Lecture 24

Note Title 4/9/2012

* BEC Capacity - Approaching irregular LDPC code ensemble.

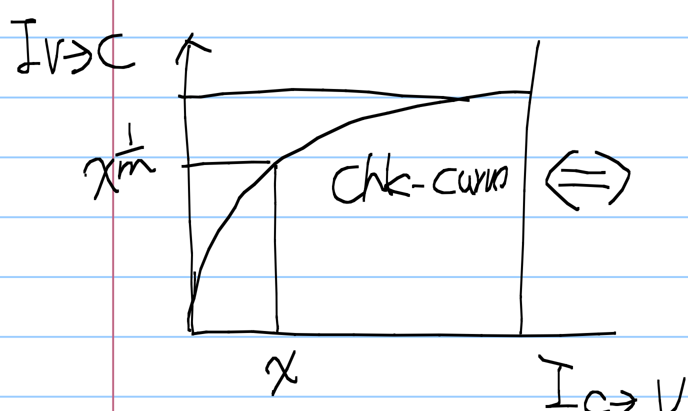
Goal: Construct a series of $(\lambda^{(m)}, \rho^{(m)})$ LDPC codes, such that they can all decode BEC of erasure prob ε & $R^{(m)} \nearrow 1-\varepsilon$ when $m \to \infty$

* Note: each λ^{m}, ρ^{m} can only have finite max degrees

* Construction inspired by the area theorem

$$\rho(x) = x^{m}$$

(a chk-node regular LDPC code with deg m



$\Longleftrightarrow$ $I_{C\to V} = \rho(I_{V\to C})$

$x = (I_{V\to C})^{m}$ —— ①

$I_{V\to C} = x^{\frac{1}{m}}$

If the var curve

$$1 - \varepsilon\lambda(1 - I_{C\to V}) = I_{V\to C}$$

is pointwise above ①, then decodability is guaranteed.
the chk curve

let $I_{C \to V} = x$

$\Leftrightarrow$ if $1 - \varepsilon\lambda(1-x) \geq (x)^{\frac{1}{m}}$ for all x

then decodability is guaranteed.

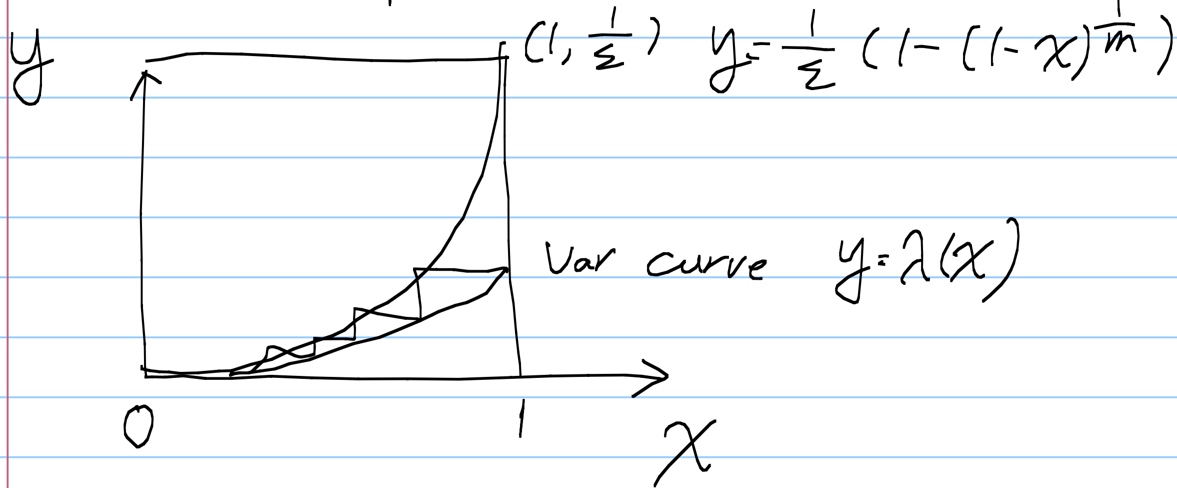
$\Leftrightarrow$ if $\lambda(1-x) \leq \frac{1}{\varepsilon}(1 - x^{\frac{1}{m}})$

then decodability is guaranteed.

$\Leftrightarrow$ if $\lambda(x) \leq \frac{1}{\varepsilon}(1 - (1-x)^{\frac{1}{m}})$

then decodability is guaranteed.

Let us plot $y = \frac{1}{\varepsilon}(1 - (1-x)^{\frac{1}{m}})$



(It is actually the flipped EXIT Chart curves)

by the area theorem, we would like to design $\lambda(x) = \sum_{k=2}^{\boxed{\text{finite}}} \lambda_k x^{k-1}$ s.t.

the area of the open tunnel is as small as possible.

Solution: Use Taylor's expansion to

fit $f(x) = \frac{1}{\varepsilon}\left(1-(1-x)^{\frac{1}{m}}\right)$

$f(0) = 0$

$f'(x) = \frac{1}{\varepsilon} \times \frac{1}{m}(1-x)^{\frac{1}{m}-1}$ $\quad f'(0) = \frac{1}{\varepsilon} \times \frac{1}{m}$

$f^{(d)}(x) = \frac{1}{\varepsilon}\frac{1}{m} \cdot \prod_{j=1}^{d-1}\left(j-\frac{1}{m}\right)(1-x)^{\frac{1}{m}-d}$

$f^{(d)}(0) = \frac{1}{\varepsilon}\frac{1}{m} \cdot \prod_{j=1}^{d-1}\left(j-\frac{1}{m}\right)$

$$\Rightarrow f(x) = \sum_{d=1}^{\infty}\left[\frac{1}{\varepsilon} \times \frac{1}{m} \times \prod_{j=1}^{d-1}\left(j-\frac{1}{m}\right)\right] \times \frac{1}{d!} x^d$$

$$= \sum_{k=2}^{\infty}\underbrace{\left[\frac{1}{\varepsilon} \times \frac{1}{m} \times \prod_{j=1}^{k-2}\left(j-\frac{1}{m}\right)\right] \times \frac{1}{(k-1)!}}_{\text{each coeff} \geq 0} x^{k-1}$$

$\Rightarrow$ if we choose

$$\lambda_k = \frac{1}{\varepsilon} \times \frac{1}{m} \prod_{j=1}^{k-2}\left(j-\frac{1}{m}\right) \times \frac{1}{(k-1)!}$$

then $\lambda(x)$ fits $\frac{1}{\varepsilon}(1-(1-x)^{\frac{1}{m}})$ perfectly

& we achieve the cap $R = 1-\varepsilon$.

However, this new $\lambda(x)$ is not a valid deg polynomial.

$$\because \sum_{k=2}^{\infty} \lambda_k = \lambda(1) = f(1) = \frac{1}{\varepsilon} > 1,$$

& its max deg is ∞. Violating the goal that $\lambda(x)$ has to have finite max deg.

Q: How to solve these two problems?

Ans: Truncate $\lambda(x)$

That is, given m &

$$\lambda_k = \frac{1}{\varepsilon} \cdot \frac{1}{m} \prod_{j=1}^{k-2} (j - \frac{1}{m}) \times \frac{1}{(k-1)!}$$

Find a K s.t.

$\sum_{k=2}^{K} \lambda_k = 1$ and choose our $\lambda(x)$

as $\lambda(x) = \sum_{k=2}^{K} \lambda_k x^{k-1}$

Remark ① Such K can always be found.

$\because \sum_{k=2}^{0} \lambda_k = 0 \le 1 \le \frac{1}{\varepsilon} = \sum_{k=2}^{\infty} \lambda_k$

② the truncated $\lambda(x) = \sum_{k=2}^{K} \lambda_k x^{k-1}$

is $\le \frac{1}{\varepsilon}(1-(1-x)^{\frac{1}{m}}) = \sum_{k=2}^{\infty} (\quad) x^{k-1}$

$\Rightarrow$ the truncated (λ, ρ) can decode BEC with erasure prob ε.

③ When $m \to \infty$. $K \to \infty$.

Pf:

$\because$ When $m \to \infty$, $\sum_{k=2}^{K} \frac{1}{\varepsilon} \frac{1}{m} \prod_{j=1}^{k-2} (j - \frac{1}{m}) \frac{1}{(k-1)!}$

For any K,

$\approx \sum_{k=2}^{K} \frac{1}{\varepsilon} \frac{1}{m} (k-2)! \frac{1}{(k-1)!}$

$= \sum_{k=2}^{K} \frac{1}{\varepsilon} \frac{1}{m} \times \frac{1}{k-1} = \frac{1}{\varepsilon} \times \frac{1}{m} \sum_{k=2}^{K} \frac{1}{k-1}$

$\Rightarrow$ when $m \to \infty$ $K \approx e^{\varepsilon \cdot m}$ $\approx \frac{1}{\varepsilon} \cdot \frac{1}{m} \times \ln(K)$

$$\Rightarrow K \approx e^{\varepsilon m} \Rightarrow \left(m \to \infty \Rightarrow K \to \infty \right)$$

* So the remaining question is when

$$\lambda(x) = \sum_{k=2}^{K} \frac{1}{\varepsilon} \frac{1}{m} \prod_{j=1}^{k-2} \left(j - \frac{1}{m}\right) \frac{1}{(k-1)!}$$

& $\rho(x) = x^m$, whether the rate $R(m) \longrightarrow (1-\varepsilon)$ when $m \to \infty$.

Since
$$R(m) = 1 - \frac{\int_0^1 \rho(x)\,dx}{\int_0^1 \lambda(x)\,dx}$$

$$= 1 - \frac{\frac{1}{m+1}}{\sum_{k=2}^{K} \frac{\lambda_k}{k}} = 1 - \frac{\frac{m}{m+1}}{\sum_{k=2}^{K} m \cdot \frac{\lambda_k}{k}} \overset{?}{=} 1 - \varepsilon$$

$\Rightarrow$ We are interested in whether

$$m \cdot \sum_{k=2}^{K} \frac{\lambda_k}{k} = \sum_{k=2}^{K} \frac{1}{\varepsilon} \frac{m}{m} \prod_{j=1}^{k-2} \left(j - \frac{1}{m}\right) \cdot \frac{1}{k!}$$

$$\longrightarrow \frac{1}{\varepsilon} \text{ when } m \to \infty$$

We prove the limit by a simple sandwiching argument

$$\underbrace{LB \leq \sum_{k=2}^{K} \frac{1}{\varepsilon} \prod_{j=2}^{k-2} \left(j - \frac{1}{m}\right) \times \frac{1}{k!}}_{\triangleq \text{ target}} \leq UB$$

$$UB : \sum_{k=2}^{\infty} \frac{1}{\varepsilon} \prod_{j=2}^{k-2} (j) \times \frac{1}{k!}$$

$$= \sum_{k=2}^{\infty} \frac{1}{\varepsilon} \times \frac{1}{k(k-1)}$$

$$= \sum_{k=2}^{\infty} \frac{1}{\varepsilon} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{\varepsilon}(1-0).$$

$$LB(A, m) = \sum_{k=2}^{A} \frac{1}{\varepsilon} \prod_{j=2}^{k-2} \left(j - \frac{1}{m}\right) \left(\frac{1}{k!}\right)$$

where $A \leq K$ is a constant does not depend on m.

Note: K depends on m, as K is chosen s.t.

$$\sum_{k=2}^{K} \lambda_k = 1$$

$$\Rightarrow \lim_{m\to\infty} LB(A,m) = \sum_{k=2}^{A} \frac{1}{\varepsilon} \prod_{j=1}^{k-2} j \cdot \frac{1}{k!}$$

$$= \sum_{k=2}^{A} \frac{1}{\varepsilon} \times \left(\frac{1}{k(k-1)}\right)$$

$$= \frac{1}{\varepsilon}\left(1 - \frac{1}{A}\right)$$

$$\Rightarrow \lim_{m\to\infty} LB(A,m) \le \lim_{m\to\infty} (\text{target}) \le \frac{1}{\varepsilon}$$

$$= \frac{1}{\varepsilon}\left(1 - \frac{1}{A}\right)$$ for any finite A.

$\Rightarrow$ let $A \to \infty$

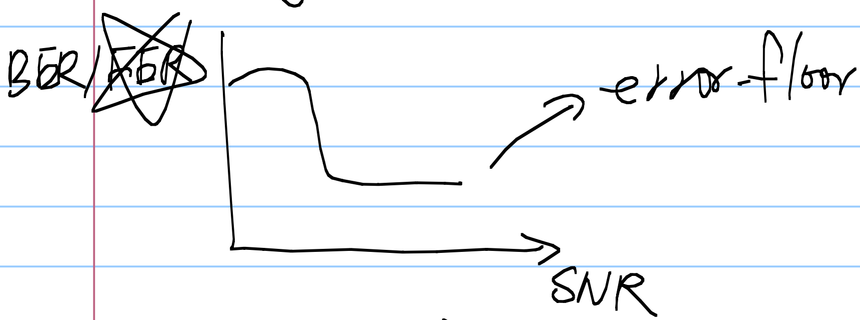
$$\frac{1}{\varepsilon} \le \lim_{m\to\infty} (\text{target}) \le \frac{1}{\varepsilon}$$

$$\Rightarrow \lim_{m\to\infty} R(m) = \lim_{m\to\infty}\left(1 - \frac{\frac{m}{m+1}}{\sum_{k=2}^{K} m \cdot \frac{\lambda_k}{k}}\right) = 1-\varepsilon$$

$\Rightarrow$ the rate of (λ^m, ρ^m) approaches $1-\varepsilon$

Unfortunately, for non-BEC, the existence of capacity approaching λ, ρ is still an open problem.

* Lowering the error-floor



* By DE, for any decodable threshold, we should not have any error-floor when the codeword length $n \to \infty$

That is when $n \to \infty$, the DE analysis is exact. The bit error rate $P_{ber}^{(t)} \longrightarrow 0$ when the noise level is below the threshold.

* However, ① all codeword lengths are finite.

② The average ber does not tell the story. In reality, a few bits (K bits) have much higher error prob than the other bits.

But when taking the average

ber, $\frac{K}{N}$ vanishes, which makes the average ber to be small.

* A better metric is the frame error rate FER, which counts all the cases when decoding is not successful.

* Q: What is the cause of frame errors?

A: Let's focus on the simplest channel: BECs.

We again rely on matrix representation

$$\boxed{H_1} \vec{x}_{erasure} = \boxed{H_0} \vec{x}_{received}$$

$$\hookrightarrow \boxed{H_1} \vec{x}_{erasure} = \boxed{H_0} x$$

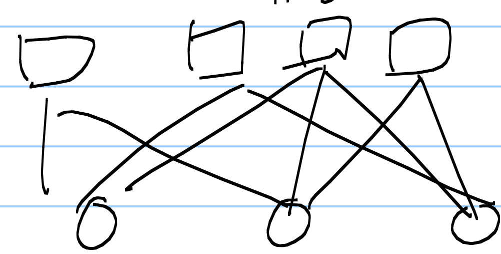
Q: When does the Message-Passing decoder stop improving?

A: When all rows in the remaining H_1 has either deg $=0$ or deg ≥ 2

Based on the above observation, we have the following definition:

Defn: A $\boxed{\text{stopping set}}$ S_{stopping} is a subset of variable nodes such that all neighbors of S_{stopping} are connected to S_{stopping} at least twice

check node



By relating the matrix-based intuition and the stopping set definition, we have

Lemma: Suppose we pass the LDPC code through a BEC & let V_{erased} denote the variable nodes that are erased. Run MP until it terminates. Then the remaining erasure bits form the largest stopping set satisfying $S^*_{\text{stopping}} \subseteq V_{\text{erased}}$

Corollary: When the erasure prob ε is small. The asymptote of the frame error rate is

$$M_s \cdot \varepsilon^{d_s}$$

where d_s is the minimal stopping distance (the minimal size of non-empty stopping sets.)

& M_s is the multiplicity of the minimum stopping sets.

This asymptote is the cause of the error floor.

For comparison, if we use the optimal decoding for BEC, then the FER asymptote becomes

$$M_H \cdot \varepsilon^{d_H}$$

where d_H: is the minimal Hamming distance

M_H: is the multiplicity of the minimal codewords

Lemma: $d_H \geq d_S$ (the optimal decoder performs better than MP)

pf 1: By contradiction ($\because$ optimal decoder $\succ$ MP decoder)

pf 2: For any <u>support set</u> of the codeword $S_{codeword}$ (those variables correspond to the ones in a valid codeword). Then its check node neighbors are connected to $S_{codeword}$ even number of times, $(0, 2, 4, 6, 8 ..)$

$\because$ We have to satisfy $Hx = 0$.

$\Rightarrow$ any $S_{codeword}$ is also a $S_{stopping}$

$$\Rightarrow d_H = \min |S_{codeword}| \geq \min |S_{stopping}| = d_S$$

Q: What is the cause of Sstopping?

A: A heuristic answer is "circle"

Lemma: Suppose $\min_v d_v \geq 2$. Then any Sstopping must contain a cycle. (But not vice-versa)