CE573 – Structural Dynamics

SDOF – HARMONIC EXCITATION

UNDAMPED SDOF:

$$m\ddot{x}(t) + kx(t) = F_0 \sin(wt)$$

Solution: $x(t) = x_h(t) + x_p(t)$

$\rightarrow \quad x_h(t) = A\sin(w_n t) + B\cos(w_n t)$

$\rightarrow \quad x_p(t) = ?$

Let's guess a particular solution be in the form of $x_p(t) = D\sin(wt) + E\cos(wt)$

$$\dot{x}_p(t) = w\left[D\cos(wt) - E\sin(wt)\right]$$
$$\ddot{x}_p(t) = -w^2\left[D\sin(wt) + E\cos(wt)\right] = -w^2 x_p(t)$$

Substitute into the dynamic equilbrium eqn.

$$m\ddot{x}_p + kx_p = F_0 \sin(wt)$$

$$\left(-mw^2 + k\right)x_p(t) = F_0 \sin(wt)$$

$$\boxed{x_p(t) = \frac{F_0 \sin(wt)}{k - mw^2} = \frac{F_0}{k}\frac{\sin(wt)}{1 - \frac{w^2}{w_n^2}}}$$

Note that the maximum amplitude of $x_p(t)$ is $\dfrac{F_0/k}{1-\left(\dfrac{w}{w_n}\right)^2}$ where F_0/k is the static deflection.

Sketching

Solution: $x(t) = x_h(t) + x_p(t)$

$$x(t) = A\sin(w_n t) + B\cos(w_n t) + \frac{F_0}{k}\frac{\sin(wt)}{1-\left(\frac{w}{w_n}\right)^2}$$

Initial conditions: $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

$$x(0) = B = x_0$$

$$\dot{x}(0) = Aw_n + \frac{F_0}{k}\frac{w}{1-\left(\frac{w}{w_n}\right)^2} = \dot{x}_0$$

$$A = \frac{1}{w_n}\dot{x}_0 - \frac{F_0}{k}\frac{\left(\frac{w}{w_n}\right)}{1-\left(\frac{w}{w_n}\right)^2}$$

Solution in full form:

$$\boxed{x(t) = x_0\cos(w_n t) + \frac{\dot{x}_0}{w_n}\sin(w_n t) - \frac{F_0}{k}\frac{\left(\frac{w}{w_n}\right)}{1-\left(\frac{w}{w_n}\right)^2}\sin(w_n t) + \frac{F_0}{k\left(1-\left(\frac{w}{w_n}\right)^2\right)}\sin(wt)}$$

If $x_0 = 0, \dot{x}_0 = 0$

$$x(t) = -\frac{F_0}{k}\frac{\left(\frac{w}{w_n}\right)}{1-\left(\frac{w}{w_n}\right)^2}\sin(w_n t) + \frac{F_0}{k\left(1-\left(\frac{w}{w_n}\right)^2\right)}\sin(wt)$$

$$\boxed{x(t) = \frac{F_0}{k}\frac{1}{1-\left(\frac{w}{w_n}\right)^2}\left(\sin(wt) - \frac{w}{w_n}\sin(w_n t)\right)}$$

Case 1: $w \ll w_n \quad \Rightarrow \quad \frac{w}{w_n} \ll 1$

Case 2: $w \gg w_n \quad \Rightarrow \quad \frac{w}{w_n} \gg 1$

Case 3: $w = w_n$

Using L'Hospital's rule, we can find the form of the solution to be $x(t) = \frac{F_0}{2k}\sin(w_n t) - \frac{F_0}{2k} w_n t \cos(w_n t)$

Sketching

Case 4: $w \rightarrow w_n$

say, $w = w_n + \Delta w$ with $\Delta w \rightarrow 0$ but not zero.

$$x(t) = \frac{F_0/k}{1-\left(\frac{w}{w_n}\right)^2}\left(\sin(wt) - 1\sin(w_n t)\right) = \frac{F_0/k}{1-\left(\frac{w}{w_n}\right)^2}\left[2\sin\left(\left(\frac{w-w_n}{2}\right)t\right)\cdot\cos\left(\left(\frac{w+w_n}{2}\right)t\right)\right]$$

Sketching,

DAMPED SDOF:

A SDOF linear system subject to harmonic excitation with forcing frequency w.

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F_0\sin(wt)$$

In mass-normalized form, the differential equation of motion is

$$\ddot{x}(t)+2\xi w_n\dot{x}(t)+w_n^2x(t)=\frac{F_0}{m}\sin(wt)$$

The solution is in the form of $x(t)=x_h(t)+x_p(t)$

We know the solution to the homogeneous (unforced) problem $\ddot{x}(t)+2\xi w_n\dot{x}(t)+w_n^2x(t)=0$ with I.C.s:

$$x_h(t)=e^{-\xi w_n t}\left[\alpha_1\cos\left(w_d t\right)+\alpha_2\sin\left(w_d t\right)\right]$$

For the particular solution, assume the solution is in the form of

$$x_p(t)=\beta_1\sin(wt)+\beta_2\cos(wt)$$

Substitute into the differential equation of motion and equate like terms. The resulting system of linear equations are:

$$\left(w_n^2-w^2\right)\beta_1-2\xi w_n w\beta_2={F_0}/{m}$$
$$2\xi w_n w\beta_1+\left(w_n^2+w^2\right)\beta_2=0$$

Solve for β_1,β_2

$$\beta_1=\frac{{F_0}/{k}\left(1-\frac{w^2}{w_n^2}\right)}{\left(1-\frac{w^2}{w_n^2}\right)^2+\left(2\xi\frac{w}{w_n}\right)^2}\qquad\qquad \beta_2=\frac{-{F_0}/{k}\;2\xi\, {w}/{w_n}}{\left(1-\frac{w^2}{w_n^2}\right)^2+\left(2\xi\frac{w}{w_n}\right)^2}$$

Complete solution,

$$x(t)=e^{-\xi w_n t}\left[\alpha_1\cos\left(w_d t\right)+\alpha_2\sin\left(w_d t\right)\right]+\beta_1\sin(wt)+\beta_2\cos(wt)$$

Note that the part from homogeneous problem diminishes with time; it is “transient”. The particular solution, however, keeps going and repeating itself; it makes the system reach a “steady-state” response.

Case 1: $w < w_n$

Case 2: $w > w_n$

The steady-state solution $x_p(t) = \beta_1 \sin(wt) + \beta_2 \cos(wt)$ can be rewritten as follows.

$$x_p(t) = \frac{F_0 / k}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\xi \frac{w}{w_n}\right)^2}} \sin(wt + \gamma) \quad \text{where } \tan(\gamma) = \frac{\beta_2}{\beta_1}$$

We see that the steady-state response has a pure harmonic oscillation at a frequency w. The amplitude of the steady-state response is

$$\text{Amplitude} = \left| x_p(t) \right| = \frac{F_0 / k}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\xi \frac{w}{w_n}\right)^2}}$$

Note that F_0 / k is the static deflection, i.e. the amount of deflection the SDOF system would have assumed if the load with amplitude F_0 were applied statically.

Definition: Amplification Factor (*AF*)

$$AF = \frac{|x_p(t)|}{F_0/k} = \frac{1}{\sqrt{\left(1-\frac{w^2}{w_n^2}\right)^2 + \left(2\xi\frac{w}{w_n}\right)^2}}$$

Plotting *AF*

Maximum values of *AF*:

$$\frac{d(AF)}{d(w/w_n)} = 0 \quad \Rightarrow \quad -\frac{1}{2}\frac{2\left[1-\left(w/w_n\right)^2\right]\left(-2\,w/w_n\right) + 4\xi^2\left(w/w_n\right)2}{\left[\left(1-\frac{w^2}{w_n^2}\right)^2 + \left(2\xi\frac{w}{w_n}\right)^2\right]^{3/2}} = 0$$

Setting numerator equal to zero, i.e. $\left(\frac{w}{w_n}\right)\left[1-\left(\frac{w}{w_n}\right)^2 - 2\xi^2\right] = 0$, gives two roots:

1. $w = 0 \qquad (\xi \geq 1/\sqrt{2} \approx 70\%)$ for which $AF_{\max} = 1$
2. $\frac{w}{w_n} = \sqrt{1-2\xi^2} \qquad (\xi \leq 1/\sqrt{2} \approx 70\%)$ for which $AF_{\max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$

Note that for lightly damped systems, i.e. $\xi \ll 1$, $AF_{\max} \cong \frac{1}{2\xi}$

These results indicate that

$$|x|_{\max} = \frac{\delta_{static}}{2\xi\sqrt{1-\xi^2}} \qquad \text{if } \xi \leq 1/\sqrt{2}$$

$$\cong \frac{\delta_{static}}{2\xi} \qquad \text{if } \xi \ll 1$$

$$= \delta_{static} \qquad \text{if } \xi \geq 1/\sqrt{2}$$

Response for $w = w_n$

$$x(t) = e^{-\xi w_n t}\left[\alpha_1 \cos\left(w_d t\right) + \alpha_2 \sin\left(w_d t\right)\right] + \beta_1 \sin(wt) + \beta_2 \cos(wt)$$

$$\beta_1 = 0 \qquad \beta_2 = \frac{-F_0/k}{2\xi} = -\frac{\delta_{static}}{2\xi}$$

Let's consider a system that is at rest initially: $x(0) = 0, \dot{x}(0) = 0$.

$$\alpha_1 = \frac{\delta_{static}}{2\sqrt{1-\xi^2}} \qquad \alpha_2 = \frac{\delta_{static}}{2\xi}$$

$$x(t) = \delta_{static}\,\frac{1}{2\xi}\left[e^{-\xi w_n t}\left(\cos\left(w_d t\right) + \frac{\xi}{\sqrt{1-\xi^2}}\sin\left(w_d t\right)\right) - \cos\left(w_n t\right)\right]$$

Note that as time increases, $x(t) \to -\frac{\delta_{static}}{2\xi}\cos\left(w_n t\right)$ [steady-state response]

Sketching,