ECE 695C, Homework #2, due date: 2/9/2012

https://engineering.purdue.edu/∼chihw/12ECE695C/12ECE695C.html

Question 1: Consider the following inference problem:

$$P_X(0) = 1/3, P_X(1) = 2/3,$$
$$P_{Y|X}(\cdot|x) \sim Gsn(0, x + \sigma^2).$$

Find the divergence $D(P_0||P_1)$ between $P_0 = P_{Y|X}(\cdot|0)$ and $P_1 = P_{Y|X}(\cdot|1)$.

Question 2: Consider the following inference problem specified by the joint probability mass function of X and Y:

	$X = 0$	$X = 1$
$Y = 0$	$\frac{5}{36}$	$\frac{5}{18}$
$Y = 1$	$\frac{5}{18}$	$\frac{5}{36}$
$Y = 2$	$\frac{1}{48}$	$\frac{1}{16}$
$Y = 3$	$\frac{1}{16}$	$\frac{1}{48}$

1. What is the corresponding p_e error probability of a MAP detector?
2. If we define $b = E(\sqrt{\frac{P_{X|Y}((1-X)|Y)}{P_{X|Y}(X|Y)}})$, find out the b value for this problem. Hint: you should first decide the function $f(x, y) = \sqrt{\frac{P_{X|Y}(1-x|y)}{P_{X|Y}(x|y)}}$, and then evaluate the $E(f(X, Y))$. In communications, this b is termed the Bhattacharyya noise parameter for a given channel.
3. Consider the following hypothesis testing problem. H_0: A sequence of n 1s $(111\cdots, 1)$ is transmitted over the above binary-input/quaternary-output channel. H_1: A sequence of n 0s $(000\cdots, 0)$ is transmitted over he above binary-input/ternary-output channel and Y_1 to Y_n are the corresponding output.

   Use the Chernoff bound to bound the average error probability of a MAP detector. Verify that the Chernoff bound is indeed the Bhattacharyya noise parameter.
4. [Optional] You might notice that $2p_e \leq b$. Prove that for any given table,

	$X = 0$	$X = 1$
$Y = 0$	$p_{0,1}$	$p_{1,1}$
$Y = 1$	$p_{0,2}$	$p_{1,2}$
$Y = 2$	$p_{0,3}$	$p_{1,3}$

   $2p_e \leq b$ always holds.

5. [Optional] You might notice that $b \leq 2\sqrt{p_e(1-p_e)}$. Prove that for any given table,

	$X=0$	$X=1$
$Y=0$	$p_{0,0}$	$p_{1,0}$
$Y=1$	$p_{0,1}$	$p_{1,1}$
$Y=2$	$p_{0,2}$	$p_{1,2}$

$b \leq 2\sqrt{p_e(1-p_e)}$ always holds. Hint: Note that $f(p) = \sqrt{p(1-p)}$ is a concave function with respect to p.

Question 3: Show that entropy $H(X) = E_X\left\{\log(\frac{1}{P_X(X)})\right\}$ is a concave function. Hint: You can use the convexity of the divergence.

Question 4: Consider a special binary erasure channel as follows. Given a bit string of length n, at most ϵn bits will be completely erased. For example: for $n = 6$, $\epsilon = 1/3$, if one sends 6 bits 010010, then at most 2 bits are erased. One possible received bit string is 01*0*0.

If we are allowed to use the channel exactly n times, design three binary codes that can achieve *error-free* transmission for the following three difference scenarios, respectively.

1. $n = 2$, $\epsilon = 1/2$.
2. $n = 3$, $\epsilon = 1/3$.
3. $n = 4$, $\epsilon = 1/2$.

In your schemes, how many codewords can you pack into the space. The code rate of yours is then defined as

$$\frac{\log_2(\text{the number of codewords})}{n}. \tag{1}$$

Hint: the X vector takes values in $\{0,1\}^n$, while the observation Y vector takes values in $\{0,1,*\}^n$.