ECE 695C, Homework #1, due date: 1/26/2012

https://engineering.purdue.edu/∼chihw/12ECE695C/12ECE695C.html

Question 1: Suppose X takes values in $\{0,1\}$ and $P_X(0) = 1/3$ and $P_X(1) = 2/3$. The conditional probability of $P_{Y|X}(\cdot|x)$ is Laplacian distributed with probability density function $p_{Y|X}(y|x) = \frac{1}{2}e^{-|y-(-1)^x|}$.

1. Find the maximum *a posteriori* probability (MAP) detector as a function of y.
2. Find the maximum likelihood (ML) detector as a function of y.
3. Are these two detectors the same?
4. Find out the mis-detection probability of the MAP detector.
5. Find out the false-alarm probability of the following naive detector:

$$\hat{X}(y) = \begin{cases} 0 & \text{if } y > 1/4 \\ 1 & \text{if } y \leq 1/4 \end{cases} \tag{1}$$

6. Find out the overall/average error probability of the MAP detector.
7. Find out the overall/average error probability of the ML detector.

Question 2: [From Poor, "An introduction to signal detection and estimation"] Suppose Y is a random variable that under hypothesis H_0 has pdf

$$p_0(y) = \begin{cases} \frac{2(y+1)}{3} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and under hypothesis H_1, has pdf

$$p_1(y) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

Repeat the previous question.

Question 3: Consider the following CDMA communication system (if you are not familiar with the CDMA system, please simply treat it as a detection/estimation problem).

When a bit 0 is transmitted, we transmit a signature sequence $X_i = (-1)^i$ for $i = 1$ to 10. I.e., $X_1 X_2 X_3 \cdots X_{10} = (-1)1(-1)\cdots 1$. When a bit 1 is transmitted, we transmit a signature sequence $X_i = 1$ for $i = 1$ to 10. I.e., $X_1 X_2 X_3 \cdots X_{10} = 111\cdots 1$. The ten symbols X_1 to X_{10} are then passed through an i.i.d. Gaussian channel. The observation $Y_i = X_i + N_i$ where N_i is i.i.d. Gaussian with mean 0 and variance σ^2.

Assuming the message bit value is equally likely to be 0 or 1, solve the following question.

1. Model this CDMA system as a hypothesis testing problem.
2. Find the MAP detector and express the MAP detector in the form of "log-likelihood-ratio" test.
3. Find the average error probability.
4. Use the Chernoff bound to derive/approximate the average error probability for general N-bit signature sequence. I.e., $X_1 X_2 X_3 \cdots X_N = (-1)1(-1)\cdots(-1)^N$.
5. [Optional for those who has learned CDMA/digital communication before.] Discuss its relationship to the matched filter in digital communication. Is the matched filter optimal?

Question 4: Consider the following CDMA communication system but with color Gaussian noise (if you are not familiar with the CDMA system, please simply treat it as a detection/estimation problem).

When a bit 0 is transmitted, we transmit a signature sequence $X_1 X_2 = 1(0.5)$ of length 2. When a bit 1 is transmitted, we transmit a signature sequence $X_1 X_2 = (-1)(-0.5)$ of length 2. The two symbols X_1 and X_2 are then passed through a *correlated* Gaussian channel. The observation $Y_i = X_i + N_i$ where (N_1, N_2) is jointly Gaussian with mean $(0, 0)$, marginal variance σ^2, and correlation coefficient $\rho = 0.5$.

Assuming the bit value is equally likely to be 0 or 1, solve the following question.

1. Model this CDMA system as a hypothesis testing problem.
2. Find the MAP detector and express the MAP detector in the form of "log-likelihood-ratio" test.
3. [Optional for those who has learned CDMA/digital communication before.] Discuss its relationship to the matched filter in digital communication. Is the matched filter optimal?

Question 5: Consider the following hypothesis testing problem. H_0: A sequence of n 1s $(111\cdots, 1)$ is transmitted over a binary symmetric channel with crossover probability

p and Y_1 to Y_n are the corresponding output. H_1: A sequence of n 0s $(000\cdots,0)$ is transmitted over a binary symmetric channel with crossover probability p and Y_1 to Y_n are the corresponding output.

Assuming both hypotheses H_0 and H_1 are equally likely to be true, use the Chernoff bound to bound the minimal average error probability. Your answer should be a function of n and p.