Lecture 18

Note Title 3/19/2012

* Density Evolution (DE) analysis of LDPC code ensembles.

* Tracing the prob density function (pdf) of the messages $m^{[t]}$

More explicitly

$$P(m_{i_0}^{[t]} \mid \vec{X} = \vec{0})$$

$$P_{b,i_0}^{[t_0]} = P(m_{i_0}^{[t_0]} < 0 \mid \vec{X} = \vec{0}) + \frac{1}{2} P(m_{i_0}^{[t_0]} = 0 \mid \vec{X} = \vec{0})$$

where

$$m_i^{(0)} = \log\left(\frac{P_{Y_i|X_i}(y_i|0)}{P_{Y_i|X_i}(y_i|1)}\right) = f(y_i)$$

Example $Y_i = (-1)^{X_i} + \sigma N_i$, N_i: standard Gsn

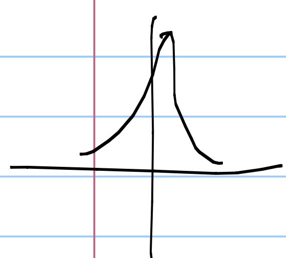
$$m_i^{(0)} = \log\left(\frac{P_{Y_i|X_i}(y_i|0)}{P_{Y_i|X_i}(y_i|1)}\right)$$

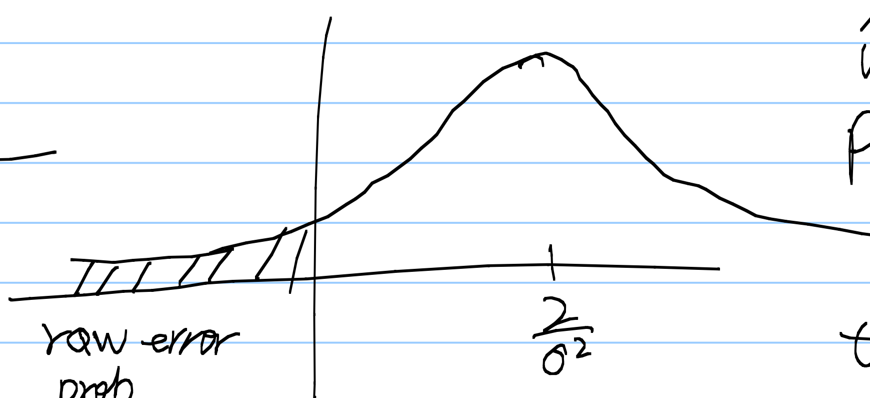
$$= \log\left(\frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i-1)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i+1)^2}{2\sigma^2}}}\right)$$

$$= \frac{2}{\sigma^2} y_i$$

$$P_{Y_i|\vec{X}=\vec{0}} \sim \text{Gsn}(1, \sigma^2)$$

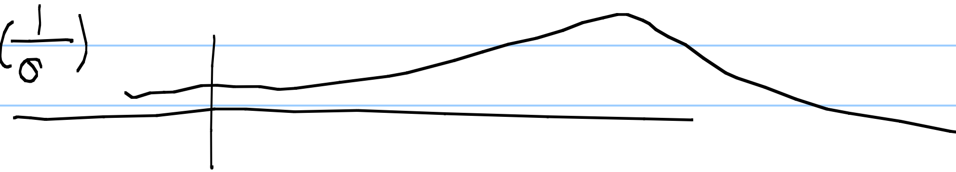
$$\Rightarrow P_{m_i^{(0)}|\vec{X}=\vec{0}} \sim \text{Gsn}\left(\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right)$$





when $\sigma \to 0$ $P_{m_i^{(0)}|\vec{X}=0}$ keeps moving to the right.

$$Q\left(\frac{\frac{2}{\sigma^2}}{\sqrt{\frac{4}{\sigma^2}}}\right) = Q\left(\frac{1}{\sigma}\right)$$



Example: The mismatch ~~of~~ the channel models.

Suppose the LDPC decoder is initialized / designed by assuming the channel being a Gsn

will $Y_i = (-1)^{X_i} + \sigma N_i$ but the

true channel model being

$$Y_i = (-1)^{X_i} + \sqrt{2}\,\sigma N_i$$ (underestimate the noise level)

Q: $P_{m_i^{(0)}}(X_i=0 = ?$

Ans: $m_i^{(0)} = \frac{2}{\sigma^2} y_i$

$\because$ Given $X_i=0$ $\quad Y_i \sim N(1, 2\sigma^2)$

$\therefore$ Given $X_i=0$

$$P_{m_i^{(0)}|X_i=0} \sim N\left(\frac{2}{\sigma^2}, \frac{8}{\sigma^2}\right)$$

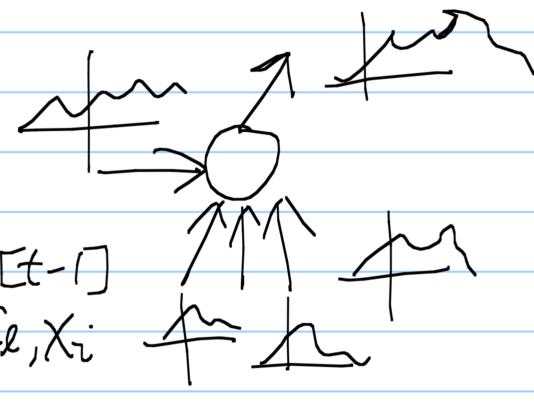
The raw error prob

$$Q\left(\frac{\frac{2}{\sigma^2}}{\sqrt{\frac{8}{\sigma^2}}}\right) = Q\left(\frac{1}{\sqrt{2}\,\sigma}\right)$$

② Iterative compute the density of $P_{m|\vec{x}=0}$ for the variable & check node map.

Recall,

Variable node map



$$ {}^{[t]}_{x_i,c_j} = m_i^{(0)} + \sum_{l \in \partial i \setminus j} m^{[t-1]}_{c_l,x_i} $$

* All m's are random variables.
* Moreover, Given $\vec{x}=\vec{0}$, each m depends on a disjoint set of independent observation (different sub-trees, no overlap by Thm 1)

* All m's are independent R.V.
* $$P_{m^{[t]}_{x_i,c_j}} = P_{m_i^{(0)}} * \left(\mathop{\Large *}_{l \in \partial i \setminus j} P_{m^{[t-1]}_{c_l,x_i}} \right)$$

convolution

* Similarly, for the check node map

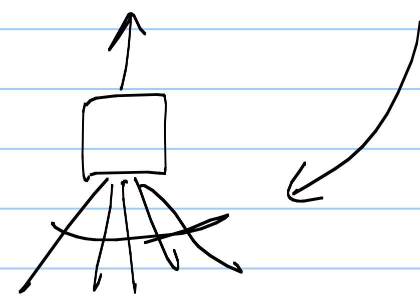
$$m^{[t]}_{c_j x_i} = \Phi_c\left(m^{[t]}_{x_k, c_j} : k \in \partial j \setminus i \right)$$

is a function of the input messages.

* Each input messages are independent R.V.

* The marginal input message distributions

$$P_{m^{[t]}_{x_k, c_j}} \Longrightarrow \text{Joint distribution}$$



$\Rightarrow$ We can compute the marginal of the output $P_{m^{[t]}_{c_j x_i}}$ from $P_{m^{[t]}_{x_k, c_j}} : k \in \partial j \setminus i$

Summary: Given ① the message maps Φ_v, Φ_c of the variable & check nodes.

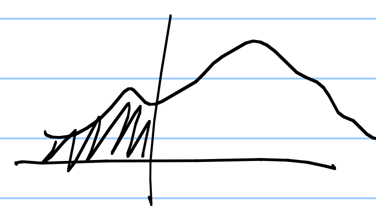
② the marginal of $m_i^{(0)}$ computed from the i.i.d. channel distribution

③ All incoming messages of the support tree are indep (by Thm 1)

$\Rightarrow$ We can iteratively compute the "density" of the messages on the support tree

Once the density of the final LLR $m_i^{[t_0]}$ is computed, the final bit error rate after t_0 iterations is

$$\text{ber} = P(m_{i_0}^{[t_0]} < 0 \mid \vec{X} = \vec{0}) + \frac{1}{2} P(m_{i_0}^{[t_0]} = 0 \mid \vec{X} = \vec{0})$$

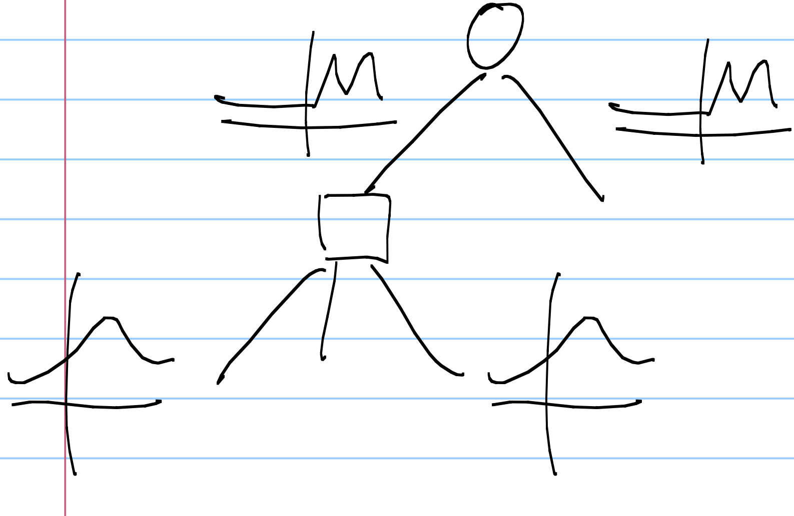


Simplification

* Assume that the channel noise is not only memoryless (independen) but also identically distributed

$\Rightarrow$ There is no need to consider a fixed root x_{i_0}. All bits will have the same ber.

$\Rightarrow$ Moreover, there is no need to distinguish the incoming messages, all of them will have the same marginal distribution



As a result, we need only

$P^{(0)}$: The marginal of $m_i^{(0)}$

$P^{(t)}$: The marginal of $V \to C$ messages

$$m^{[t]}_{X_i, C_j}$$

$Q^{(t]}$: The marginal of $C \to V$ messages

$$m^{[t]}_{C_j X_i}$$

$$\Rightarrow \quad P^{(1)} = P^{(0)}$$

$$Q^{(t)} = F_{chk}\left(P^{(t)}, d_c\right)$$

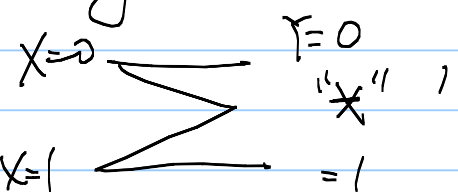
$$P^{(t+1)} = F_{var}\left(Q^{(t)}, d_v, P^{(0)}\right)$$

where F_{chk}, F_{var} are functionals that take densities as input & output a new density.

Even with the above simplification

Usually, the "density evolution" does not admit any closed-form expression, except for the simple setting of BECs.

Example: the density evolution formula for BECs.



, the erasure prob $= \varepsilon$

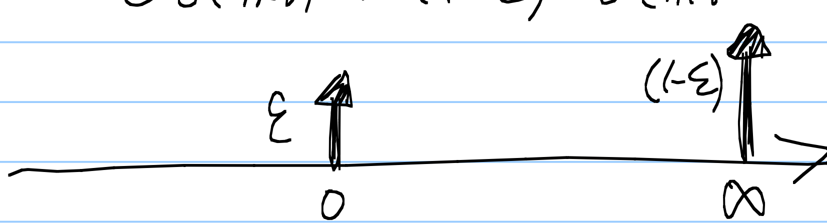
① Suppose the all-zero $\vec{X}$ is transmitted

② $$m_i^{(0)} = \log\left(\frac{P(y_i|0)}{P(y_i|1)}\right)$$

$$= \begin{cases} \infty & \text{if } y_i = 0 \\ 0 & \text{if } y_i = \text{"*"} \\ -\infty & \text{if } y_i = 1. \end{cases}$$

Since the all-zero codeword is transmitted

$$P^{(0)} \triangleq P_{m_i}^{(0)} = \varepsilon\,\delta(m_i) + (1-\varepsilon)\cdot\delta(m_i - \infty)$$

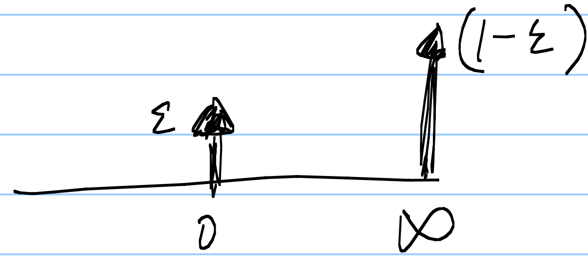


$$m_{c_j x_i}^{(0)} = 0 \qquad Q^{(0)} \triangleq Q_{m_{c_j x_i}}^{(0)} = \delta(m)$$

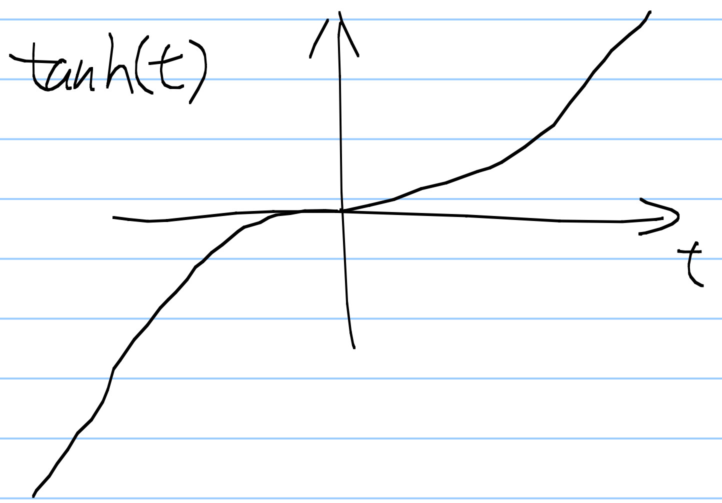
$$m^{(1)}_{x_i c_j} = m^{(0)} + \sum_{l \in \partial i \setminus j} m^{(0)}_{c_l x_i}$$

$$P^{(1)} = P^{(0)} * \left(* \, Q^{(0)} \right)$$

$$=$$



$$m^{[t]}_{c_j x_i} = 2 \tanh^{-1} \left(\prod_{k \in \partial j \setminus i} \tanh\left(\frac{m^{[t]}_{x_k c_j}}{2} \right) \right)$$

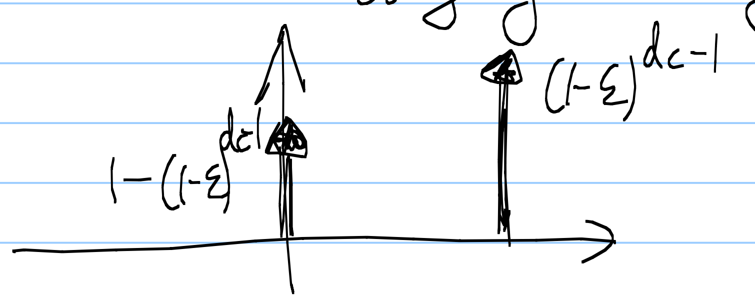


$\Rightarrow$ any one of incoming edge is 0

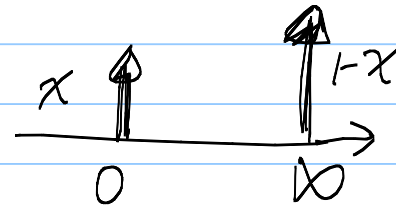
$\Rightarrow$ the outgoing message is zero.

otherwise the outgoing message is ∞

$$\Rightarrow Q^{(1)} =$$



Notice that P, Q are of the form



we thus need to only trace the scalar $\underline{\underline{x}}$. (or p. q)

$$p^{(0)} = \varepsilon \qquad q^{(0)} = 1.$$

$$p^{(1)} = \varepsilon \qquad q^{(1)} = 1-(1-\varepsilon)^{d_c-1}$$

$$p^{(2)} = ?$$

$$m^{(2)}_{x_i c_j} = m^{(0)} + \sum_{l \in \partial i \setminus j} m^{(2)}_{c_l x_i}$$

Note that any one of $m^{(1)}_{c_l x_i}$ & $m^{(0)}_i$ is ∞

$$\Rightarrow m^{(2)}_{x_i c_j} = \infty.$$

otherwise

$$m^{(2)}_{x_i c_j} = 0$$

$$\Rightarrow p^{(2)} = p^{(1)} \cdot \left(q^{(0)}\right)^{d_v-1}$$

$$q^{(2)} = 1-(1-p^{(0)})^{d_c-1}$$

The final answer

$$\begin{cases} p^{(0)} = \varepsilon \qquad q^{(0)} = 1. \\ p^{(t)} = p^{(0)} \cdot \left(q^{(t-1)}\right)^{d_v - 1} \\ q^{(t)} = 1 - \left(1 - p^{(t)}\right)^{d_c - 1} \end{cases}$$

The error prob after t_0 iterations becomes

$$P\left(m_{i_0}^{(t_0)} < 0 \mid X_{i_0} = 0\right)$$

$$+ \frac{1}{2} P\left(m_{i_0}^{(t_0)} = 0 \mid X_{i_0} = 0\right)$$

$$= \frac{1}{2}\left(p^{(0)}\left(q^{(t_0)}\right)^{d_v}\right)$$

Observation of the results of the DE for BECs & for all other channels.

* Experiencing a threshold behavior, Ex: for BECs if $\varepsilon < 0.4293$, $\Rightarrow$ converges to zero.

$\varepsilon > 0.4994 \Rightarrow$ bounded away from zero.

The threshold behavior is provable for BECs.

* Thm: for any i.i.d. symmetric channel, the error prob $P_{ber}^{(t)}$ is non-increasing function with respect to t.

Pf: The density evolution is actually computing the error of a "support tree code". When the support tree is larger, the $P_{ber}^{(t)}$ decreases.