Lecture 10

Note Title 2/13/2012

* The VA is an efficient codeword-based ML decoder. 6 different variants.

Q: Can we also design an efficient bit-based ML decoder?

Recall that

$$\prod_{t=1}^{T} f_t(\overline{p[t]}) = P(\vec{Y}=y \mid \vec{V}=\vec{p}) = \frac{P(\vec{Y}=u, \vec{V}=\vec{p})}{2^{-\#\text{ u bits}}}$$

A codeword-based decoder finds

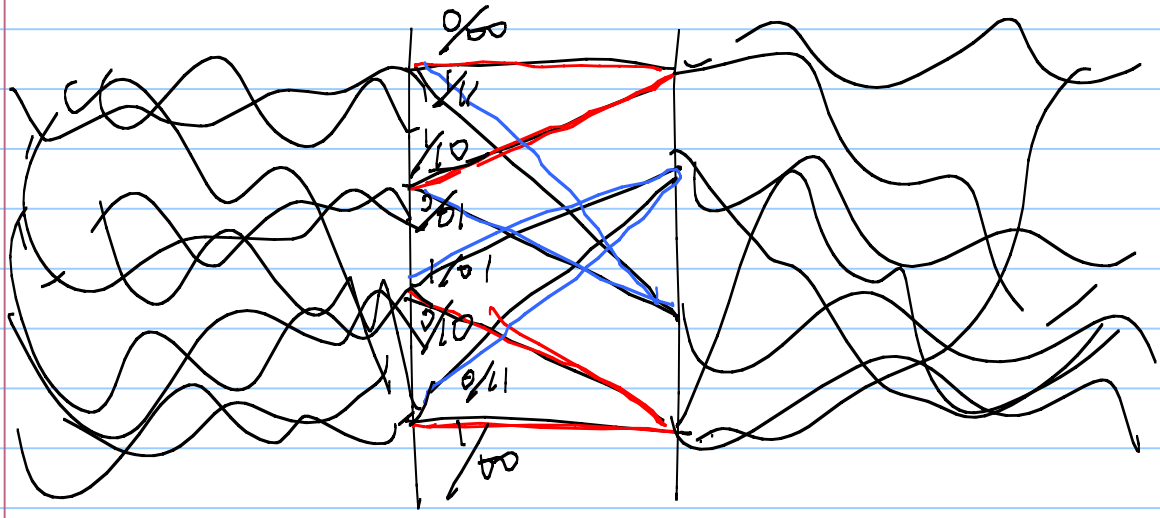
$$\underset{\vec{p}}{\operatorname{argmax}} \prod_{t=1}^{T} f_t(\overline{p[t]})$$

A bit-based decoder of the ith bit finds

$$\underset{x=0,1}{\operatorname{argmax}} \left(\sum_{\vec{p}:\vec{p}|_i=x} \prod_{t=1}^{T} f_t(\overline{p[t]}) \right)$$

t

Illustration V_{t-1}, V_t



If we are interested in

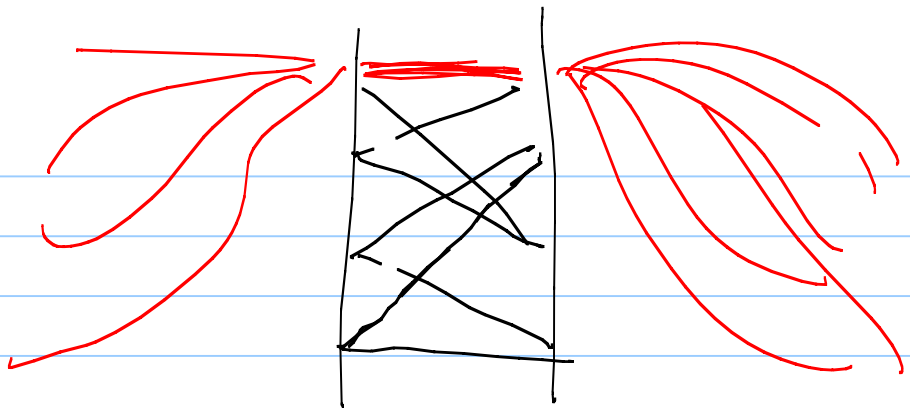
$V_t = 0$ vs. $V_t = 1$

$$\sum_{\text{all paths using}} \prod_{t=1}^{T} f_t(p^{[t]}) \quad \text{vs.} \quad \sum_{\text{all paths using}} \prod_{t=1}^{T} f_t(p^{[t]})$$

How to compute the sum efficiently?

We first look at each segment respectively.

Sum of



$$= \left(\sum_{\text{○ ○}} \prod_{s=1}^{t-1} f_s(\overline{p[s]}) \right) f_t(\text{○ ○})$$

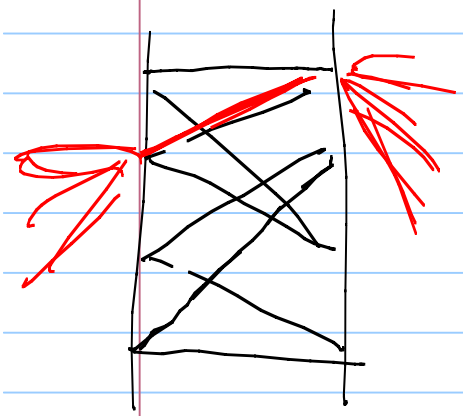
A generalized form of the distributive rule

$$a_1 b c_1 + a_2 b c_2 + a_1 b c_2 + a_2 b c_1 = (a_1 + a_2) b (c_1 + c_2)$$

$$\left(\sum_{\text{○ ○}} \prod_{s=t+1} f_s(\overline{p[s]}) \right)$$

Similarly

$$= \left(\sum_{\text{○ ○}} \prod_{s=1}^{t-1} f_s(\overline{p[s]}) \right) f_t(\text{○ ○})$$



$$\left(\sum_{\text{○ ○}} \prod_{s=t+1} f_s(\overline{p[s]}) \right)$$

In summary, if we know

$$\left(\sum_{\text{0 0}}^{\text{0 0}} \prod_{s=1}^{t-1} f_s(\overline{p[s]})\right), \left(\sum_{\text{0 0}}^{\text{0 0}} \prod_{s=1}^{t-1} f_s(\overline{p[s]})\right) \ldots$$

& $\left(\sum_{\text{0 0}}^{\text{0 0}} \prod_{s=t+1}^{T} f_s(\overline{p[s]})\right) \left(\sum_{\text{0 0}}^{\text{0 0}} \prod_{s=1}^{T} f_s(\overline{p[s]})\right) \ldots$

then we can compute

$$\sum_{\text{all paths using } \rightarrow} \prod_{t=1}^{T} f_t(\overline{p[t]}) \quad \text{vs.} \quad \sum_{\text{all paths using } \not\rightarrow} \prod_{t=1}^{T} f_t(\overline{p[t]})$$

$>$

efficiently. Q: How to find

The BCJR algorithm 1974

* Again we rely on iterative computation.
* Iteration must be performed in two ways.

** The forward iteration computes

$$\left(\sum_{\text{0 0 / 0 0 / 0 0 / 0 0}} \prod_{s=1}^{t-1} f_s(\overline{p[s]}) \right), \left(\sum_{\text{0 0 / 0 0 / 0 0 / 0 0}} \prod_{s=1}^{t-1} f_s(\overline{p[s]}) \right) \ldots$$

** The backward iteration computes

$$\left(\sum_{\text{0 0 / 0 0 / 0 0 / 0 0}} \prod_{s=t+1}^{T} f_s(\overline{p[s]}) \right) \left(\sum_{\text{0 0 / 0 0 / 0 0 / 0 0}} \prod_{s=t+1}^{T} f_s(\overline{p[s]}) \right).$$

Detailed description of the BCJR algorithm

① Given the observation $\vec{y}_{obs}$
Compute the metric $m_{s,s'}$ for each path segment from state $s \to$ state s'. (The same step as in the VA.)

② The forward iteration. Compute the forward state metric

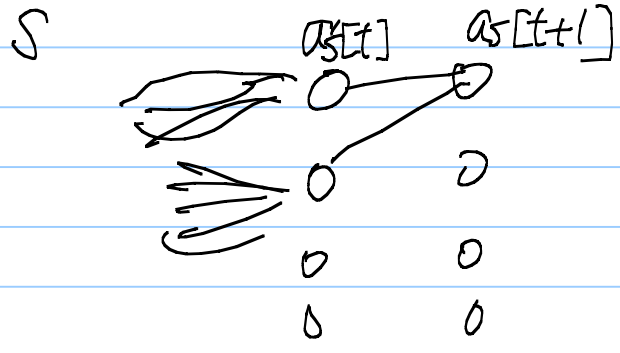
$$a_s[t]$$

Initialization

$$a_{00}[0] = 1, \quad a_s[0] = 0 \text{ for all } s \neq 00$$

$$\forall t = 0, \cdots, T-1,$$

$$a_s[t+1] = \sum_{s' \text{ that reaches } s} a_{s'}[t] \cdot m_{s',s}$$



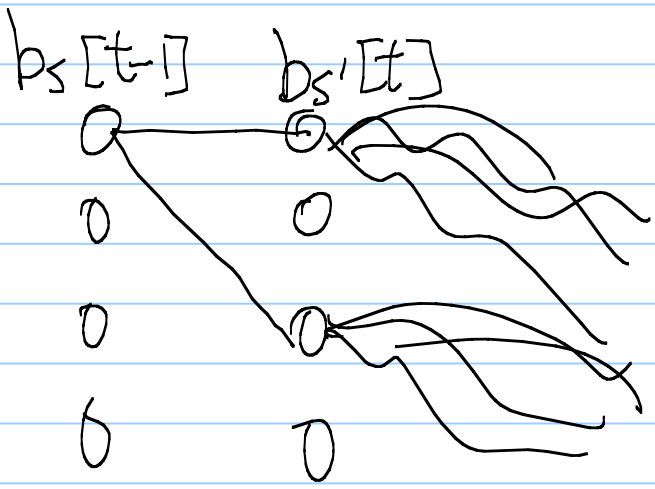
③ Backward iteration. Compute $b_s[t]$

Initialization

$b_s[T] = 1$ for all s

$\forall\ t = T, \cdots, 1.$

$$b_s[t-1] = \sum_{\substack{s' \text{ that is} \\ \text{reachable from } s}} b_{s'}[t]\, m_{s,s'}$$



④ Make the decision for the i-th bit by $a_s[t]\ b_{s'}[t+1]$.

$$\sum_{\overline{p}|_i = x} \prod_{t=1}^{T} f_t(\overline{p}[t])$$

$$= \sum_{\substack{s, s' \text{ s.t.} \\ \text{path } s \to s' |_i = x}} a_s[t_0]\, m_{s,s'}\, b_{s'}[t_0+1]$$

where the i-th bit is in the t_0-th path segment.

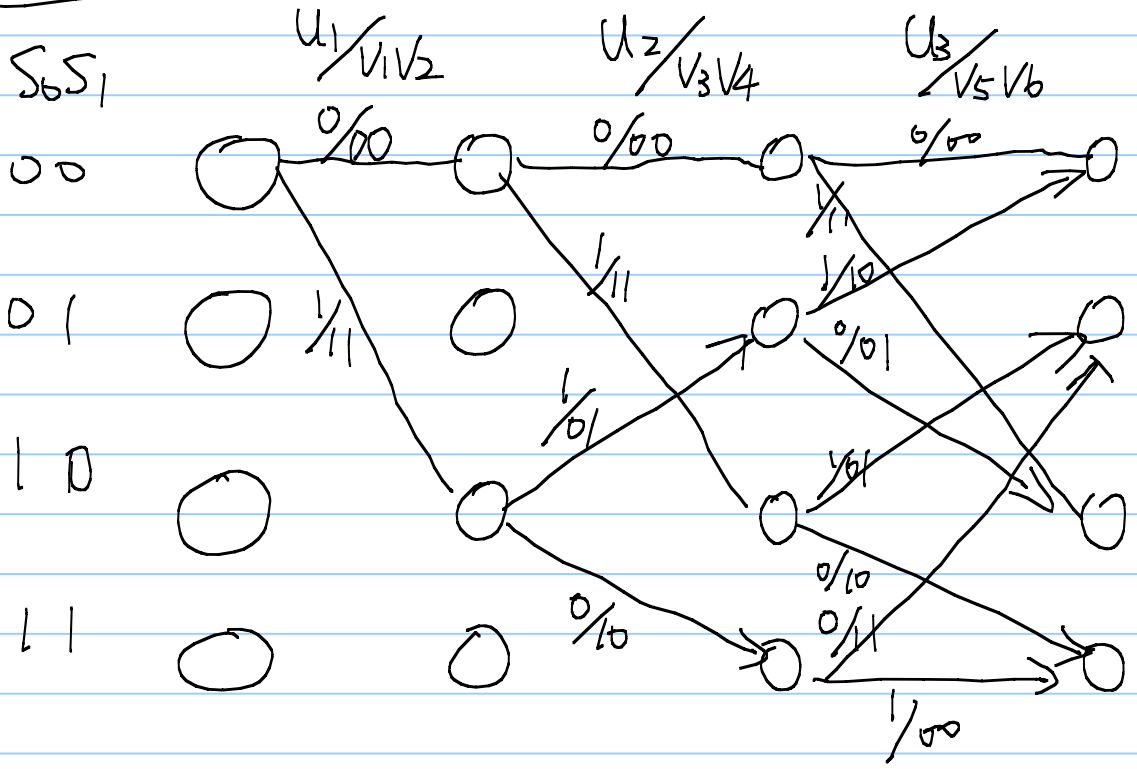
Then we can find

$$\underset{x=0,1}{\operatorname{argmax}} \sum_{\bar{p}|_i = x} \prod_{t=1}^{I} f_t(\bar{p}[t])$$

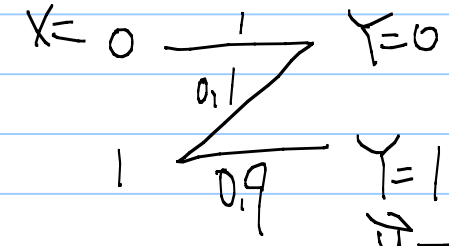
* BCJR finds the optimal bit-level ML decoder.
* The BCJR algorithm can be easily modified to be a ML decoder for the info bit u & for the coded bits v. by carefully counting the paths in the final comparison stage.

The principle of the BCJR decoder can be extended to take into account of different prior distributions.

An illustrating example With the addition into of the prior distri P_U



Channel : Z-channel



$\vec{y}=$ The observation is 00 10 01

the input info bits

u_1, u_2, u_3

but suppose we know that are not uniform Bernoulli distributed but have $P(U=1)=0.05$ $P(U=0)=0.95$

Q: Use the BCJR algorithm to find the MAP u_2 value.

Ans: We are interested in maximizing

$$\underset{u_2=0 \text{ or } 1}{\operatorname{argmax}} \sum_{\text{all } \vec{u} \text{ w. the given } u_2 \text{ value}} P_{\vec{Y}}(\vec{y}, \vec{u})$$

↳ The joint prob.

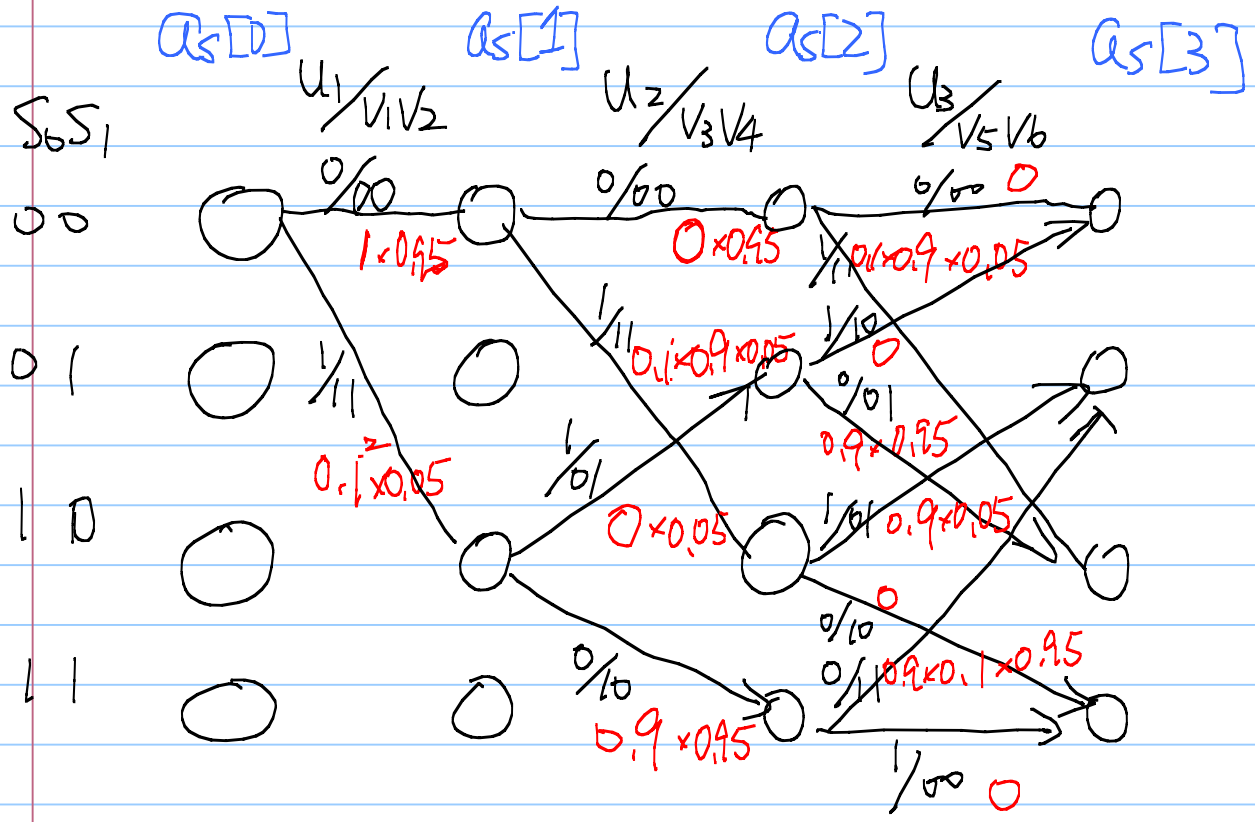
$$= \underset{u_2=0 \text{ or } 1}{\operatorname{argmax}} \sum_{\substack{\text{all paths} \\ \text{with the} \\ \text{given } u_2 \\ \text{value}}} \left(\prod_{t=1}^{3} P_{Y_{2t-1}Y_{2t}|V_{2t-1}V_{2t}} \right) \cdot P_{u_1 u_2 u_3}$$

$$= \underset{u_2=0 \text{ or } 1}{\operatorname{argmax}} \sum_{\substack{\text{all paths} \\ \text{with the} \\ \text{given } u_2 \\ \text{value}}} \left(\prod_{t=1}^{3} \left(P_{Y_{2t-1}Y_{2t}|V_{2t-1}V_{2t}} \cdot P_{U_t} \right) \right)$$

⇓ use $f_t(\cdot)$ instead

$$\equiv \underset{u_2=0 \text{ or } 1}{\operatorname{argmax}} \sum_{\text{all paths using } u_2} \prod_{t=1}^{3} f_t\left(\overline{p[t]}\right)$$

Observation: 00 10 01



Step 1: Initialize the path metrics

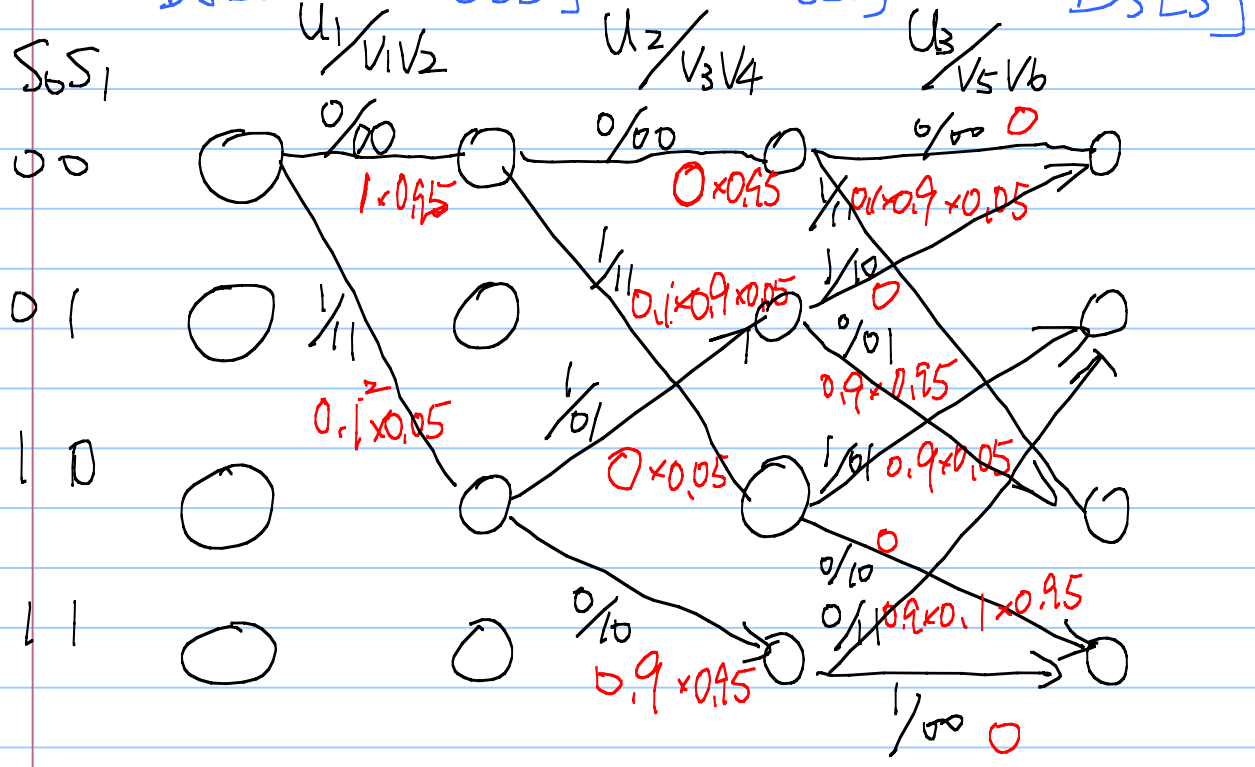
Step 2: Compute the forward metrics.

$$\alpha_S[t+1] = \sum_{S': S' \to S} \alpha_{S'}[t] \cdot m_{S',S}$$

$\alpha_S[t]$			
1	0.95	0	0
0	0	0	5.578875×10^{-4}
0	5×10^{-4}	4.215×10^{-3}	0
0	0	4.275×10^{-4}	0

Observation: 00 10 01

$b_s[0]$ $b_s[1]$ $b_s[2]$ $b_s[3]$

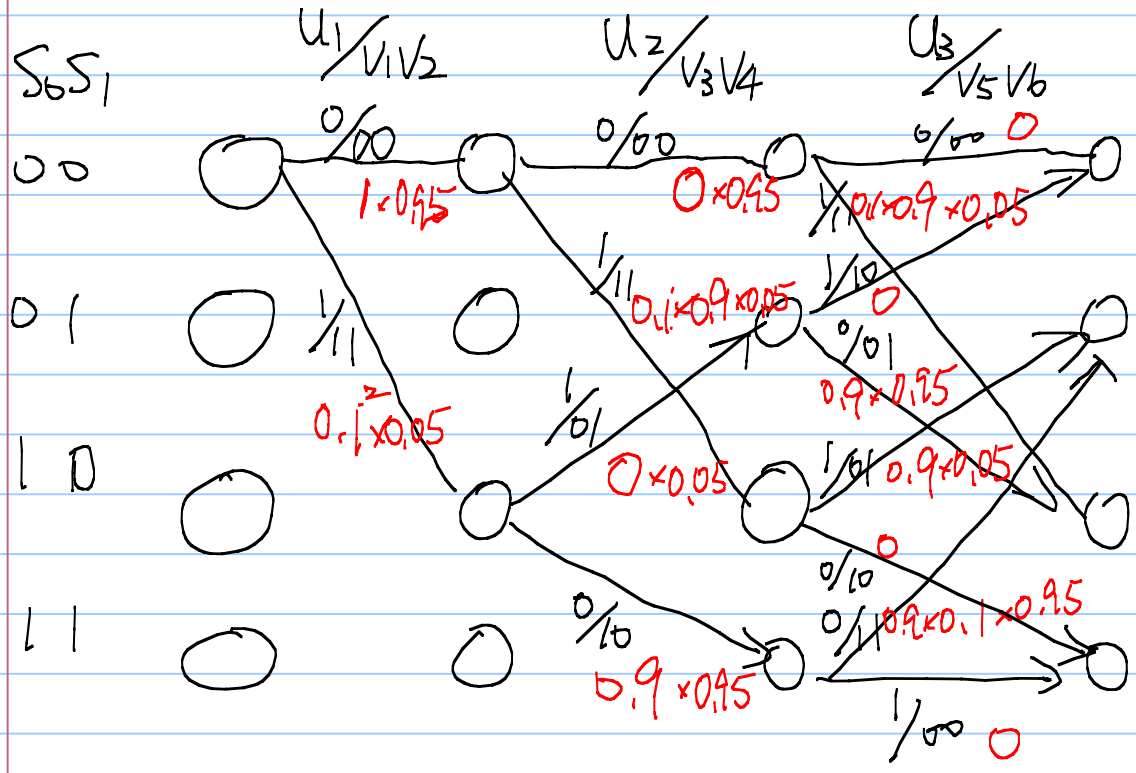


Step 3: Compute the backward metrics.

$$b_s[t-1] = \sum b_{s'}[t]\, m_{s,s'}$$

$b_s[t]$			$b_s[3]$
2.2925×10^{-4}	2.025×10^{-4}	4.5×10^{-3}	1
0	0	8.55×10^{-4}	1
0	7.3125×10^{-2}	4.5×10^{-2}	1
0	0	8.55×10^{-2}	1

Observation: 00 10 01



Step 4: Make a decision on U_2

Two cases: $U_2=0$ $U_2=1$

$$\alpha_{00}[1]\cdot m_{00,00}\, b_{00}[2] + \alpha_{10}[1]\cdot m_{10,11}\, b_{11}[2] = 3.65125\times10^{-5}$$

$$\alpha_{00}[1]\cdot m_{00,10}\, b_{10}[2] + \alpha_{10}[1]\cdot m_{10,01}\, b_{01}[2] = 1.92375\times10^{-4}$$

$\Rightarrow$ The MAP for U_2 is 1.

With high posterior prob.

$$\frac{1.92375\times10^{-4}}{3.65125\times10^{-5}+1.92375\times10^{-4}} \doteq 84\%$$

* Although both being linear with respect to T, the complexity of BCJR $\geq$ VA (needs a lot of multiplication & summation)

* In practice, BCJR is used less frequently as the bit error rate improvement over the VA does not justify the complexity.

* Recently, BCJR is getting more attention due to the invention of the "turbo codes."

* In practice, we simply "approximate" the iteration of the VA & the BCJR.

Common simplification techniques.

① quantized $a_S[t]$ & $b_S[t]$ values

② Incomplete sum / max / product: The best k rule.

③ Proper rescaling. $a'_S[t] \leftarrow \text{const} \cdot a_S[t]$

$b'_S[t] \leftarrow \text{const} \cdot b_S[t]$

since our decision, based on the relative values of $a_S[t]$ ($b_S[t]$) for different S value rescaling does not change the decision

④ Implementation by table look-up.