Lecture 17

Note Title 3/7/2012

* Density Evolution Analysis for LDPC codes (Richardson Urbanke 01)

* It is generally very difficult to analyze individual codes. But it is easier to analyze the averaged behavior of a set of codes, also termed the code ensemble.

* DE focuses on the following special code ensemble.

* The regular bipartite-graph-based ensemble, which has three parameters

n: the codeword length.

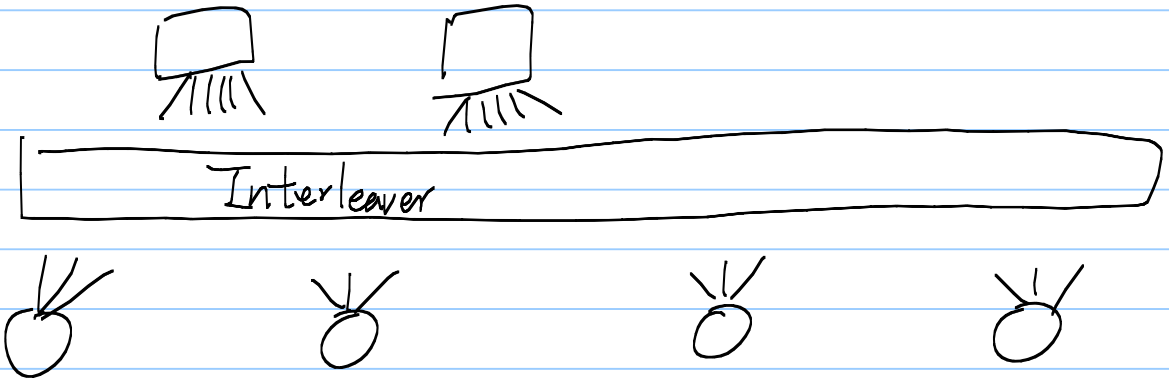
d_v: the variable node degree

d_c: the check node degree

The construction:

① n variable nodes, $n \times \frac{d_v}{d_c}$ check nodes.

② Each variable node has d_v sockets
Each check node has d_c sockets
Ex: $d_v = 3$ $d_c = 6$ $n = 4$

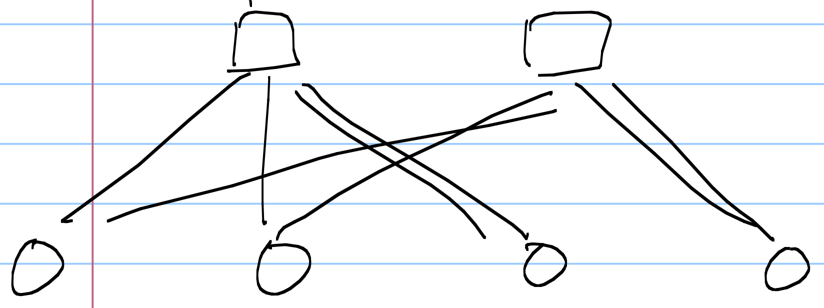


③ A uniform interleaver that maps the var. sockets to the check node sockets.

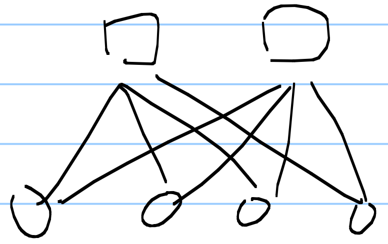
④ $H = \left(h_{j,i} \right)$

Each entry $h_{j,i} = 1$ if there is an odd # of edges connecting X_i, C_j
$= 0$ otherwise

Example $d_v=2$ $d_c=4$ $N=4$



$$H=\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$



$$H=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

⑤ Any permutation / interleaver will lead to a new code.
There are $(nd_v)!$ different permutation $\Rightarrow$ $(nd_v)!$ different codes.

⑥ We are interested in the bit error rate averged over $x_1, \cdots, x_N$ and over all $(nd_v)!$ codes, when we stop the LDPC decoder after a fixed # of T iterations.

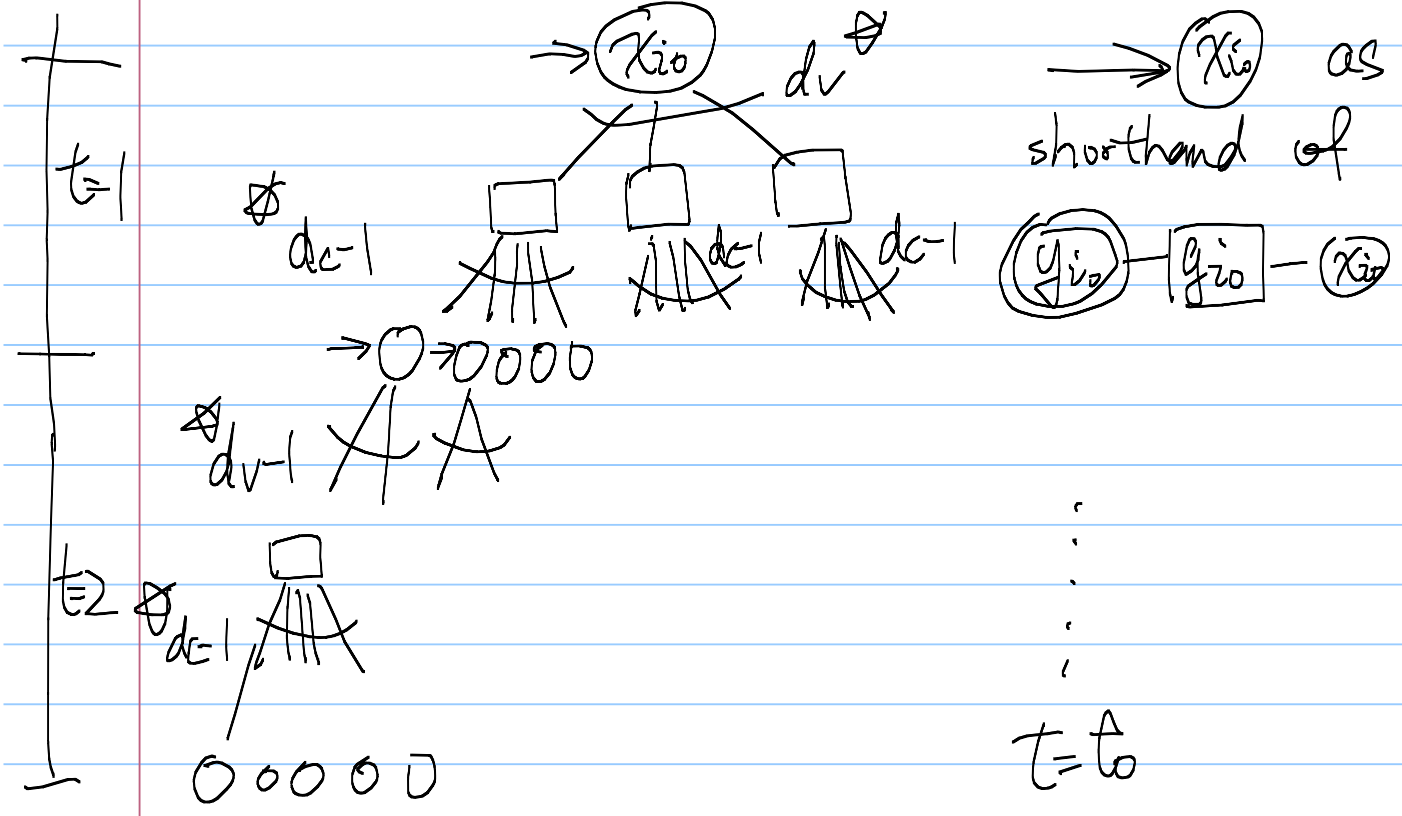
⑦ For finite n, the problem is still too hard. We let $n \to \infty$ while keeping T, d_v, d_c fixed.

Recall our "free of short cycles" observation.

* Thm 1: Fix i_0 (focusing on a given bit x_{i_0}) and fix t_0, d_v, d_c

$$\lim_{n\to\infty} \text{Prob}(x_{i_0} \text{ is involved in a cycle of length} \leq 4t_0) = 1.$$

* Thm 1 implies that the neighborhood of x_{i_0} within distance $2t_0$ must be a tree.



Use x_{i_0} as shorthand of y_{i_0} — g_{i_0} — x_{i_0}

There is no repeated appearance of any variable node x_i

* The above subgraph is termed the "support tree" of the message passing decoder after t_0 iterations.

Namely, the decision of X_{i_0} after t_0 iterations depends only on the observation y_i for those x_i in the support tree

* Theorem 2: ① The structure of a linear code is symmetric. (No special preference between 0 & 1)

② The sumprodct, or belief propagation decoder is symmetric. (No special ⋯ between 0, 1)

③ If the channel is also symmetric

ex: BSC, $X = \begin{smallmatrix}1\\-1\end{smallmatrix} \longrightarrow \overset{N\downarrow}{\oplus} \longrightarrow Y$

BiAWGNC

then the BER/FER performances of sending any two codewords $\vec{x}_1$, $\vec{x}_2$ are identical

Corollary: to evaluate the performance of an LDPC code, we can assume the all-zero $\vec{x} = 00000\cdots0$ is sent.

The bit error rate after t_0 iteration

$$P_{b,i_0}^{[t_0]} = P(\hat{X}_{i_0}^{[t_0]} \neq X_{i_0})$$

$$= P(\hat{X}_{i_0}^{[t_0]} \neq X_{i_0} \mid \vec{X} = \vec{0})$$

$$= P(\hat{X}_{i_0}^{[t_0]} \neq 0 \mid \vec{X} = \vec{0})$$

$$= P(m_{i_0}^{[t_0]} < 0 \mid \vec{X} = \vec{0})$$

$$+ \frac{1}{2} P(m_{i_0}^{[t_0]} = 0 \mid \vec{X} = \vec{0})$$

For the following, we focus only on symmetric channels

* Observation 3: $m_{i_0}^{[t_0]}$ is a random variable, since it depends on the random observation y_i within the support tree. Exercise: how many obseration bits y_i are involved in the t_0-level support tree of x_{i_0}?

* Let us trace the prob "density" function (pdf) of the LLR messages. In particular, the $P_{m|\vec{x}=\vec{0}}(dm)$

$\Rightarrow$ Density evolution

①

Density of the initial message $m_i^{(0)}$

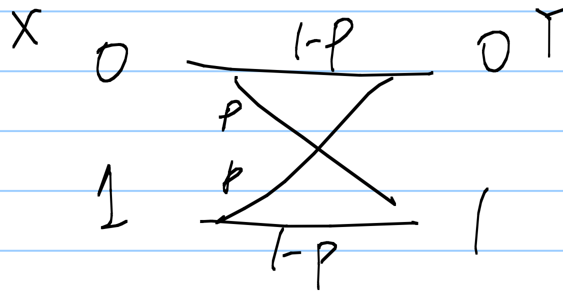
Recall

$$m_i^{(0)} = \log\left(\frac{P_{Y_i|X_i}(y_i|0)}{P_{Y_i|X_i}(y_i|1)}\right)$$

$$= f(y_i)$$ a function of observation y_i

$\Rightarrow$ Given the $P_{Y_i|X_i}(\cdot|0)$, we can derive $P_{m_i^{(0)}|X_i}(\cdot|0)$

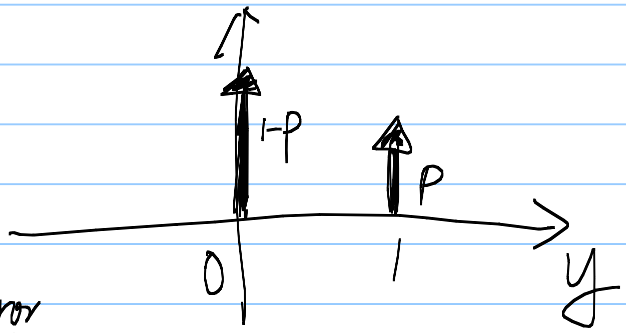
Example: BSC w. cross-over prob p.



$$m_i^{(0)} = \log\left(\frac{P_{Y_i|X_i}(y_i|0)}{P_{Y_i|X_i}(y_i|1)}\right)$$

$$= \begin{cases} \log\left(\frac{1-p}{p}\right) & \text{if } y_i = 0 \\ \log\left(\frac{p}{1-p}\right) & \text{if } y_i = 1 \end{cases}$$

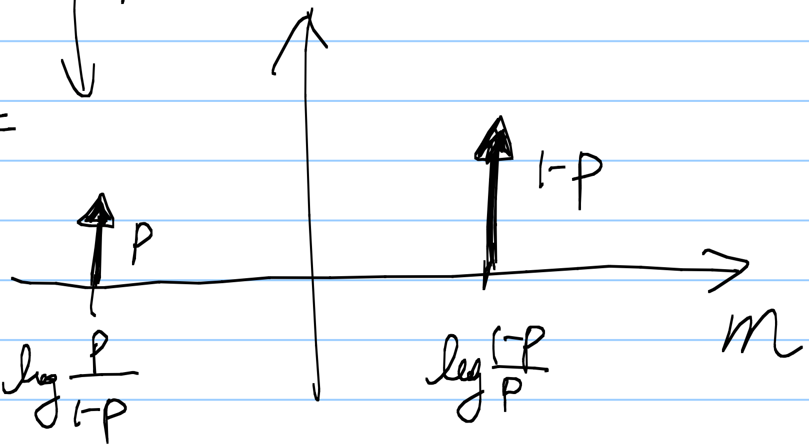
$P_{Y_i|\vec{X}=\vec{0}}(\cdot|\vec{0}) =$



$\Downarrow$

raw error prob.

$P_{m_i^{(0)}|\vec{X}=0} =$



The density of our interest $\longrightarrow$

Remark: The confusion between the "$P_{Y_i|X_i}$" inspired $f(Y_i)$ and the governing density $P_{Y_i|X_i=0}$

Example $Y_i = (-1)^{X_i} + \sigma N_i$, N_i: standard Gsn

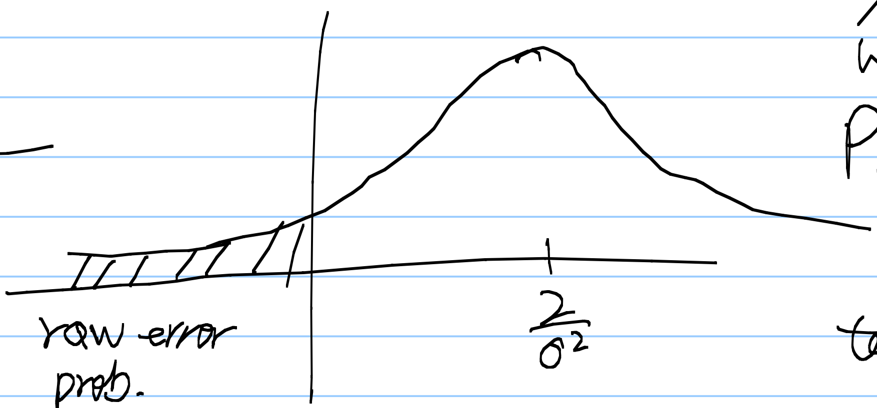
$$m_i^{(0)} = \log\left(\frac{P_{Y_i|X_i}(y_i|0)}{P_{Y_i|X_i}(y_i|1)}\right)$$

$$= \log\left(\frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i-1)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i+1)^2}{2\sigma^2}}}\right)$$

$$= \frac{2}{\sigma^2} y_i$$

$$P_{Y_i|\vec{X}=\vec{0}} \sim Gsn(1, \sigma^2)$$

$$\Rightarrow P_{m_i^{(0)}|\vec{X}=\vec{0}} \sim Gsn\left(\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right)$$



when $\sigma \to 0$ $P_{m_i^{(0)}|\vec{X}=0}$ keeps moving to the right.

$$Q\left(\frac{\frac{2}{\sigma^2}}{\sqrt{\frac{4}{\sigma^2}}}\right) = Q\left(\frac{1}{\sigma}\right)$$