CE573 – Structural Dynamics

SDOF – HARMONIC EXCITATION

UNDAMPED SDOF:

\[ m\ddot{x}(t) + kx(t) = F_0 \sin(wt) \]

Solution:

\[ x(t) = x_h(t) + x_p(t) \]

\[ x_h(t) = A \sin(w_n t) + B \cos(w_n t) \]

\[ x_p(t) = ? \]

Let's guess a particular solution be in the form of 
\[ x_p(t) = D \sin(w t) + E \cos(w t) \]

\[ \dot{x}_p(t) = w [D \cos(w t) - E \sin(w t)] \]

\[ \ddot{x}_p(t) = -w^2 [D \sin(w t) + E \cos(w t)] = -w^2 x_p(t) \]

Substitute into the dynamic equilibrium eqn.

\[ m\ddot{x}_p + kx_p = F_0 \sin(wt) \]

\[ (-mw^2 + k)x_p(t) = F_0 \sin(wt) \]

\[ x_p(t) = \frac{F_0 \sin(wt)}{k - mw^2} = \frac{F_0 \sin(wt)}{k} \frac{1}{1 - \frac{w^2}{w_n^2}} \]

Note that the maximum amplitude of \( x_p(t) \) is \( \frac{F_0}{\sqrt{k}} \) where \( \frac{F_0}{k} \) is the static deflection.

Sketching
Solution: $x(t) = x_h(t) + x_p(t)$

$$x(t) = A \sin(w_n t) + B \cos(w_n t) + \frac{F_0}{k} \frac{\sin(w t)}{1 - \left(\frac{w}{w_n}\right)^2}$$

Initial conditions: $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

$$x(0) = B = x_0$$
$$\dot{x}(0) = A w_n + \frac{F_0}{k} \frac{w}{1 - \left(\frac{w}{w_n}\right)^2} = \dot{x}_0$$
$$A = \frac{1}{w_n} \dot{x}_0 - \frac{F_0}{k} \frac{w}{1 - \left(\frac{w}{w_n}\right)^2}$$

Solution in full form:

$$x(t) = x_0 \cos(w_n t) + \frac{\dot{x}_0}{w_n} \sin(w_n t) - \frac{F_0}{k} \frac{w}{1 - \left(\frac{w}{w_n}\right)^2} \sin(w_n t) + \frac{F_0}{k} \frac{w}{1 - \left(\frac{w}{w_n}\right)^2} \sin(w t)$$

If $x_0 = 0, \dot{x}_0 = 0$

$$x(t) = -\frac{F_0}{k} \frac{w}{1 - \left(\frac{w}{w_n}\right)^2} \sin(w_n t) + \frac{F_0}{k} \frac{w}{1 - \left(\frac{w}{w_n}\right)^2} \sin(w t)$$

$$x(t) = \frac{F_0}{k} \frac{1}{1 - \left(\frac{w}{w_n}\right)^2} \left(\sin(w t) - \frac{w}{w_n} \sin(w_n t)\right)$$
**Case 1:** \( w \ll w_n \Rightarrow \frac{w}{w_n} \ll 1 \)

**Case 2:** \( w \gg w_n \Rightarrow \frac{w}{w_n} \gg 1 \)

**Case 3:** \( w = w_n \)

Using L'Hospital’s rule, we can find the form of the solution to be

\[
x(t) = \frac{F_0}{2k} \sin(w_n t) - \frac{F_0}{2k} \frac{w_n}{w_n} t \cos(w_n t)
\]

Sketching
**Case 4:** \( w \rightarrow w_n \)

say, \( w = w_n + \Delta w \) with \( \Delta w \rightarrow 0 \) but not zero.

\[
x(t) = \frac{F_0}{k} \left( \sin(wt) - 1 \sin(w_n t) \right) = \frac{F_0}{k} \left[ 2 \sin \left( \frac{w - w_n}{2} t \right) \cos \left( \frac{w + w_n}{2} t \right) \right]
\]

Sketching,
**DAMPED SDOF:**

A SDOF linear system subject to harmonic excitation with forcing frequency $w$.

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \sin(wt)$$

In mass-normalized form, the differential equation of motion is

$$\ddot{x}(t) + 2\xi w_n \dot{x}(t) + w_n^2 x(t) = \frac{F_0}{m} \sin(wt)$$

The solution is in the form of $x(t) = x_h(t) + x_p(t)$

We know the solution to the homogeneous (unforced) problem $\ddot{x}(t) + 2\xi w_n \dot{x}(t) + w_n^2 x(t) = 0$ with I.C.s:

$$x_h(t) = e^{-\xi w_n t} \left[ \alpha_1 \cos(w_n t) + \alpha_2 \sin(w_n t) \right]$$

For the particular solution, assume the solution is in the form of

$$x_p(t) = \beta_1 \sin(wt) + \beta_2 \cos(wt)$$

Substitute into the differential equation of motion and equate like terms. The resulting system of linear equations are:

$$\begin{align*}
(w_n^2 - w^2)\beta_1 - 2\xi w_n w\beta_2 &= \frac{F_0}{m} \\
2\xi w_n w\beta_1 + (w_n^2 + w^2)\beta_2 &= 0
\end{align*}$$

Solve for $\beta_1, \beta_2$

$$\beta_1 = \frac{\frac{F_0}{k} \left( 1 - \frac{w^2}{w_n^2} \right)}{\left( 1 - \frac{w^2}{w_n^2} \right)^2 + \left( 2\xi \frac{w}{w_n} \right)^2}$$

$$\beta_2 = -\frac{\frac{F_0}{k} \left( \frac{2\xi w}{w_n} \right)}{\left( 1 - \frac{w^2}{w_n^2} \right)^2 + \left( 2\xi \frac{w}{w_n} \right)^2}$$

Complete solution,

$$x(t) = e^{-\xi w_n t} \left[ \alpha_1 \cos(w_n t) + \alpha_2 \sin(w_n t) \right] + \beta_1 \sin(wt) + \beta_2 \cos(wt)$$

Note that the part from homogeneous problem diminishes with time; it is "transient". The particular solution, however, keeps going and repeating itself; it makes the system reach a "steady-state" response.
Case 1: \( w < w_n \)

Case 2: \( w > w_n \)

The steady-state solution \( x_p(t) = \beta_1 \sin(wt) + \beta_2 \cos(wt) \) can be rewritten as follows.

\[
x_p(t) = \frac{F_0 / k}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\xi \frac{w}{w_n}\right)^2}} \sin(wt + \gamma) \quad \text{where} \quad \tan(\gamma) = \frac{\beta_2}{\beta_1}
\]

We see that the steady-state response has a pure harmonic oscillation at a frequency \( w \). The amplitude of the steady-state response is

\[
\text{Amplitude} = \left| x_p(t) \right| = \frac{F_0 / k}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\xi \frac{w}{w_n}\right)^2}}
\]

Note that \( \frac{F_0}{k} \) is the static deflection, i.e. the amount of deflection the SDOF system would have assumed if the load with amplitude \( F_0 \) were applied statically.
Definition: Amplification Factor (AF)

\[ AF = \frac{|x_p(t)|}{F_0/t_k} = \frac{1}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\zeta \frac{w}{w_n}\right)^2}} \]

Plotting AF

Maximum values of AF:

\[ \frac{d(AF)}{d(w/w_n)} = 0 \Rightarrow -\frac{1}{2} \left[ 1 - \left(\frac{w}{w_n}\right)^2 \right] \left(-2 \frac{w}{w_n} + 4\zeta^2 \left(\frac{w}{w_n}\right)^2\right) = 0 \]

Setting numerator equal to zero, i.e. \[ \left(\frac{w}{w_n}\right) \left[1 - \left(\frac{w}{w_n}\right)^2 - 2\zeta^2\right] = 0 \], gives two roots:

1. \( w = 0 \) \((\zeta \geq \frac{1}{\sqrt{2}} \approx 70\%)\) for which \( AF_{\text{max}} = 1 \)

2. \( \frac{w}{w_n} = \sqrt{1 - 2\zeta^2} \) \((\zeta \leq \frac{1}{\sqrt{2}} \approx 70\%)\) for which \( AF_{\text{max}} = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \)

Note that for lightly damped systems, i.e. \( \zeta \ll 1 \), \( AF_{\text{max}} \approx \frac{1}{2\zeta} \)

These results indicate that

\[ |x|_{\text{max}} = \frac{\delta_{\text{static}}}{2\zeta \sqrt{1 - \zeta^2}} \quad \text{if} \quad \zeta \leq \frac{1}{\sqrt{2}} \]

\[ \approx \frac{\delta_{\text{static}}}{2\zeta} \quad \text{if} \quad \zeta \ll 1 \]

\[ = \delta_{\text{static}} \quad \text{if} \quad \zeta \geq \frac{1}{\sqrt{2}} \]
Response for $w = w_n$

$$x(t) = e^{-\xi w_n t} \left[ \alpha_1 \cos(w_d t) + \alpha_2 \sin(w_d t) \right] + \beta_1 \sin(w t) + \beta_2 \cos(w t)$$

$$\beta_1 = 0 \quad \beta_2 = \frac{-F_0}{k} = -\frac{\delta_{\text{static}}}{2 \xi}$$

Let's consider a system that is at rest initially: $x(0) = 0, \dot{x}(0) = 0$.

$$\alpha_1 = \frac{\delta_{\text{static}}}{2 \sqrt{1 - \xi^2}} \quad \alpha_2 = \frac{\delta_{\text{static}}}{2 \xi}$$

$$x(t) = \delta_{\text{static}} \frac{1}{2 \xi} \left[ e^{-\xi w_n t} \left( \cos(w_d t) + \frac{\xi}{\sqrt{1 - \xi^2}} \sin(w_d t) \right) - \cos(w_n t) \right]$$

Note that as time increases, $x(t) \to -\frac{\delta_{\text{static}}}{2 \xi} \cos(w_n t)$ [steady-state response]

Sketching,