Lecture 21

Note Title 3/28/2012

* Gaussian approximation of DE

Motivation 4: Gaussian distri can be described by a single parameter, which enables a "scalar evolution" rather than "density evolution"

Remark: Since the message of a Gsn channel has density $N(\frac{2}{\sigma^2}, \frac{4}{\sigma^2})$, people generally focuses on Gsn densitie of the form $N(a, 2a)$

A typical Gsn approximation thus becomes

$$P_{v\to c}^{(t)} \xrightarrow[\text{approx}]{\boxed{\text{Step 1}}} \tilde{P}_{v\to c}^{(t)} \sim N(a^{(t)}, 2a^{(t)}) \text{ for}$$

some $a^{(t)}$ $\boxed{\text{Step 2}}$

Find $Q_{c\to v}^{(t)} = F_{chk}(\tilde{P}_{v\to c}^{(t)}, d_c - 1)$

or $Q_{c\to v}^{(t)} = \sum_{k=2}^{\max d_c} \rho_k F_{chk}(\tilde{P}_{v\to c}^{(t)}, k-1)$

$\boxed{\text{Step 3}}$ → Approximate $Q^{(t)}_{c\to v}$ by $\tilde{Q}^{(t)}_{c\to v} \sim N(b^{(t)}, 2b^{(t)})$

$\boxed{\text{Step 4}}$ use

$\tilde{Q}^{(t)}_{c\to v}$ to generate $P^{(t+1)}_{v\to c}$ by

$$P^{(t+1)}_{v\to c} = \sum_{k=2}^{\max d_v} \lambda_k F_{var}\left(\tilde{P}^{(0)}, \tilde{Q}^{(t)}_{c\to v}, k-1\right)$$

If we precompute $\boxed{\text{Steps } 2, 3}$, then we can have a scalar transformation map

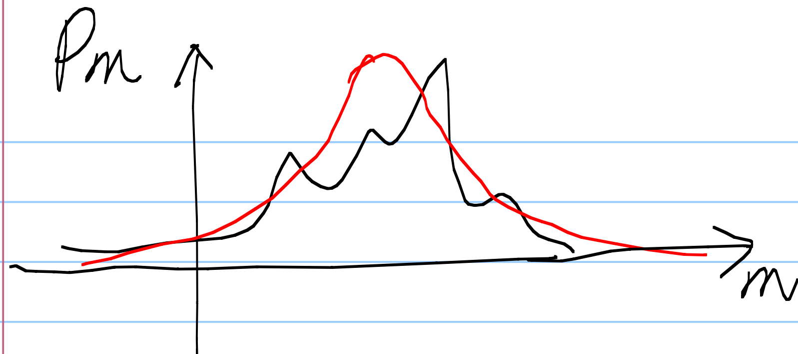
$b^{(t)} = f_{chk}(a^{(t)}, d_c - 1)$ from $a^{(t)}$ to $b^{(t)}$

Similary $\boxed{\text{Steps } 4, 1}$ gives from $a^{(0)}, b^{(t)}$ to $a^{(t+1)}$

$a^{(t+1)} = f_{var}(a^{(0)}, b^{(t)}, d_v - 1)$, which

can be easily precomputed & store in a table. (Two tables: One $a \to b$ table, one $b \to a$ table)

* The remaining question is how to "approximate" a $P^{(t)}_{v\to c}$ by $N(a^{(t)}, 2a^{(t)})$



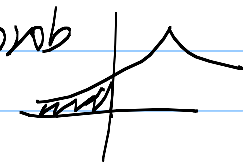
There are many approaches. Generally, we try to match some properties of $P^{(t)}_{v\to c}$

& $N(a^{(t)}, 2a^{(t)})$

Ex: ① Match the mean.

or ② Match the variance

③ Match the error prob



④ Match the mutual information

$$I(X;Y) = E\left(\log \frac{P_{Y|X}(Y|X)}{P_Y(Y)}\right)$$

symmetric CH.

$$= E_{X=0}\left(\log\left(\frac{P_{Y|X}(Y|0)}{P_Y(Y)}\right)\right)$$

$$= E_{X=0}\left(\log \frac{e^m P_{Y|X=1}}{\frac{1}{2} e^m P_{Y|X=1} + \frac{1}{2} P_{Y|X=1}}\right)$$

$$= E_{X=0}\left(\log\left(\frac{2e^m}{e^m+1}\right)\right) = \int_{-\infty}^{\infty} \log\left(\frac{2e^m}{e^m+1}\right) dP_m$$

Q: How good is this iteratively computed approximation?

A: Empirically very good. But not easily tractable analytically

* Another analysis tool that is in parallel with DE

* EXIT Chart analysis.

$\boxed{\text{EXtrinsic Information}}$

An analysis method that is slightly more <u>empirical</u> than DE, but is widely popular in the turbo code community.

* We will first consider its application to <u>BEC w. LDPC codes</u>, & then discuss its application to <u>AWGNC w. turbo codes.</u>

* EXIT chart + BEC + LDPC codes

We are now interested in the iteration of $\left(I(X;Y)\right)^{(t)}$ instead of $p_n^{(t)}$.

* For BEC w. erasure prob ε, & uniform prior

$I(X;Y) = 1 - \varepsilon$ (exercise)

therefore the var. & check node iteration becomes.

Var

$$I_{v\to c} = 1 - \varepsilon_{v\to c}$$
$$= 1 - \varepsilon^{(0)} \lambda(\varepsilon_{c\to v})$$
$$= 1 - (1 - I^{(0)}) \cdot \lambda(1 - I_{c\to v})$$

Information-based Map of var

Chk

$$I_{c\to v} = 1 - \varepsilon_{c\to v}$$
$$= 1 - (1 - \rho(1 - \varepsilon_{v\to c}))$$
$$= \rho(I_{v\to c})$$

Info-based map of chk

* We deliberately omit the superscript of time index

* We can then plot $I_{v\to c}$ vs $I_{c\to v}$ for the variable node, & $I_{c\to v}$ vs. $I_{v\to c}$ for the check nodes.

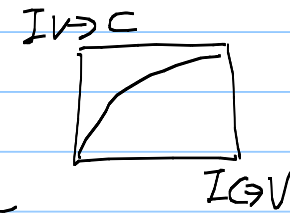
* If we flip the horizontal & vertical axis of the check node. Then we can plot two figures together.

* Let us try different initial erasure values:

$$\varepsilon^{(0)} = 1 - I^{(0)}$$

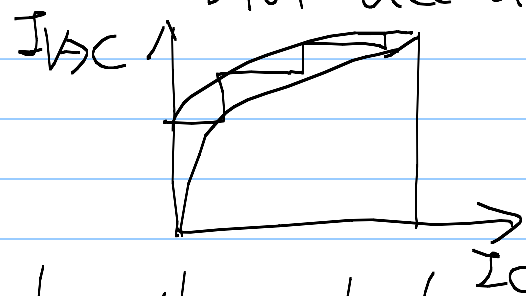
for the regular (3,6) code & the irregular rate 1/2 code in HW6

Ans: $\lambda(x) = x^2$, $\rho(x) = x^5$



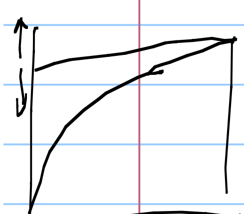
Q:
* When is successful decoding guaranteed?

* Ans: When there exists an "open" tunnel from the lower-left corner to the upper right corner, $\Rightarrow$ successful decoding is guaranteed.



we can "exit/escape" to the upper right corner (mutual info $\equiv 1$)

* Remark: the check node map does not depend on $\varepsilon^{(0)}$. When we change the $\varepsilon^{(0)}$ value, only the upper curve (the var) changes, which closes or open the tunnel



* EXIT chart is an excellent visiualization tool

Ex: the decodable threshold $I^{(0)} = 1 - \varepsilon^{(0)}$ is the smallest s.t.

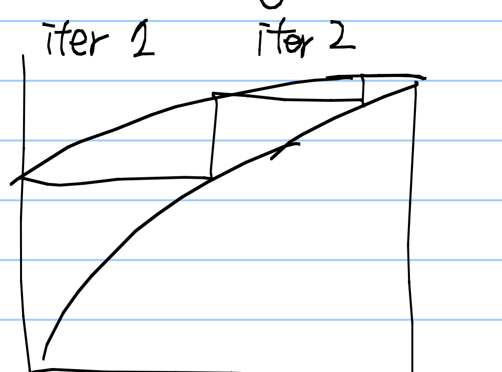
$$\underbrace{1-(1-I^{(0)})\lambda(1-x)}_{\text{upper curve}} = \underbrace{\rho^{-1}(x)}_{\text{lower curve}}$$

has no root except when $x=1$.

(To ensure the open tunnel.)

* How many iterations do we need?

Ans: the # of stairs to reach the upper right corner

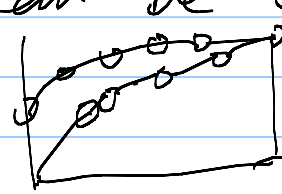


$t=3$

Need only 10-50 pts

* Where is the bottleneck?

*** EXIT Chart can be computed very efficiently Ex



ex: each curve on

then check intersection

* In addition to the benefits of efficiency/visualization, we can have some provable results as follows.

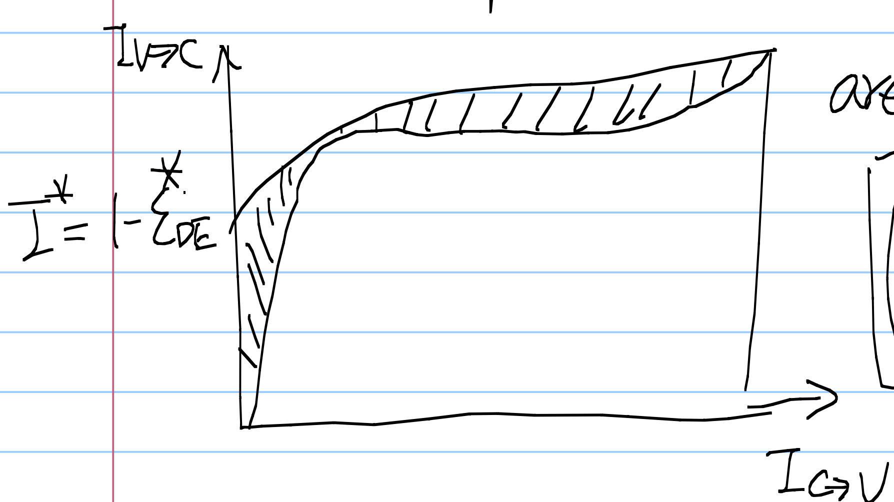
Theorem: $\boxed{\text{The area theorem}}$

for LDPC codes with rate R on BECs. Define the deficiency of the code by

$$\varepsilon_{cap} \triangleq 1-R$$

$$\varepsilon_{cap} - \varepsilon^*_{DE} = 1-R-\varepsilon^*_{DE} \triangleq \text{deficiency}$$

then the open tunnel at ε^*_{DE} has



area of the tunnel is

$$\boxed{\Big(\underbrace{\sum_{k=2} \frac{\lambda_k}{k}}\Big)\left(1-R-\varepsilon^*_{DE}\right)}$$

not depending on ε

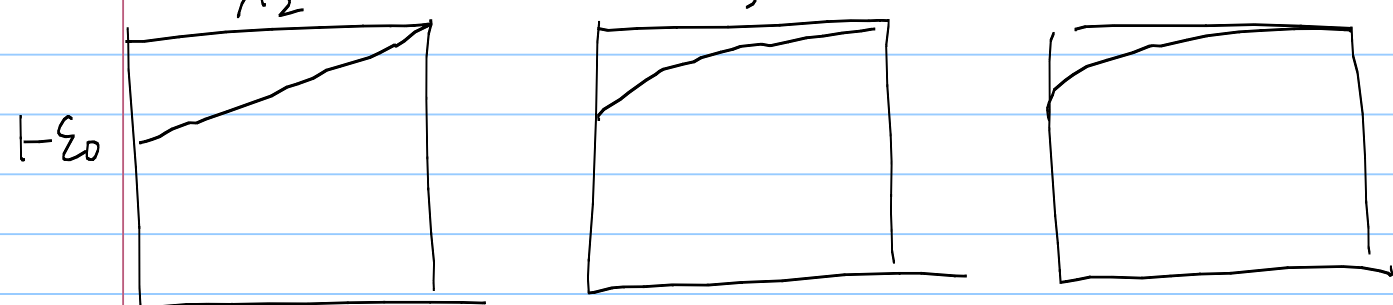
can be viewed as a conversion from edge to node

Namely, the smaller the area of the open tunnel, the closer the performance is to the capacity...

⇒ Designing an optimal code thus becomes a <u>curve-fitting</u> problem that would like to minimize the the area of the open tunnel.

That is the var curve $I_{v\to c} = 1-\varepsilon_0 \lambda(1-I_{c\to v})$ is the weighted average of

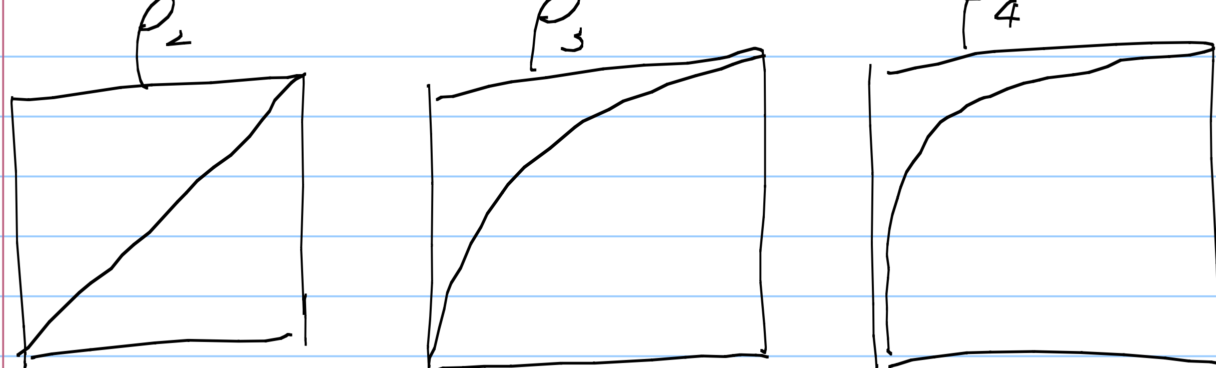
λ_2 λ_3 λ_4



$1-\varepsilon_0(1-x)$ $1-\varepsilon_0(1-x)^2$ $1-\varepsilon_0(1-x)^3$

the check nod map $\rho(I_{v\to c}) = I_{c\to v}$

ρ_2 ρ_3 ρ_4



satisfying $1-\dfrac{\sum \rho_k}{\sum \lambda_k} = R$ is a fixed value