sY}\right\}}{\partial s} &= (-1 + \sigma^2 s)e^{-s + \frac{\sigma^2}{2}s^2} = 0 \Rightarrow s^* = \frac{1}{\sigma^2}.
\end{aligned}
$$

MD:

$$
\begin{aligned}
p_1\left(\bar{Y} < 0\right) = p_1\left(-\bar{Y} > 0\right) &\lessapprox \min_{s \geq 0} \frac{\mathbb{E}\left\{e^{-s\bar{Y}}\right\}}{e^{s \cdot 0}} \\
&= \left(\min_{s \geq 0} \mathbb{E}\left\{e^{-sY}\right\}\right)^5 \\
&= e^{-\frac{5}{2\sigma^2}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E}\left\{e^{-sY}\right\} &= e^{-s + \frac{\sigma^2}{2}s^2} \\
\frac{\partial \mathbb{E}\left\{e^{sY}\right\}}{\partial s} &= (-1 + \sigma^2 s)e^{-s + \frac{\sigma^2}{2}s^2} = 0 \Rightarrow s^* = \frac{1}{\sigma^2}.
\end{aligned}
$$

Average error probability:

$$
\begin{aligned}
p(f(\bar{Y}) \neq X) &= p(f(\bar{Y}) = 1|X = 0)p(X = 0) + p(f(\bar{Y}) = 0|X = 1)p(X = 1) \\
&= e^{-\frac{5}{2\sigma^2}} \times \frac{1}{2} + e^{-\frac{5}{2\sigma^2}} \times \frac{1}{2} \\
&= e^{-\frac{5}{2\sigma^2}}
\end{aligned}
$$

5. [Optional for those who has learned CDMA/digital communication before.] Discuss its relationship to the matched filter in digital communication. Is the matched filter optimal?

Matched filter

$$\begin{aligned}\mathbf{u}_0 &:= [-1\ 1 - 1 \cdots 1];\\ \mathbf{u}_1 &:= [\ 1\ 1\ 1 \cdots 1];\\ \bar{\mathbf{u}} &:= \mathbf{u}_0 - \mathbf{u}_1 = [-2\ 0 - 2 \cdots 0];\end{aligned}$$

Front-end processing

$$\bar{Y} := \sum_{i=1}^{N} \bar{\mathbf{u}}(i) Y_i$$

Hypothesis testing

$$H_0 : \bar{Y} \sim \mathcal{N}(10, 20\sigma^2) \quad \text{versus} \quad H_1 : \bar{Y} \sim \mathcal{N}(-10, 20\sigma^2)$$

FA:

$$\begin{aligned}p(f(Y) = 1 | X = 0) &= \int_{-\infty}^{0} p_0(y) dy\\ &= Q\left(\frac{\sqrt{5}}{\sigma}\right)\end{aligned}$$

MD:

$$\begin{aligned}p(f(Y) = 0 | X = 1) &= \int_{0}^{\infty} p_1(y) dy\\ &= Q\left(\frac{\sqrt{5}}{\sigma}\right)\end{aligned}$$

Average error probability:

$$\begin{aligned}p(f(Y) \neq X) &= p(f(Y) = 1 | X = 0) p(X = 0) + p(f(Y) = 0 | X = 1) p(X = 1)\\ &= Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2} + Q\left(\frac{\sqrt{5}}{\sigma}\right) \times \frac{1}{2}\\ &= Q\left(\frac{\sqrt{5}}{\sigma}\right)\end{aligned}$$

The matched filter is optimal because the performance of matched filter is the same to that of MAP in white noise.

Question 4

1. Model this CDMA system as a hypothesis testing problem.

Consider the following two hypotheses concerning a real-valued observation $\mathbf{y} := [Y_1, Y_2]^T$:

$$\mathcal{H}_0 : \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}) \quad \text{versus} \quad \mathcal{H}_1 : \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$$

where $\boldsymbol{\mu}_0 = [1, 0.5]^T$, $\boldsymbol{\mu}_1 = [-1, -0.5]^T$, and $\boldsymbol{\Sigma} = \sigma^2 \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$.

2. Find the MAP detector and express the MAP detector in the form of log-likelihoodratio test.

$\hat{x}_{\mathrm{MAP}}(\mathbf{y}) = \arg\max_x p(x|\mathbf{y})$

$$\begin{aligned}
\frac{p(\mathrm{bit}=0|\mathbf{y})}{p(\mathrm{bit}=1|\mathbf{y})} &\gtreqless 1 \\
\frac{p(\mathbf{y}|\mathrm{bit}=0)}{p(\mathbf{y}|\mathrm{bit}=1)} &\gtreqless \frac{p(\mathrm{bit}=1)}{p(\mathrm{bit}=0)} = 1 \\
\frac{\exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}_0)\right\}}{\exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}_1)\right\}} &\gtreqless 1 \\
-\frac{1}{2}\left\{(\mathbf{y}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}_0) - (\mathbf{y}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}_1)\right\} &\gtreqless 0 \\
\underbrace{(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)^T}_{=:\boldsymbol{\mu}^T}\boldsymbol{\Sigma}^{-1}\mathbf{y} &\lesseqgtr \frac{\boldsymbol{\mu}_1^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0}{2} \\
\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{y} &\lesseqgtr 0 \\
-\frac{2}{\sigma^2}Y_1 &\lesseqgtr 0 \\
Y_1 &\gtreqless 0
\end{aligned}$$

where $\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1} = \frac{1}{\sigma^2}[-2\ 0]$.

i) for $Y_1 < 0$
$\hat{\mathrm{bit}}_{\mathrm{MAP}}(Y_1) = 1$,

ii) for $0 < Y_1$
$\hat{\mathrm{bit}}_{\mathrm{MAP}}(Y_1) = 0$,

iii) for $Y_1 = 0$
$\hat{\mathrm{bit}}_{\mathrm{MAP}}(Y_1) = 0$ or 1.

3. [Optional for those who has learned CDMA/digital communication before.] Discuss its relationship to the matched filter in digital communication. Is the matched filter optimal?

Matrix decomposition:

$$\begin{aligned}
\boldsymbol{\Sigma} &= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sigma^2}{2} & 0 \\ 0 & \frac{3\sigma^2}{2} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\
\boldsymbol{\Sigma}^{-1} &= \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^T = \mathbf{D}^T\mathbf{D}
\end{aligned}$$

where $\mathbf{D} = \sqrt{\boldsymbol{\Lambda}^{-1}}\mathbf{U}^T$.
Front-end processing:

$$\begin{aligned}
\tilde{\mathbf{y}} &:= \mathbf{D}\mathbf{y} \quad \text{(whitening noise)} \\
\tilde{\boldsymbol{\mu}} &:= \mathbf{D}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1) = \mathbf{D}\boldsymbol{\mu}.
\end{aligned}$$

The test statistic $T(\mathbf{y}) = \tilde{\boldsymbol{\mu}}^T\tilde{\mathbf{y}}$ has pdf $\mathcal{N}\left(\boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu}_0, \boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu}\right)$ under $\mathcal{H}_0$, and pdf $\mathcal{N}\left(\boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu}_1, \boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu}\right)$ under $\mathcal{H}_1$, then we see that in correlated noise, the shape of the signal can affect the performance:

$$\mathcal{H}_0 : T(\mathbf{y}) \sim \mathcal{N}\left(\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right) \quad \text{versus} \quad \mathcal{H}_1 : T(\mathbf{y}) \sim \mathcal{N}\left(-\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right).$$

FA for the Matched filter:

$$\begin{aligned} p(f(Y) = 1|X = 0) &= \int_{-\infty}^{0} p_0(y)dy \\ &= Q\left(\frac{1}{\sigma}\right) \end{aligned}$$

MD for the Matched filter:

$$\begin{aligned} p(f(Y) = 0|X = 1) &= \int_{0}^{\infty} p_1(y)dy \\ &= Q\left(\frac{1}{\sigma}\right) \end{aligned}$$

Average error probability for the Matched filter:

$$\begin{aligned} p(f(Y) \neq X) &= p(f(Y) = 1|X = 0)p(X = 0) + p(f(Y) = 0|X = 1)p(X = 1) \\ &= Q\left(\frac{1}{\sigma}\right) \times \frac{1}{2} + Q\left(\frac{1}{\sigma}\right) \times \frac{1}{2} \\ &= Q\left(\frac{1}{\sigma}\right) \end{aligned}$$

The matched filter is optimal because the performance of matched filter is the same to that of MAP in color noise.

Question 5

FA:

$$p_0\left(\sum_{i=1}^{n} Y_i < \frac{n}{2}\right) = p_0\left(-\sum_{i=1}^{n} Y_i > -\frac{n}{2}\right) \lessapprox \min_{s\geq 0} \frac{\mathbb{E}\left\{e^{-s\sum_{i=1}^{n} Y_i}\right\}}{e^{-s\frac{n}{2}}}$$
$$= \left(\min_{s\geq 0} \underbrace{\frac{\mathbb{E}\left\{e^{-sY}\right\}}{e^{-\frac{s}{2}}}}_{=:f(s)}\right)^n$$
$$= 2^n(p(1-p))^{\frac{n}{2}},$$

where

$$f(s) = \frac{(1-p)+pe^{-s}}{e^{-\frac{s}{2}}}$$
$$\frac{\partial f(s)}{\partial s} = \frac{1-p}{2}e^{\frac{s}{2}} - \frac{p}{2}e^{-\frac{s}{2}} = 0 \quad \Rightarrow \quad s^* = \ln\frac{p}{1-p}.$$

MD:

$$p_1\left(\sum_{i=1}^{n} Y_i > \frac{n}{2}\right) \lessapprox \min_{s\geq 0} \frac{\mathbb{E}\left\{e^{s\sum_{i=1}^{n} Y_i}\right\}}{e^{s\frac{n}{2}}}$$
$$= \left(\min_{s\geq 0} \underbrace{\frac{\mathbb{E}\left\{e^{sY}\right\}}{e^{\frac{s}{2}}}}_{=:f(s)}\right)^n$$
$$= 2^n(p(1-p))^{\frac{n}{2}},$$

where

$$f(s) = \frac{(1-p)e^{s}+p}{e^{\frac{s}{2}}}$$
$$\frac{\partial f(s)}{\partial s} = \frac{1-p}{2}e^{\frac{s}{2}} - \frac{p}{2}e^{-\frac{s}{2}} = 0 \quad \Rightarrow \quad s^* = \ln\frac{p}{1-p}.$$

Average error probability:

$$p(f(Y)\neq X) = p(f(Y)=1|X=0)p(X=0) + p(f(Y)=0|X=1)p(X=1)$$
$$= 2^n(p(1-p))^{\frac{n}{2}} \times \frac{1}{2} + 2^n(p(1-p))^{\frac{n}{2}} \times \frac{1}{2}$$
$$= 2^n(p(1-p))^{\frac{n}{2}}$$