# Review

Note Title 2/20/2012

* The <u>factor graph</u> representation of the convolutional code.

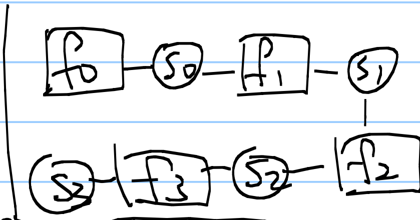
* Each circle is a state variable that can take different values.

* Each rectangle is a <u>factor function</u> that takes input of the variables that are connected to the rectangle.

* The function $\boxed{f}$ is usually decided by the underlying code structure & by the observation. & the objective function of interest.

* The objective function is thus

$$\arg\max_{\vec{S}} f_0(S_0) \prod_{t=1}^{T} f_t(S_{t-1}, S_t)$$



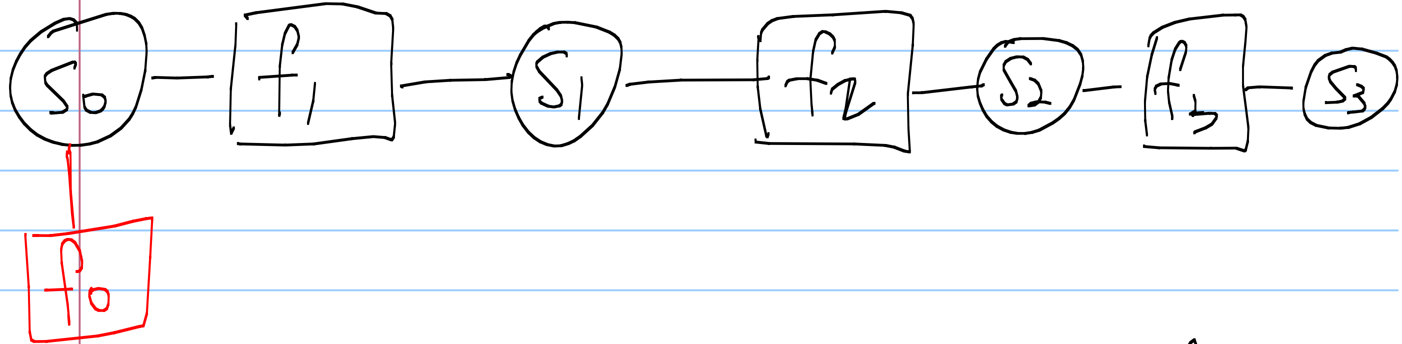

or $$\arg\max_{x} \sum_{\vec{S}: \vec{S}|_i = x} f_0(S_0) \prod_{t=1}^{T} f_t(S_{t-1}, S_t)$$

Note: Each $f_t$ may be different due to different $\vec{y}$

Lecture 12

Note Title 2/20/2012

Back to our discussion of the convolutional codes




Let us rederive the VA based on the factor graph representation

Obj: $\max\limits_{\vec{S}} f_0(S_0) \cdot \prod\limits_{t=1}^{T} f_t(S_{t-1}, S_t)$

Forward metrics $a_{S_t} = \max\limits_{S_{t-1}} a_{S_{t-1}} f_t(S_{t-1}, S_t)$

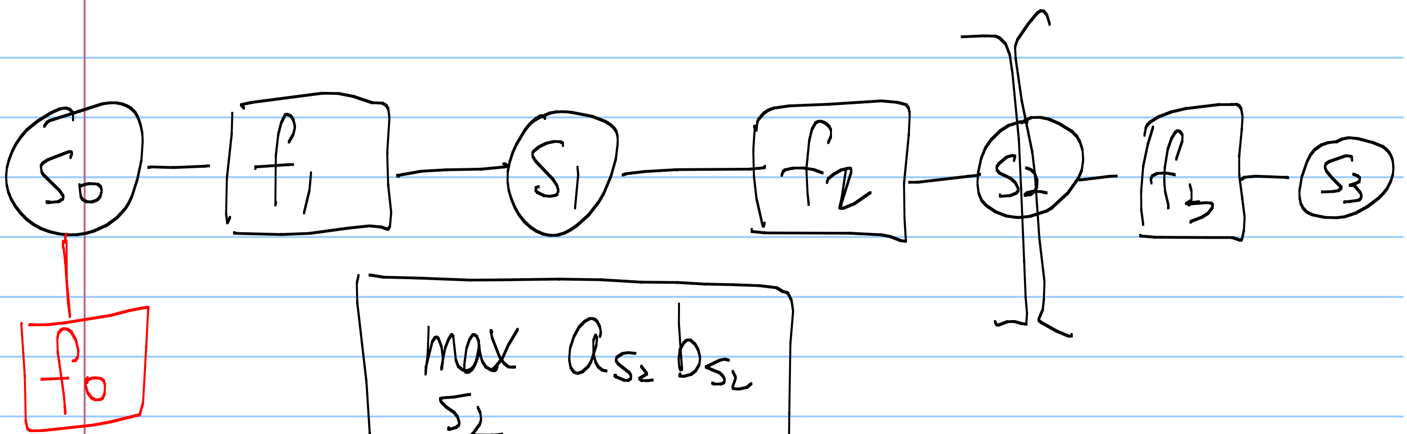
$a_{S_0} = f_0(S_0)$

Backward metrics

$b_{S_t} = \max\limits_{S_{t+1}} b_{S_{t+1}} f_{t+1}(S_t, S_{t+1})$

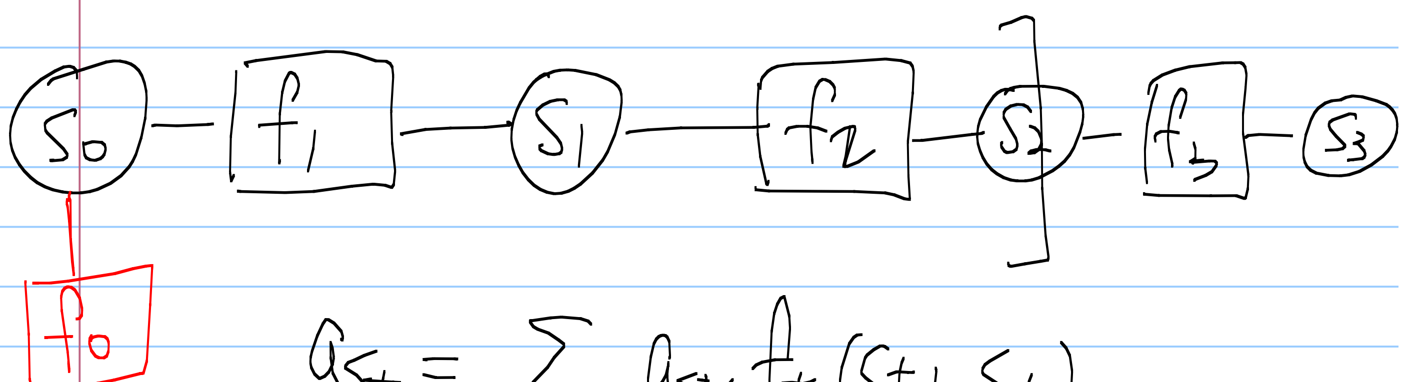
$b_{S_T} = 1$ for all $S_T$ values.

To combine the $a_{s_t}$ & $b_{s_t}$.



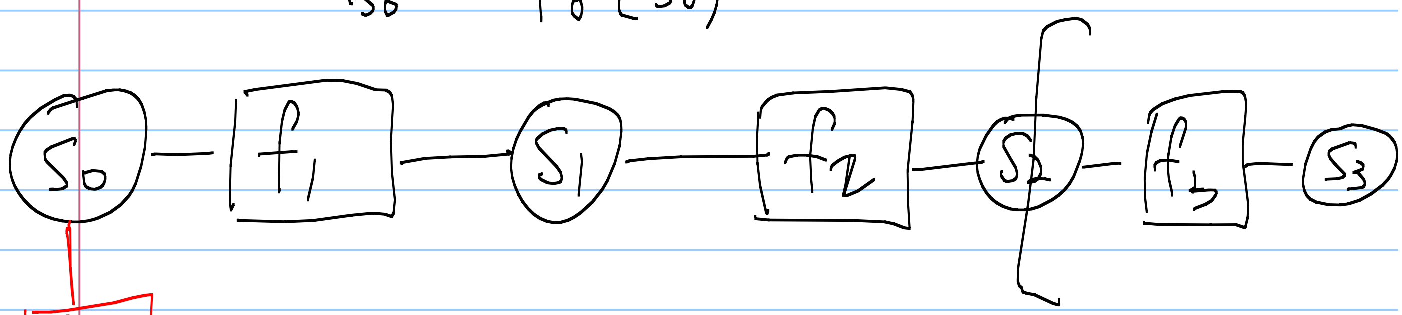

$$\boxed{\max_{s_2} a_{s_2} b_{s_2}}$$

The BCJR algorithm can be derived similarly




$$a_{s_t} = \sum_{s_{t-1}} a_{s_{t-1}} f_t(s_{t-1}, s_t)$$

$$a_{s_0} = f_0(s_0)$$




$$b_{s_t} = \sum_{s_{t+1}} f_{t+1}(s_t, s_{t+1})\, b_{s_{t+1}}$$

This state based generation leads to new application of the VA & BCJR decoders:

Example 1: Consider a simple linear code with codeword length 2, and parity-check matrix $H = (1, 1)$.
Both coderwords are equally likely to be chosen.

Suppose we observe $\vec{y}_{obs} = (1, 1)$ through a BSC with cross-over prob $p$.

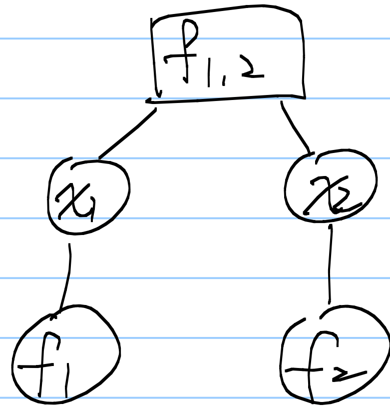
$$\therefore \; P_{\vec{X},\vec{Y}}\big((0,0),(1,1)\big)$$

$$= P_{\vec{X}}(0,0) \cdot P_{\vec{Y}|\vec{X}}\big((1,1)\,|\,(0,0)\big)$$

$$= \underbrace{P_{\vec{X}}(\underline{x_1}, \underline{x_2}) \cdot P_{Y_1|X_1}(y_1|\underline{x_1}) \cdot P_{Y_2|X_2}(y_2|\underline{x_2})}_{\text{3 factors.}}$$

2 variables

Ans. The factor graph is




$$f_1(x_1) = P_{Y_1|X_1}(y_1|x_1) = \begin{cases} p & \text{if } x_1 = 0 \\ 1-p & \text{if } x_1 = 1 \end{cases}$$

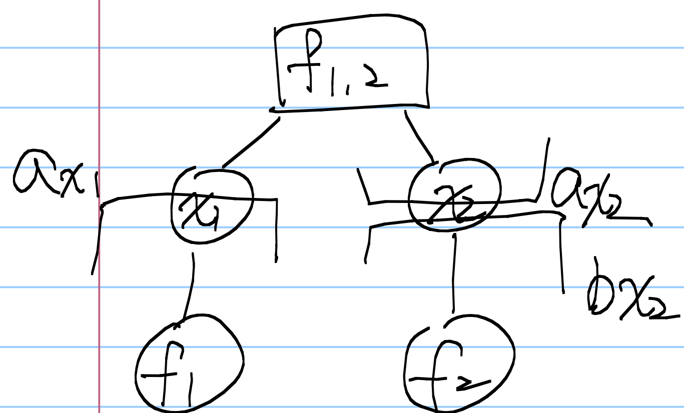
$$f_2(x_2) = P_{Y_2|X_2}(y_2|x_2) = \begin{cases} p & \text{if } x_2 = 0 \\ 1-p & \text{if } x_2 = 1 \end{cases}$$

$$f_{1,2}(x_1, x_2) = P_{X_1 X_2}(x_1, x_2) = \begin{cases} \frac{1}{2} & \text{if } x_1 + x_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Q: Use the VA to find the most likely codeword.

We use

The 2-WAY VA.




$$a_{x_1} = f_1(x_1) = \begin{cases} p & \text{if } x_1 = 0 \\ 1-p & \text{if } x_1 = 1 \end{cases}$$

$$a_{x_2} = \max_{x_1} a_{x_1} f_{1,2}(x_1, x_2)$$

$$= \begin{cases} \max\left(\underset{x_1=0}{p \times \tfrac{1}{2}}, \underset{x_1=1}{(1-p) \times 0}\right) = \dfrac{p}{2} & \text{if } x_2 = 0 \\ \max\left(p \times 0, (1-p) \times \tfrac{1}{2}\right) = \dfrac{1-p}{2} & \text{if } x_2 = 1 \end{cases}$$

$$b_{x_2} = f_2(x_2) = \begin{cases} p & \text{if } x_2 = 0 \\ 1-p & \text{if } x_2 = 1 \end{cases}$$

The final decision

$$\max_{x_2} a_{x_2} b_{x_1} = \max\Big(\overset{x_2=0}{\frac{p^2}{2}}, \overset{x_2=1}{\frac{(1-p)^2}{2}}\Big)$$

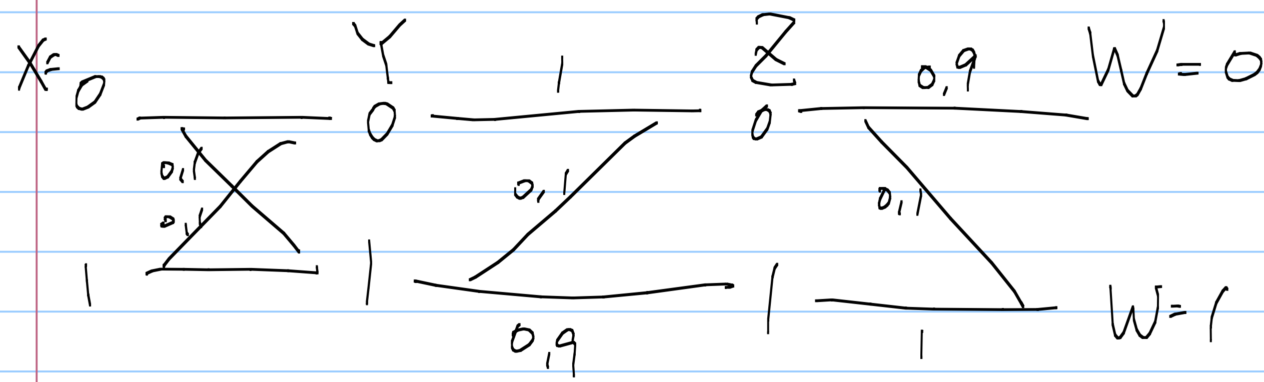
+ backward tracking for both directions.

Q: Does the answer make sense?

A: Yes. it is indeed the MAP decoder

Example 2: A Markov chain example

The channel model:



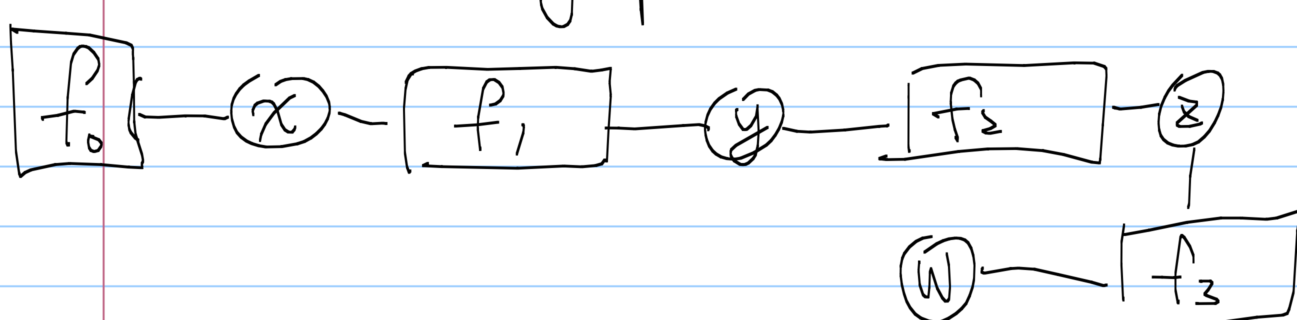

Q: Suppose we know $X=0$, what is the most likely value of $W$?

Ans: Let us use the factor graph + BCJR to solve this problem

Obj: $P_{XYZW} = \underbrace{P_X}_{} \cdot \underbrace{P_{Y|X}}_{} \cdot \underbrace{P_{Z|Y}}_{} \cdot \underbrace{P_{W|Z}}_{}$

factors.

The factor graph becomes

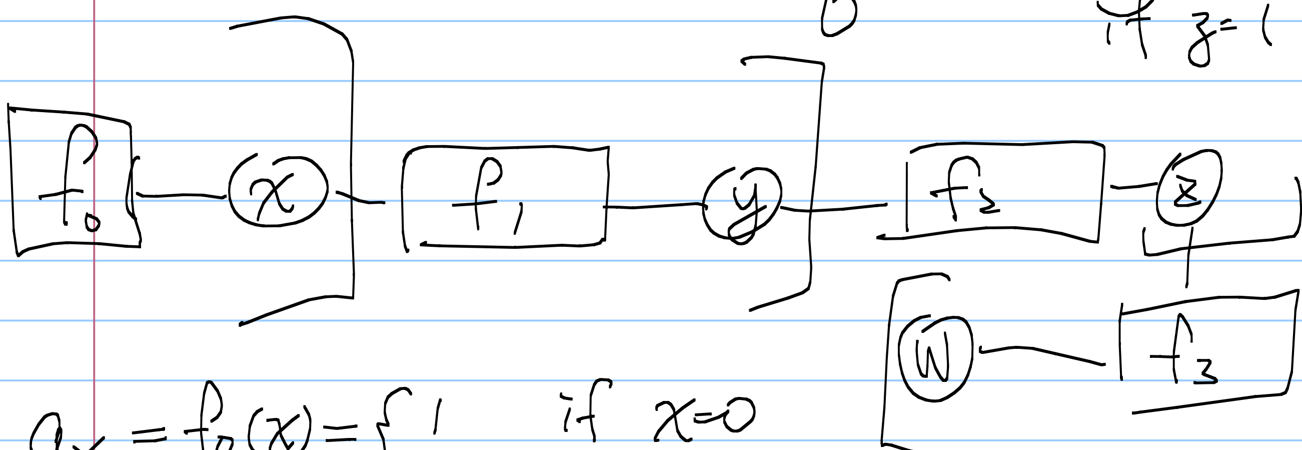


$$f_0(x) = P_X(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(x,y) = P_{Y|X}(y|x) = \begin{cases} 0.9 & \text{if } x=y \\ 0.1 & \text{if } x \neq y \end{cases}$$

$$f_2(y,z) = P_{Z|Y}(z|y) = \begin{cases} 1 & \text{if } y=z=0 \\ 0.9 & \text{if } y=z=1 \\ 0.1 & \text{if } y=1 \; z=0 \\ 0 & \text{if } y=0 \; z=1 \end{cases}$$

$$f_3(z,w) = P_{W|Z}(w|z) = \begin{cases} 1 & \text{if } z, w=1 \\ 0.9 & \text{if } z=0=w \\ 0.1 & \text{if } z=0 \; w=1 \\ 0 & \text{if } z=1 \; w=0 \end{cases}$$




$$a_x = f_0(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & x=1 \end{cases}$$

$$a_y = \sum_x a_x f_1(x,y) = \begin{cases} 0.9 & \text{if } y=0 \\ 0.1 & \text{if } y=1 \end{cases}$$

$$a_z = \sum_y a_y f_2(y,z) = \begin{cases} \underbrace{0.9\times 1 + 0.1\times 0.1}_{=0.91} & \text{if } z=0 \\ \underbrace{0.9\times 0 + 0.1\times 0.9}_{=0.09} & \text{if } z=1 \end{cases}$$

$$a_w = \sum_z a_z f_3(z,w) = \begin{cases} 0.91\times 0.9 + 0.09\times 0 \\ = 0.819 & \text{if } w=0 \\ 0.91\times 0.1 + 0.09\times 1 & \text{if } w=1 \\ = 0.181 \end{cases}$$

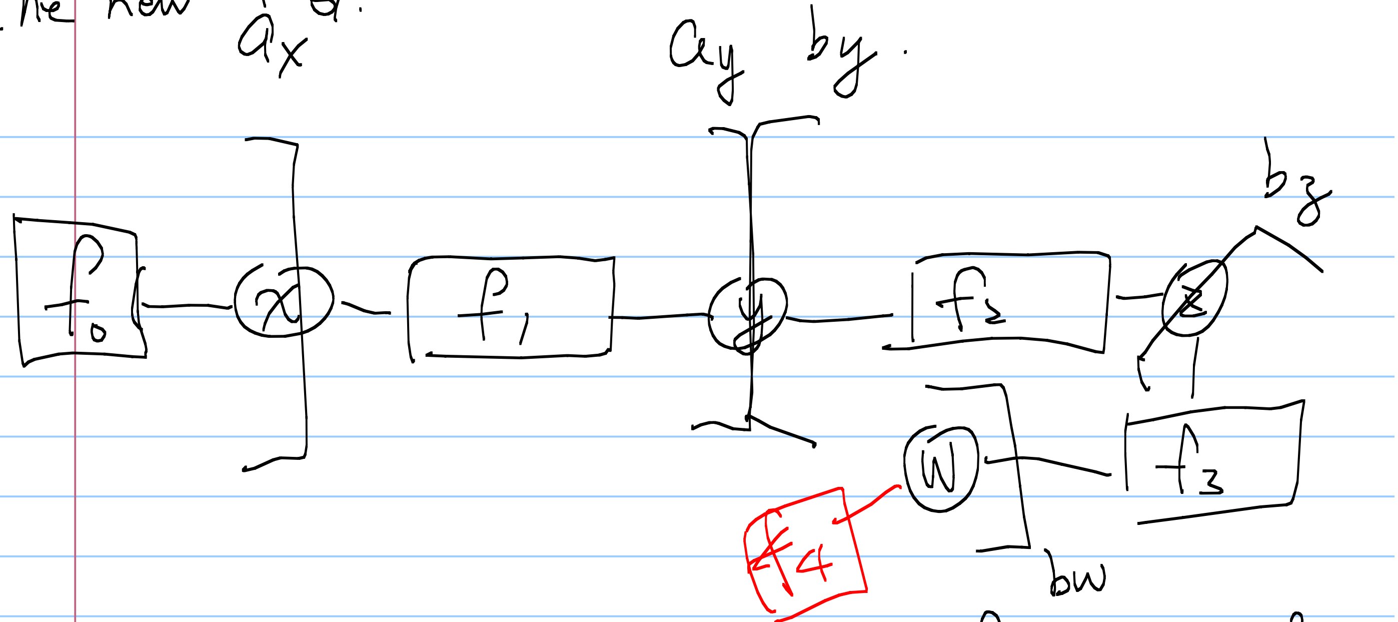
$$\boxed{\arg\max_w a_w = 0 = \text{a MAP decision}}$$

* If you trace the computation in terms of probability, we are actually computing the marginals of $X, Y, Z, W$ respectively.

Q: Suppose we know $X=0$, and $W=1$. What is the MAP decision of $Y$.

A: We need to modify the factor graph to take into account that $W=1$ is given.

The new FG.




$$a_x = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x=1 \end{cases}$$

$$b_w = f_4(w) = \begin{cases} 0 & \text{if } w=0 \\ 1 & \text{if } w=1 \end{cases}$$

$$b_z = \sum_w f_3(z,w)\, b_w$$

$$= \begin{cases} 0.9\times 0 + 0.1\times 1 = \boxed{0.1} & \text{if } z=0 \\ 0\times 0 + 1\times 1 = \boxed{1} & \text{if } z=1 \end{cases}$$

$$a_y = \begin{cases} 0.9 & \text{if } y=0 \\ 0.1 & \text{if } y=1 \end{cases}$$

Remark: The connection to the traditional prob computation is not clear

$$b_y = \sum_z f_2(y,z)\, b_z$$

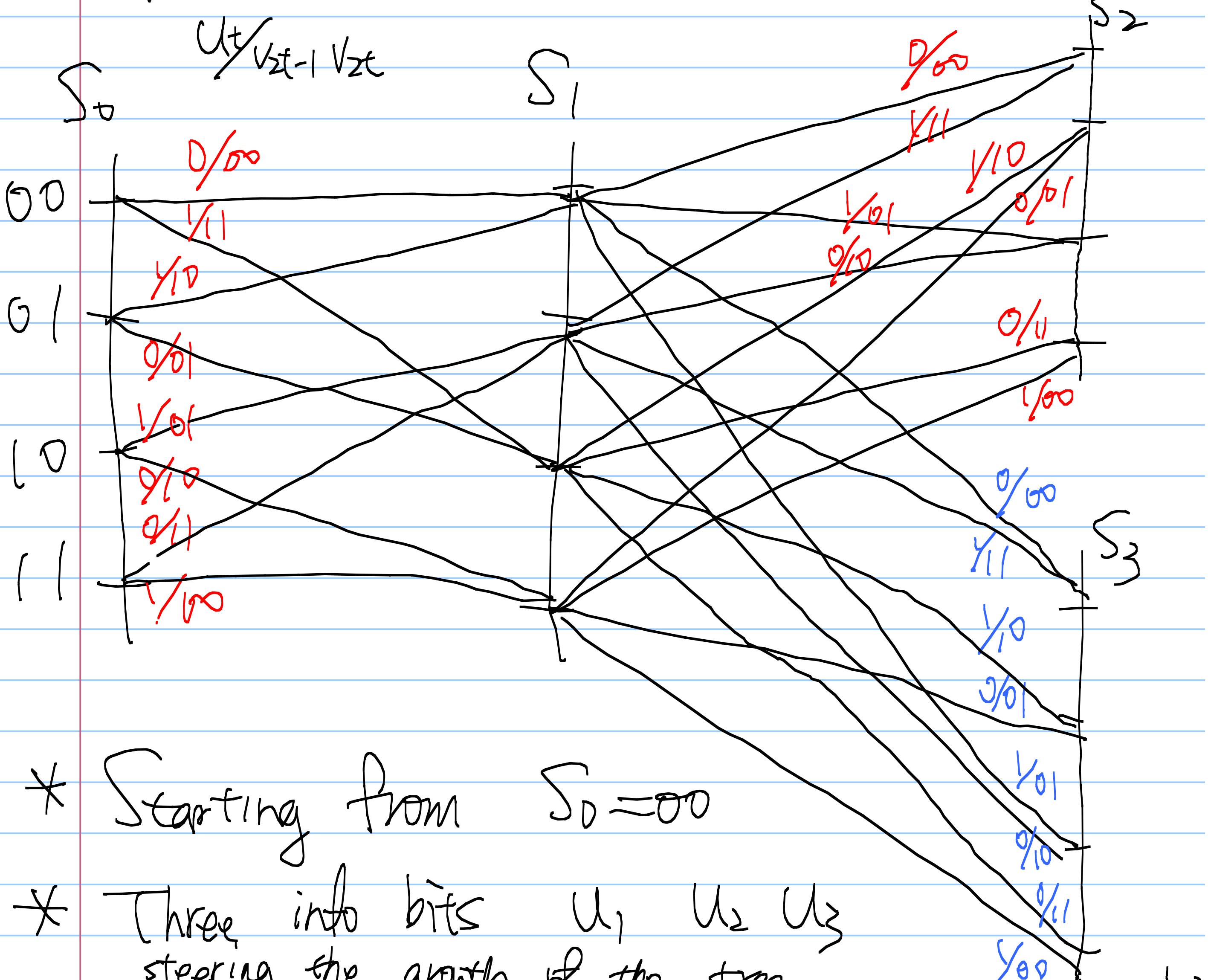
$$= \begin{cases} 1\times 0.1 + 0\times 1 = 0.1 & y=0 \\ 0.1\times 0.1 + 0.9\times 1 = 0.91 & y=1 \end{cases}$$

decision:

$$\max_y a_y b_y = \max_y \begin{cases} 0.09 & y=0 \\ 0.091 & y=1 \end{cases}$$

$\Rightarrow$ MAP decision of $y$ is $1$

## Example 3: A tree-based "convolution" code




* Starting from $S_0 = 00$

* Three info bits $u_1\ u_2\ u_3$ steering the growth of the tree

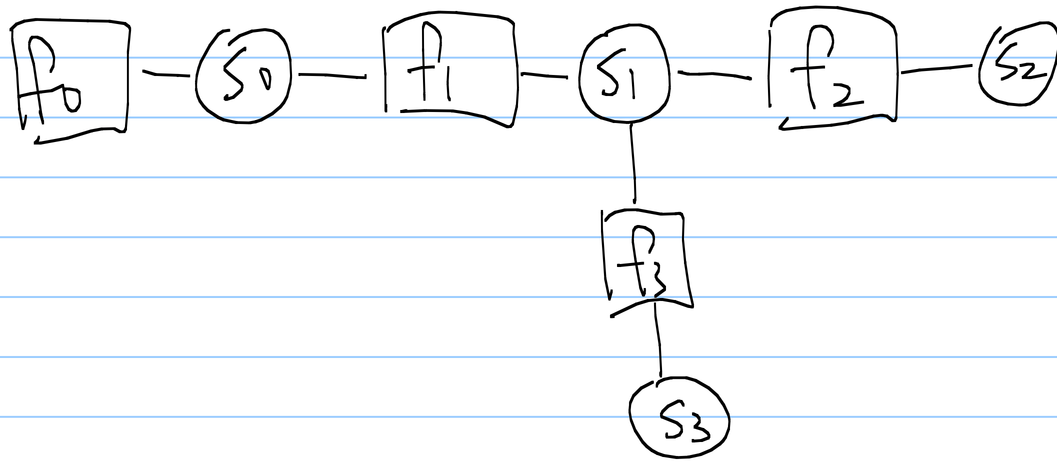
* Six coded bits $v_1 v_2\ v_3 v_4\ v_5 v_6$ (rate $\frac{1}{2}$)

* Encoding is straightforward.

* The channel is BSC with crossover prob 0.1
* observation: $\vec{y} = 001110$

Q: Use the VA to decide the ML codeword.

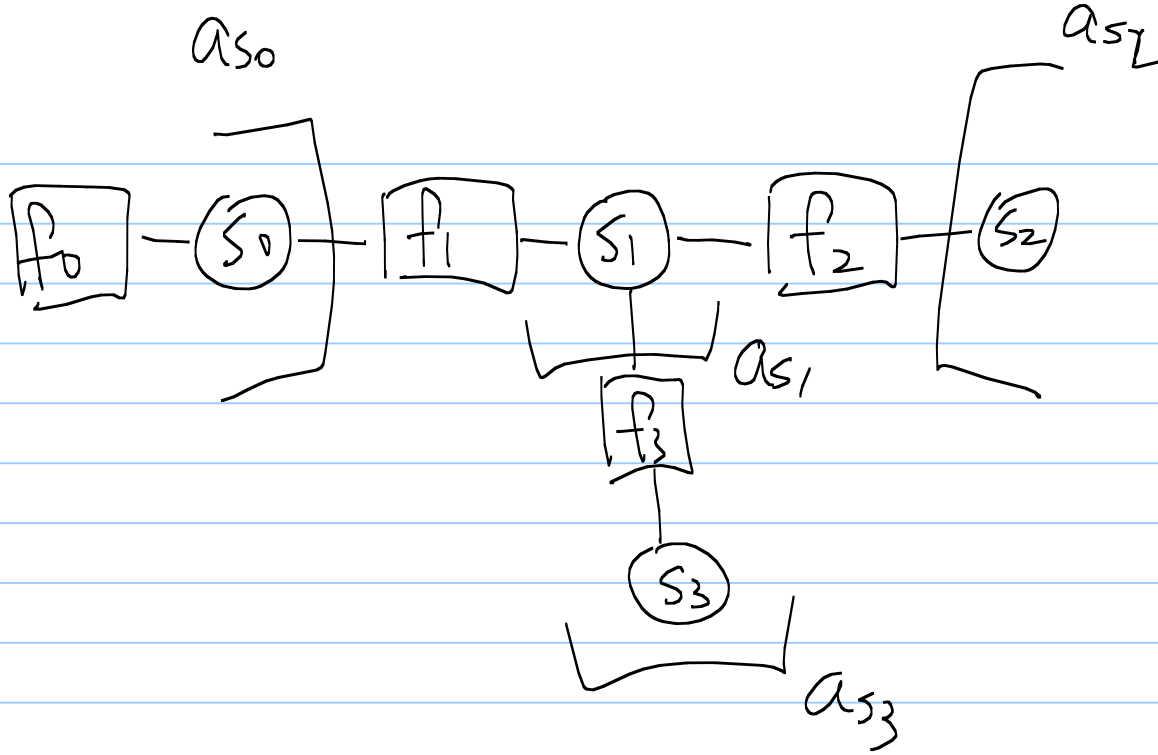
Ans: ① Convert it to the factor graph representation:




② Initialize the $f_0 \cdots f_3$ functions by the trellis structure & by the observation $\vec{y}_{obs}$

$$f_0(S_0) = \begin{cases} 1 & \text{if } S=00 \\ 0 & \text{otherwise} \end{cases}$$

$f_1 \cdots f_3$ are initialized in the same way as the prev. example




Each $a_s$ is a summarizing "function" with respect to $S$.

$$a_{s_0} = f_0(s_0)$$

$$a_{s_2} = 1$$

$$a_{s_1} = \max_{\underbrace{s_0, s_2}_{4\times4=16 \text{ choices.}}} a_{s_0} f_1(s_0, s_1)\, a_{s_2} f_2(s_1, s_2)$$

$$a_{s_3} = \max_{s_1} a_{s_1} f_3(s_1, s_3)$$

Final decision:

$$\max_{s_3} a_{s_3} \quad + \text{ backward tracking}$$