Response of MDOF structures to ground motion

$$[M]\{\ddot{\underset{\sim}{x}}(t)\}+[C]\{\dot{\underset{\sim}{x}}(t)\}+[K]\{\underset{\sim}{x}(t)\}=-[M]\{\underset{\sim}{1}\}\ddot{x}_g(t)$$

If damping is well-behaving, or can be approximated using equivalent viscous damping, we can decouple the equations of motion using modal decomposition:

$$\{\underset{\sim}{x}(t)\}=\begin{bmatrix}\underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 & \cdots & \underset{\sim}{\phi}_N\end{bmatrix}\begin{Bmatrix}q_1(t)\\ q_2(t)\\ \vdots\\ q_N(t)\end{Bmatrix}=[\Phi]\{\underset{\sim}{q}(t)\}$$

and separate the system into its natural modes.

$$[M]\{\ddot{\underset{\sim}{x}}(t)\}+[C]\{\dot{\underset{\sim}{x}}(t)\}+[K]\{\underset{\sim}{x}(t)\}=-[M]\{\underset{\sim}{1}\}\ddot{x}_g(t)$$

becomes

$$\{\underset{\sim}{\phi_i}\}^T[M]\{\underset{\sim}{\phi_i}\}\ddot{q}_i(t)+\{\underset{\sim}{\phi_i}\}^T[C]\{\underset{\sim}{\phi_i}\}\dot{q}_i(t)+\{\underset{\sim}{\phi_i}\}^T[K]\{\underset{\sim}{\phi_i}\}q_i(t)=-\{\underset{\sim}{\phi_i}\}^T[M]\{\underset{\sim}{1}\}\ddot{x}_g(t)$$
$$i=1,2,\cdots,N$$

or when normalized with respect to modal mass $\{\underset{\sim}{\phi_i}\}^T[M]\{\underset{\sim}{\phi_i}\}$

$$\ddot{q}_i(t)+2\xi_i w_i\dot{q}_i(t)+w_i^2 q_i(t)=-\alpha_i\ddot{x}_g(t) \qquad i=1,2,\cdots,N$$

$$\ddot{q}_i(t)+2\xi_i w_i\dot{q}_i(t)+w_i^2 q_i(t)=-\alpha_i\ddot{x}_g(t) \qquad i=1,2,\cdots,N$$

where $\alpha_i=\dfrac{\{\underset{\sim}{\phi_i}\}^T[M]\{\underset{\sim}{1}\}}{\{\underset{\sim}{\phi_i}\}^T[M]\{\underset{\sim}{\phi_i}\}}$, called **modal participation factor** for mode *i*.

$$\alpha_i=\frac{\{\underset{\sim}{\phi_i}\}^T[M]\{\underset{\sim}{1}\}}{\{\underset{\sim}{\phi_i}\}^T[M]\{\underset{\sim}{\phi_i}\}}=\frac{\sum_{j=1}^{N}m_j\phi_{ji}}{\sum_{j=1}^{N}m_j\phi_{ji}^2}$$

$$\ddot{q}_i(t) + 2\xi_i w_i \dot{q}_i(t) + w_i^2 q_i(t) = -\alpha_i \ddot{x}_g(t) \qquad i = 1, 2, \cdots, N$$

For a lightly damped (underdamped) system that is initially at rest, solution can be found using the convolution/Duhamel's integral from

$$q_i(t) = -\frac{\alpha_i}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

Or using a numerical solution algorithm, once you compute $q_i(t)$

you can find the contribution of the *i*-th mode to the response of the structure.

Using the modal response, we can find various responses in each mode.

Contribution of the *i*-th mode to the displacement $x_j(t)$ at the *j*-th floor:

$$x_{ji}(t) = \phi_{ji} q_i(t) \qquad j = 1, 2, \ldots, N$$

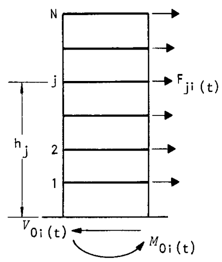
Interstory drift, i.e. story distortion, in story *j* is given by the difference of displacements of the floor above and floor below:

$$\Delta_{ji}(t) = x_{ji}(t) - x_{(j-1)i}(t)$$

To find internal forces (story shears, moments, etc.) associated with deformations convenient, we can introduce the concept of **equivalent lateral forces**.

Equivalent lateral forces are external forces $\boldsymbol{F}$ which, if applied as static forces, would cause structural displacements $\boldsymbol{x}$.

At any instant of time, the equivalent lateral forces associated with displacements due to contribution by mode *i* :



$$\left\{\underset{\sim}{F}_i(t)\right\} = \begin{Bmatrix} F_{1i}(t) \\ F_{2i}(t) \\ \vdots \\ F_{Ni}(t) \end{Bmatrix}$$

$$\left\{\underset{\sim}{F}_i(t)\right\} = \left[K\right]\left\{\underset{\sim}{x}_i(t)\right\} \quad \text{undamped}$$

Similarly, we can use inertial forces to find the equivalent lateral forces:

$$\left\{\underset{\sim}{F}_i(t)\right\} = -\left[M\right]\left\{\underset{\sim}{\ddot{x}}_i^{abs}(t)\right\}$$

see next page

$$\left\{\underset{\sim}{\ddot{x}}^{abs}(t)\right\} = \left\{\underset{\sim}{\ddot{x}}(t)\right\} + \left\{\underset{\sim}{1}\right\}\ddot{x}_g(t) = \sum_{i=1}^{N}\left\{\underset{\sim}{\phi}_i\right\}\ddot{q}_i(t) + \sum_{i=1}^{N}\left\{\underset{\sim}{\phi}_i\right\}\alpha_i\ddot{x}_g(t)$$

$$\left\{\underset{\sim}{\ddot{x}}_i^{abs}(t)\right\} = \left\{\underset{\sim}{\phi}_i\right\}\left(\ddot{q}_i(t) + \alpha_i\ddot{x}_g(t)\right) = \left\{\underset{\sim}{\phi}_i\right\}(-2\xi_i w_i \dot{q}_i(t) - w_i^2 q_i(t))$$

$$\left\{\underset{\sim}{\ddot{x}}_i^{abs}(t)\right\} \cong -w_i^2\left\{\underset{\sim}{\phi}_i\right\}q_i(t)$$

the velocity term is at least an order of magnitude smaller than the displacement term, and as such, neglected.

$$\left\{\underset{\sim}{F}_i(t)\right\} \cong -\left[M\right]\left(-w_i^2\left\{\underset{\sim}{\phi}_i\right\}q_i(t)\right) = w_i^2\left[M\right]\left\{\underset{\sim}{\phi}_i\right\}q_i(t)$$

$$\left\{\underset{\sim}{F}_i(t)\right\} = \begin{Bmatrix} F_{1i}(t) \\ F_{2i}(t) \\ \vdots \\ F_{Ni}(t) \end{Bmatrix} \cong \begin{Bmatrix} w_i^2 m_1\phi_{1i}q_i(t) \\ w_i^2 m_2\phi_{2i}q_i(t) \\ \vdots \\ w_i^2 m_N\phi_{Ni}q_i(t) \end{Bmatrix} = \begin{Bmatrix} w_i^2 m_1 x_{1i}(t) \\ w_i^2 m_2 x_{2i}(t) \\ \vdots \\ w_i^2 m_N x_{Ni}(t) \end{Bmatrix}$$

$$\{\underset{\sim}{x}(t)\} = [\Phi]\{\underset{\sim}{q}(t)\}$$

$$\{\underset{\sim}{\phi}_i\}^T [M]\{\underset{\sim}{x}(t)\} = \{\underset{\sim}{\phi}_i\}^T [M][\Phi]\{\underset{\sim}{q}(t)\} = \{\underset{\sim}{\phi}_i\}^T [M]\{\underset{\sim}{\phi}_i\} q_i(t)$$

$$q_i(t) = \frac{\{\underset{\sim}{\phi}_i\}^T [M]\{\underset{\sim}{x}(t)\}}{\{\underset{\sim}{\phi}_i\}^T [M]\{\underset{\sim}{\phi}_i\}}$$

$$\text{if } \{\underset{\sim}{x}(t)\} = \{\underset{\sim}{1}\} \qquad q_i(t)\Big|_{\{\underset{\sim}{x}(t)\}=\{\underset{\sim}{1}\}} = \frac{\{\underset{\sim}{\phi}_i\}^T [M]\{\underset{\sim}{1}\}}{\{\underset{\sim}{\phi}_i\}^T [M]\{\underset{\sim}{\phi}_i\}} = \alpha_i$$

$$\{\underset{\sim}{1}\} = [\Phi]\begin{Bmatrix}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N\end{Bmatrix} = \sum_{i=1}^{N}\{\underset{\sim}{\phi}_i\}\alpha_i$$

As $\quad q_i(t) = -\frac{\alpha_i}{w_{d,i}}\int_0^t \ddot{x}_g(\tau)e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau)d\tau$

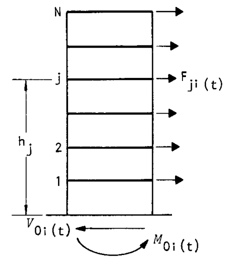
the equivalent lateral force at the *j*-th floor can be found from

$$F_{ji}(t) = -w_i^2 m_j \phi_{ji} \frac{\alpha_i}{w_{d,i}}\int_0^t \ddot{x}_g(\tau)e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau)d\tau$$

Internal forces can be determined by static analysis of the structure loaded by the equivalent lateral forces.

Story shear *at j*-th story due to response in *i*-th mode may be calculated by summing the modal inertial forces above and at story *j* :

$$V_{ji}(t) = \sum_{k=j}^{N} F_{ki}(t)$$



Total shear force at the foundation level (“**base shear**”) due to response in *i*-th mode:

$$V_{0i}(t) = \sum_{j=1}^{N} F_{ji}(t)$$

Total overturning moment at the foundation level (“**base overturning moment**”) due to response in *i*-th mode:

$$M_{0i}(t) = \sum_{j=1}^{N} F_{ji}(t) \cdot h_j$$

h_j : elev. of story *j* above the base

We can write the base shear for *i*-th mode as

$$V_{0i}(t) = \sum_{j=1}^{N} F_{ji}(t) = \sum_{j=1}^{N} -m_j \left(\ddot{x}_{ji}(t)\right)_e = \sum_{j=1}^{N} w_i^2 m_j x_{ji}$$

$$V_{0i}(t) = \sum_{j=1}^{N} -w_i^2 m_j \phi_{ji} \frac{\alpha_i}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

$$V_{0i}(t) = -\{\underset{\sim}{1}\}^T [M] \{\underset{\sim}{\phi_i}\} \alpha_i \frac{w_i^2}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

$$V_{0i}(t) = -\{\underset{\sim}{1}\}^T [M] \{\underset{\sim}{\phi_i}\} \frac{\{\underset{\sim}{\phi_i}\}^T [M] \{\underset{\sim}{1}\}}{\{\underset{\sim}{\phi_i}\}^T [M] \{\underset{\sim}{\phi_i}\}} \frac{w_i^2}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

But $\{\underset{\sim}{1}\}^T [M] \{\underset{\sim}{\phi_i}\} = \left(\{\underset{\sim}{1}\}^T [M] \{\underset{\sim}{\phi_i}\}\right)^T = \{\underset{\sim}{\phi_i}\}^T [M] \{\underset{\sim}{1}\}$

$$V_{0i}(t) = -\frac{\left(\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{1}\}\right)^2}{\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{\phi_i}\}} \frac{w_i^2}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

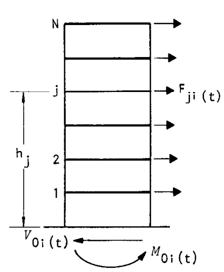
$$V_{0i}(t) = -\frac{\left(\sum_{j=1}^{N} \phi_{ji} m_j\right)^2}{\sum_{j=1}^{N} m_j \left(\phi_{ji}\right)^2} \frac{w_i^2}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

The term $\frac{\left(\sum_{j=1}^{N} \phi_{ji} m_j\right)^2}{\sum_{j=1}^{N} m_j \left(\phi_{ji}\right)^2}$ is called the "**effective modal mass**" of mode *i*.

The overturning base moment for *i*-th mode could be written as

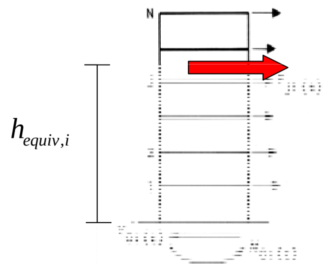
$$M_{0i}(t) = \sum_{j=1}^{N} F_{ji}(t) \cdot h_j$$

$$M_{0i}(t) = -\frac{\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{1}\}}{\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{\phi_i}\}} \left(\sum_{j=1}^{N} h_j m_j \phi_{ji}\right) \frac{w_i^2}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$



$$\text{Equivalent Height}_{\text{mode } i} = \frac{\text{Base Overturning Moment}_{\text{mode } i}}{\text{Base Shear}_{\text{mode } i}}$$

$$h_{equiv,i} = \frac{M_{0i}(t)}{V_{0i}(t)} = \frac{\sum_{j=1}^{N} F_{ji}(t) \cdot h_j}{\sum_{j=1}^{N} F_{ji}(t)} = \frac{\sum_{j=1}^{N} h_j m_j \phi_{ji}}{\sum_{j=1}^{N} m_j \phi_{ji}}$$



"**Effective modal mass**" of mode i = $\dfrac{\left(\sum_{j=1}^{N} \phi_{ji} m_j\right)^2}{\sum_{j=1}^{N} m_j \left(\phi_{ji}\right)^2}$

$$h_{equiv,i} = \frac{M_{0i}(t)}{V_{0i}(t)} = \frac{\sum_{j=1}^{N} h_j m_j \phi_{ji}}{\sum_{j=1}^{N} m_j \phi_{ji}}$$

Building		Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9	Mode 10
1-Story	Effective Modal Mass	1.00									
	Equivalent Height	1.00									
2-Story	Effective Modal Mass	0.95	0.05								
	Equivalent Height	0.81	-0.31								
3-Story	Effective Modal Mass	0.91	0.07	0.01							
	Equivalent Height	0.75	-0.27	0.19							
4-Story	Effective Modal Mass	0.89	0.08	0.02	0.00						
	Equivalent Height	0.72	-0.25	0.16	-0.13						
5-Story	Effective Modal Mass	0.88	0.09	0.02	0.01	0.00					
	Equivalent Height	0.70	-0.24	0.15	-0.12	0.10					
6-Story	Effective Modal Mass	0.87	0.09	0.03	0.01	0.00	0.00				
	Equivalent Height	0.69	-0.24	0.15	-0.11	0.09	-0.09				
7-Story	Effective Modal Mass	0.86	0.09	0.03	0.01	0.01	0.00	0.00			
	Equivalent Height	0.68	-0.23	0.14	-0.11	0.09	-0.08	0.07			
8-Story	Effective Modal Mass	0.86	0.09	0.03	0.01	0.01	0.00	0.00	0.00		
	Equivalent Height	0.68	-0.23	0.14	-0.10	0.08	-0.07	0.07	-0.06		
9-Story	Effective Modal Mass	0.85	0.09	0.03	0.01	0.01	0.00	0.00	0.00	0.00	
	Equivalent Height	0.67	-0.23	0.14	-0.10	0.08	-0.07	0.06	-0.06	0.06	
10-Story	Effective Modal Mass	0.85	0.09	0.03	0.01	0.01	0.00	0.00	0.00	0.00	0.00
	Equivalent Height	0.67	-0.22	0.14	-0.10	0.08	-0.07	0.06	-0.06	0.05	-0.05

The total response of the structure is obtained by combining the modal responses in all the modes of vibration.

The displacement at the *j*-th floor, the lateral force at the *j*-th floor, the base shear, and the base moment are given by

$$x_j(t) = \sum_{i=1}^{N} x_{ji}(t)$$

$$F_j(t) = \sum_{i=1}^{N} F_{ji}(t)$$

$$V_0(t) = \sum_{i=1}^{N} V_{0i}(t)$$

$$M_0(t) = \sum_{i=1}^{N} M_{0i}(t)$$

MODAL DECOMPOSITION APPROACH TO ANALYSE BASE-EXCITED STRUCTURES

The response of an idealized multistory building to earthquake ground motion can be computed by the following procedure:

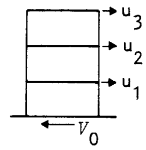
1. Find the ground acceleration.

2. Define the structural properties:
   a. Compute mass and stiffness distribution (i.e. $[M]$ and $[K]$)
   b. Estimate modal damping ratios.

3. Find natural frequencies $w_1, w_2, \cdots, w_N$ and modeshapes $\{\underset{\sim}{\phi}_1\}, \{\underset{\sim}{\phi}_2\}, \cdots, \{\underset{\sim}{\phi}_N\}$ of vibration (i.e. solve the eigenvalue problem).

4. Compute the response of individual modes of vibration by repeating the following steps for each mode:
   a. Compute the modal response $q_i(t)$ by numerical evaluation of the Duhamel integral.
   b. Compute the floor displacements.
   c. Compute story drifts from the floor displacements.
   d. Compute the equivalent lateral forces.
   e. Compute internal forces –story shears and moments– by static analysis of the structure subject to the equivalent forces.
5. Determine the total value of any response quantity by combining the modal contributions that response quantity.

Examples



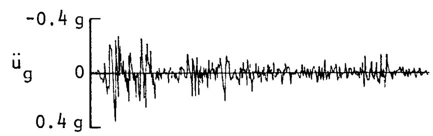
(a) Idealized three-story building

T_1=1.2 sec

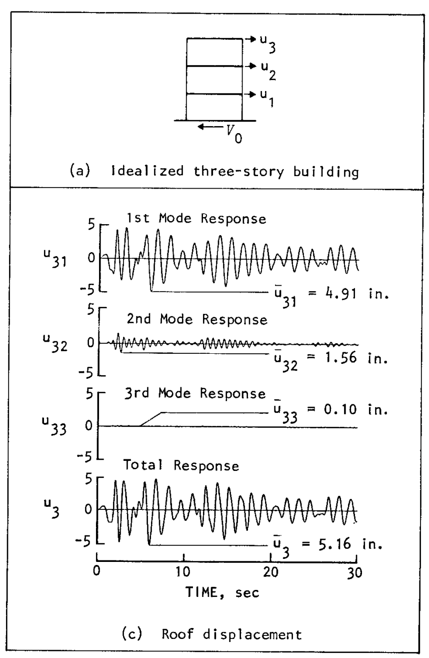
T_2=0.7 sec

T_3=0.4 sec

Total wt=900 kip

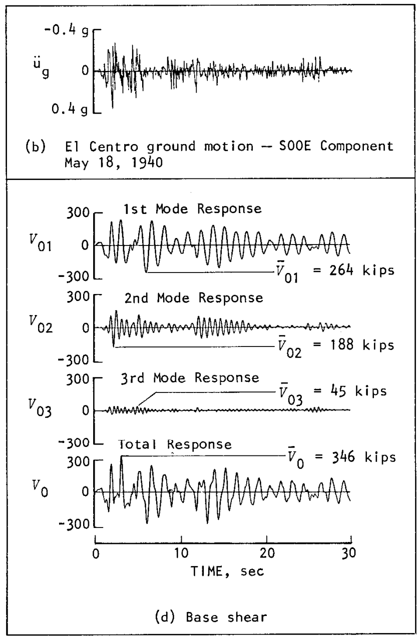


(b) El Centro ground motion — S00E Component
May 18, 1940



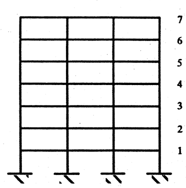
(a) Idealized three-story building

(c) Roof displacement



(b) El Centro ground motion — S00E Component May 18, 1940

(d) Base shear



Calculate the displacement response of the seven–story building when subjected to the NS component of the ground motion recorded in El Centro during the 1940 earthquake.

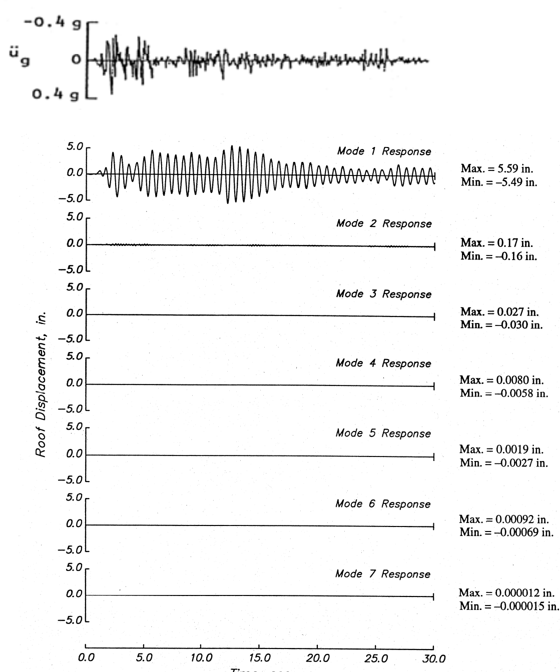
Story properties:
k = 6000 kip/ft
mg = 100 kip

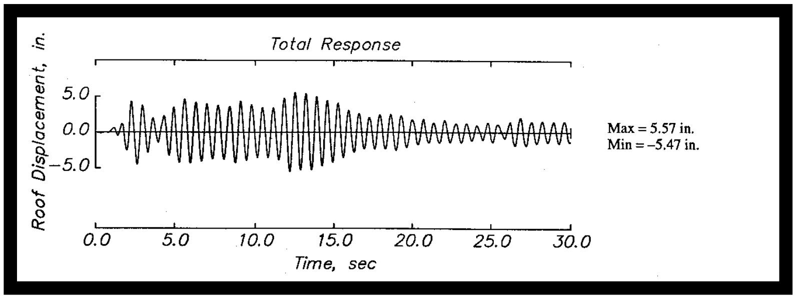
Assume damping factors of 0.02 for all modes.

SEVEN–STORY BUILDING

Mode	1	2	3	4	5	6	7
Period, sec	0.684	0.231	0.143	0.107	0.088	0.078	0.073
Frequency, Hz	1.46	4.32	7.00	9.36	11.32	12.78	13.69
Omega, rad/sec	9.19	27.17	43.95	58.82	71.12	80.31	85.99
STORY	MODE SHAPES						
7	0.2914	0.2787	0.2538	0.2178	0.1722	0.1192	−0.0609
6	0.2787	0.1722	0.0000	−0.1722	−0.2787	−0.2787	0.1722
5	0.2538	0.0000	−0.2538	−0.2538	0.0000	0.2538	−0.2538
4	0.2178	−0.1722	−0.2538	0.1192	0.2787	−0.0609	0.2914
3	0.1722	−0.2787	0.0000	0.2787	−0.1722	−0.1722	−0.2787
2	0.1192	−0.2787	0.2538	−0.0609	−0.1722	0.2914	0.2178
1	0.0609	−0.1722	0.2538	−0.2914	0.2787	−0.2178	−0.1192

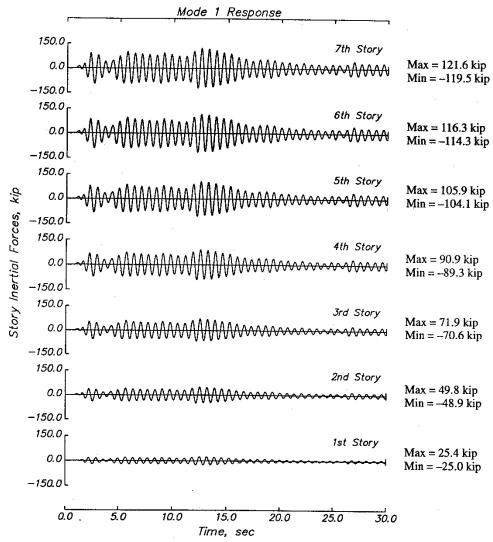
El Centro 5/18/1940 EQ ground motion – NS component. The length of the record is 30 sec.



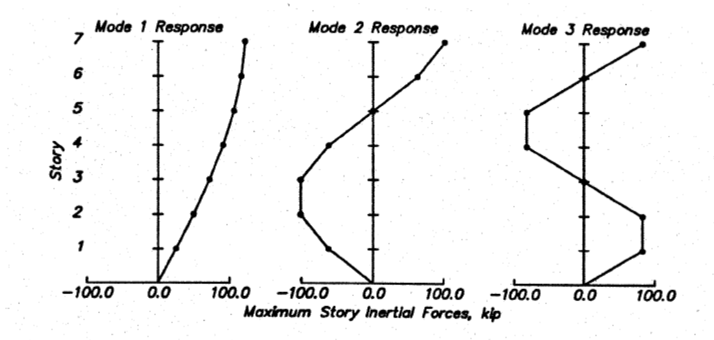


Use the response records to compute inertia forces developed in the structure.

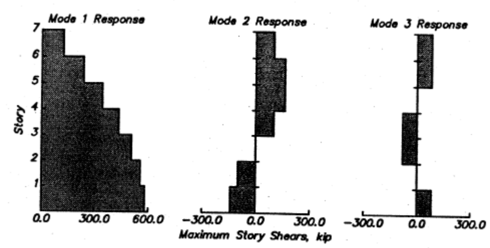
Ex: Inertial forces that develop in the structure during 1st mode response .



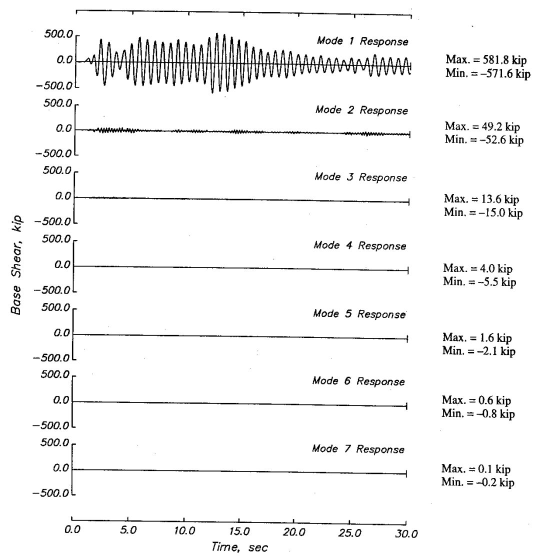
Distribution of the modal inertial forces follow the respective modeshape

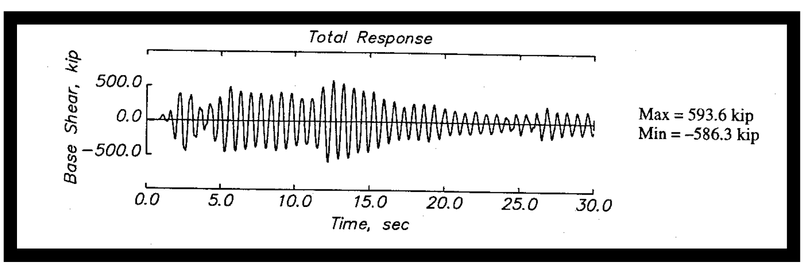


Distribution of shear forces in the structure for the first three modes:



Modal base shear demand





Question: Is there an easier way to estimate maximum response?

YES!

use response spectra

$$q_i(t) = -\frac{\alpha_i}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau = -\frac{\alpha_i}{w_{d,i}} W_i(t, \ddot{x}_g, \xi_i, w_i)$$

where $W_i(t, \ddot{x}_g, \xi_i, w_i) = \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$

For lightly damped structures $w_{d,i} \approx w_i$, so we can approximate

$$q_i(t) = -\frac{\alpha_i}{w_i} W_i(t, \ddot{x}_g, \xi_i, w_i)$$

For example, displacements are

$$\left\{ \underset{\sim}{x}(t) \right\} = \sum_{i=1}^{N} q_i(t) \left\{ \underset{\sim}{\phi}_i(t) \right\} = \sum_{i=1}^{N} -\frac{\alpha_i}{w_i} W_i(t, \ddot{x}_g, \xi_i, w_i) \left\{ \underset{\sim}{\phi}_i(t) \right\}$$

Interested in the "maxima" – the absolute maximum quantities, such as peak displacement, peak interstory drift (story distortion), and such.

<u>Ex</u>: Maximum displacement of floor *j*.

We find the maximum story displacement for each story and in each mode.

Say, we want to find, $x_{ji}(t)$ the displacement of *j*-th story due to response in *i*-th mode.

$$x_{ji}(t) = \phi_{ji} q_i(t)$$

$$x_{ji}^{\max} = \max_t \left| x_{ji}(t) \right| = \left| \phi_{ji} \right| \max_t \left| q_i(t) \right| = \left| \phi_{ji} \right| q_i^{\max}(t) = \left| \phi_{ji} \right| \frac{\left| \alpha_i \right|}{w_i} \max_t \left| W_i(t, \ddot{x}_g, \xi_i, w_i) \right|$$

$$\frac{1}{w_i} \max_t \left| W_i(t, \ddot{x}_g, \xi_i, w_i) \right| = SD(T_i, \xi_i \mid \ddot{x}_g)$$

The maximum displacement (relative to ground) of a single-degree-of-freedom system

with period T_i and damping ratio ξ_i when excited with the given ground motion $\ddot{x}_g$.

$$x_{ji}^{\max} = \max_t \left| x_{ji}(t) \right| = \left| \phi_{ji} \right| \max_t \left| q_i(t) \right| = \left| \phi_{ji} \right| q_i^{\max}(t) = \left| \phi_{ji} \right| \frac{\left| \alpha_i \right|}{w_i} \max_t \left| W_i(t, \ddot{x}_g, \xi_i, w_i) \right|$$

$$x_{ji}^{\max} = \left| \phi_{ji} \right| \left| \alpha_i \right| SD(T_i, \xi_i \mid \ddot{x}_g)$$

How do we find total response?

$$x_j(t) = x_{j1}(t) + x_{j2}(t) + \cdots + x_{jN}(t)$$

$$x_j^{\max} = \max_t \left| x_{j1}(t) + x_{j2}(t) + \cdots + x_{jN}(t) \right|$$

$$\max_t \left| x_{j1}(t) + x_{j2}(t) + \cdots + x_{jN}(t) \right| \le \max_t \left| x_{j1}(t) \right| + \max_t \left| x_{j2}(t) \right| + \cdots + \max_t \left| x_{jN}(t) \right|$$

Absolute Sum approach (Absolute combination)

$$x_j^{\max} = \max_t \left| x_{j1}(t) \right| + \max_t \left| x_{j2}(t) \right| + \cdots + \max_t \left| x_{jN}(t) \right| = \sum_{i=1}^{N} \max_t \left| x_{ji}(t) \right| = \sum_{i=1}^{N} x_{ji}^{\max}$$

Square-root of Sum of Squares (SRSS) combination

$$x_j^{\max} = \sqrt{\sum_{i=1}^{N} \left(x_{ji}^{\max} \right)^2}$$

$$\Delta_{ji}(t) = x_{ji}(t) - x_{(j-1)i}(t)$$

$$\Delta_j(t) = \sum_{i=1}^{N} \Delta_{ji}(t)$$

$$\Delta_j^{\max}(t) = \sqrt{\sum_{i=1}^{N} \left(\Delta_{ji}^{\max} \right)^2}$$

CAUTION: When you want to combine the effects of the modes to estimate a reasonable value for the maximum of a response parameter (story displacement, interstory drift, story force, etc.), you need to **find value of the response parameter for each mode and then combine using any of the combination rules**.

Do not use already-combined response parameters (say, story displacement estimates that considered contributions from all modes) to estimate other response parameters (say, story forces); such an approach will result in erroneous estimates.

Base shear

$$V_{0i}(t) = -\frac{\left(\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{1}\}\right)^2}{\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{\phi_i}\}} \frac{w_i^2}{w_{d,i}} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

$$= -\frac{\left(\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{1}\}\right)^2}{\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{\phi_i}\}} w_i \int_0^t \ddot{x}_g(\tau) e^{-\xi_i w_i (t-\tau)} \sin w_{d,i}(t-\tau) d\tau$$

$$V_{0i}^{\max} = \max_t \left|V_{0i}(t)\right| = \frac{\left(\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{1}\}\right)^2}{\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{\phi_i}\}} w_i \max_t \left|W_i(t, \ddot{x}_g, \xi_i, w_i)\right|$$

$$w_i \max_t \left|W_i(t, \ddot{x}_g, \xi_i, w_i)\right| = PSA(T_i, \xi_i \mid \ddot{x}_g)$$

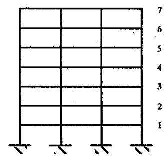
$$V_{0i}^{\max} = \frac{\left(\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{1}\}\right)^2}{\{\underset{\sim}{\phi_i}\}^T [M]\{\underset{\sim}{\phi_i}\}} PSA(T_i, \xi_i \mid \ddot{x}_g)$$

$$V_{0i}^{\max} = \text{Effective modal mass}_i \cdot PSA(T_i, \xi_i \mid \ddot{x}_g)$$

How do we find total base shear?

$$V_0^{\max} = \sqrt{\sum_{i=1}^{N} \left(V_{0i}^{\max}\right)^2} = \sqrt{\sum_{i=1}^{N} \left[\left(\text{effective modal mass}\right)_i \bullet PSA(T_i, \xi_i \mid \ddot{x}_g)\right]^2}$$

Example: 7-story building



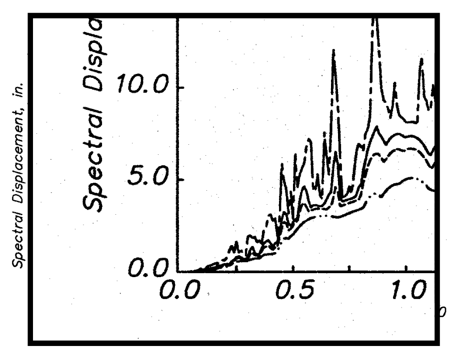
Calculate the displacement response of the seven–story building when subjected to the NS component of the ground motion recorded in El Centro during the 1940 earthquake.

Story properties:
$k = 6000$ kip/ft
$mg = 100$ kip

Assume damping factors of 0.02 for all modes.

SEVEN–STORY BUILDING

Mode	1	2	3	4	5	6	7
Period, sec	0.684	0.231	0.143	0.107	0.088	0.078	0.073
Frequency, Hz	1.46	4.32	7.00	9.36	11.32	12.78	13.69
Omega, rad/sec	9.19	27.17	43.95	58.82	71.12	80.31	85.99
STORY	MODE SHAPES						
7	0.2914	0.2787	0.2538	0.2178	0.1722	0.1192	−0.0609
6	0.2787	0.1722	0.0000	−0.1722	−0.2787	−0.2787	0.1722
5	0.2538	0.0000	−0.2538	−0.2538	0.0000	0.2538	−0.2538
4	0.2178	−0.1722	−0.2538	0.1192	0.2787	−0.0609	0.2914
3	0.1722	−0.2787	0.0000	0.2787	−0.1722	−0.1722	−0.2787
2	0.1192	−0.2787	0.2538	−0.0609	−0.1722	0.2914	0.2178
1	0.0609	−0.1722	0.2538	−0.2914	0.2787	−0.2178	−0.1192



The spectral displacement values at the first four periods of our 7-story structure are

$T_1 = 0.68\,\text{sec} \rightarrow S_{d1} = 4.43\text{ in}$
$T_2 = 0.23\,\text{sec} \rightarrow S_{d2} = 0.44\text{ in}$
$T_3 = 0.14\,\text{sec} \rightarrow S_{d3} = 0.15\text{ in}$
$T_4 = 0.11\,\text{sec} \rightarrow S_{d4} = 0.07\text{ in}$

giving

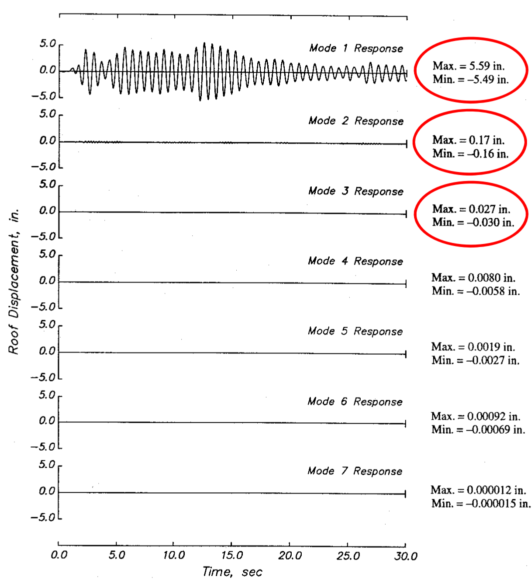
$q_1^{\max} = \alpha_1 SD_1 = 4.33 \text{ x } 4.43\text{ in} = 19.2\text{ in}$
$q_2^{\max} = 0.62\text{ in}$
$q_3^{\max} = 0.12\text{ in}$
$q_4^{\max} = 0.04\text{ in}$

$$T_1 = 0.68\,\text{sec} \quad \rightarrow \quad S_{d1} = 4.43\,\text{in} \quad \rightarrow \quad q_1^{\max} = 19.2\,\text{in}$$
$$T_2 = 0.23\,\text{sec} \quad \rightarrow \quad S_{d2} = 0.44\,\text{in} \quad \rightarrow \quad q_2^{\max} = 0.62\,\text{in}$$
$$T_3 = 0.14\,\text{sec} \quad \rightarrow \quad S_{d3} = 0.15\,\text{in} \quad \rightarrow \quad q_3^{\max} = 0.12\,\text{in}$$
$$T_4 = 0.11\,\text{sec} \quad \rightarrow \quad S_{d4} = 0.07\,\text{in} \quad \rightarrow \quad q_4^{\max} = 0.04\,\text{in}$$

Roof displacement

$$u_{71}^{\max} = \phi_{71} q_1^{\max} = 0.2914 \text{ x } 19.2 \text{ in} = 5.6 \text{ in}$$
$$u_{72}^{\max} = \phi_{72} q_2^{\max} = 0.17 \text{ in} \qquad \text{(3.1\% of the first-mode response.)}$$
$$u_{73}^{\max} = \phi_{73} q_3^{\max} = 0.03 \text{ in} \qquad \text{(0.6\% of the first-mode response.)}$$

Note that these maxima match the maxima in the corresponding response histories.



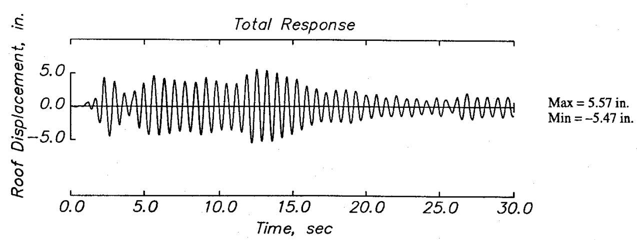
Roof total displacement estimate

Absolute Sum approach (Absolute combination)

$$u_7^{\max} \approx u_{71}^{\max} + u_{72}^{\max} + u_{73}^{\max} + u_{74}^{\max} + u_{75}^{\max} + u_{76}^{\max} + u_{77}^{\max}$$
$$= 5.59 + 0.17 + 0.03 + 0.008 + 0.0027 + 0.00092 + 0.000015 = 5.80 \text{ in}$$

Square-root of Sum of Squares (SRSS) combination

$$u_7^{\max} \approx \sqrt{\left(u_{71}^{\max}\right)^2 + \left(u_{72}^{\max}\right)^2 + \left(u_{73}^{\max}\right)^2 + \left(u_{74}^{\max}\right)^2 + \left(u_{75}^{\max}\right)^2 + \left(u_{76}^{\max}\right)^2 + \left(u_{77}^{\max}\right)^2}$$
$$= 5.59 \text{ in}$$



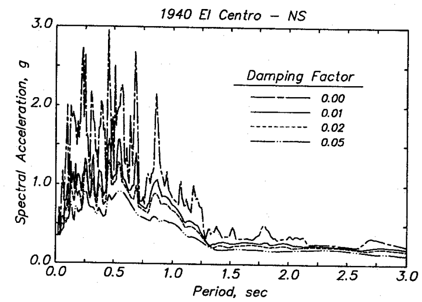
Rule of thumb:

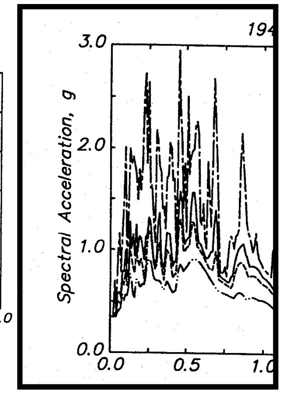
Maximum displacement at the roof is 1.2~1.5 times the spectral displacement of the fundamental mode.

More like 1.2 for frame and 1.5 for shearwall buildings.

Base Shear Force

$$V_{0i}^{\max} = \left(\text{Effective Modal Mass}\right)_i \cdot PSA(T_i, \xi_i \mid \ddot{x}_g)$$





Mode	Effective Modal Mass (kip-sec²/ft)	Effective Weight (kip)	Period (sec)	Spectral Pseudo-Acceleration, SA (g)
1	**18.74**	**603.4**	**0.68**	**0.96**
2	**1.96**	**63.1**	**0.23**	**0.84**
3	**0.62**	**20.0**	**0.14**	**0.75**
4	**0.26**	**8.4**	**0.11**	**0.65**
5	**0.11**	**3.5**	**0.09**	-
6	**0.04**	**1.3**	**0.08**	-
7	**0.01**	**0.3**	**0.07**	-

Total : 700 kip

$V_{01}^{max} = 18.74 \times 0.96g = 579.3$ kip

$V_{02}^{max} = 1.96 \times 0.84g = 53.0$ kip (9% of first-mode response)

$V_{03}^{max} = 0.62 \times 0.75g = 15.0$ kip (3% of first-mode response)

$V_{04}^{max} = 0.26 \times 0.65g = 5.5$ kip (1% of first-mode response)

Absolute Sum approach (Absolute combination)

$V_0^{max} = 658$ kip

Square-root of Sum of Squares (SRSS) combination

$V_0^{max} = 584$ kip

Modal base shear demand