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A METHOD OF COMPUTATION FOR STRUCTURAL DYNAMICS

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SUMMARY

This paper describes a general procedure for the solution of problems in structural dynamics. The method is capable of application to structures of any degree of complication, with any relationship between force and displacement ranging from linear elastic behavior through various degrees of inelastic behavior or plastic response, up to failure. Any type of dynamic loading such as that due to shock or impact, vibration, earthquake motion, or blast from a nuclear weapon, can be considered.

A method of numerical integration is described which for simpler cases and for a relatively small number of degrees of freedom is suitable for use with desk calculators. However the method is developed particularly for use with high-speed digital computers. Consideration is given to various types of damping, and to nonlinear behavior. A description is given of a method of treatment of elasto-plastic members in flexure, including the development of yield hinges. By suitable means of application of the loading, and with the introduction of enough damping to prevent indefinite oscillatory motion, the procedure can be used to determine the "static" behavior of a structure as it progresses through various degrees of inelastic behavior up to collapse.

Although the general procedure described is suitable for use in the study of the response of structures to earthquake motions, a modified procedure is described for handling the problem of a structure having time-dependent boundary conditions, which permits the direct calculation of displacements in

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the structure relative to the ground displacement, for convenience in interpreting earthquake phenomena.

The paper discusses the problem of structures with curved members and describes in detail a method of dealing with arched or curved structures in which the members are not permitted to change in length. The treatment of such inextensible structures presents some difficulties in the general treatment but methods are available for modifying the general procedure by introducing a series of constraint relations which reduces the number of degrees of freedom permitted in the motion of the structure.

The methods described herein have been used for the computation of the dynamic response of structures of various degrees of complexity including arches, domes, stiffened rings, framed structures, and simple spring-mass systems, subjected to various types of loading including nuclear bomb blast, earthquake foundation motions, random shock disturbances, wave action, and impact and dynamic effects from moving vehicles. However, examples of these applications are not given.

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INTRODUCTION

Only general principles and methods for dynamic analysis are considered in this paper. The basic method of analysis is a general step-by-step method of integration of the equations of motion, and is applicable to any structure consisting of a group or series of concentrated or "lumped" masses supported on a deformable structure. For convenience, the structure may be considered to be a framework with joints or "nodes" at which forces may be applied, or at which masses may be placed. Any finite number of degrees of freedom may be considered, but it is essential in the procedure that the forces required to produce a pattern of deflection of the framework must be determinable when the deflections of the nodes at which masses are placed are specified. It is not necessary that the framing behave elastically.

Because in an actual structure the mass is not really divided into separate component parts connected together by flexible elements, the structure which is analyzed is only an approximation to the actual structure. However, a reasonable approximation can usually be made. For the actual structure, with distributed masses, the number of degrees of freedom of motion of the structure is infinitely large. The replacement of the distributed mass by a number of concentrated masses reduces the number of degrees of freedom and affects the response of the structure in the higher modes.

The simplification of a structure for the purpose of making a dynamic analysis involves engineering judgment. One must select the essential elements of resistance of the structure so as to arrive at an adequate and reasonably accurate expression of the resistance of the structure to motion. Also one must evaluate the masses of the structure and "lump" them together at places where they can be considered to act. In general the number of masses which should be considered depends on the accuracy which one wishes to achieve in the calculations.

The structure may be made up of individual members connected together at joints which would then be considered as the nodes of the structure.

However, the structure may be an integral one, either a solid body or an assemblage of plates and other elements, in which the nodes may only define points on the structure for convenience in the placement of loads and masses. It is necessary that the nodes be so chosen that the resistance of the structure to displacement of the nodes can be determinable, and that, if desired, the influence of inelastic behavior or plastic behavior can be taken into account. The structure is considered to be supported at its foundations at nodes also.

Although the method of analysis can take into account a situation in which the masses change with time or with displacement, the procedure is presented herein for masses which remain constant. No major change in concept or in procedure is required to deal with the case of a variable mass-time relationship.

Method of Analysis

Step-by-Step Integration Procedure

Consider the plane structure shown in Fig. 1 which is made up of weightless but deformable elements supporting lumped masses. The deformable elements of the structure in this particular instance are shown as either beams or bars which act under axial loading. However, much more complex structural types can be considered. All the elements shown are deformable and consequently each mass such as M, C, D, E, can move in both the vertical and horizontal direction. The points of support, such as at A and B, may also move both vertically and horizontally. In particular, these points may move in such a way that deformations and stresses are introduced in the structure by relative motion of the points of support on the foundation. Any of the masses may have acting on it a force in any direction, or component forces in the horizontal and vertical directions.

The mass at M in Fig. 1 is shown removed from the structure in Fig. 2. In Fig. 2a the point of attachment of the mass is indicated, and the positive directions of the resisting force exerted by the structure, R, and the displacement of the structure, x, are indicated. In Fig. 2b the mass is shown as a free body with the positive directions of P, R, and x indicated. Although this mass is shown as being acted on by only a horizontal force, in general it could have a vertical component of force acting on it also in which case it would have a component of motion and a component of resistance in the vertical direction. The sign convention that is chosen is determined by the arbitrary choice of the positive direction of the force P. The positive directions of x and of the resisting force R acting on the structure are taken as the same as for P, and the positive direction for the resisting force R acting on the free body mass M is opposite in sense to that of the resisting force R acting on the structure.

The positive directions of the acceleration a, the velocity v, and the displacement x are all the same. In general we will have for each possible component of direction of motion of the mass a displacement, velocity, acceleration, resisting force, and applied force. The resisting forces R at any instant of time are defined in such a way as to be that system of forces which are required to pull the weightless deformable structure into a deflection configuration defined by the instantaneous values of the displacement x at the same instant of time.

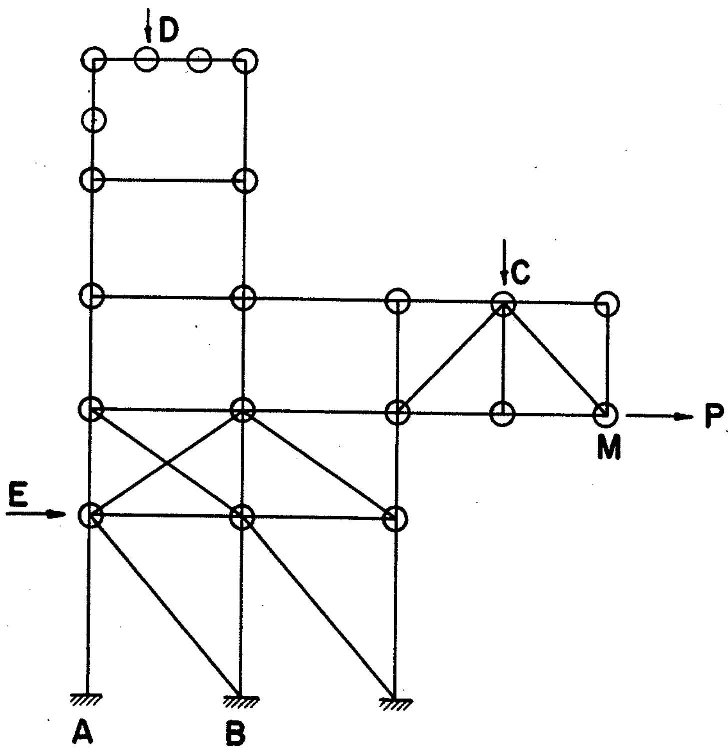
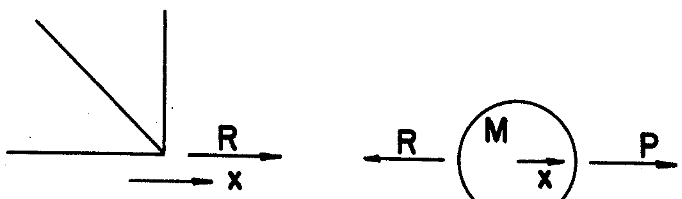


Fig. 1. General Type of Structure and Loading



(a) Structure with Mass Removed

(b) Mass Removed from Structure

Fig. 2. Detail of Mass as Free Body

The sign convention and notation are chosen so as to make it apparent that the masses M modify or filter the forces P and transmit them to the structure in modified form as R. If the forces P are applied very slowly there is only a small acceleration and R is approximately equal to P. If the forces are applied quickly the difference between R and P can be very large. All the stresses in the structure are defined by the system of forces R. The structure can be analyzed statically for these forces. In general the forces R may continue to exist even after the forces P have dropped to zero. Similarly, R may be defined by the foundation displacements as well as by the deflections of the masses, even when no force P acts on the masses.

In general all of the quantities so far defined are functions both of position on the structure and of time. If we consider a time t_n and designate the values of all of the various parameters including displacements and forces at that time with a subscript n, such as R_n, our problem becomes that of defining the displacements x_{n+1} and forces R_{n+1} as well as the velocities v_{n+1} and accelerations a_{n+1} at a time t_{n+1}, which differs from t_n by the time interval h.

In the argument which follows we shall deal with a single mass in a single direction, but we might just as well deal with the whole set of masses and their possible displacements and designate the various directions with additional subscripts m such as in $R_{n,m}$. However in the discussion which follows we shall drop the second subscript for convenience, remembering that for each of the degrees of freedom for each mass we have a set of equations similar to the general set presented here. The derivation which immediately follows is described in terms of a situation where damping does not exist, for convenience. Later the procedure will be revised to include damping forces.

In general at any time, (consequently omit the subscript n), the acceleration is given by the relation:

$$a = (P - R)/M \tag{1}$$

It is assumed that at time t_n we know the values of the displacement and the velocity as well as the acceleration, but we know nothing about the situation at time t_{n+1}. Although there are methods of numerical integration which permit us to make estimates, at least for small time intervals, of the displacements and velocities at the later time knowing only the situation at the earlier time, these methods which do not take account of the change in resisting force during the time interval are not as accurate as the method described herein.

The method presented here was derived by the writer and first presented in Ref. 1. The relations which follow are given in terms of the acceleration at the end of the interval, a_{n+1}, although this is not in general known. A discussion of how the procedure can be handled in spite of this fact will be given subsequently. Two parameters, γ and β, are introduced to indicate how much of the acceleration at the end of the interval enters into the relations for velocity and displacement at the end of the interval. The relations which are adopted are given below:

$$v_{n+1} = v_n + (1 - \gamma)\, a_n h + \gamma\, a_{n+1} h \tag{2}$$

$$x_{n+1} = x_n + v_n h + (\tfrac{1}{2} - \beta)\, a_n h^2 + \beta\, a_{n+1} h^2 \tag{3}$$

It will be shown subsequently that unless the quantity γ is taken as 1/2, there is a spurious damping introduced, proportional to the quantity

$\gamma - 1/2$

It can be seen that if γ is taken as zero a negative damping results, which will involve a self-excited vibration arising solely from the numerical procedure. Similarly, if γ is greater than 1/2 a positive damping is introduced which will reduce the magnitude of the response even without real damping in the problem. Taking account of the fact that γ must equal 1/2, we can rewrite Eq. (2) as follows:

$$v_{n+1} = v_n + a_n h/2 + a_{n+1} h/2 \tag{4}$$

Application of the General Procedure

In general unless β is 0 we may proceed with our calculation as follows:

(1) Assume values of the acceleration of each mass at the end of the interval.

(2) Compute the velocity and the displacement of each mass at the end of the interval from Eqs. (4) and (3), respectively. (Unless damping is present it is not necessary to compute the velocity at the end of the interval until step (5) is completed.)

(3) For the computed displacements at the end of the interval compute the resisting forces R which are required to hold the structural framework in the deflected configuration.

(4) From Eq. (1) and the applied loads and resisting forces at the end of the interval recompute the acceleration at the end of the interval.

(5) Compare the derived acceleration with the assumed acceleration at the end of the interval. If these are the same the calculation is completed. If these are different, repeat the calculation with a different value of assumed acceleration. It will usually be best to use the derived value as the new acceleration for the end of the interval.

The rate of convergence of the process toward equality of the derived and assumed accelerations is a function of the time interval h. For a single-degree-of-freedom system having a circular frequency of vibration ω, the ratio of the error in derived acceleration to the error in assumed acceleration, (where the error is the difference between a value and the "correct" value), is given by the relation:

$$\frac{\text{error in derived acceleration}}{\text{error in assumed acceleration}} = \rho = -\beta \omega^2 h^2 \tag{5}$$

It is convenient to define the quantity ωh by the symbol θ according to Eq. (6):

$$\omega h = \theta \tag{6}$$

With this notation Eq. (5) can be rewritten as:

$$\rho = -\beta\theta^2 \tag{7}$$

Since the circular frequency ω is related to the period T by the relation

$$\omega = \frac{2\pi}{T} \tag{8}$$

Eq. (7) can be rewritten as follows:

$$\rho = -\beta\left(\frac{2\pi h}{T}\right)^2 \tag{9}$$

Now for convergence in a sequence of calculations the quantity ρ must be numerically less than 1. The critical value, for convergence, of the time interval h_c can then be computed from Eq. (9) by setting the right-hand side numerically equal to 1, with the result:

$$\frac{h}{T} = \frac{1}{2\pi}\sqrt{\frac{1}{\beta}} \tag{10}$$

Critical values of the convergence limit as a function of β are contained in Table 1.

For practical purposes the time interval would ordinarily be taken as smaller than that which corresponds to pure oscillation, or $\rho = -1$, in order to insure rapid enough convergence. If $\rho = -0.32$, the errors will be reduced to one per cent of their original value in four steps or four cycles of iteration. It would appear that, since for small values of β the convergence is most rapid, the lower values of β would be best to use. However other considerations affect the choice. The most important of these is the matter of stability which is discussed in the next section.

For a complex system, it can be shown that the rate of convergence is dependent upon the frequency or the period of the highest mode of the system. Consequently, the time interval used must be related to the shortest period of vibration, or the period in the highest mode of vibration, for the lumped mass system. Since stability also depends on a similar criterion, it appears that the greater the number of masses into which a system is broken down the shorter will be the permissible time interval for numerical calculation of the dynamic response of the system.

Stability and Errors in Numerical Computation

In order to study the stability of the numerical integration procedure, let us consider the special case of a simple system, a mass with one-degree-of-freedom without external force acting on it. For such a condition, and for some initial displacement and velocity, the motion of the system should be a pure oscillation, with a circular frequency of vibration as given by the relation

$$\omega^2 = \frac{K}{M} \tag{11}$$

in which K is the spring constant and M the mass. The relationship between the acceleration and the displacement is given by:

$$a = -\omega^2 x \tag{12}$$

With the above relations, and with the use of the symbol α^2 as defined by

$$\alpha^2 = \frac{\theta^2}{1 + \beta\theta^2} \tag{13}$$

we can derive a difference equation relating the values of three successive displacements of the system. The equation in general terms is:

$$x_{n+1} - (2 - \alpha^2)\, x_n + x_{n-1} + (\gamma - \tfrac{1}{2})\, \alpha^2 (x_n - x_{n-1}) = 0 \tag{14}$$

From the general relations between finite differences and derivatives, it can be seen that the last term on the left of this equation corresponds to a factor times the velocity of the system, and consequently can be interpreted as a viscous damping term even though the system was defined as having no damping. This spurious last term can be eliminated by the choice of $\gamma = 1/2$. In this case the general difference equation can be rewritten as:

$$x_{n+1} - (2 - \alpha^2)\, x_n + x_{n-1} = 0 \qquad (15)$$

The general solution of the finite difference equation, Eq. (15), can be written in the case where the quantity α^2 is less than 4. In this case, define a quantity ϕ by the relation:

$$\alpha = 2 \sin \varphi/2 \qquad (16)$$

The solution of Eq. (15) can then be written in the form

$$x = A \cos \varphi\, t/h + B \sin \varphi\, t/h \qquad (17)$$

This can also be stated in terms of a pseudo period T_s, and an initial displacement x_o and a parameter B which is of the same form as a velocity. The result is:

$$x = x_o \cos 2\pi t/T_s + B \sin 2\pi t/T_s \qquad (18)$$

This may be compared with the exact solution, $\bar{x}$, which is given by Eq. (19):

$$\bar{x} = x_o \cos 2\pi t/T + \frac{v_o}{\omega} \sin 2\pi t/T \qquad (19)$$

It can be seen that the approximate solution, Eq. (18), is similar to the exact solution and gives precisely the same response for an initial displacement, but gives a different period from that of the actual system. The value of the pseudo period T_s is given by the relation:

$$T_s = 2\pi h/\varphi \qquad (20)$$

The relation between the pseudo period T_s and the true period T is:

$$T_s/T = \theta/\varphi \qquad (21)$$

The following approximate formula may be used for a simpler definition of the relative value of the pseudo and the real period of vibration:

$$T_s/T = \sim 1 - (1 - 12\beta)\, \theta^2/24 - (17 - 120\beta + 720\beta^2)\, \theta^4/5760 - \ldots \qquad (22)$$

The response of the system to an initial velocity is given by the second term in Eq. (18). The relationship between this response and the true response to an initial velocity, as shown by the second term in Eq. (19), is indicated in Eq. (23):

$$\frac{B}{v_o/\omega} = \left[1 + (\beta - \tfrac{1}{4})\, \theta^2\right]^{-1/2} = \sim 1 + \frac{1/4 - \beta}{2}\, \theta^2 + \ldots \qquad (23)$$

If β is exactly equal to 1/4, the maximum velocity response is correct but if it is different from 1/4 there is an incorrect maximum velocity response.

Values of the errors in the period and of the errors in maximum response to an initial velocity are given in Tables 2a and b for several values of β and for a range in values of h/T. There is also given in Table 2c the rate of convergence for the corresponding tabular entries. For a system with a number of degrees of freedom, the limits are expressed in terms of the shortest period of the system.

When $\alpha^2 > 4$, the solution of Eq. (15) oscillates without bounds, and the calculation does not yield results even in remote agreement with the exact solution. The solution is said to be "unstable".

TABLE 1

CONVERGENCE AND STABILITY LIMITS

Item	Values of β				
	0	1/12	1/8	1/6	1/4
Convergence Limit, h/T	inf.	0.551	0.450	0.389	0.318
Stability Limit, h/T	0.318	0.389	0.450	0.551	inf.

The stability limit criterion, corresponding to a value of $\alpha^2 = 4$, can be expressed in terms of the time interval also. The relation between α and θ in Eq. (13) can be expressed as:

$$\theta^2 = \frac{\alpha^2}{1 - \beta\alpha^2} \qquad (24)$$

from which the stability limit h_s can be written as:

$$\frac{2\pi h_s}{T} = \frac{2}{\sqrt{1 - 4\beta}} \qquad (25)$$

which can be simplified to the form:

$$\frac{h_s}{T} = \frac{1/\pi}{\sqrt{1 - 4\beta}} \qquad (26)$$

Values of the stability limit are shown in Table 1 as a function of β.

From Table 1, it can be seen that for values of β greater than 1/8, if the time interval is chosen for convergence the numerical procedure will always be stable. However, for values of β less than 1/8, convergence does not insure stability. Lack of stability gives no warning of difficulty, but introduces a spurious increasing oscillation into a system which may be in oscillation anyway. Therefore an inexperienced computer may not recognize the difficulty. Moreover, an instability in the higher modes only may not even be apparent to an experienced computer. Consequently it appears that unless other steps are taken to insure stability, one should limit the time interval by the stability criterion rather than by the convergence criterion.

Interpretation of Parameter β

A method very much similar to that described here for $\beta = 0$ has been discussed in Ref. 2. A method corresponding in many respects to that for $\beta = 1/12$ has been given in Ref. 3. However, the general treatment previously presented is different from that given here, particularly in the treatment of the starting of the motion. A method similar to that for $\beta = 1/4$ was first presented by S. Timoshenko in Ref. 4. However, he did not carry the procedure to the point of generalizing it for other than simple one-degree-of-freedom systems, nor did he develop the conditions on stability and convergence.

TABLE 2

EFFECTS OF LENGTH OF INTERVAL ON ERRORS DUE TO NUMERICAL PROCEDURE

h/T	Values of β				
	0	1/12	1/8	1/6	1/4
	(a) Relative Errors in Period				
0.05	-0.004	-0.0001	0.002	0.004	0.008
0.10	-0.017	-0.0003	0.008	0.017	0.033
0.20	-0.076	-0.006	0.028	0.059	0.121
0.25	-0.130	-0.015	0.038	0.087	0.179
0.318	-0.363	-0.045	0.047	0.129	0.273
0.389	*	-0.220	0.035	0.170	0.382
0.450	*	*	-0.100	0.195	0.480
	(b) Relative Errors in Maximum Response to an Initial Velocity				
0.05	0.012	0.008	0.006	0.004	0
0.10	0.052	0.034	0.025	0.017	0
0.20	0.209	0.166	0.116	0.073	0
0.25	0.614	0.306	0.202	0.122	0
0.318	inf.	0.732	0.414	0.225	0
0.389	*	inf.	1.000	0.414	0
0.450	*	*	inf.	0.732	0
	(c) Rate of Convergence				
0.05	0	0.008	0.012	0.016	0.025
0.10	0	0.033	0.049	0.066	0.099
0.20	0	0.132	0.197	0.263	0.395
0.25	0	0.206	0.308	0.411	0.617
0.318	0	0.333	0.500	0.667	1.000
0.389	*	0.500	0.750	1.000	1.500
0.450	*	*	1.000	1.333	2.000

* Values indicated are beyond limit for stability.

It is interesting to note the correspondence between β and the variation in acceleration during the time interval. Although a physical relationship is not possible for all values, for at least four values of β it is possible to define consistent variations of acceleration in the time interval. Three of these are shown in Fig. 3. It appears that a choice of $\beta = 1/6$ corresponds to a linear variation of acceleration in the time interval; a choice of $\beta = 1/4$ corresponds to a uniform value of acceleration during the time interval equal to the mean of the initial and final values of acceleration; and a value of $\beta = 1/8$ corresponds to a step function with a uniform value equal to the initial value for the first half of the time interval and a uniform value equal to the final value for the second half of the time interval.

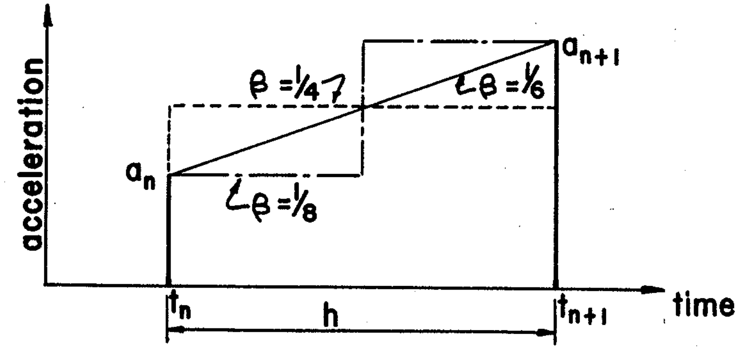


Fig. 3. Consistent Variations of Acceleration in a Time Interval

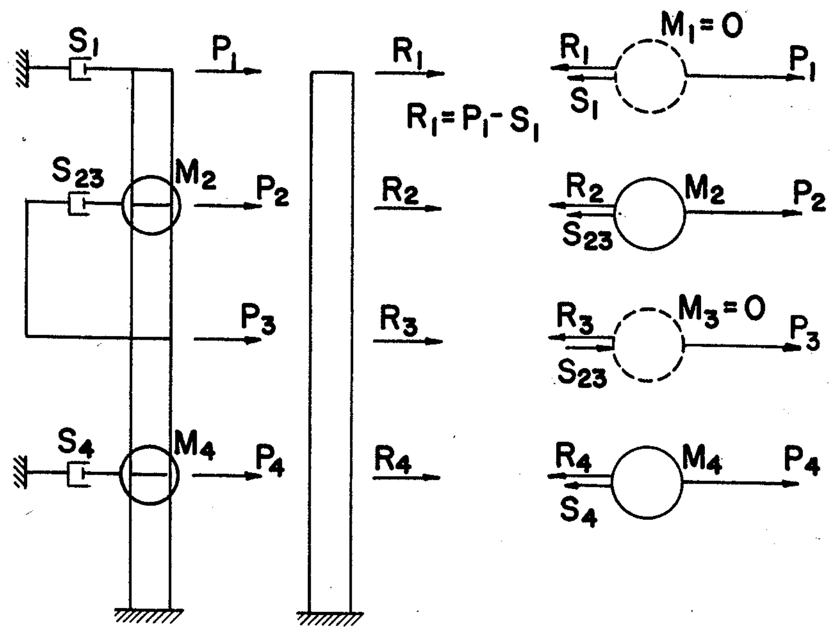


Fig. 4. Structure with Absolute and Relative Damping

It can also be shown that $\beta = 0$ corresponds to double pulses of acceleration at the beginning and end of the time interval with each double pulse consisting of a part equal to 1/2 of the acceleration times the time interval, one occurring just before the end of the preceding interval and the other just after the beginning of the next interval.

Treatment of Damping

Damping of various kinds may exist in a structural system. Damping forces may be proportional to velocity, or to some power of the velocity, or they may be of a frictional type, or in some cases they may be even proportional to displacement or relative displacement. Damping may be "absolute", and depend on the particular motion or velocity of the mass with respect to ground, or "relative" and depend on the motion or velocity of the mass with respect to some other point on the structure. A structure with several points having damping forces is shown in Fig. 4. If masses M_1 and M_3 are different from zero, then the damping forces can be applied to the masses, as indicated on the right-hand side of the figure, and the calculation is made in the following way:

After the acceleration at the end of the interval is assumed, the velocity at the end of the interval and the displacement are computed from Eqs. (4) and (3). The complete motion of the system is now defined at the end of the interval, and regardless of the kind of damping, the damping force is determinable. With the damping force placed on the mass in the proper direction, the acceleration at the end of the interval is now computed taking into account the damping force as well as the applied force and the resisting force. If this is different from the assumed acceleration the calculation is repeated.

As in the case with zero damping, the resisting forces acting on the structure are computed from the displacement of the structure at the end of the interval. The damping forces acting on the masses are computed from the displacements or velocities of the masses in accordance with the damping law used.

It can be seen that in the case of damping a value of $\beta = 0$ presents no real advantages over any other choice of β because of the fact that the acceleration still has to be assumed at the end of the interval in order to compute the velocity or other parameters which determine the damping forces at the end of the interval.

For those cases where there are no masses at points where damping forces or external forces are applied, the situation is more complex. Fig. 4 illustrates this by applying forces to the two zero masses, M_1 and M_3. It is necessary in these cases that the resisting force be equal to the algebraic sum of the external load and the damping force, and this resisting force is then applied to the structure and except for its dependence on the damping force remains at this particular value, regardless of the displacements. However, it is now necessary to compute the accelerations and velocities of the point on the structure in order to find the damping force. This can be done in the following way for the points where the mass is zero:

(1) Assume a value of acceleration of the mass point even though the mass is zero.
(2) Compute the velocity and displacement of the point and determine the damping force.

(3) Now apply the net resistance, corresponding to the difference between the applied force and the damping force, to the structure and determine the displacements of the structure at all points when the prescribed displacements are put in at the points where masses exist.
(4) Compute the acceleration at the end of the interval that is required to give the displacement determined in the preceding step.
(5) Compare the acceleration so computed with the one initially assumed and repeat if necessary.

This procedure is considerably more complex than that which is used when masses exist at all nodes. Consequently, it may be better to put in an actual small mass than to make the mass zero. However, if damping forces are not acting at points where there are zero masses, then it is relatively simple to handle the problem directly.

Recommended Time Interval and Choice of β

In any set of calculations in which an error in any one step makes all subsequent steps incorrect, it is desirable to have a self checking procedure. Consequently the procedure described herein works best where it is used with a converging system of approximations because under these conditions the calculations in any one time interval are repeated several times with slightly different values of the numbers involved. A close agreement between the results of successive calculations is in general a sufficient check on the accuracy of the numerical work when the work is done on a desk calculator. (Such a check may not be necessary on a high-speed computer.) It can be seen, therefore, that a value of $\beta = 0$, where there is no damping, is not necessarily a good choice because a separate check will be required in these cases to insure accuracy.

Studies of the effect of damping and of negative spring constants such as those corresponding to a decrease in load with increase in displacement, indicate that better results are obtained with values of β in the range from 1/6 to 1/4 than in the range below 1/8. In general, with a time interval of the order of 1/5 to 1/6 of the shortest natural period of vibration, the rate of convergence will be rapid enough for all practical purposes, and the errors will be small enough to be tolerable for every combination of damping or negative spring constant that appears to be practicable.

As a structure goes into the inelastic range, in general the periods of vibration all become longer and the shortest period becomes longer as well. For purely plastic resistance at the limit condition of an elasto-plastic structure, the period is infinitely long. Consequently the time interval can become considerably longer as plastic action develops in the structure. It is generally desirable to increase the time interval in accordance with the change in the structural rigidity as the structure becomes inelastic. However, it is not always convenient to compute the shortest period of vibration for a structure which goes partly into the plastic range. Consequently it is desirable to establish the conditions which govern the choice of time interval on some measurable behavior of the structure which is a natural function of the method of calculation. Since in general it is not practicable to consider modal behavior for structures which are not elastic, it is not desirable nor convenient to separate the individual modes of action of the structure.

With the use of a value of β greater than 1/8, the more convergence of the sequence of calculations is sufficient to insure stability and the rate of convergence will be an adequate criterion for the time interval. Consequently, one can establish a rate of convergence based on the number of iterations which it is desirable to make in a time interval and then examine the rate of convergence of the actual calculations as the calculations progress. For example, if it is desired to have three significant figure accuracy in displacement and velocity, three significant figures in the acceleration are also desirable. A rate of convergence such that the error is less than one part in a thousand at the end of three cycles would imply a rate of convergence of 10 per cent or of 0.10. By reference to Table 1c it can be seen that a ratio of time interval to natural period of the order of 0.10 for $\beta = 1/4$ will insure this rate of convergence. Similarly, a time interval slightly longer will be adequate for $\beta = 1/6$. However, it appears that these time intervals will introduce errors of the order of 2 per cent to 3 per cent in the period. If this is not admissible, then a faster rate of convergence can be established to keep the error in the period down as well.

Let us say for example that a time interval consistent with a rate of convergence of 10 per cent is desirable and that three cycles of iteration will normally be considered convenient so as to bring the results to an error of less than one part in a thousand at the end of the third cycle. Then a fourth cycle will verify that this is in fact the case. Under these conditions, then, we can assume a time interval and run through the calculations several times. If the desired rate of convergence is not obtained, we have obviously assumed too long a time interval and we can shorten the length. Since the rate of convergence is generally a function of the square of the time interval, the next estimate can usually be decided on from the course of the preceding calculations. If the time interval is made smaller in accordance with the estimate of convergence, and if convergence is now in fact obtained with about three to four cycles of iteration, then this time interval can be used in subsequent steps. If however the time interval chosen leads to convergence in only two or three cycles, we have probably taken too short a time interval and in the next cycle in the calculations we may choose to use a longer time interval.

With this type of procedure, and with continual reexamination of the time interval in terms of the rate of convergence, one can take account of the change in the characteristics of the structure without loss in accuracy and without supplementary calculations. The procedure described above can readily be programmed for a high-speed digital computer. However, an upper limit on the time interval is usually desirable to avoid difficulty in terms of accounting appropriately for the variations in the applied loading.

Calculation of Resistance

For the general case of a framed structure with either flexural or axial resistance, and with deformation in all of the members, it is a relatively simple matter to perform the calculation for the resisting forces if the displacements are given. All that is required is to take the set of displacements at a given time instant, pull the joints of the structure into the corresponding configuration, compute the axial forces in the members from the changes in length between the nodes, and compute the moments in the members by a process which involves two stages as follows:

(1) Consider the joints locked against rotation and determine the fixed end moments in all of the members corresponding to the deformations.

(2) Relax the joints by permitting them to rotate. Either moment distribution or some other technique of calculation can be used to determine the final moments in all the members.

The shears and axial forces can now be computed in all the members, and one can compute the horizontal and vertical components of the forces on the pins which hold the structure in the displaced configuration. These forces, reversed in direction, are the resisting forces acting on the structure.

If the structure is not elastic, it is possible to make the calculation for any known load-deformation or axial-displacement deformation or curvature-moment relationship. The technique of the calculations is unchanged. However, for nonlinear behavior it is essential that one take account properly of the direction of loading in a member or element so as to be sure that the element is continuing to deform in the same direction with the appropriate reduced stiffness, and to determine when it is recovering or unloading with the appropriate increased stiffness. A test of the direction of relative deformation can readily be made to determine this.

In a structure such as shown in Fig. 5, where masses might be considered placed at all the joints of the structure, it is usually neither convenient nor desirable to consider the most general situation in which the masses can be displaced horizontally as well as vertically. The reason for this is that the period of vibration of the masses in the horizontal direction is very short compared with the period of vibration in the vertical direction. If one were to solve the problem with the general procedure, one would have to use an extremely short time interval which would make the calculation tedious.

In such cases it is usually desirable to consider a restricted type of deformation of the structure to reduce the number of degrees of freedom. This can be done by connecting the masses to the structure by means of vertical links as shown in Fig. 5, and specifying that only vertical forces can act on the masses which in turn are considered to move only vertically. The links are considered to remain vertical by making them long enough so that the horizontal motions of the truss joints do not introduce an angle into the link between the truss joint and the mass.

In the structure shown in Fig. 5 the number of degrees of freedom with a mass at each node would be 20. However for the system shown only 5 degrees of freedom are needed. Both the difficulty in the calculation and the number of repetitions of calculations for a given duration of motion will be greatly increased if the 20 degree-of-freedom structure is used. However, some difficulty in computing the resistance functions in Fig. 5 is encountered because it is not possible directly to determine the changes in length of all the members from only the vertical component of displacement of the lower chord of the truss.

In cases of this sort, it is possible to proceed in a slightly different fashion by determining either directly, or by inversion of the influence matrix for the structure, the set of forces required to produce the given set of vertical displacements. In the general case, one can summarize the calculations as follows:

(1) Compute the vertical displacements of each of the masses for individual unit values of vertical force at each of the masses in turn. The set of values obtained is designated as the influence matrix for deflection of the structure.

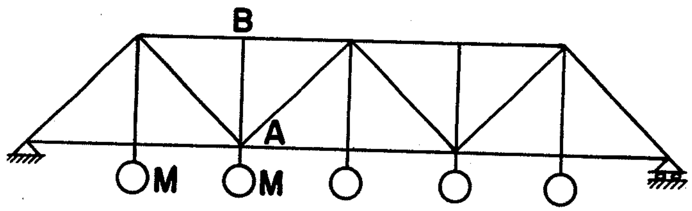
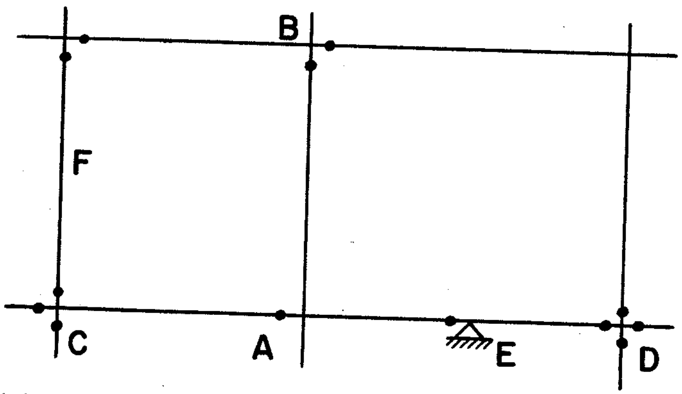
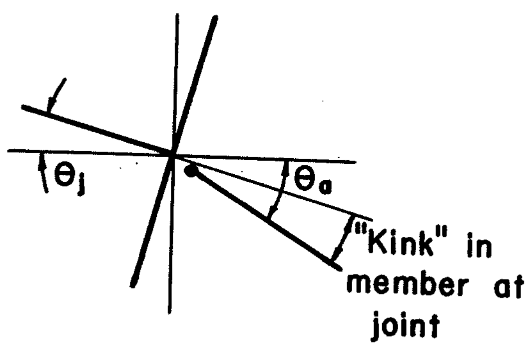


Fig. 5. Structure with Reduced Number of Degrees of Freedom



(a) Frame with Plastic Hinges



(b) Detail of Joint

Fig. 6. Elasto-Plastic Frame

(2) From these influence coefficients, compute the set of forces on the structure for given values of the displacements. The calculation involves solving a set of simultaneous equations for each set of values of displacement. This can be systematized by "inverting" the influence matrix to obtain the stiffness matrix for the structure, which represents the forces acting on each of the masses for a unit displacement of each of the masses in turn. If this inversion is performed once, the matrix of values can be applied to any set of displacements to determine the required resisting forces.

(3) The coefficients in the influence matrix will change when any elements in the structure become plastic. Consequently the calculation of the influence matrix and the inversion of the matrix to obtain the stiffness matrix must be performed for each change in stiffness that occurs during the history of the structure. It may be desirable to make the calculations by a "relaxation" procedure for more complex structures of this type. However, in principle the structure can always be analyzed to find the forces consistent with a given set of deformations.

It might be pointed out that for a beam of the same span as in Fig. 5, with a series of masses acting on it, it is not necessary to use the influence matrix and invert it to find the stiffness matrix. The structure can be analyzed directly by pulling the joints into the deflected configuration, locking them against rotation, and then distributing the fixed end moments corresponding to these deformations to find the shears in the members and the consequent forces acting on the masses. Although in such a case there would be theoretically 10 degrees of freedom, 5 of the degrees of freedom are associated with horizontal axial deformations in the beam, and these are not "coupled" with the vertical deformations because of the nature of the system. This observation is true only for a straight beam. In a later discussion, the procedure applicable to an arched beam will be described.

The structure in Fig. 5 is one of several special cases which require a slightly modified form of treatment. Another case concerns any structure with constraints on the deformation of some of the members. Such a structure might be a polygonal arch which is considered to deform only in flexure and which is not considered to have any axial deformation of the arched members. In this case also the number of degrees of freedom are reduced, but the treatment must be somewhat different from that for the truss because the number of constraints may reduce the number of degrees of freedom even below the number corresponding to a mass at each node prescribed to move in a particular direction. Further discussion of this topic is contained in the last chapter of this paper.

Both Figs. 4 and 5 show structures with zero masses at some of the nodes. A general procedure for such structures can be discussed. In general, one can handle the problem in the following way when damping forces are not present. The modification for damping is readily made as has been discussed previously.

(1) Assume values of acceleration only at the joints where masses exist and in the direction in which the masses can move.

(2) Compute the displacements of the masses, but not the displacements of the joints not having masses.

(3) Let the joints without masses be free to deflect but apply to these joints the external forces which are applied at these points. If these external

forces are zero, no restraints and no loads are applied at the massless nodes.

(4) We now have a system with prescribed deflections at some points only and with prescribed loads at other points. It may be pointed out that at some particular joint for example there may be a prescribed vertical displacement and a prescribed zero horizontal load, if we consider that only vertical displacement of the mass is to be considered at that point.

(5) We now require the calculation of the resisting forces at the masses in the direction of motion. In general this can be done by an influence coefficient procedure in which we apply unit forces at the masses which are displaced, one at a time, and compute the displacements at these masses. We use these influence coefficients to write a set of equations which in effect says that the product of each unknown reaction force at each mass multiplied by the influence coefficient for deflection at each of the masses, and summed for each of the masses individually, leads to a deflection at each mass equal to the prescribed deflection minus the deflection in that direction at that mass due to the prescribed forces acting at the nodes where no masses exist.

(6) The solution of this set of equations yields the desired results. We may now have to complete the calculation by determining the displacements at the massless nodes of the structure if these are required for any other purpose. In most instances they will not be needed and need not be determined. Where damping forces exist, however, we will need these displacements so that we can determine the velocities and accelerations at these points in order to check on the values of the damping forces dependent on these quantities.

Non-Elastic Behavior

General Comments

For trusses the problem of inelastic behavior of the members is a relatively simple and straightforward one. One need take into account only the change in deformation of the members in each time interval to determine whether the plastic action is continuing or the member is unloading elastically. Having knowledge of the load-deformation characteristics of the member, the required force in each member can be determined and the restraining forces or the resisting forces in the structure are directly determinable. Although the general concept for framed members is the same, there are some difficulties which require consideration.

Elasto-Plastic Frames

The method for dealing with elasto-plastic behavior in frames is described here. Consider a general framework consisting of members acting in flexure as shown in Fig. 6a. At some stage in the deformation, the moment at the ends of the members shown as black dots in the figure has reached the plastic limit moment. One or several of the members meeting at a joint may be loaded to the plastic limit moment.

It is general enough to consider members which are not loaded internally and in which the moments arise from a prescribed set of displacements of the joints. In a situation such as that shown at E in the figure we might have had an interior load applied dynamically. Then we will in effect have a prescribed displacement of joint E which can be considered to be an external hinged support as shown. Consequently, our method of treatment is general and can deal with internal loads as well as loads applied at the joints.

At some time t_n the structure has been analyzed with the result that we have determined the plastic hinges which exist at that time and, as a consequence, we have determined the velocities, displacements, and resisting forces in the structure. We can also determine from the final configuration and moments in the structure at that time the rotations of the joints θ_j and the rotations at the plastic hinges θ_a. If we have analyzed the structure by moment distribution, the fixed end moments can be computed and distributed and from a comparison of the fixed end moments and final moments the rotations at each end of each member can be determined. The rotation of a joint is the same as the rotation of the ends of all members meeting at that joint which do not have plastic hinges. The "kink" or relative angle between a member having a plastic hinge at the joint and the angle of the joint is measured by the difference between θ_a and θ_j. The kink angle is of importance in determining whether the member is loading or unloading at that joint.

It is considerably simpler to deal with the structure in each interval by making calculations of the change in the structure. In other words, one works with the change in displacements during the interval as defining the fixed end moments at the end of the interval and the final moments so computed are the changes in moment to be added to the original moments in the various members. In this process one can deal with the elasto-plastic hinges as actual hinges if the structure is loading during the interval and the hinges will be removed if the structure is unloading during the interval.

Three situations require consideration:

(1) If the total moment, including the original moment plus the increment in moment during the interval, at the end of any member exceeds the plastic hinge moment, then a hinge must be placed in that member.

(2) If a hinge exists in a member and the increment in kink angle in the member at the joint at the end of the interval is in the same direction as the preceding total kink, then the hinge remains.

(3) If the kink angle in what was originally a plastic hinge should decrease, then the plastic hinge must be removed.

The structure is modified by adding, keeping, or removing the plastic hinges originally assumed for the calculation at the end of the interval, in accord with the above criteria, and the calculation repeated until the conditions obtained are consistent.

Only in unusual cases can there be plastic hinges in all members meeting at a joint. Such a case is shown at D in Fig. 6. In such an instance the joint rotation can be assumed to remain at its value at the time the last members at the joint become plastic. (It is clear that in such a joint at least two members must become plastic finally, and at least two members must simultaneously "unload" and have the hinges removed.)

The only other special case that needs consideration is that shown by the member F in which both ends develop plastic hinges. Such a member can be considered to have end rotations the same as those which existed prior to the development of the last hinge in the member.

In some cases it may be desirable to study the formation of plastic hinges in a structure as the loading on the structure is applied, more or less "statically", or applied and released in various ways. This can be done very simply by specifying a relatively slow rate of application of the loading, corresponding to a maximum being reached in not less than twice the natural period of vibration in the fundamental mode. It is usually convenient to introduce damping so as to avoid oscillatory motion during or following the loading cycle. The masses can be taken arbitrarily small in such a calculation if we are not concerned with the dynamic response, so as to make the fundamental period relatively short.

Time-Dependent Boundary Conditions

In general if the boundaries or supports of a structure move with time no change is required in the method of calculation and the general procedures described herein are directly applicable. However, there are instances where it is desirable, or convenient, to deal with the motion of the elements of the structure relative to the base or foundation of the structure, rather than in absolute terms. Such situations arise commonly in earthquake motions. The method described herein is useful primarily in those cases where the foundation moves as a single unit and where motion of the foundation of the structure with all masses equal to zero would not introduce stresses in the structural framework.

Consider the structure shown in Fig. 7 where the base can move as a unit in the horizontal direction. Let the motion of the base be defined as a function of time by the quantity y. The motions of any of the masses in the structure are defined by x. Let us assume that the axial deformations of the members can be neglected and that we have only flexural deformations to consider. In such cases, we can lump the masses at each floor level at one point as shown. The more general case offers no difficulties, however.

If now we apply to the structure with masses on it a force F, in accordance with the relation

$$F = M\ddot{y} \qquad (27)$$

then the structure and the foundation will move as a unit with no relative displacement between the two.

If now we apply to the structure considered supported on a fixed foundation a force $\bar{P}$, defined by the relation

$$\bar{P} = P - F \qquad (28)$$

we have changed the problem from one in which there are external forces and a time-dependent boundary motion into one in which we have external forces and no boundary motion. The displacements for the modified forces $\bar{P}$ will be designated by u, where

$$u = \bar{x} - y \qquad (29)$$

it can be seen, of course, that the total motions of the masses, x, are the sum of the relative motions u plus the foundation motions y, and the total loads applied are the external forces P. Consequently, the procedure is valid.

For more complex systems, one must modify this procedure by taking into account the displacements at each of the several points of support, and defining the quantity y for the motion of the masses as being the motion at that point on a massless structure consistent with the foundation motions. In the

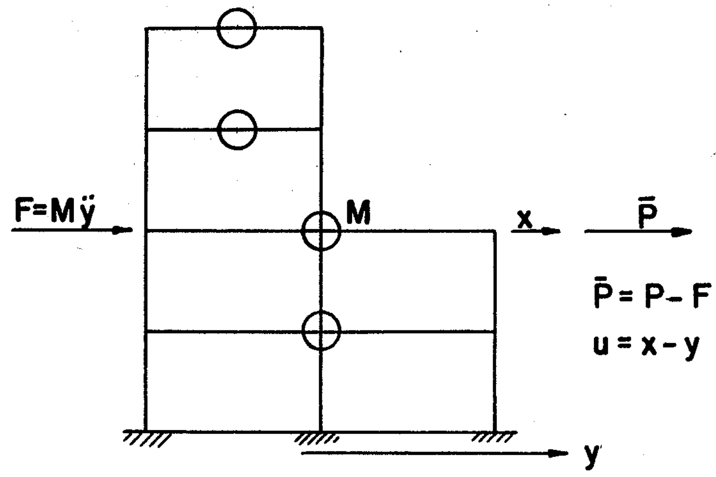
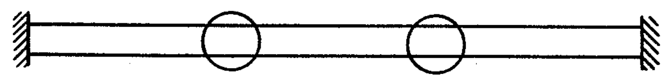
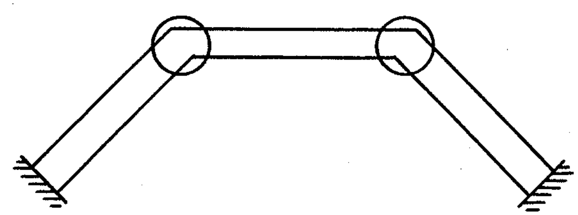


Fig. 7. Structure with Time-Dependent Boundary Conditions



(a) Structure with Weak or Zero Coupling Between Sets of Modes



(b) Structure with All Modes Coupled

Fig. 8. Examples of Different Types of Coupling Between Modes

case of a beam, for example, simply supported at one end and hinged at the other where the hinged end is constrained to move vertically in accordance with some time-dependent condition, the displacements of the masses along the length of the beam are proportional to the end displacement and are equal to that end displacement multiplied by the ratio of the distance from the simple support to the mass divided by the length of the beam. The same method of analysis applies, and the modified forces will give the relative displacements of the beam, relative to boundaries fixed in position.

In this case, and in other cases where no stresses are introduced in the weightless beam by the boundary motions, the entire solution is straightforward and all of the stresses can be obtained from the modified system. However, in more complex cases where the boundary motions may introduce stresses in the massless system, the general principles are applicable but the total stresses must then be determined by adding the stresses in the massless system with boundary motions to the stresses in the system with fixed boundaries and with modified forces. In such instances, the problem of plastic behavior introduces complexities which may be unwieldy and it is not desirable to work with the replacement system. In all cases it is possible to work with the original system directly with the general methods that have been described.

Curved Frames and Arches

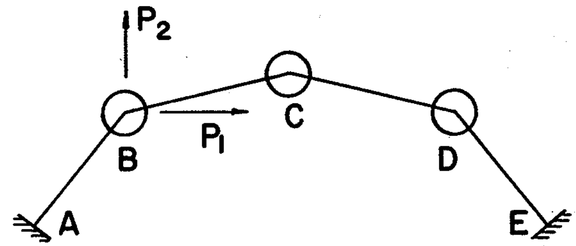
General Procedure

Consider the two structures shown in Fig. 8. In Fig. 8a, a beam is shown with two masses. These masses may move vertically with consequent flexure of the beam or horizontally with consequent axial deformation. There is no coupling between these sets of motions. There are two degrees of freedom in each set or type of motion and four degrees of freedom altogether. The vertical motions can be dealt with as if in a two-degree-of-freedom system and the axial motions can be dealt with also as in a two-degree-of-freedom system with neither being considered to have influence on the other. However, in the structure shown in Fig. 8b, neither pure vertical nor pure horizontal motion of each mass is possible. In general, all four modes of motion or all four degrees of freedom are coupled, and the structure requires treatment as if it were a four-degree-of-freedom system.

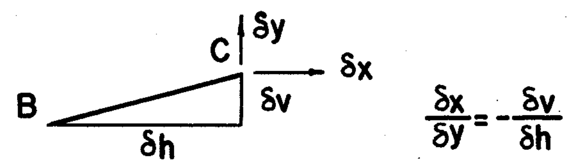
However, in the structure shown in Fig. 8b, if it is desired to avoid axial deformation of the bars, only one mode of motion is possible, with the mass to the left moving down to the right and the mass to the right moving up and to the left. The system has only one degree of freedom, and can be analyzed as such although the method of analysis requires some care in order to avoid introducing motions that are inadmissible.

Inextensible Members—Constraint Relations

Consider the structure shown in Fig. 9a. Here there are three interior masses and if the members are considered inextensible there will be two degrees of freedom. We arrive at this number by considering the total number of degrees of freedom of the system if the members were capable of deformation, namely 6, and subtracting from this number the number of constraint conditions or in this case the number of inextensible members, namely 4.



(a) Arch Considered



(b) Constraint Relation for Zero Extension of BC

Fig. 9. Extensible and Inextensible Arched Structures

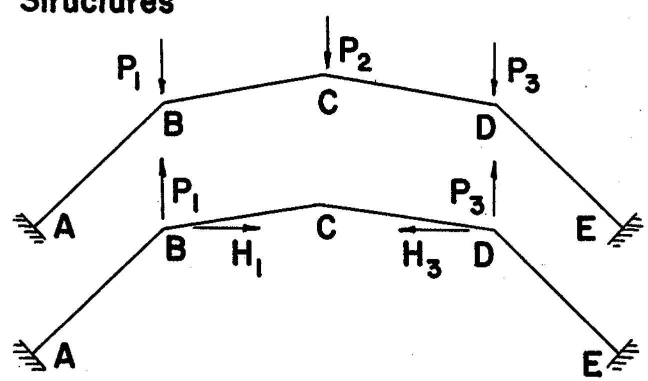


Fig. 10. Symmetrical Loading Producing No Displacement in an Inextensible Arch

For any one of these members, such as BC, the condition that the deformations at the joints correspond to no change in length of member BC can be expressed in terms of the relative displacement of joint C compared with joint B and the slope of the member BC. For small deformations, where δx and δy represent the horizontal and the vertical motion of point C with respect to point B, and where δV and δH represent respectively the vertical difference in elevation between C and B and the horizontal distance between C and B, the relationship that the relative deformation involve no change in length can be expressed as follows:

$$\frac{\delta x}{\delta y} = - \frac{\delta V}{\delta H} \tag{30}$$

This equation can be interpreted as meaning that the deformation is such that point C moves perpendicularly to member BC.

A similar constraint relation can be written for each of the bars. If the total deformations are expressed in terms of the increments in the deformation, one can find four relations among the three horizontal and three vertical displacements of point B, C, and D. These will be somewhat more complex but the relations can be written in any case. The number of independent relations which it is possible to write in this case will be equal to the number of constraint conditions, namely four. With four relations among six displacements, it appears that there are only two independent displacements remaining. These can be chosen as any two components of displacement, and the calculations carried through in the usual way, but with only two masses and two directions of motion considered. By the use of the constraint relationships, when the two deflections which are independent are assumed the remaining deflections can be readily computed, and the entire pattern of deformation of the structure established.

However, it is possible to develop a more direct procedure for determining the relationships for the resisting force components and for the displacements. This can be done by noting that there are certain patterns of loading on the structure which produce only axial stresses in the members. For these loading components no deformations of the arch can take place. It is obvious that one type of loading would correspond to axial forces directed along the length of any one of the members. This can be expressed also in terms of the vertical and horizontal components at the two ends of the member such that the resultant lies along the length of the member.

By considering four component loadings corresponding to loadings producing axial force in only one member at a time, one arrives at four independent load conditions which produce no displacement in the arch in Fig. 9. These can be combined in various ways. One obvious combination consists in a pattern of only vertical forces at the interior nodes. These of course would be resisted by vertical and horizontal reactions at the ends but these do not enter into the picture if the ends do not move. This pattern of loading is shown in the upper part of Fig. 10. A second pattern of loading in the lower part of Fig. 10 consists of the two sets of loadings which produce tension only in the outer members of the arch. If the structure is symmetrical, and if the loads B and D are made equal, then the loading pattern will be symmetrical. The loading pattern in the upper part of Fig. 10 will always be symmetrical for a symmetrical structure. These two loadings represent the two independent types of symmetrical loadings producing no displacement in the inextensible arch of Fig. 9. Of course any combination of these two loadings will also be a symmetrical loading producing no displacement. Consequently, one can arrive at a loading consisting of a downward load at C and equal horizontal loads with no upward load at joints B and D. There are two other independent loadings which are anti-symmetrical. From combinations of the four loadings, various types of loading patterns can be derived.

The advantage of having the constraint conditions stated in this way is that now we can add any of these types of loadings to any set of resistances or of external loadings so as to preserve the number of degrees of freedom of the structure at the proper number, in this particular case two. In other words, after the two independent displacements are decided on such as the horizontal and vertical displacement at joint B in Fig. 9, and the corresponding displacements at C and at D are determined consistent with these, then both the external forces and the resistance can be modified by adding force patterns similar to those in Fig. 10, (plus the two anti-symmetrical loadings), so as to make the accelerations in the various directions of all of the masses consistent.

Consistency in the vertical displacements during the interval requires consistency in the vertical accelerations at the end of the interval. Since the displacements at the beginning of the interval have been made consistent, and since the velocities and accelerations at the beginning of the interval are consistent, we need only insure that the accelerations at the end of the interval are properly consistent. In order for these to be properly consistent, they must obey the same relationship that the displacements obey. In other words, we have the same relationships among the accelerations in the various directions at the joints or nodes as we have constraint relations corresponding to zero extension of the members. It is a simple matter to adjust the resisting forces by adding the proper components of loadings so as to make the net acceleration at all of the masses consistent.

As a by-product of the results shown in Fig. 10, one can apply the principle of virtual work to find relationships among the displacements at the various joints. For any pattern of loading which produces no displacement, the product of the loadings in the pattern times the displacements in any set of consistent displacements of the structure in the direction of the loadings in the pattern must be zero. This follows from the principle of virtual work directly. In the case of an arch the vertices of which lie in a parabola, as in Fig. 9, the loadings at the three vertices, as shown in the upper part of Fig. 10, will be equal. This implies that the sum of the vertical deflections at B, C, and D, taken as positive downward, must be zero. A similar relationship can be arrived at from the loadings in the lower part of Fig. 10 or from each of the load conditions considered.

Reduction in Number of Effective Degrees of Freedom

The operations above suggest a modification for the approximate dynamic analysis of an arch which is inextensible. Consider the arch shown in Fig. 11 which is the same as in Fig. 9 except that the masses are carried by vertical links, which remain vertical, in the same manner as in the truss in Fig. 5. Here now we constrain the masses to move only in the vertical direction. There are then only three degrees of freedom for the structure. If the structure were extensible, we have reduced the number of degrees of freedom from 6 to 3, but we have lost the horizontal motion of the masses. If we know that these are unimportant or feel that they might be, or if we wish to permit

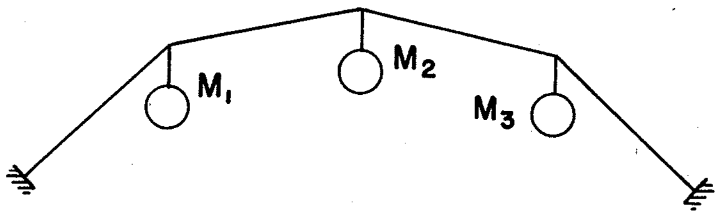


Fig. 11. Scheme For Approximate Dynamic Analysis of an Arch

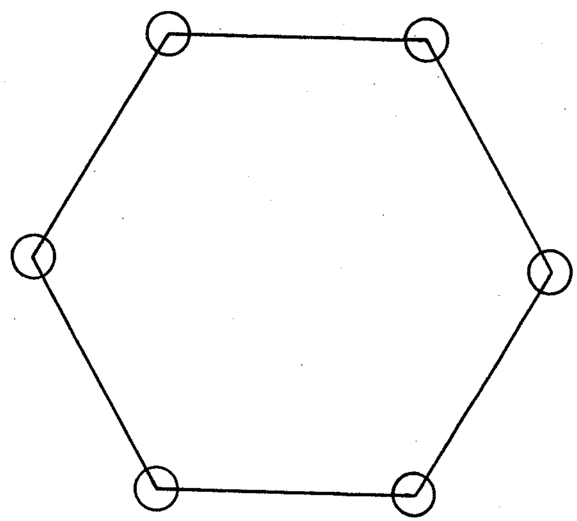


Fig. 12. Hexagonal Ring

this approximation in order to find a quicker solution to the problem, as we might in the case of a very flat arch, we can work with this problem in the same fashion that we solved the problem in Fig. 5 earlier. We need now find only the influence matrix and the corresponding stiffness matrix in order to find the solution.

If, however, the arch is inextensible, there are really only two degrees of freedom of the structure and we have a system which apparently has three. However, by use of the fact that for the arch shown a set of loadings consisting of equal vertical loads at each of the three joints will produce no deformation, we arrive at the condition that the sum of the three vertical displacements must be zero. Then we could use this as a constraint condition which will permit us to work with only two independent vertical displacements. We can define the third vertical displacement in terms of the other two, and arrive at a consistent set of forces by adding to any set that we compute at the two independent joints a proper combination of three equal vertical loads so as to make the vertical accelerations at the three points consistent. Consistency in the vertical accelerations means consistency in the increments in deflections, which means that the vertical accelerations must also obey the rule that the sum of the vertical accelerations at points 1, 2, and 3 as shown in Fig. 11 must be zero.

It can be seen that in general for an arch with N nodes, the number of degrees of freedom for an extensible arch will be 2N, and for an inextensible arch, N-1. The number of degrees of freedom for vertical motion at the nodes will also be N-1.

Extensible and Inextensible Rings

If the ends of the arch are free to move, additional degrees of freedom are introduced. These represent no real difficulty and can be considered either for the extensible or inextensible case in a relatively simple manner. However, more complexities are introduced in the case of a complete ring.

Consider, for example, the hexagonal ring in Fig. 12. There are six masses at the nodes of the ring. For other shapes or number of sides, the relationships can readily be determined. In this particular case, it is clear that the number of degrees of freedom for an extensible ring is 12. However three of these are the rigid body motions for an object in a plane and produce no internal stresses. In the general extensible case of deformation, with two degrees of freedom for each mass, one would compute ordinarily 12 displacements, one in each of the two directions at each of the 6 masses. The statically required forces needed for consistent accelerations will automatically be achieved, and the system can be treated as if it had in fact 12 degrees of freedom.

In the case of the inextensible ring, there are in the case of the hexagon six constraint relations corresponding to the six bars. This means that there are essentially only 3 degrees of freedom for the system rather than 9 for the inextensible case. Here again the 3 rigid body motions correspond to no stress in the system. However the system can be treated as if it had in fact six degrees of freedom since the rigid body motions will automatically be achieved. The six constraint relations must be used to insure consistent values of acceleration and displacement.

Miscellaneous Problems

Space Frames and Space Structures

Space frameworks or space structures of a more complex character than those generally described herein can be handled by the same general principles. For a structure in space, there are in general 3 degrees of freedom for each mass at a node, and consequently 3 components of external force and resisting force which must be considered. This introduces complication only in the order of magnitude of the calculations which are involved.

Buckling

For structures in which axial loading produces secondary effects because of the deflections of the parts of the structure, thereby increasing the moments and the deflections for a given loading, the calculations are more complex. In general, the resisting forces for a given amount of prescribed deflection are reduced, in about the same ratio that the deflections for a given loading are increased when buckling loads are present. This reduction in resisting force means generally a greater acceleration and a consequent greater dynamic displacement in a structure in which buckling tendencies exist. These tendencies can be taken into account but the methods for doing so become relatively unwieldy. One can compute the resisting forces on the assumption that the buckling tendencies are negligible, and then for these resisting forces determine the relative deflections corresponding to the secondary effects of the forces produced by the deflection of the structure. Then by reducing the resisting forces in a more or less arbitrary manner, one can arrive at a set of resistances which would account for the prescribed deflections. Although this method is not entirely satisfactory, it does give a means of taking into account approximately the buckling tendencies for complex structures. For simple structures, the problems can sometimes be handled directly. The real difficulty in the case of buckling problems is that buckling is a nonlinear effect and the methods of calculation used to handle the calculations in each interval in the process depend on linearity, within that interval at least.

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ULTIMATE STRENGTH CRITERIA FOR REINFORCED CONCRETE

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SYNOPSIS

Criteria for ultimate strength of structural members are derived by determining analytically the value of extreme compression edge strain which results in a maximum value of moment or load. The following rectangular members are considered: (a) homogeneous beams, (b) reinforced concrete beams, and (c) eccentrically loaded reinforced concrete columns. In all three cases, ultimate strength so derived is in agreement with tests, and is a function of only the stress at the extreme compression edge and properties of the cross-section involved.

INTRODUCTION

Ultimate strength of structural members is commonly determined on the basis of the assumptions that plane sections remain plane during bending and that stress is a function of strain only. Within the range of linear stress-strain relationship the stress distribution throughout a member is determined from the equations of equilibrium of forces and of moments together with the requirement of linear distribution of strains. If the stress-strain relationship is non-linear, simplifying assumptions, such as the parabolic, trapezoidal or rectangular stress distribution, are commonly introduced, or else the ultimate strength design equations are derived empirically from test data.

By another approach, the problem of ultimate strength may be studied analytically by finding the maximum value of certain load functions expressed in terms of the internal resisting forces of the loaded member. This can be done without defining mathematically the stress-strain relationship of the inelastic material. In this manner, criteria for ultimate strength may be derived which

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