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ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES BY THE SLOPE DEFLECTION METHOD

BY

W. M. WILSON

F. E. RICHART AND CAMILLO WEISS

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ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES BY THE SLOPE DEFLECTION METHOD

I. Preliminary

1. *Object and Scope of Investigation.*—Frames composed of rectangular elements must in general be designed with stiff connections between the members at the joints, in order that loads may be carried. These connections must be capable of transferring not only direct axial tensile and compressive forces, but also bending moments. It follows that frames made up of rectangular elements are usually statically indeterminate; that is, the stresses in them can be found only by taking into account the relative stiffness and deformations of the various members. The common use of rectangular frames in engineering structures makes it highly desirable that the most convenient methods of analyzing their stresses should be developed. The stresses in a number of such rectangular frames have been analyzed by the writers. This bulletin describes the methods used and presents the formulas derived.

The bulletin is divided into two parts: the first part is devoted to the derivation of fundamental equations; in the second part, methods and equations are derived for use in determining moments, stresses, and deflections for a variety of typical structures.

2. *Acknowledgments.*—The investigation here reported was made under the auspices of the Department of Civil Engineering of which Dr. F. H. Newell is the head. A portion of the work was done in 1915 in connection with the development of a thesis in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering by F. E. Richart. Many of the analyses have been checked by W. L. Parish, graduate student in Architectural Engineering, and Yi Liu, graduate student in Civil Engineering, to whom the authors gratefully acknowledge their indebtedness.

PART I

DERIVATION OF FUNDAMENTAL EQUATIONS

II. Propositions upon which Fundamental Equations Are Based

3. *Statement of Propositions.*—The fundamental equations used in these investigations are derived from the principal propositions of the moment-area method.* These may be expressed as follows:

(1) *When a member is subjected to flexure, the difference in the slope of the elastic curve between any two points is equal in magnitude to the area of the* $\frac{M}{EI}$ *diagram for the portion of the member between the two points.*

(2) *When a member is subjected to flexure, the distance of any point Q on the elastic curve, measured normal to initial position of member, from a tangent drawn to the elastic curve at any other point P is equal in magnitude to the first or statical moment of the area of the* $\frac{M}{EI}$ *diagram between the two points, about the point Q.*

The $\frac{M}{EI}$ diagram is a graph in which the ordinate at any point is obtained by dividing the resisting moment, M, by the product of modulus of elasticity of material, E, and the moment of inertia of the section, I, at that point. If E and I are constant, the diagram will be similar in shape to the moment diagram for the member.

4. *Proof of Propositions.*—The line AB, Fig. 1, represents the elastic curve of a member in flexure. Consider the elementary length, ds, of the member shown in Fig. 2. The angle between radii at the ends of ds will be denoted by $d\theta$. The linear deformation of a fibre at a distance c from the neutral surface is $cd\theta$, and the unit deformation of the same fibre is $\frac{cd\theta}{ds}$. From the well known flexure formula the

*The principles of the moment-area method were given in an article by O. Mohr, Beitraege zur Theorie der Holz-und Eisenkonstruktionen, Zeitschrift des Arch.-und Ing. Ver. zu Hannover, 1868, p. 19. About the same time the method was presented by C. E. Greene in lectures at the University of Michigan. Several modern textbooks on mechanics give the method; see, for instance, Strength of Materials, by J. E. Boyd, Second Ed., 1917.

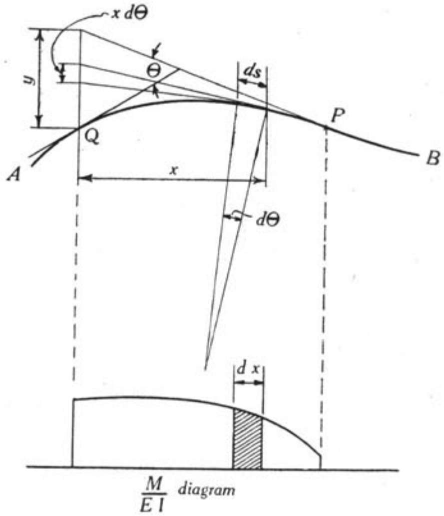


FIGURE 1

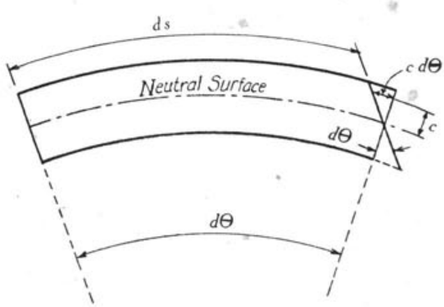


FIGURE 2

accompanying unit stress in the fibre is $S=\frac{Mc}{I}$, in which M is the resisting moment and I moment of inertia of section.

Since the modulus of elasticity is the ratio of unit stress to unit deformation, E is equal to $\frac{Mc}{I}$ divided by $\frac{cd\theta}{ds}$. Hence $d\theta=\frac{M}{EI}ds$. Since in a well designed beam, the curvature and slope are small, dx may be substituted for ds without material error, and $d\theta=\frac{M}{EI}dx$.

In the $\frac{M}{EI}$ diagram of Fig. 1, $\frac{M}{EI}dx$ represents the area of the diagram for the length dx. The area of the diagram between points Q and P on the elastic curve is then equal to $\int_Q^P \frac{M}{EI}dx$. But the difference of slope of the tangent to the elastic curve is also represented by

$$\theta = \int d\theta = \int_Q^P \frac{M}{EI}dx \qquad (1)$$

Hence proposition (1) of the preceding section is proved. It may be noted that if M is taken as the resisting moment acting on the portion of the member to the left of any section, by applying the conventions of section 5 the area of the $\frac{M}{EI}$ diagram is positive; also the direction of integration from Q to P is positive, and the difference in slope θ is positive. Other terms involved may be considered as scalar quantities. These conventions apply to any case, as, for instance, difference in slope from P to Q is negative, since the direction of integration is negative.

In Fig. 1 the tangents at the extremities of the element of the elastic curve, ds, are extended until they intersect the vertical line through the point Q. The intercept on this vertical line between the two consecutive tangents is $xd\theta$. The total vertical distance, y, of Q from the tangent drawn at P is the algebraic sum of all the intercepts between tangents for the portion of the curve between Q and P; that is, $y = \int_Q^P xd\theta$. Substituting the value of $d\theta$ found previously,

$$y = \int_Q^P \frac{M}{EI}x\,dx \qquad (2)$$

In the $\frac{M}{EI}$ diagram of Fig. 1, $\frac{M}{EI}dx$ represents the area of the diagram for the length dx, and $\frac{M}{EI}dx$ times x represents the moment of this area about the point Q. The moment of the entire area of the $\frac{M}{EI}$ diagram between points Q and P about the point Q may now be expressed by $\int_{Q}^{P} \frac{M}{EI} x dx$. Since this expression is identical with the right-hand member of equation (2), proposition (2) of the preceding section is proved. The conventions of section 5 apply here as explained in the proof of proposition (1).

III. Derivation of Fundamental Equations

5. *Conventional Signs.*—The signs of the quantities used in the equations in this bulletin are determined by the following conventional rules:

When the tangent to the elastic curve of a member has been turned through a clockwise direction, measured *from* its initial position, the change in slope, or the angular deformation, is positive.

When the line joining the ends of a member is rotated, the movement of one end of the member relative to the other, measured perpendicular to the initial position of member is called a deflection, and is so used throughout the following discussion. The deflection is positive when such rotation is in a clockwise direction from the initial position of member.

The resisting moment or moment of the internal stresses on a section is positive when the internal or resisting couple acts in a clockwise direction upon *the portion of the member considered.* According to this rule the portion of the member considered must always be specified, and will be indicated by the subscripts used with the moments. For example, if C is a point on a member between the ends A and B, M_{CA} is equal to $-M_{CB}$.

The moment of an external force or couple is positive if it tends to cause a clockwise rotation.

6. *Derivation of Equations for Moments at Ends of Members in Flexure—Member Restrained at the Ends with No Intermediate Loads.*—The line AB in Fig. 3 represents the elastic curve of a member which is not acted upon by any external forces or couples except at the ends. The resisting moment at A is represented by M_{AB} and at B by M_{BA}. The change in the slope of the elastic curve at A from its initial position is represented by θ_A, and that at B by θ_B. The deflection of A from its original position A' is d. The distance of B from the tangent drawn to the curve at A is equal to $(d-l\theta_A)$.

From proposition (2), section 3, $(d-l\theta_A)$ may be expressed as the statical moment of the $\frac{M}{EI}$ diagram for member AB about the end B.

The quantities E and I will here be considered as constant throughout the length AB. If M represents the resisting moment on the portion of member to the left of a section, M is equal to $-M_{AB}$ at A, and to $+M_{BA}$ at B. The $\frac{M}{EI}$ diagram of Fig. 3 can best be treated as the

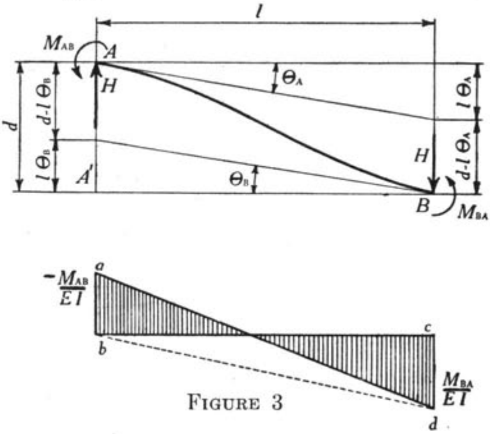


FIGURE 3

algebraic sum of the two triangles *bad* and *bcd*. Hence the statical moment of the $\frac{M}{EI}$ diagram about B is equal to the area of triangle *bad* times the distance to its centroid, $\frac{2}{3}l$, plus the area of triangle *bcd* times the distance to its centroid, $\frac{1}{3}l$. This gives

$$d-l\theta_A=\frac{-M_{AB}l^2}{3EI}+\frac{M_{BA}l^2}{6EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From proposition (1), section 3, $\theta_B-\theta_A$ is equal to the area of the $\frac{M}{EI}$ diagram for member AB, or the algebraic sum of areas *bad* and *bcd*. This gives

$$\theta_B-\theta_A=\frac{-M_{AB}l}{2EI}+\frac{M_{BA}l}{2EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Combining equations (3) and (4) to eliminate M_{BA}, letting $\frac{I}{l}=K$ and $\frac{d}{l}=R$, gives

$$M_{AB}=2EK(2\theta_A+\theta_B-3R) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly combining equations (3) and (4) to eliminate M_{AB} gives

$$M_{BA}=2EK(2\theta_B+\theta_A-3R) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since the signs of all quantities in equations (3) and (4) are independent of the sense of the quantities themselves it follows that equations (3) and (4) are general; and they give the sense as well as magnitude of the moments, no matter what the senses of θ_A, θ_B, and R may be, provided the method of determining signs given in section 5 is followed. As before noted, M_{AB} is the resisting moment acting at the end A of the member AB. The moment which AB exerts upon the support at A is equal in magnitude but opposite in sense to M_{AB}. A and B are not necessarily supports of a member but may be any two points along the length of a member, provided there is no intermediate load on the member between them.

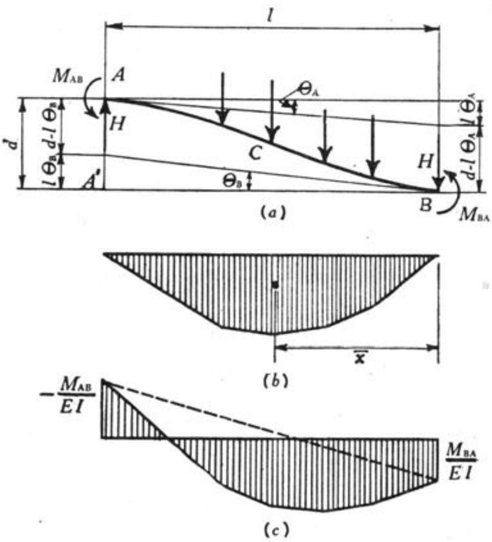


FIGURE 4

Equations (5) and (6) are fundamental equations.* They may be expressed as follows:—The moment at the end of any member carrying no intermediate loads is equal to $2EK$ times the quantity: Twice the change in slope at the near end plus the change in slope at the far end minus three times the ratio of deflection to length. E is the modulus

*The slope-deflection equations for a member acted upon only by forces and couples at the ends were deduced by Manderla in 1878. See Annual Report of the Technische Hochschule, Munich, 1879, and Allgemeine Bauzeitung, 1880. The use of these equations has been developed by several writers, among whom are:

Mohr, Otto, "Abhandlungen aus dem Gebiete der Technischen Mechanik," Second Ed., 1914.

Kunz, F. C. "Secondary Stresses," Engineering News, Vol. 66, p. 397, Oct. 5, 1911.

Wilson and Maney, "Wind Stresses in the Steel Frames of Office Buildings," Univ. of Ill. Eng. Exp. Sta., Bul. 80, 1915.

of elasticity of the material, and K is the ratio of moment of inertia to length of member.

7. *Derivation of Equations for Moments at Ends of Members in Flexure—Member Restrained at the Ends with Any System of Intermediate Loads.*—The line AB, Fig. 4-a, represents the elastic curve of the member of Fig. 3, but acted upon by a system of intermediate loads. The moments, slopes, and deflections at A and B are similar to those of Fig. 3. The $\frac{M}{EI}$ diagram, however, is affected by the intermediate loads. The quantity EI will again be considered constant. From well known principles of mechanics, the $\frac{M}{EI}$ diagram of Fig. 4-c may be obtained by superimposing the $\frac{M}{EI}$ diagram for a simple beam under the same intermediate loads (see Fig. 4-b) upon the $\frac{M}{EI}$ diagram of Fig. 3. This is merely the algebraic addition of the different moments at any section, just as in an algebraic analysis the moment at the end of a girder is combined with the moment of the shear at the end and of the external loads about the given section. Denote the area of the simple beam diagram of Fig. 4-b by F, and the distance of its centroid from B by $\bar{x}$. Then, using the propositions of section 3 as before, the statical moment of the $\frac{M}{EI}$ diagram about B is equal to $(d-\theta_A l)$.

$$(d-\theta_A l)=\frac{-M_{AB}l}{3EI}+\frac{M_{BA}l}{6EI}-\frac{F\bar{x}}{EI} \quad \ldots \ldots \ldots \quad (7)$$

The area of the $\frac{M}{EI}$ diagram is equal to $\theta_B-\theta_A$.

$$(\theta_B-\theta_A)=\frac{-M_{AB}l}{2EI}+\frac{M_{BA}l}{2EI}-\frac{F}{EI} \quad \ldots \ldots \ldots \quad (8)$$

Combining equations (7) and (8) to eliminate M_{BA}, letting $\frac{I}{l}=K$ and $\frac{d}{l}=R$, gives

$$M_{AB}=2EK\ (2\theta_A+\theta_B-3R)\ -\ \frac{2F}{l^2}(3\bar{x}-l) \quad \ldots \ldots \quad (9)$$

Similarly, combining equations (7) and (8) to eliminate M_{AB} gives

$$M_{BA}=2EK\ (2\theta_B+\theta_A-3R)+\frac{2F}{l^2}\ (2l-3\bar{x}) \quad \ldots \ldots \quad (10)$$

It is seen that equations (9) and (10) are identical with equations (5) and (6) except that they contain an additional term in the right-hand members of the equations. This additional term is independent of the slopes and deflections of the member, and depends solely upon the intermediate loads. Further significance is given to this term if the slopes and deflections are made equal to zero, as is true in a fixed beam with supports on same level. The last term then becomes the resisting moment acting on the end of the fixed beam. Hence it is seen that in general *the resisting moment at the end of a member with any system of intermediate loads can be expressed as the algebraic sum of the resisting moment at the end of a member with no intermediate loads, as given by equations* (5) *and* (6), *and the resisting moment at the end of a fixed beam with an equal span and carrying the same system of intermediate loads.*

If the resisting moment at the end of a fixed beam with supports on same level be expressed by C, with subscripts similar to those used for moments, M, equations (9) and (10) may be written in the following general form

$$M_{AB}=2EK(2\theta_A+\theta_B-3R)-C_{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$M_{BA}=2EK(2\theta_B+\theta_A-3R)+C_{BA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

These are the general *slope deflection* equations which apply to any condition of loading and restraint.

The sign of the constant C may be determined as follows: *In a fixed beam the sign of the resisting moment at the end of a member is opposite to that of the moment of external loads.* For instance, in Fig. 4 the moment of external loads about the end A is clockwise, so the resisting moment C_{AB} is counter clockwise or negative; and since the moment of the loads is counter clockwise about B, C_{BA} is clockwise or

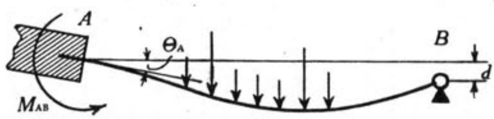


FIGURE 5

positive. If the loads were upward instead of downward, the signs of C_{AB} and C_{BA} would be reversed. With signs thus treated, C becomes merely a numerical, or scalar, quantity.

It has been noted that equations (11) and (12) apply to any condition of restraint of the ends of a member. Fig. 5 shows a member restrained at A and hinged to the support at B, so that the resisting moment at B is zero. Equations (11) and (12) may be written:

$$M_{AB}=2EK(2\theta_A+\theta_B-3R)-C_{AB}$$
$$0=M_{BA}=2EK(2\theta_B+\theta_A-3R)+C_{BA}$$

Combining these two equations to eliminate θ_B gives

$$M_{AB}=EK(3\theta_A-3R)-(C_{AB}+\frac{C_{BA}}{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If the beam is fixed at A and hinged at B, with the supports on the same level, θ_A and R in equation (13) are zero, and therefore the term $-(C_{AB}+\frac{C_{BA}}{2})$ represents the resisting moment at the end A, and can be readily calculated for any given loading.

By similar reasoning, when a beam is restrained at the end B and hinged to support at A, it is found that

$$M_{BA}=EK(3\theta_B-3R)+(C_{BA}+\frac{C_{AB}}{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

For more convenient reference let the quantity $(C_{AB}+\frac{C_{BA}}{2})$ be denoted by H_{AB}, and the quantity $(C_{BA}+\frac{C_{AB}}{2})$ by H_{BA}.

Equations (13) and (14) then take the general form

$$M_{AB}=EK(3\theta_A-3R)-H_{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$M_{BA}=EK(3\theta_B-3R)+H_{BA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The term H represents the resisting moment at the fixed end of a beam which is fixed at one end and hinged to the support at the other, with supports at same level. The sign of H is determined in the same way as the sign of C in equations (11) and (12). That is, the sign of H is always opposite to the sign of the moment of the external loads about the fixed end of the member. If the external loads act upward instead of downward, the values of H in equations (15) and (16) must be reversed.

Equations (11) and (12) are the general equations for the ends of a member in flexure. Equations (15) and (16) are special forms of equations (11) and (12), applicable to members having one end hinged. For convenience in reference these four equations are given in Table 1 where they are denoted as equations (A), (B), (C), and (D), respectively.

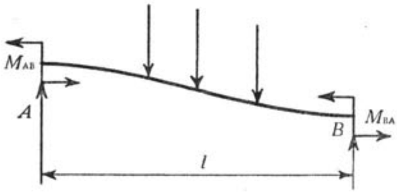


FIGURE 6

TABLE 1

GENERAL EQUATIONS FOR THE MOMENTS AT THE ENDS OF A MEMBER AB IN FIG. 6

$$\mathbf{M_{AB} = 2EK(2\theta_A + \theta_B - 3R) \mp C_{AB}} \quad \text{(A)}$$

$$\mathbf{M_{BA} = 2EK(2\theta_B + \theta_A - 3R) \pm C_{BA}} \quad \text{(B)}$$

If end B is hinged,

$$\mathbf{M_{AB} = EK(3\theta_A - 3R) \mp H_{AB}} \quad \text{(C)}$$

If end A is hinged,

$$\mathbf{M_{BA} = EK(3\theta_B - 3R) \pm H_{BA}} \quad \text{(D)}$$

NOTE.—The signs of the quantities used in these equations are determined by the following rules:

θ is positive (+) when the tangent to the elastic curve is turned in a clockwise direction.

R is positive (+) when the member is deflected in a clockwise direction.

The moment of the internal stresses on a section is positive (+) when the internal couple acts in a clockwise direction upon the *portion of the member considered*.

If the moment of the external forces on the member about the end at which the moment is to be determined is positive (+), the sign before the constant is minus (—); if the moment of the external forces about the end at which the moment is to be determined is negative (—), the sign before the constant is plus (+). With the forces acting downward as shown in the sketch, for the moment at A, C_{AB} and H_{AB} are preceded by a minus (—) sign, but for the moment at B, C_{BA} and H_{BA} are preceded by a plus (+) sign.

8. *Derivation of Equations for Moments at Ends of Members in Flexure—Member Restrained at the Ends, with Special Cases of Loading.*— One method of determining the quantities C and H in equations (A), (B), (C), and (D) of Table 1 has been explained. To illustrate the method, some special cases will be considered here.

Fig. 7 shows a member restrained at the ends with a concentrated load at a distance a from A, and a distance b from B. In the simple beam moment diagram, the maximum ordinate is $\frac{-Pab}{l}$, the area F is $\frac{-Pab}{2}$, and the distance $\bar{x}$ of the centroid of the area F from B is $\frac{1}{3}(l+b)$.

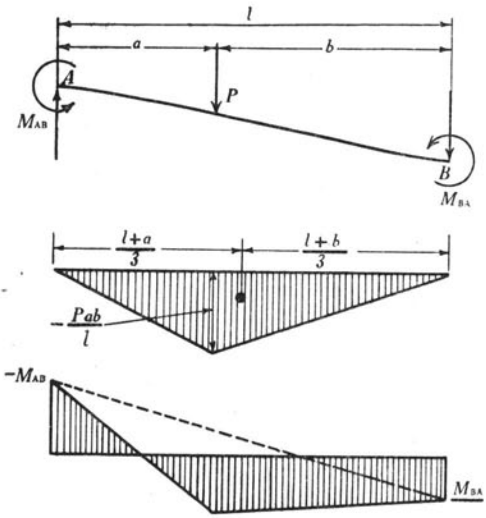


FIGURE 7

Hence putting these values in the last terms of equations (9) and (10) gives

$$-\frac{2F}{l^2}(3\bar{x}-l)=\frac{-Pab^2}{l^2}=-C_{AB} \quad . . . . (17)$$

and

$$\frac{2F}{l^2}(2l-3\bar{x})=\frac{+Pa^2b}{l^2}=+C_{BA} \quad . . . . (18)$$

If the member had been hinged instead of being restrained at B, the value of H_{AB} could have been found from the last term of equation (13), in which

$$-(C_{AB}+\frac{C_{BA}}{2})=\frac{-Pab}{2l^2}(l+b)=-H_{AB} \quad . . . . (19)$$

Similarly, if the member had been hinged at A and restrained at B, the value of H_{BA} could have been found from the last term of equation (14) in which

$$+(C_{BA}+\frac{C_{AB}}{2})=\frac{Pab(l+a)}{2l^2}=H_{BA} \quad . . . . (20)$$

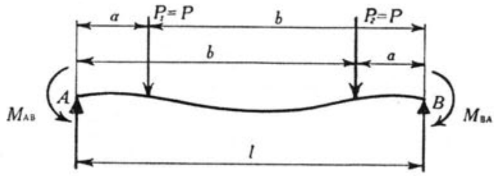


FIGURE 8

As another common case, consider a loading which is symmetrical about the middle of the member, as shown by Fig. 8. It is obvious that the centroid of the simple beam moment diagram will be at the middle of the member, so that $\bar{x}=\frac{l}{2}$. Substituting this in the last term of equations (9) and (10),

$$-\frac{2F}{l^2}(3\bar{x}-l)=-\frac{F}{l}=-C_{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and

$$+\frac{2F}{l^2}(2l-3\bar{x})=+\frac{F}{l}=+C_{BA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Similarly, for a member having the end A hinged, the last term of equation (13) gives

$$-(C_{AB}+\frac{C_{BA}}{2})=-\frac{3}{2}\,\frac{F}{l}=-H_{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

For a member having the end B hinged, the last term of equation gives

$$+(C_{BA}+\frac{C_{AB}}{2})=+\frac{3}{2}\frac{F}{l}=+H_{BA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

A geometrical meaning is attached to the term $\frac{F}{l}$ since it represents the average ordinate of the moment diagram for a simple beam under the given loading.

From these illustrations it is seen that values of C and H may be found by the use of equations (9), (10), (13), and (14). Values are also given for the more common cases of loading in text books on strength of materials, but when so determined, the sign must be fixed in accordance with the rules of section 5.

Another method for determining C and H may be readily applied to any kind of loading. For a member carrying a single concentrated load P, as shown in Fig. 7, the value of C_{AB} is $\frac{Pab^2}{l^2}$, and the value of C_{BA} is $\frac{Pa^2b}{l^2}$, as given in equations (17) and (18). If there are several concentrated loads on the member, by summation $C_{AB}=\sum\frac{Pab^2}{l^2}$, and $C_{BA}=\sum\frac{Pa^2b}{l^2}$.

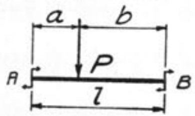
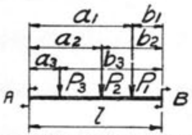
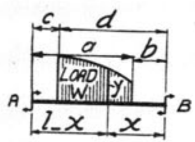
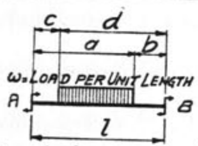
If there is a distributed load on the member the same method may be used, by performing an integration in place of the summation. Let w be the unit loading on an element of length dx, which is at a distance x from the left end, and a distance $l-x$ from the right end of the member. In the expression $\frac{Pab^2}{l^2}$, replace P by wdx, a by x, and b by $l-x$, whence $C_{AB}=\int\frac{wx(l-x)^2dx}{l^2}$. Similarly, $C_{BA}=\int\frac{wx^2(l-x)dx}{l^2}$. The limits of the definite integral are fixed by the length of the member under load.

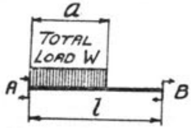
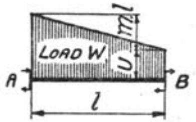
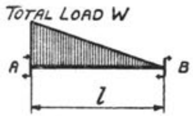
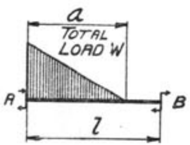
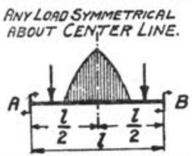
If the unit load w is not constant, its variation may be expressed in terms of x, and the general value for the total load on a length dx thus found substituted for P in the given expression for a single concentrated load, after which the integration may be performed as just indicated.

Values of C and H for different systems of loads are given for reference in Tables 2 and 3.

TABLE 2

VALUES OF CONSTANTS C AND H FOR DIFFERENT SYSTEMS OF LOADS TO BE USED IN THE EQUATIONS OF TABLE 1

No.	Condition of Loading	C_{AB}	C_{BA}	H_{AB}	H_{BA}
1	 No intermediate external loads.	O	O	O	O
2	 Single concentrated load at any point.	$\frac{Pab^2}{l^2}$	$\frac{Pba^2}{l^2}$	$\frac{Pab}{2l^2}(l+b)$	$\frac{Pab}{2l^2}(l+a)$
3	 Any number of concentrated loads.	$\frac{1}{l^2}\sum Pab^2$	$\frac{1}{l^2}\sum Pba^2$	$\frac{1}{2l^2}\sum Pab\ (l+b)$	$\frac{1}{2l^2}\sum Pab(l+a)$
4	 Any distributed load over any portion of the member.	$\frac{1}{l^2}\int_b^d yx^2(l-x)dx$	$\frac{1}{l^2}\int_c^a yx(l-x)^2dx$	$\frac{1}{2l^2}\int_b^d yx(l^2-x^2)dx$	$\frac{1}{2l^2}\int_c^a yx(l-x)(2l-x)dx$
5	 Uniform load over any portion of the member.	$\frac{w}{12l^2}\left[d^3(4l-3d)-b^3(4l-3b)\right]$	$\frac{w}{12l^2}\left[a^3(4l-3a)-c^3(4l-3c)\right]$	$\frac{w}{8l^2}(d^2-b^2)(2l^2-b^2-d^2)$	$\frac{w}{8l^2}(a^2-c^2)(2l^2-a^2-c^2)$

6	 Uniform load over portion of member at one end.	$\frac{Wa}{12l^2}(3a^2-8al+6l^2)$	$\frac{Wa^2}{12l^2}(4l-3a)$	$\frac{Wa}{8l^2}(2l-a)^2$	$\frac{Wa}{8l^2}(2l^2-a^2)$
7	 Uniformly varying load over entire member.	$\frac{l^2}{60}(5u+3ml)$	$\frac{l^2}{60}(5u+2ml)$	$\frac{l^2}{120}(15u+8ml)$	$\frac{l^2}{120}(15u+7ml)$
8	 Load varying uniformly from zero at one end to a maximum at the other end.	$\frac{Wl}{10}$	$\frac{Wl}{15}$	$\frac{2}{15}Wl$	$\frac{7}{60}Wl$
9	 Load varying uniformly from zero at any point to a maximum at one end.	$\frac{Wa}{30l^2}(3a^2-10al+10l^2)$	$\frac{Wa^2}{30l^2}(5l-3a)$	$\frac{Wa}{60l^2}(3a^2-15al+20l^2)$	$\frac{Wa}{60l^2}(10l^2-3a^2)$
10	 Any load symmetrical about center of member.	$\frac{F}{l}$	$\frac{F}{l}$	$\frac{3}{2}\cdot\frac{F}{l}$	$\frac{3}{2}\cdot\frac{F}{l}$

F is the area of the moment diagram of a simple beam having the same length and carrying the same load as the member in question.

Values of $\frac{F}{l}$ for different loads are given in Table 3.

TABLE 3

VALUES OF CONSTANTS C AND H FOR DIFFERENT SYSTEMS OF LOADS TO BE USED IN THE EQUATIONS OF TABLE 1

All Loads Symmetrical about Center of Member

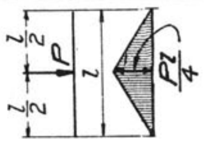
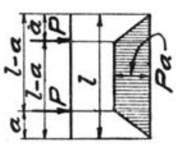
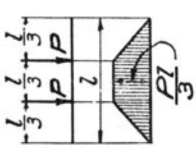
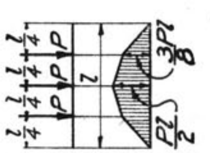
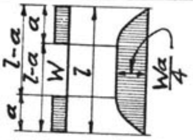
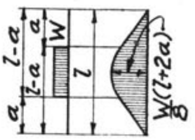
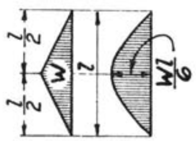
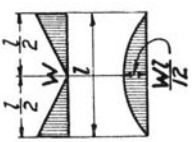
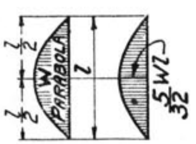
No.	Condition of Loading / Moment Diagram	$C_{AB}=C_{BA}=\frac{F}{l}$	$H_{AB}=H_{BA}=\frac{3}{2}\frac{F}{l}$
1	 Single load at the center.	$\frac{1}{8}Pl$	$\frac{3}{16}Pl$
2	 Two equal loads.	$\frac{Pa}{l}(l-a)$	$\frac{3}{2}\frac{Pa}{l}(l-a)$
3	 Equal loads at the third points.	$\frac{2}{9}Pl$	$\frac{1}{3}Pl$
4	 Equal loads at the quarter points.	$\frac{5}{16}Pl$	$\frac{15}{32}Pl$
5	 Uniform load over entire span.	$\frac{1}{12}Wl$	$\frac{1}{8}Wl$

TABLE 3—CONTINUED

No.	Condition of Loading / Moment Diagram	$C_{AB}=C_{BA}=\frac{F}{l}$	$H_{AB}=H_{BA}=\frac{3}{2}\frac{F}{l}$
6	 Equal uniform loads at the ends.	$\frac{Wa}{12l}(3l-2a)$	$\frac{Wa}{8l}(3l-2a)$
7	 Uniform load at the center.	$\frac{W}{12l}(l^2+2al-2a^2)$	$\frac{W}{8l}(l^2+2al-2a^2)$
8	 Load increasing uniformly from zero at the ends.	$\frac{5}{48}Wl$	$\frac{5}{32}Wl$
9	 Load increasing uniformly from zero at the center.	$\frac{1}{16}Wl$	$\frac{3}{32}Wl$
10	 Load varying as the ordinates of a parabola.	$\frac{1}{10}Wl$	$\frac{3}{20}Wl$

PART II

DETERMINATION OF STRESSES IN STATICALLY INDETERMINATE STRUCTURES

9. *Assumptions upon which the Analyses are Based.*—The analyses in this bulletin are based upon the following assumptions:

(1) That the connections are perfectly rigid.
(2) That the length of a member is not changed by axial stress.
(3) That the shearing deformation is zero.

Recent tests by Abe* show that the first assumption is approximately true for reinforced concrete frames, and tests by Wilson and Moore† show that this assumption is also approximately true for certain types of riveted connections of steel frames.

The error due to assumptions (2) and (3) depends upon the geometrical properties of the frame, but for frames of usual proportions the error is not large. These assumptions are discussed in detail in sections 67 and 68. The error due to slip in connections is discussed in section 69.

10. *Notation.*—The following notation has been used:

a = distance from end A of a member to a load.

b = distance from end B of a member to a load.

d = deflection of one end of a member with respect to the other end, measured perpendicular to initial position of member.

e = eccentricity of load.

h = vertical height of a structure.

k = error in resisting moment due to neglect of shearing strain.

l = length of a member.

m = change in the rate of loading in a unit distance.

n = ratio of K of top member to K of left-hand column for a four-sided frame.

p = ratio of K of top member to K of bottom member for a four-sided frame.

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*Abe, Mikishi, "Analysis and Tests of Rigidly Connected Reinforced Concrete Frames," Univ. of Ill. Eng. Exp. Sta., Bul. 107, 1918.

†Wilson, W. M., and Moore, H. F., "Tests to Determine the Rigidity of Riveted Joints of Steel Structures," Univ. of Ill. Eng. Exp. Sta., Bul 104, 1917.

q = ratio of the length of the left-hand column to the length of the right-hand column of a two-legged bent.

s = ratio of K of top member to K of right-hand column for a four-sided frame.

u = load per unit of length (variable).

w = uniformly distributed load per unit of length.

A = area of section of a member.

C_{AB} = resisting moment at end A of a member AB fixed at both ends and having both ends at the same level.

E = modulus of elasticity in tension and compression.

F = area of the moment diagram of a simple beam.

G = modulus of elasticity in shear.

H = reaction.

H_{AB} = resisting moment at end A of a member AB fixed at A and hinged at B and having both ends at the same level.

I = moment of inertia of section of a member.

K = ratio of moment of inertia of section to length of a member.

M = moment of an external couple.

M_A = statical moment of external forces about point A.

M_{AB} = resisting moment acting at the end A of a member AB.

M_{BA} = resisting moment acting at the end B of a member AB.

N = restraint factor, depending on manner in which the ends of a member are held.

P = concentrated load.

$R = \frac{d}{l}$ = ratio of the deflection of one end of a member (with respect to the other end) to the length of the member.

S = shear.

W = total distributed load on a member.

$\alpha = n^2 + 2pn + 2n + 3p$, for a symmetrical four-sided frame.

$\beta = 6n + p + 1$, for a symmetrical four-sided frame.

$\Delta = 22(pns + ps + ns + np) + 2(p^2s + ps^2 + pn^2 + p^2n + s^2 + s + n^2 + n) + 6(n^2s + ns^2 + p^2 + p)$, for a rectangular frame.

$\Delta_o = 2[ns(4 + 3q + 4q^2) + (s^2 + s) + q^2(n^2 + n) + 3(q^2sn^2 + s^2n)]$, for a two-legged rectangular bent with unequal legs.

$\Delta_o = 2(3ns^2 + 11ns + s^2 + s + 3n^2s + n^2 + n)$, for a two-legged rectangular bent.

θ = change in the slope of the tangent to the elastic curve of a member.

IV. Girders Having Restrained Ends

11. *Moments at the Ends of a Girder Having Fixed Ends—Both Supports on the Same Level.*—If a girder is fixed at the ends and if both supports are on the same level, θ_A, θ_B, and R of equations (A) and (B), Table 1, equal zero. This being the case, $M_{AB}=\mp C_{AB}$ and $M_{BA}=\pm C_{BA}$. Values of C_{AB} and C_{BA} for different systems of loads are given in Table 2.

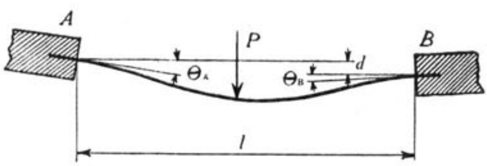

Figure 9

12. *Moments at the End of a Girder Having One End Fixed and the Other End Hinged, Both Supports on the Same Level.*—If a girder is fixed at one end, θ for that end equals zero. Likewise if both supports are on the same level, $R=0$. This being the case, the moment at the fixed end, as given by equations (C) and (D), Table 1, is $\mp H_{AB}$ or $\pm H_{BA}$. Values of H_{AB} and H_{BA} for different systems of loads are given in Table 2.

13. *Moments at the Ends of a Girder Having Ends Restrained but not Fixed.*—Fig. 9 represents a girder restrained at A and B. P represents the resultant of any system of forces on AB. The change in slope at A is θ_A and at B is θ_B. The deflection of B relative to A is d $\quad R=\frac{d}{l}$.

Applying equations (A) and (B), Table 1, gives

$$M_{AB}=2EK(2\theta_A+\theta_B-3R)-C_{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$M_{BA}=2EK(2\theta_B+\theta_A-3R)+C_{BA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In order to determine M_{AB} and M_{BA}, θ_A, θ_B, and R must be known. As shown in Fig. 9, θ_A and R are positive and θ_B is negative. If P

had been upward instead of downward, C_{AB} would have been preceded by a plus (+) sign and C_{BA} by a minus (−) sign. Values of C_{AB} and C_{BA} for different systems of loads to be substituted in equations (25) and (26) are given in Table 2.

14. *Moment at End of a Girder Having One End Hinged and the Other End Restrained but not Fixed.*—Fig. 10 represents a girder hinged

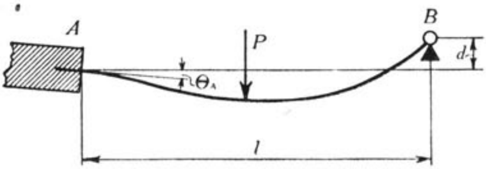

FIGURE 10

at B and restrained but not fixed at A. P represents the resultant of any system of forces on AB. The change in slope at A is θ_A, and the deflection of B relative to A is d. $R=\frac{d}{l}$.

Applying equation (C) of Table 1 gives

$$M_{AB}=EK(3\theta_A-3R)-H_{AB} \qquad (27)$$

As shown in Fig. 10, θ_A is positive (+) and R is negative (−). If P had been upward, H_{AB} would have been preceded by a plus (+) sign.

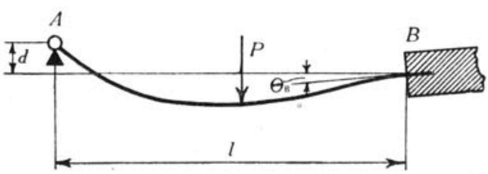

FIGURE 11

For the girder represented by Fig. 11

$$M_{BA}=EK(3\theta_B-3R)+H_{BA} \qquad (28)$$

in which R is positive (+) and θ_B is negative (−).

Values of H_{AB} and H_{BA} for different systems of loads to be used in equations (27) and (28) are given in Table 2.

V. Continuous Girders

15. *Girder Continuous over Any Number of Supports and Carrying Any System of Vertical Loads. Supports all on the Same Level. General Equation of Three Moments—Two Intermediate Spans.*—Although the

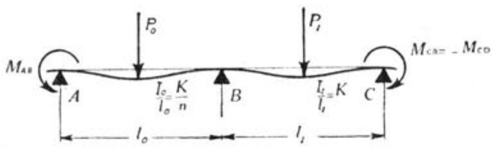

Figure 12

results of this section are included in the following section, detailed procedure is given here to show how the slope-deflection equations are to be applied to a continuous girder. Fig. 12 represents two intermediate spans of a continuous girder extending over a number of spans. All supports are on the same level. P_o represents the resultant of the forces on AB, and P_1 represents the resultant of the forces on BC. $\frac{I}{l}$ for span BC is K. $\frac{I}{l}$ for span AB is $\frac{K}{n}$. Since all the supports are on the same level, $R=0$ for all spans.

Applying the equations of Table 1 gives

$$M_{AB}=\frac{2EK}{n}(2\theta_A+\theta_B)-C_{AB} \quad . \quad . \quad . \quad (29)$$

$$M_{BA}=\frac{2EK}{n}(2\theta_B+\theta_A)+C_{BA} \quad . \quad . \quad . \quad (30)$$

$$M_{BC}=2EK(2\theta_B+\theta_C)-C_{BC} \quad . \quad . \quad . \quad (31)$$

$$M_{CB}=2EK(2\theta_C+\theta_B)+C_{CB} \quad . \quad . \quad . \quad (32)$$

$$M_{BA}+M_{BC}=0 \quad . \quad . \quad . \quad (33)$$

Substituting the value of θ_A from equation (29) in equation (30) gives

$$2M_{BA}-M_{AB}=\frac{6EK}{n}\theta_B+2C_{BA}+C_{AB} \quad . \quad . \quad . \quad (34)$$

Substituting the value of θ_C from equation (31) in equation (32) gives

$$M_{CB}-2M_{BC}=-6EK\theta_B+C_{CB}+2C_{BC} \quad . \quad . \quad . \quad (35)$$

Substituting the value of θ_B from equation (35) in equation (34), and substituting $-M_{BC}$ for M_{BA} gives

$$nM_{AB}+2M_{BC}(n+1)+M_{CD}=-[n(2C_{BA}+C_{AB})+(C_{CB}+2C_{BC})] \quad (36)$$

In determining the values of C and H given in Table 3, it was found that

$$H_{AB}=\frac{2C_{AB}+C_{BA}}{2} \text{ and } H_{BA}=\frac{2C_{BA}+C_{AB}}{2}$$

Equation (36) can, therefore, be written in the form

$$nM_{AB}+2M_{BC}(n+1)+M_{CD}=-2[nH_{BA}+H_{BC}] \quad . \quad . \quad . \quad . \quad (37)$$

This is the general form of the well-known "Equation of Three Moments."* It may be applied to a continuous girder having all supports on the same level, no matter what the type of loading to which the girder is subjected. As applied to two adjacent spans, $K=\frac{I}{l}$ for the right-hand span and $\frac{K}{n}=\frac{I}{l}$ for the left-hand span; that is, $n=\frac{I}{l}$ for the right-hand span divided by $\frac{I}{l}$ for the left-hand span. Values of H for different types of loading are given in Table 2.

16. *Girder Continuous over Any Number of Supports and Carrying Any System of Vertical Loads. Supports All on the Same Level. General Equation of Three Moments—Two Adjacent Spans at One End. End of Girder Hinged.*—Fig. 13 represents the two spans at the left-hand end of a continuous girder. All supports are on the same level. P_o represents the resultant of the loads on AB, and P_1 represents the

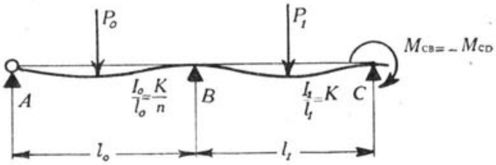

FIGURE 13

*The Equation of Three Moments was first deduced for a girder carrying uniform loads by Clapeyron, in 1857, and was published in Comptes Rendus des Séances de l'Académie des Sciences, Paris, Vol. 45, p. 1076. It has been extended and generalized for other loadings by Bresse, Cours de Mécanique Appliquée, Paris, 1862; Winkler, Die Lehre von der Elasticitat und Festigkeit, Prague, 1867, and others.

resultant of the loads on BC. The girder is hinged at A. Equation (37), having been derived for the general case, is applicable. As the girder is hinged at A, $M_{AB}=0$. Equation (37), therefore, takes the form

$$2M_{BC}(n+1)+M_{CD}=-2[nH_{BA}+H_{BC}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

for two adjacent spans at the left-hand end of the girder when the left-hand end is hinged. Likewise,

$$nM_{AB}+2M_{BC}(n+1)=-2\,[n\,H_{BA}+H_{BC}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

for two adjacent spans at the right-hand end of the girder when the right-hand end is hinged.

17. *Girder Continuous over Any Number of Supports and Carrying Any System of Vertical Loads. Supports All on the Same Level. General Equation of Three Moments—Two Adjacent Spans at One End. End of Girder Restrained.*—Fig. 14 represents the two spans at the left-

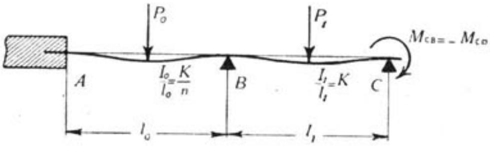

FIGURE 14

hand end of a continuous girder. All supports are on the same level. P_o represents the resultant of the loads on AB, and P_1 represents the resultant of the loads on BC. The girder is restrained at A.

The values of the moments depend upon the restraint of the point A, and therefore the moments cannot be determined unless either the moment at A or the slope of the elastic curve of the girder at A is known. If the moment at A is known, equation (37) is applicable. If the slope at A is known, θ_A is a known quantity.

Substituting the value θ_B from equation (29) in equation (30) gives

$$2n\,M_{AB}=6EK\theta_A-n\,M_{BC}-2n\,H_{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Substituting M_{AB} from equation (40) in equation (37) gives

$$M_{BC}(4+3n)+2M_{CD}=2n\,H_{AB}-4(nH_{BA}+H_{BC})-6EK\,\theta_A \quad . \quad (41)$$

Equation (41) is applicable to the two adjacent spans at the left-hand end of a continuous girder restrained at the left-hand end.

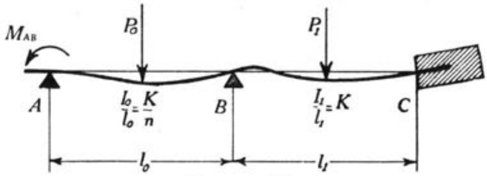

FIGURE 15

Fig. 15 represents the two spans at the right-hand end of a continuous girder. All supports are on the same level. P_o represents the resultant of the loads on AB, and P_1 represents the resultant of the loads on BC. The girder is restrained at C.

From equation (37)

$$2nM_{AB}+4M_{BC}(n+1)-2M_{CB}=-4(nH_{BA}+H_{BC}) \quad . \quad . \quad . \quad . \quad (42)$$

Applying the equations of Table 1 gives

$$M_{BC}=2EK(2\theta_B+\theta_C)-C_{BC} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$M_{CB}=2EK(2\theta_C+\theta_B)+C_{CB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Eliminating θ_B from equations (43) and (44) gives

$$-2M_{CB}=-M_{BC}-6EK\theta_C-2H_{CB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Substituting the value of M_{CB} from equation (45) in equation (42) gives

$$2nM_{AB}+M_{BC}(4n+3)=2H_{CB}-4[nH_{BA}+H_{BC}]+6EK\theta_C \quad . \quad (46)$$

Equation (46) is applicable to the two adjacent spans at the right-hand end of a continuous girder.

18. *Girder Continuous over Any Number of Supports and Carrying Any System of Vertical Loads. Supports All on the Same Level. General Equation of Three Moments—Girder Fixed at the Ends.*—If the girder is fixed at the ends, θ_A of equation (41) and θ_C of equation (46) equal zero. Equations (41) and (46) then take the form

$$M_{BC}(4+3n)+2M_{CD}=2nH_{AB}-4[nH_{BA}+H_{BC}] \quad . \quad . \quad . \quad . \quad (47)$$

$$2nM_{AB}+M_{BC}(4n+3)=2H_{CB}-4[nH_{BA}+H_{BC}] \quad . \quad . \quad . \quad . \quad (48)$$

Equation (47) is applicable to the two adjacent spans at the left-hand end of a continuous girder fixed at the left-hand end, and equation (48) is applicable to the two adjacent spans at the right-hand end of a continuous girder fixed at the right-hand end. Values of H for different systems of loads are given in Table 2.

19. *Girder Continuous over Three Supports and Carrying Any System of Vertical Loads. Supports on Different Levels. General*

Equation of Three Moments.—In section 15, the equations of Table 1 have been used to derive a general "Equation of Three Moments." The derivation is seen to be merely the combination of four linear equations involving slopes, deflections, and moments at the supports for each span into an equation involving the same quantities for any two adjacent spans.

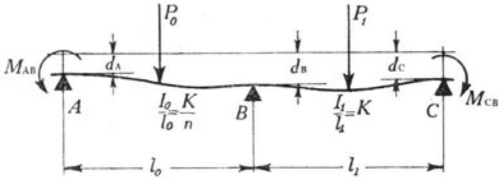

FIGURE 16

Fig. 16 represents two adjacent intermediate spans of a girder continuous over a number of supports. Some of the supports have settled. The vertical distances of the points A, B, and C from a horizontal base line are d_A, d_B, and d_C. P_o represents the resultant of all loads on AB, and P_1 represents the resultant of all loads on BC.

The moments in the girder may be considered as made up of two parts, one part due to the loads represented by P_o and P_1 when the supports are on the same level, and the other part due to the settlement of the supports, it being considered that the girder remains in contact with all supports.

Applying the equations of Table 1 gives

$$M_{AB}=\frac{2EK}{n}(2\theta_A+\theta_B-3R_o)-C_{AB} \quad \text{. . . . .} \quad (49)$$

$$M_{BA}=\frac{2EK}{n}(2\theta_B+\theta_A-3R_o)+C_{BA} \quad \text{. . . . .} \quad (50)$$

$$M_{BC}=2EK(2\theta_B+\theta_C-3R_1)-C_{BC} \quad \text{. . . . .} \quad (51)$$

$$M_{CB}=2EK(2\theta_C+\theta_B-3R_1)+C_{CB} \quad \text{. . . . .} \quad (52)$$

$$M_{BA}+M_{BC}=0 \quad \text{. . . . .} \quad (53)$$

in which $\frac{I}{l}$ for the right-hand span equals K and $\frac{I}{l}$ for the left-hand span equals $\frac{K}{n}$.

From Fig. 16 $R_o=\frac{d_B-d_A}{l_o}$ and $R_1=\frac{d_C-d_B}{l_1}$.

Equations (49) to (53) are identical with equations (29) to (33) of section 15, except for the additional terms R_o and R_1. Hence the method of solving for the moments is the same as in section 15, and will not be repeated here in detail. The general equation of three moments for a continuous girder carrying any system of vertical loads and with supports on different levels, together with some special forms of the equation, are given in Table 4.*

20. *Girder Continuous over Three Supports and Carrying Any System of Vertical Loads. Supports on Different Levels. Various Conditions of Restraint of Ends.*—Table 4, section 19, gives the general equation of three moments which is to be applied here. Various cases of restraint of the ends of girder and the effect upon the quantities in the general equation will be considered. It is seen that if the end of the girder is hinged, the resisting moment there is zero. If it is fixed, the slope at that point is zero. If the end is restrained, or partially fixed, the moments at the other supports can be found, if either the slope or the moment at the end is known. Modifications of the equations of Table 4, for these various cases of restraint, are given in Table 5.

21. *Girder Continuous over Four Supports and Carrying Any System of Vertical Loads. Supports on Different Levels. Various Conditions of Restraint of Ends.*—Table 4, section 19, gives the general equation of three moments which may be applied here. The method of using the equation of three moments is to apply it successively to each pair of adjacent spans in the continuous girder, thus deriving one less equation than the number of spans. If the conditions of restraint at the ends of the continuous girder are known, these equations may be solved simultaneously for the unknown moment at each support.

When the end of the girder is hinged, the moment there is zero. When the end is fixed, the slope there is zero. When the end is restrained, if either the slope or the moment at the end is known, the moments at the other supports may be found.

The equations of Table 4 have been applied to the girder described in this section, and the values of moments obtained for various cases of restraint of ends are given in Table 6.

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*Johnson, Bryan, and Turneaure, "The Theory and Practice of Modern Framed Structures," Part II, p. 19, Ninth Ed. 1911, give the general Equation of Three Moments for a continuous girder with supports on different levels and carrying concentrated loads. The pocketbook, Die Huette, and Lanza, Applied Mechanics, have similar equations.

A. Ostenfeld, Teknisk Statik, Vol. 2, Second Ed., Copenhagen, 1913, pp. 98–162, gives a comprehensive treatment of continuous girders, including the case of varying moment of inertia from point to point, and of girder resting upon elastic supports. Both analytical and graphical methods are presented.

TABLE 4

CONTINUOUS GIRDERS

Equations of Three Moments.

Supports on Different Levels.[1] $K=\frac{I}{l}$ for right-hand span.

Any System of Vertical Loads.[2] $\frac{K}{n}=\frac{I}{l}$ for left-hand span.

No.	Portion of Girder Considered	Equations of Three Moments
(a)	Fig. 16 Intermediate spans	$nM_{AB}+2M_{BC}(n+1)+M_{CD}=\frac{6EK}{l_o l_1}\Big[l_1(d_B-d_A)-l_o(d_C-d_B)\Big]-2\Big[H_{BC}+nH_{BA}\Big]$
(b)	Fig. 17 Two adjacent spans at left-hand end. End of girder hinged.	$2M_{BC}(n+1)+M_{CD}=\frac{6EK}{l_o l_1}\Big[l_1(d_B-d_A)-l_o(d_C-d_B)\Big]-2\Big[nH_{BA}+H_{BC}\Big]$
(c)	Fig. 18 Two adjacent spans at right-hand end. End of girder hinged.	$nM_{AB}+2M_{BC}(n+1)=\frac{6EK}{l_o l_1}\Big[l_1(d_B-d_A)-l_o(d_C-d_B)\Big]-2\Big[nH_{BA}+H_{BC}\Big]$
(d)	Fig. 19 Two adjacent spans at left-hand end. End of girder restrained.	If M_{AB} is known $2M_{BC}(n+1)+M_{CD}=\frac{6EK}{l_o l_1}\Big[l_1(d_B-d_A)-l_o(d_C-d_B)\Big]-2\Big[H_{BC}+nH_{BA}\Big]-nM_{AB}$ If θ_A is known $M_{BC}(4+3n)+2M_{CD}=\frac{6EK}{l_o l_1}\Big[3l_1(d_B-d_A)-2l_o(d_C-d_B)\Big]+2nH_{AB}-4\Big[H_{BC}+nH_{BA}\Big]-6EK\,\theta_A$ If the girder is fixed at A, $\theta_A=0$
(e)	Fig. 20 Two adjacent spans at right-hand end. End of girder restrained.	If M_{CB} is known $nM_{AB}+2M_{BC}(n+1)=\frac{6EK}{l_o l_1}\Big[l_1(d_B-d_A)-l_o(d_C-d_B)\Big]-2\Big[H_{BC}+nH_{BA}\Big]+M_{CB}$ If θ_C is known $2nM_{AB}+M_{BC}(4n+3)=\frac{6EK}{l_o l_1}\Big[2l_1(d_B-d_A)-3l_o(d_C-d_B)\Big]-4\Big[H_{BC}+nH_{BA}\Big]+2H_{CB}+6EK\,\theta_C$ If the girder is fixed at C, $\theta_C=0$

[1] If there is no settlement of supports, let all values of d in these equations equal zero.

[2] If there are no loads on girder except at supports, let all values of H in these equations equal zero.

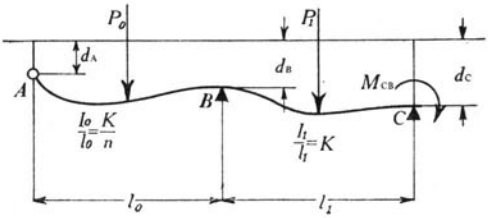


FIGURE 17

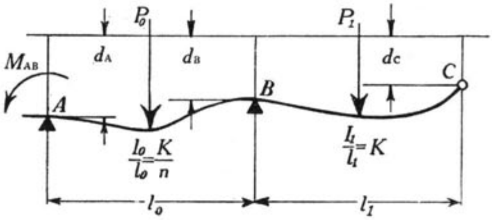


FIGURE 18

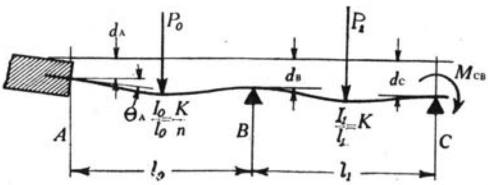


FIGURE 19

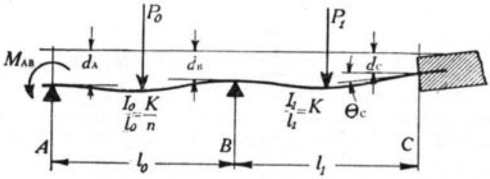


FIGURE 20

TABLE 5

GIRDER CONTINUOUS OVER THREE SUPPORTS

Supports on Different Levels.[1] $K=\frac{I}{l}$ for right-hand span.

Any System of Vertical Loads.[2] $\frac{K}{n}=\frac{I}{l}$ for left-hand span.

(a)	Fig. 21 Ends of girder hinged	General Case $M_{BC}=\frac{6EK}{2l_o l_1(n+1)}\left[l_1(d_B-d_A)-l_o(d_C-d_B)\right]-\frac{2}{n+1}\left[nH_{BA}+H_{BC}\right]$ Spans Identical Except for Loads $M_{BC}=\frac{3EK}{2l^2}\left[l(2d_B-d_A-d_C)\right]-\left[H_{BA}+H_{BC}\right]$
(b)	Fig. 22 Ends of girder restrained	General Case θ_A and M_{CB} known $M_{BC}=\frac{1}{3n+4}\left\{\frac{6EK}{l_o l_1}\left[3l_1(d_B-d_A)-2l_o(d_C-d_B)\right]+2nH_{AB}-4(H_{BC}+nH_{BA})-6EK\theta_A+2M_{CB}\right\}$ $M_{AB}=\frac{3EK}{n}\left(\theta_A-\frac{d_B-d_A}{l_o}\right)-\frac{M_{BC}}{2}-H_{AB}$ θ_C and M_{AB} known $M_{BC}=\frac{1}{4n+3}\left\{\frac{6EK}{l_o l_1}\left[2l_1(d_B-d_A)-3l_o(d_C-d_B)\right]-4(H_{BC}+nH_{BA})+2H_{CB}+6EK\theta_C-2nM_{AB}\right\}$ $M_{CB}=\frac{M_{BC}}{2}+3EK\left[\theta_C-\frac{d_C-d_B}{l_1}\right]+H_{CB}$ θ_A and θ_C known $M_{BC}=\frac{1}{n+1}\left\{\frac{6EK}{l_o l_1}\left[l_1(d_B-d_A)-l_o(d_C-d_B)\right]-nC_{BA}-C_{BC}-2EK(\theta_A-\theta_C)\right\}$ $M_{AB}=\frac{3EK}{n}\left[\theta_A-\frac{d_B-d_A}{l_o}\right]-\frac{M_{BC}}{2}-H_{AB}$ $M_{CB}=3EK\left[\theta_C-\frac{d_C-d_B}{l_1}\right]+\frac{M_{BC}}{2}+H_{CB}$

[1] If there is no settlement of supports, let all values of d in these equations equal zero.

[2] If there are no loads on girder except at supports, let all values of H and C in these equations equal zero.

If an end is fixed, θ for that end is zero.

If an end is hinged, M for that end is zero.

TABLE 5—CONTINUED

(b) | Fig. 22 | Ends of girder restrained

General Case—Continued

M_{AB} and M_{CB} known

$$M_{BC}=\frac{1}{2(n+1)}(M_{CB}-nM_{AB})+\frac{3EK}{l_o l_1(n+1)}\Big[l_1(d_B-d_A)-l_o(d_C-d_B)\Big]-\frac{nH_{BA}+H_{BC}}{n+1}$$

Spans Identical Except for Load

θ_A and M_{CB} known

$$M_{BC}=\frac{1}{7}\left\{\frac{6EK}{l}\Big[3(d_B-d_A)-2(d_C-d_B)\Big]+2H_{AB}-4(H_{BC}+H_{BA})-6EK\theta_A+2M_{CB}\right\}$$

$$M_{AB}=3EK\left[\theta_A-\frac{d_B-d_A}{l}\right]-\frac{M_{BC}}{2}-H_{AB}$$

θ_C and M_{AB} known

$$M_{BC}=\frac{1}{7}\left\{\frac{6EK}{l}\Big[2(d_B-d_A)-3(d_C-d_B)\Big]-4(H_{BC}+H_{BA})+2H_{CB}+6EK\theta_C-2M_{AB}\right\}$$

$$M_{CB}=\frac{M_{BC}}{2}+3EK\left[\theta_C-\frac{d_C-d_B}{l}\right]+H_{CB}$$

θ_A and θ_C known

$$M_{BC}=\frac{1}{2}\left\{\frac{6EK}{l}\Big[2d_B-d_C-d_A\Big]-C_{BA}-C_{BC}-2EK(\theta_A-\theta_C)\right\}$$

$$M_{AB}=3EK\left[\theta_A-\frac{d_B-d_A}{l}\right]-\frac{M_{BC}}{2}-H_{AB}$$

$$M_{CB}=3EK\left[\theta_C\quad\frac{d_C-d_B}{l}\right]+\frac{M_{BC}}{2}+H_{CB}$$

M_{AB} and M_{CB} known

$$M_{BC}=\frac{1}{4}\Big[M_{CB}-M_{AB}\Big]+\frac{3EK}{2l}\Big[2d_B-d_A-d_C\Big]-\frac{H_{BA}+H_{BC}}{2}$$

TABLE 6

GIRDER CONTINUOUS OVER FOUR SUPPORTS

Supports on Different Levels.[1] Any System of Vertical Loads.[2]

$\frac{I}{l}$ for span $AB = \frac{K_1}{n_o}$. $\frac{I}{l}$ for span $BC = \frac{K_2}{n_1} = K_1$. $\frac{I}{l}$ for span $CD = K_2$.

(a)	Fig. 23 Ends of girder hinged	General Case
		$M_{BC} = \frac{6EK_1\left[2(n_1+1)l_1l_2(d_B-d_A)-(3n_1+2)l_ol_2(d_C-d_B)+n_1l_ol_1(d_D-d_C)\right]}{l_ol_1l_2\left[4(n_o+1)(n_1+1)-n_1\right]}$
		$-\left[\frac{4(n_1+1)(n_oH_{BA}+H_{BC})-2(n_1H_{CB}+H_{CD})}{4(n_o+1)(n_1+1)-n_1}\right]$
		$M_{CD} = \frac{6EK_2\left[(2n_o+3)l_ol_2(d_C-d_B)-2(n_o+1)l_ol_1(d_D-d_C)-l_1l_2(d_B-d_A)\right]}{l_ol_1l_2\left[4(n_o+1)(n_1+1)-n_1\right]}$
		$-\left[\frac{4(n_o+1)(n_1H_{CB}+H_{CD})-2n_1(n_oH_{BA}+H_{BC})}{4(n_o+1)(n_1+1)-n_1}\right]$
		All Spans Identical Except for Loads
		$M_{BC} = \frac{2EK}{5l}(9d_B-6d_C+d_D-4d_A) - \frac{2}{15}\left[4(H_{BA}+H_{BC})-(H_{CB}+H_{CD})\right]$
		$M_{CD} = \frac{2EK}{5l}(9d_C-6d_B-4d_D+d_A) - \frac{2}{15}\left[4(H_{CB}+H_{CD})-(H_{BA}+H_{BC})\right]$

[1] If there is no settlement of supports, let all values of d in these equations equal zero.

[2] If there are no loads on the girder except at supports, let all values of H in these equations equal zero.

If an end is fixed, θ for that end is zero. If an end is hinged, M for that end is zero.

TABLE 6—CONTINUED

(b)	Fig. 24 Ends of girder restrained	General Case θ_A and M_{DC} known $M_{BC} = \dfrac{6EK_1\Big[3(n_1+1)l_1l_2(d_B-d_A)-(3n_1+2)l_ol_2(d_C-d_B)+n_1l_ol_1(d_D-d_C)\Big]}{l_ol_1l_2\Big[3(n_o+1)(n_1+1)+1\Big]}$ $-\dfrac{2\Big[(n_1+1)(2n_oH_{BA}-n_oH_{AB}+2H_{BC}+3EK_1\theta_A)-(n_1H_{CB}+H_{CD})\Big]+M_{DC}}{3(n_o+1)(n_1+1)+1}$ $M_{CD} = \dfrac{3EK_2\Big[3(n_o+2)l_ol_2(d_C-d_B)-(3n_o+4)l_ol_1(d_D-d_C)-3l_1l_2(d_B-d_A)\Big]}{l_ol_1l_2\Big[3(n_o+1)(n_1+1)+1\Big]}$ $+\dfrac{2n_1(2n_oH_{BA}-n_oH_{AB}+2H_{BC}+3EK_1\theta_A)-(3n_o+4)(2n_1H_{CB}+2H_{CD}-M_{DC})}{2\Big[3(n_o+1)(n_1+1)+1\Big]}$ $M_{AB} = -\dfrac{M_{BC}}{2}-\dfrac{H_{AB}+3EK_1}{n_ol_o}\Big[l_o\theta_A-(d_B-d_A)\Big]$ θ_D and M_{AB} known $M_{BC} = \dfrac{3EK_1\Big[(4n_1+3)l_1l_2(d_B-d_A)-3(2n_1+1)l_ol_2(d_C-d_B)+3n_1l_ol_1(d_D-d_C)\Big]}{l_ol_1l_2\Big[3(n_o+1)(n_1+1)+n_on_1\Big]}$ $-\left[\dfrac{(4n_1+3)(2n_oH_{BA}+2H_{BC}+n_oM_{AB})+2(H_{DC}-2H_{CD}-2n_1H_{CB}+3EK_2\theta_D)}{2\Big[3(n_o+1)(n_1+1)+n_on_1\Big]}\right]$

TABLE 6—CONTINUED

GIRDER CONTINUOUS OVER FOUR SUPPORTS

Supports on Different Levels. Any System of Vertical Loads.

$\frac{I}{l}$ for span $AB = \frac{K_1}{n_o}$. $\frac{I}{l}$ for span $BC = \frac{K_2}{n_1} = K_1$. $\frac{I}{l}$ for span $CD = K_2$

(b)	Fig. 24 Ends of girder restrained (Continued)	General Case—Continued
		$M_{CD} = \frac{6EK_2\left[(2n_o+3)l_ol_2(d_C-d_B)-3(n_o+1)l_ol_1(d_D-d_C)-l_1l_2(d_B-d_A)\right]}{l_ol_1l_2\left[3(n_o+1)(n_1+1)+n_on_1\right]}$
		$+\frac{2(n_o+1)(H_{DC}-2H_{CD}-2n_1H_{CB}+3EK_2\theta_D)+n_1(2n_oH_{BA}+2H_{BC}+n_oM_{AB})}{3(n_o+1)(n_1+1)+n_on_1}$
		$M_{DC} = \frac{M_{CD}}{2}+H_{DC}+3EK_2\left[\theta_D-\frac{(d_D-d_C)}{l_2}\right]$
		θ_A and θ_D known
		$M_{BC} = \frac{6EK_1\left[(4n_1+3)l_1l_2(d_B-d_A)-2(2n_1+1)l_ol_2(d_C-d_B)+2n_1l_ol_1(d_D-d_C)\right]}{l_ol_1l_2\left[4(n_o+1)(n_1+1)-n_o\right]}$
		$-\frac{2\left\{(4n_1+3)(2n_oH_{BA}-n_oH_{AB}+2H_{BC})+2(H_{DC}-2H_{CD}-2n_1H_{CB})+3EK_1\left[(4n_1+3)\theta_A+2n_1\theta_D\right]\right\}}{3\left[4(n_o+1)(n_1+1)-n_o\right]}$
		$M_{CD} = \frac{6EK_2\left[2(n_o+2)l_ol_2(d_C-d_B)-(3n_o+4)l_ol_1(d_D-d_C)-2l_1l_2(d_B-d_A)\right]}{l_ol_1l_2\left[4(n_o+1)(n_1+1)-n_o\right]}$
		$+\frac{2\left\{(3n_o+4)(H_{DC}-2H_{CD}-2n_1H_{CB})+2n_1(2n_oH_{BA}-n_oH_{AB}+2H_{BC})+3EK_2\left[(3n_o+4)\theta_D+2\theta_A\right]\right\}}{3\left[4(n_o+1)(n_1+1)-n_o\right]}$

TABLE 6—CONTINUED

(b)	Fig 24 Ends of girder restrained (Continued)	$M_{AB}= -\frac{M_{BC}}{2} - H_{AB} + \frac{3EK_1}{n_o}\left[\theta_A - \frac{(d_B-d_A)}{l_o}\right]$ $M_{DC}= \frac{M_{CD}}{2} + H_{DC} + 3EK_2\left[\theta_D - \frac{(d_D-d_C)}{l_2}\right]$ M_{AB} and M_{DC} known $M_{BC} = \frac{6EK_1\left[2(n_1+1)l_1l_2(d_B-d_A)-(3n_1+2)l_ol_2(d_C-d_B)+n_1l_ol_1(d_D-d_C)\right]}{l_ol_1l_2\left[4\,(n_o+1)\,(n_1+1)-n_1\right]}$ $-\left[\frac{4(n_1+1)\,(n_oH_{BA}+H_{BC})-2\,(n_1H_{CB}+H_{CD})+2n_o\,(n_1+1)M_{AB}+M_{DC}}{4(n_o+1)\,(n_1+1)-n_1}\right]$ $M_{CD}= \frac{6EK_2\left[(2n_o+3)\,l_ol_2\,(d_C-d_B)-2\,(n_o+1)l_ol_1(d_D-d_C)-l_1l_2(d_B-d_A)\right]}{l_ol_1l_2\left[4(n_o+1)\,(n_1+1)-n_o\right]}$ $-\left[\frac{4(n_o+1)\,(n_1H_{CB}+H_{CD})-2n_1(n_oH_{BA}+H_{BC})-2(n_o+1)\,M_{DC}-n_on_1M_{AB}}{4(n_o+1)\,(n_1+1)-n_1}\right]$

TABLE 6—CONTINUED

GIRDER CONTINUOUS OVER FOUR SUPPORTS

Supports on Different Levels. Any System of Vertical Loads.

$\frac{I}{l}$ for span $AB=\frac{K_1}{n_o}$ $\frac{I}{l}$ for span $BC=\frac{K_2}{n_1}=K_1$. $\frac{I}{l}$ for span $CD=K_2$.

(b)	Fig. 24 Ends of girder restrained (Continued)	General Case—Continued All Spans Identical Except for Loads θ_A and M_{DC} known $M_{BC}=\dfrac{6EK(11d_B-6d_A-6d_C+d_D)-2l\left[2\,(2H_{BA}-H_{AB}+2H_{BC}+3EK\theta_A)-(H_{CB}+H_{CD})\right]-lM_{DC}}{13l}$ $M_{CD}=\dfrac{3EK(16d_C-7d_D-12d_B+3d_A)+l\left[(2H_{BA}-H_{AB}+2H_{BC}+3EK\theta_A)-\frac{7}{2}(2H_{CB}+2H_{CD}-M_{DC})\right]}{13l}$ $M_{AB}=-\dfrac{M_{BC}}{2}-H_{AB}+\dfrac{3EK}{l}\left[l\theta_A-d_B+d_A\right]$ θ_D and M_{AB} known $M_{BC}=\dfrac{3EK(16d_B-7d_A-12d_C+3d_D)-l\left[\frac{7}{2}\,(2H_{BA}+2H_{BC}+M_{AB})+H_{DC}-2H_{CD}-2H_{CB}+3EK\theta_D\right]}{13l}$ $M_{CD}=\dfrac{6EK(11d_C-6d_D-6d_B+d_A)+l\left[4\,(H_{DC}-2H_{CD}-2H_{CB}+3EK\theta_D)+2H_{BA}+2H_{BC}+M_{AB}\right]}{13l}$ $M_{DC}=\dfrac{M_{CD}}{2}+H_{DC}+3EK\left[\theta_D-\dfrac{(d_D-d_C)}{l}\right]$

TABLE 6—CONTINUED

(b)	Fig. 24 Ends of girder restrained (Continued)	θ_A and θ_D known $M_{BC}=\frac{2EK}{5l}(13d_B-8d_C-7d_A+2d_D)-\frac{2}{45}\Big[7(2H_{BA}-H_{AB}+2H_{BC})+2(H_{DC}-2H_{CD}-2H_{CB})$ $+3EK(7\theta_A+2\theta_D)\Big]$ $M_{CD}=\frac{2EK}{5l}(13d_C-8d_B-7d_D+2d_A)+\frac{2}{45}\Big[7(H_{DC}-2H_{CD}-2H_{CB})+2(-H_{AB}+2H_{BA}+2H_{BC})$ $+3EK(7\theta_D+2\theta_A)\Big]$ $M_{AB}=-\frac{1}{2}M_{BC}-H_{AB}+3EK\left[\theta_A-\frac{(d_B-d_A)}{l}\right]$ $M_{DC}=\frac{1}{2}M_{CD}+H_{DC}+3EK\left[\theta_D-\frac{(d_D-d_C)}{l}\right]$ M_{AB} and M_{DC} known $M_{BC}=\frac{2EK}{5l}(9d_B-6d_C-4d_A+d_D)-\frac{1}{15}\Big[8(H_{BA}+H_{BC})-2(H_{CB}+H_{CD})+4M_{AB}+M_{DC}\Big]$ $M_{CD}=\frac{2EK}{5l}(9d_C-6d_B-4d_D+d_A)-\frac{1}{15}\Big[8(H_{CB}+H_{CD})-2(H_{BA}+H_{BC})-4M_{DC}-M_{AB}\Big]$

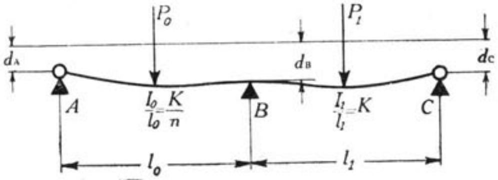


FIGURE 21

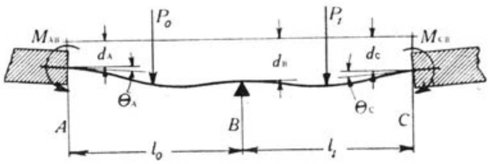


FIGURE 22

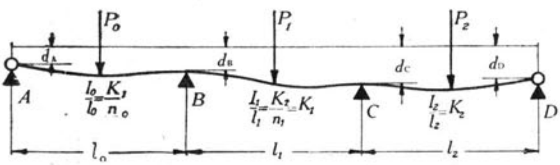


FIGURE 23

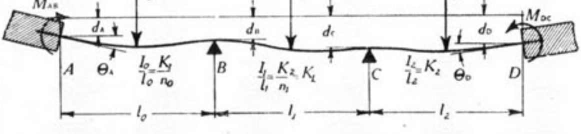


FIGURE 24

VI. Two-legged Rectangular Bent. Legs of Equal Length

22. *Two-legged Rectangular Bent. Concentrated Horizontal Force at the Top—Legs Hinged at the Bases.*—Fig. 25 represents a two-legged rectangular bent having legs hinged at the bases. A single force P is applied at the top of the bent.

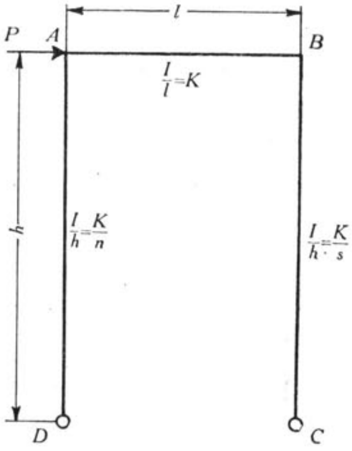


FIGURE 25

Applying the equations of Table 1

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad . \quad . \quad . \quad (54)$$

$$M_{AD}=\frac{EK}{n}(3\theta_A-3R) \quad . \quad . \quad . \quad (55)$$

$$M_{BA}=2EK(2\theta_B+\theta_A) \quad . \quad . \quad . \quad (56)$$

$$M_{BC}=\frac{EK}{s}(3\theta_B-3R) \quad . \quad . \quad . \quad (57)$$

$$M_{AD}+M_{BC}+Ph=0 \quad . \quad . \quad . \quad (58)$$

In these equations, from the definition of resisting moment, $M_{AB}=-M_{AD}$ and $-M_{BA}=+M_{BC}$. Hence M_{AB} and M_{BC} may be considered as the two unknown moments to be found. Elimination may be done as follows:

Subtracting equation 56 from equation 54 gives

$$M_{AB}+M_{BC}=2EK(\theta_A-\theta_B) \qquad (59)$$

Subtracting s times equation 57 from n times equation 55 gives

$$-nM_{AB}-sM_{BC}=EK(3\theta_A-3\theta_B) \qquad (60)$$

Combining equations 59 and 60 to eliminate the quantity $(\theta_A-\theta_B)$ gives

$$M_{AB}(3+2n)+M_{BC}(3+2s)=0 \qquad (61)$$

Combining equation 61 with equation 58 gives

$$M_{AB}=\frac{Ph}{2}\left(\frac{3+2s}{3+n+s}\right) \qquad (62)$$

$$M_{BC}=-\frac{Ph}{2}\left(\frac{3+2n}{3+n+s}\right) \qquad (63)$$

If $n=s$, that is, if the sections of AD and BC have the same moments of inertia, equations 62 and 63 take the form

$$M_{BC}=-M_{AB}=-\frac{Ph}{2} \qquad (64)$$

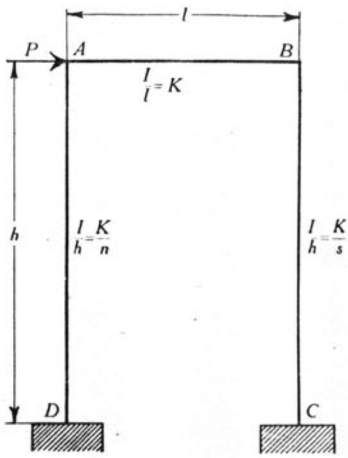


FIGURE 26

23. *Two-legged Rectangular Bent. Concentrated Horizontal Force at the Top—Legs Fixed at the Bases.*—Fig. 26 represents a two-legged rectangular bent having legs fixed at the bases. A single force P is applied at the top of the bent.

Applying equation (A), Table 1,

$$M_{AB}=2EK(2\theta_A+\theta_B) \qquad (65)$$

$$M_{AD}=\frac{2EK}{n}(2\theta_A-3R) \qquad (66)$$

$$M_{DA}=\frac{2EK}{n}(\theta_A-3R) \qquad (67)$$

$$M_{BA}=2EK(2\theta_B+\theta_A) \qquad (68)$$

$$M_{BC}=\frac{2EK}{s}(2\theta_B-3R) \qquad (69)$$

$$M_{CB}=\frac{2EK}{s}(\theta_B-3R) \qquad (70)$$

$$M_{AD}+M_{DA}+M_{BC}+M_{CB}+Ph=0 \qquad (71)$$

$$M_{AB}+M_{AD}=0 \qquad (72)$$

$$M_{BA}+M_{BC}=0 \qquad (73)$$

Substituting the values of M_{AB} and M_{AD} from equations 65 and 66 in equation 72 and simplifying gives

$$\frac{2\theta_A}{R}(1+n)+\frac{n\theta_B}{R}=3 \qquad (74)$$

Substituting the values of M_{BA} and M_{BC} from equations 68 and 69 in equation 73 and simplifying gives

$$\frac{2\theta_B}{R}(1+s)+\frac{s\theta_A}{R}=3 \qquad (75)$$

Substituting the values for the moments in equation 71 and simplifying gives

$$\frac{s\theta_A}{R}+\frac{n\theta_B}{R}=2(s+n)-\frac{n\,s\,Ph}{6EK}\cdot\frac{1}{R} \qquad (76)$$

Solving equation 74 for $\frac{\theta_B}{R}$ gives

$$\frac{\theta_B}{R}=\frac{3}{n}-\frac{2\,\theta_A}{R}\left(\frac{1+n}{n}\right) \qquad (77)$$

Substituting the value of $\frac{\theta_B}{R}$ from equation 77 in equation 75 gives

$$\frac{\theta_A}{R}=\frac{3(2s+2-n)}{3ns+4n+4s+4} \qquad (78)$$

Substituting the value of $\frac{\theta_A}{R}$ from equation 78 in equation 77 and simplifying gives

$$\frac{\theta_B}{R}=\frac{3(2n+2-s)}{3ns+4n+4s+4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Substituting the values of $\frac{\theta_A}{R}$ and $\frac{\theta_B}{R}$ from equation 78 and equation 79 in equation 76 gives

$$\frac{3s(2s+2-n)}{3ns+4n+4s+4}+\frac{3n(2n+2-s)}{3ns+4n+4s+4}=2(s+n)-\frac{ns\,Ph}{6EK}\cdot\frac{1}{R}$$

Solving this equation for $\frac{1}{R}$ gives

$$\frac{1}{R}=\frac{6EK}{ns\,Ph}\;\frac{2(3ns^2+11ns+s^2+s+3n^2s+n^2+n)}{3ns+4n+4s+4} \quad . \quad . \quad . \quad (80)$$

Substituting the value of $\frac{1}{R}$ from equation 80 in equation 79 and solving for θ_B gives

$$\theta_B=\frac{Ph}{12EK}\;\frac{3ns(2n+2-s)}{3ns^2+11ns+s^2+s+3n^2s+n^2+n} \quad . \quad . \quad . \quad . \quad (81)$$

Substituting the value of $\frac{1}{R}$ from equation 80 in equation 78 and solving for θ_A gives

$$\theta_A=\frac{Ph}{12EK}\;\frac{3ns(2s+2-n)}{3ns^2+11ns+s^2+s+3n^2s+n^2+n} \quad . \quad . \quad . \quad . \quad (82)$$

Substituting the values of θ_B and θ_A from equations 81 and 82 in equation 65 gives

$$M_{AB}=\frac{Ph}{2}\;\frac{3ns(s+2)}{3ns^2+11ns+s^2+s+3n^2s+n^2+n} \quad . \quad . \quad . \quad (83)$$

Substituting the values of θ_B and θ_A from equations 81 and 82 in equation 68 and substituting $-M_{BC}$ for M_{BA} gives

$$M_{BC}=-\frac{Ph}{2}\;\frac{3ns(n+2)}{3ns^2+11ns+s^2+s+3n^2s+n^2+n} \quad . \quad . \quad . \quad (84)$$

From equation 67

$$M_{DA}=\frac{2EKR}{n}\left[\frac{\theta_A}{R}-3\right]. \qquad (85)$$

Substituting the value of R from equation 80 and of $\frac{\theta_A}{R}$ from equation 78 gives

$$M_{DA}=-\frac{Ph}{2}\,\frac{s(2s+2+5n+3ns)}{3ns^2+11ns+s^2+s+3n^2s+n^2+n} \qquad (86)$$

From equation 70

$$M_{CB}=\frac{2EKR}{s}\left(\frac{\theta_B}{R}-3\right)$$

Substituting the value of R from equation 80 and of $\frac{\theta_B}{R}$ from equation 79 gives

$$M_{CB}=-\frac{Ph}{2}\,\frac{n(2n+2+5s+3ns)}{(3ns^2+11ns+s^2+s+3n^2s+n^2+n)} \qquad (87)$$

Letting Δ_o represent $2(3ns^2+11ns+s^2+s+3n^2s+n^2+n)$, the equation for the moments in the frame are

$$M_{AD}=-\frac{Ph}{\Delta_o}\,3ns(s+2) \qquad (88)$$

$$M_{BC}=-\frac{Ph}{\Delta_o}\,3ns(n+2) \qquad (89)$$

$$M_{CB}=-\frac{Ph}{\Delta_o}\,n(2n+2+5s+3ns) \qquad (90)$$

$$M_{DA}=-\frac{Ph}{\Delta_o}\,s(2s+2+5n+3ns) \qquad (91)$$

If $n=s$, that is, if the section of AD has the same moment of inertia as the section of BC, equations 88 to 91 take the form

$$M_{AD}=M_{BC}=-\frac{Ph}{2}\,\frac{3n}{6n+1} \qquad (92)$$

$$M_{CB}=M_{DA}=-\frac{Ph}{2}\,\frac{3n+1}{6n+1} \qquad (93)$$

If $n=s=1$, equations 92 and 93 take the form

$$M_{AD}=M_{BC}=-\frac{3}{14}Ph \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

$$M_{CB}=M_{DA}=-\frac{4}{14}Ph \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

24. *Two-legged Rectangular Bent. Any System of Horizontal Loads on One Leg. Legs Hinged at the Bases.*—Fig. 27 represents a

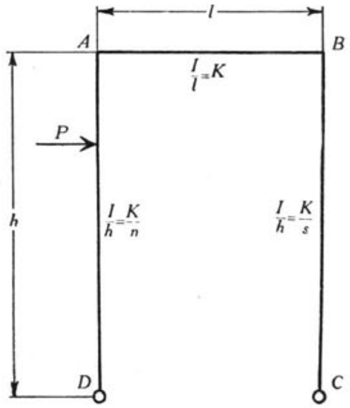


FIGURE 27

two-legged rectangular bent with any horizontal load on the leg AD. The legs are hinged at D and C. P represents the resultant of any system of horizontal forces acting on AD.

If M_D represents the moment of P about D,

$$M_{AD}+M_{BC}+M_D=0$$

Applying the equations of Table 1 gives

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$M_{AD}=\frac{EK}{n}(3\theta_A-3R)+H_{AD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$M_{BA}=2EK(2\theta_B+\theta_A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$M_{BC}=\frac{EK}{s}(3\theta_B-3R) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

$$M_{AD}+M_{BC}+M_D=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

Equations 96 to 100 are almost identical in form to equations 54 to 58 of section 22. Hence the solution for the moments by eliminating values of θ and R, as in section 22, gives

$$M_{AB}=\frac{1}{2}\;\frac{M_D(2s+3)-2n\,H_{AD}}{n+s+3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

$$M_{BC}=-\frac{1}{2}\;\frac{M_D(2n+3)+2n\,H_{AD}}{n+s+3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

If $n=s$, that is, if the sections of AD and BC have the same moments of inertia, equations 102 and 101 take the form

$$M_{BC}=-\frac{1}{2}\left[M_D+\frac{2n\,H_{AD}}{2n+3}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

$$M_{AB}=\frac{1}{2}\left[M_D-\frac{2n\,H_{AD}}{2n+3}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

If $n=s=1$, equations 103 and 104 take the form

$$M_{BC}=-\frac{1}{10}\left[5M_D+2H_{AD}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

$$M_{AB}=\frac{1}{10}\left[5M_D-2H_{AD}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

If both legs of the bent are loaded and if the bent and loads are symmetrical about a vertical center line

$$M_{AB}=M_{BC}=-\left[\frac{2n}{2n+3}\right]H_{AD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Values of H_{AD} to be used in equations 101 to 107 are given in Table 2.

25. *Two-legged Rectangular Bent. Any System of Horizontal Loads on One Leg—Legs Fixed at the Bases.*—Fig. 28 represents a two-

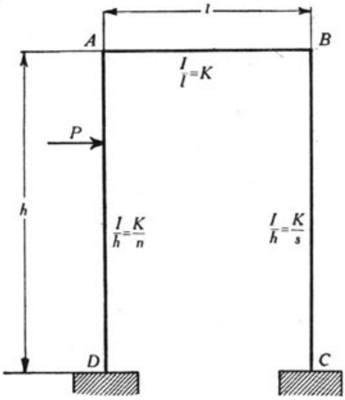


FIGURE 28

legged rectangular bent with any horizontal load on the leg AD. The legs are fixed at D and C. P represents the resultant of all the forces on AD.

If M_D represents the moment of P about D,

$$M_{AD}+M_{DA}+M_{BC}+M_{CB}+M_D=0$$

Applying the equations of Table 1 gives

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad (108)$$

$$n\,M_{AD}=2EK(2\theta_A-3R)+n\,C_{AD} \quad (109)$$

$$n\,M_{DA}=2EK(\theta_A-3R)-n\,C_{DA} \quad (110)$$

$$M_{BA}=2EK(2\theta_B+\theta_A) \quad (111)$$

$$s\,M_{BC}=2EK(2\theta_B-3R) \quad (112)$$

$$s\,M_{CB}=2EK(\theta_B-3R) \quad (113)$$

$$M_{AD}+M_{DA}+M_{BC}+M_{CB}+M_D=0 \quad (114)$$

The equations 108 to 114 are similar to equations 65 to 73 of section 23. The method of solving for the four unknown moments will be done in a different way from that of section 23. The equations will first be combined to eliminate θ and R, then the resulting equations solved simultaneously for the moments.

Adding equation 112 and two times equation 110 and subtracting equation 109 and two times equation 113 to eliminate θ_A, θ_B, and R gives

$$n\,M_{AB}+s\,M_{BC}-2s\,M_{CB}+2n\,M_{DA}=-n(C_{AD}+2C_{DA}) \quad (115)$$

Adding equation 109 and two times equation 112 and subtracting equations 110 and 111 and two times equation 113 gives

$$-n\,M_{AB}+(2s+1)M_{BC}-2s\,M_{CB}-n\,M_{DA}=n(C_{AD}+C_{DA}) \quad (116)$$

Adding equations 108, 110 and 112, and subtracting equations 109, 111 and 113 gives

$$(n+1)M_{AB}+(s+1)M_{BC}-s\,M_{CB}+n\,M_{DA}= \\ -n(C_{AD}+C_{DA}) \quad \ldots \quad (117)$$

Equations 114 to 117 are rewritten in Table 7. In this table the unknown moments are written at the heads of the columns and the coefficients are written below.

TABLE 7

EQUATIONS FOR THE TWO-LEGGED RECTANGULAR BENT OF FIG. 28

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation	How Equation was Obtained
	M_{AB}	M_{BC}	M_{CB}	M_{DA}		
114	-1	1	1	1	$-M_D$	114
115	n	s	$-2s$	$2n$	$-n(C_{AD}+2C_{DA})$	$-(109)+(112)+2(110)$ $-2(113)$
116	$-n$	$2s+1$	$-2s$	$-n$	$n(C_{AD}+C_{DA})$	$-(111)+2(112)-2(113)$ $-(110)+(109)$
117	$n+1$	$s+1$	$-s$	n	$-n(C_{AD}+C_{DA})$	$(108)-(109)+(110)$ $-(111)+(112)-(113)$

Solving these equations simultaneously and letting
$\Delta_o=2(11sn+3sn^2+3s^2n+s^2+s+n^2+n)$ gives

$$M_{AD}=-\frac{n}{\Delta_o}\left[3s\,(s+2)\,(M_D-C_{DA})-C_{AD}\,(6ns+3s^2+5s+2n)\right] \quad \ldots \quad (118)$$

$$M_{BC}=-\frac{n}{\Delta_o}\left[3s(n+2)(M_D-C_{DA})+C_{AD}(3ns-n-4s)\right] \quad (119)$$

$$M_{CB}=-\frac{n}{\Delta_o}\left[(3ns+2n+5s+2)\,(M_D-C_{DA})+C_{AD}\,(3ns+3n \\ -3s-1)\right] \quad \ldots \quad (120)$$

$$M_{DA} = -\frac{1}{\Delta_o}\Big[s\,(3ns + 2s + 5n + 2)\ (M_D - C_{DA})$$

$$+ n\,C_{AD}(3s^2 + 12s + 1)\Big] - C_{DA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

If $n = s$ equations 118 and 121 take the form

$$M_{AD} = -\frac{n}{2}\left[\frac{3(M_D - C_{DA})}{6n+1} - C_{AD}\left(\frac{1}{n+2} + \frac{3}{6n+1}\right)\right] \quad . \quad (122)$$

$$M_{BC} = -\frac{n}{2}\left[\frac{3(M_D - C_{DA})}{6n+1} + C_{AD}\left(\frac{1}{n+2} - \frac{3}{6n+1}\right)\right] \quad . \quad (123)$$

$$M_{CB} = -\frac{1}{2}\left[\frac{(3n+1)\ (M_D - C_{DA})}{6n+1} - C_{AD}\left(\frac{1}{n+2} - \frac{3n}{6n+1}\right)\right]$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

$$M_{DA} = -\frac{1}{2}\left[\frac{(3n+1)\ (M_D - C_{DA})}{6n+1} + C_{AD}\left(\frac{1}{n+2} + \frac{3n}{6n+1}\right)\right]$$

$$-C_{DA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

If both legs of the bent are loaded and if the bent and loads are symmetrical about a vertical center line

$$M_{AB} = M_{BC} = -\frac{n}{n+2}\,C_{AD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

$$M_{CB} = -M_{DA} = \frac{1}{n+2}\,C_{AD} + C_{DA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

Values of C_{AD} and C_{DA} to be used in equations 118 to 127 are given in Table 2.

26. *Two-legged Rectangular Bent. Any System of Vertical Loads on the Top—Legs Hinged at the Bases.*—Fig. 29 represents a two-legged rectangular bent having any system of vertical loads on AB. The legs are hinged at D and C. P represents the resultant of the loads on AB.

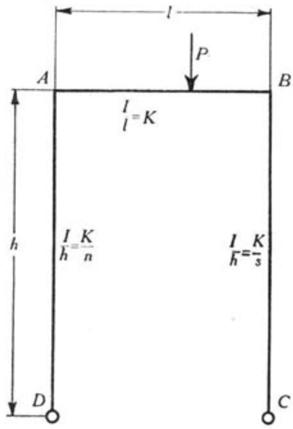


FIGURE 29

Applying the equations of Table 1 gives

$$M_{AB}=2EK(2\theta_A+\theta_B)-C_{AB} \qquad (128)$$

$$M_{AD}=\frac{EK}{n}(3\theta_A-3R) \qquad (129)$$

$$M_{BA}=2EK(2\theta_B+\theta_A)+C_{BA} \qquad (130)$$

$$M_{BC}=\frac{EK}{s}(3\theta_B-3R) \qquad (131)$$

$$M_{AD}+M_{BC}=0 \qquad (132)$$

Eliminating values of θ and R as in section 22 gives

$$M_{BC}=-\frac{3}{2}\left[\frac{C_{AB}+C_{BA}}{n+s+3}\right] \qquad (133)$$

$$M_{AD}=\frac{3}{2}\left[\frac{C_{AB}+C_{BA}}{n+s+3}\right]=-M_{BC} \qquad (134)$$

Values of C_{BA} and C_{AB} to be used in equations 133 and 134 are given in Table 2.

27. *Two-legged Rectangular Bent. Any System of Vertical Loads on the Top—Legs Fixed at the Bases.*—Fig. 30 represents a two-legged rectangular bent with any system of vertical loads on AB. The legs are fixed at C and D. P represents the resultant of all the loads on AB.

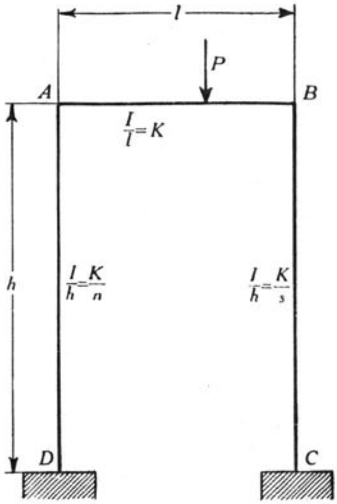


FIGURE 30

Applying the equations of Table 1 gives

$$n\,M_{DA}=2EK(\theta_A-3R) \quad (135)$$

$$n\,M_{AB}=-2EK(2\theta_A-3R) \quad (136)$$

$$M_{AB}=2EK(2\theta_A+\theta_B)-C_{AB} \quad (137)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A)-C_{BA} \quad (138)$$

$$s\,M_{BC}=2EK(2\theta_B-3R) \quad (139)$$

$$s\,M_{CB}=2EK(\theta_B-3R) \quad (140)$$

$$-M_{AB}+M_{BC}+M_{CB}+M_{DA}=0 \quad (141)$$

Combining these equations as indicated in the table to eliminate θ_A, θ_B, and R gives the equations in Table 8.

Solving these equations simultaneously gives

$$M_{AB}=-\frac{1}{\Delta_o}\left\{C_{BA}(10ns+s^2)+C_{AB}(11ns+2s^2+2s+2n)\right\} \quad (146)$$

$$M_{BC}=-\frac{1}{\Delta_o}\left\{C_{BA}(11ns+2n^2+2n+2s)+C_{AB}(10ns+n^2)\right\} \quad (147)$$

TABLE 8

EQUATIONS FOR THE TWO-LEGGED RECTANGULAR BENT REPRESENTED BY FIG. 30

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation	How Equation Was Obtained
	M_{AB}	M_{BC}	M_{CB}	M_{DA}		
142	-1	1	1	1	0	(141)
143	n	s	$-2s$	$2n$	0	(136)+(139) +2[(135)−(140)]
144	$-n$	$2s+1$	$-2s$	$-n$	$-C_{BA}$	(138)+2[(139)−(140)] −[(135)+(136)]
145	$n+1$	$s+1$	$-s$	n	$-(C_{AB}+C_{BA})$	(135)+(136)+(137)+(138) +(139)−(140)

$$M_{CB}=-\frac{1}{\Delta_o}\Big\{C_{BA}(7ns-2n^2-2n+s)+C_{AB}(8ns-n^2+3n)\Big\} \quad . \quad (148)$$

$$M_{DA}=\frac{1}{\Delta_o}\Big\{C_{BA}(8ns-s^2+3s)+C_{AB}(7ns-2s^2-2s+n)\Big\} \quad . \quad (149)$$

in which $\Delta_o=2(11sn+3sn^2+3s^2n+s^2+s+n^2+n)$

If the load is symmetrical, that is, if $C_{BA}=C_{AB}=\frac{F}{l}$, equations 146 to 149 take the form

$$M_{AB}=-\frac{F}{l\Delta_o}(21ns+3s^2+2s+2n) \quad . \quad (150)$$

$$M_{BC}=-\frac{F}{l\Delta_o}(21ns+3n^2+2n+2s) \quad . \quad (151)$$

$$M_{CB}=-\frac{F}{l\Delta_o}(15ns-3n^2+n+s) \quad . \quad (152)$$

$$M_{DA}=\frac{F}{l\Delta_o}(15ns-3s^2+s+n) \quad . \quad (153)$$

If the bent is symmetrical about a vertical center line, that is, if $n=s$, equations 146 to 149 take the form

$$M_{AB}=-\frac{1}{2}\left\{C_{BA}\left[\frac{2}{n+2}-\frac{1}{6n+1}\right]+C_{AB}\left[\frac{2}{n+2}+\frac{1}{6n+1}\right]\right\} \quad . \quad (154)$$

$$M_{BC}=-\frac{1}{2}\left\{C_{BA}\left[\frac{2}{n+2}+\frac{1}{6n+1}\right]+C_{AB}\left[\frac{2}{n+2}-\frac{1}{6n+1}\right]\right\} \quad . \quad (155)$$

$$M_{CB}=-\frac{1}{2}\left\{C_{BA}\left[\frac{1}{n+2}-\frac{1}{6n+1}\right]+C_{AB}\left[\frac{1}{n+2}+\frac{1}{6n+1}\right]\right\} \quad . \quad (156)$$

$$M_{DA}=\quad\frac{1}{2}\left\{C_{BA}\left[\frac{1}{n+2}+\frac{1}{6n+1}\right]+C_{AB}\left[\frac{1}{n+2}-\frac{1}{6n+1}\right]\right\} \quad . \quad (157)$$

If the bent and the loading are symmetrical about a vertical center line, that is, if $n=s$ and $C_{BA}=C_{AB}=\frac{F}{l}$, equations 146 to 149 take the form

$$M_{AB}=M_{BC}=-\frac{2}{(n+2)}\,\frac{F}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

$$M_{CB}=-M_{DA}=-\frac{1}{n+2}\,\frac{F}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

28. *Two-legged Rectangular Bent. External Moment at One Corner—Legs Hinged at the Bases.*—Fig. 31 represents a two-legged

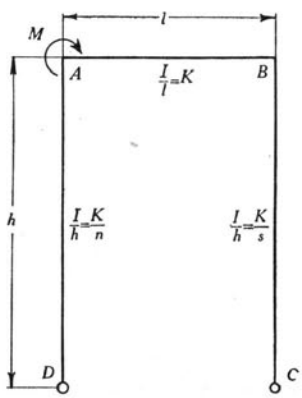


FIGURE 31

rectangular bent having legs hinged at the bases. An external couple whose moment is represented by M is applied at A.

Applying the equations of Table 1 to this bent gives four equations which are identical with the first four equations of section 22. Also, for equilibrium at A,

$$M_{AD}+M_{AB}=M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

Solving these five equations for the moments gives

$$M_{AB}=M\left[1-\frac{3}{2}\ \frac{1}{(n+s+3)}\right] \qquad (161)$$

For equilibrium the horizontal reactions at C and D must be equal and opposite. Therefore, $M_{AD}=-M_{BC}$. Combining equations 160 and 161 gives

$$M_{AD}=-M_{BC}=\frac{3}{2}\left(\frac{M}{n+s+3}\right) \qquad (162)$$

If the bent is symmetrical about a vertical center line $n=s$. Equations 161 and 162 then reduce to the form

$$M_{AB}=M\left[1-\frac{3}{2}\ \frac{1}{(2n+3)}\right] \qquad (163)$$

$$M_{AD}=-M_{BC}=\frac{3}{2}\left(\frac{M}{2n+3}\right) \qquad (164)$$

If $n=s=1$

$$M_{AB}=\frac{7}{10}M \qquad (165)$$

$$M_{AD}=-M_{BC}=\frac{3}{10}M \qquad (166)$$

29. *Two-legged Rectangular Bent. External Moment at One Corner—Legs Fixed at the Bases.*—Fig. 32 represents a two-legged rectangular bent having legs fixed at the bases. An external couple whose moment is represented by M is applied at A.

Applying the equations of Table 1 to this bent gives six equations which are identical with the first six equations of section 23. For equilibrium of the entire bent,

$$-M_{AB}+M_{DA}+M_{BC}+M_{CB}=-M \qquad (167)$$

Also,

$$M_{AB}+M_{AD}=M \qquad (168)$$

$$M_{BA}+M_{BC}=0 \qquad (169)$$

Solving these equations for the moments gives

$$M_{AB}=-\frac{M}{\Delta_o}(11ns+2s^2+2s+2n)+M \quad (170)$$

$$M_{BC}=-\frac{nM}{\Delta_o}(10s+n) \quad (171)$$

$$M_{CB}=-\frac{nM}{\Delta_o}(8s-n+3) \quad (172)$$

$$M_{DA}=\frac{M}{\Delta_o}(7ns-2s^2-2s+n) \quad (173)$$

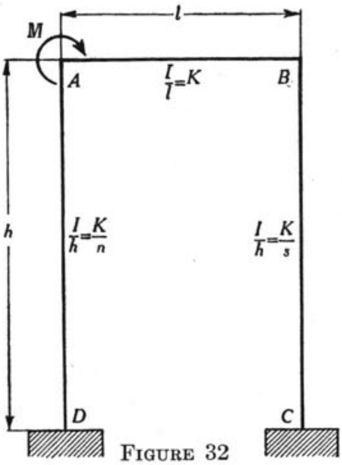


FIGURE 32

$$M_{AD}=\frac{M}{\Delta_o}(11ns+2s^2+2s+2n) \quad (174)$$

in which

$$\Delta_o=22sn+2(s^2+s+n^2+n)+6(sn^2+s^2n)$$

If the bent is symmetrical about a vertical center line $n=s$. Equations 170, 171, 172, 173, and 174 then take the form

$$M_{AB}=\frac{M}{2}\left[1+\frac{n}{n+2}-\frac{1}{6n+1}\right] \quad (175)$$

$$M_{BC}=\frac{M}{2}\left[\frac{n}{n+2}-\frac{6n}{6n+1}\right] \quad (176)$$

$$M_{CB}=-\frac{M}{2}\left[\frac{1}{n+2}+\frac{1}{6n+1}\right] \quad (177)$$

$$M_{DA}=\frac{M}{2}\left[\frac{1}{n+2}-\frac{1}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

$$M_{AD}=\frac{M}{2}\left[1-\frac{n}{n+2}+\frac{1}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

If $n=s=1$, equations 170, 171, 172, 173, and 174 take the form

$$M_{AB}=\frac{25}{42}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

$$M_{BC}=-\frac{11}{42}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

$$M_{CB}=-\frac{10}{42}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

$$M_{DA}=\frac{4}{42}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

$$M_{AD}=\frac{17}{42}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

30. *Two-legged Rectangular Bent. Settlement of Foundations—Legs Hinged at the Bases.*—Fig. 33 represents a two-legged rectangular bent. The legs of the bent are hinged at D and C. The unstrained position of the bent is represented by the broken line $DA'B'C'$. Due to settlement of the foundations the point C' has moved to C. The motion of C can be considered as made up of two parts, as follows: first, without being strained the bent rotates about D until C' is at C'', a point on CD; secondly, the bent is strained by applying a force at C'' acting along DC which moves C'' to C; that is, no matter what motion of C relative to D takes place, the stress in the bent depends only upon the change in the distance of C from D.

The change in the distance from D to C is represented by d. d is made up of two parts, d_1 due to the deflection of AD, and d_2 due to the deflection of BC.

Applying the equations of Table 1 gives

$$M_{AB}=-\frac{EK}{n}(3\theta_A-3R_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

$$M_{BC}=\frac{EK}{s}(3\theta_B-3R_1)+\frac{3EK}{s}\,\frac{d}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

$$M_{AB}-M_{BC}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

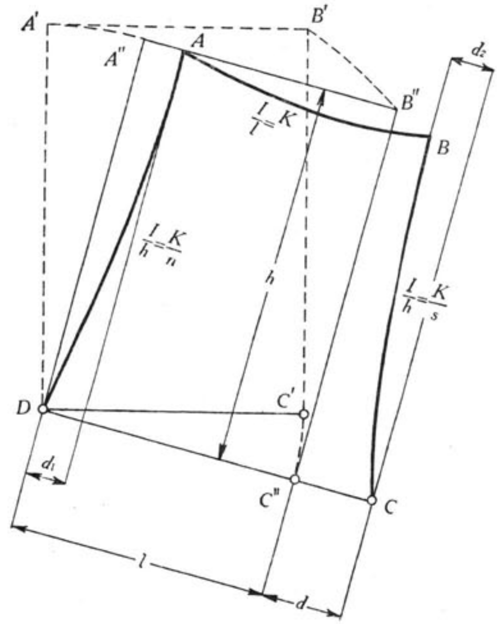


FIGURE 33

Solving these equations for the moments gives

$$M_{AB}=M_{BC}=\frac{d}{h}\,\frac{3EK}{(n+s+3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (190)$$

If the bent is symmetrical about a vertical center line $n=s$, and

$$M_{AB}=M_{BC}=\frac{d}{h}\,\frac{3EK}{2n+3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191)$$

If $n=s=1$

$$M_{AB}=M_{BC}=\frac{3}{5}\,\frac{d}{h}\,EK \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

d in equations 190, 191, and 192 represents the increase in the distance from D to C.

31. *Two-legged Rectangular Bent. Settlement of Foundations—Legs Restrained at the Bases.*—Fig. 34 represents a two-legged rectangular bent. The legs of the bent are restrained at C and D. The unstrained position of the bent is represented by the broken line $DA'B'C'$. Due to settlement of the foundations C' has moved to C.

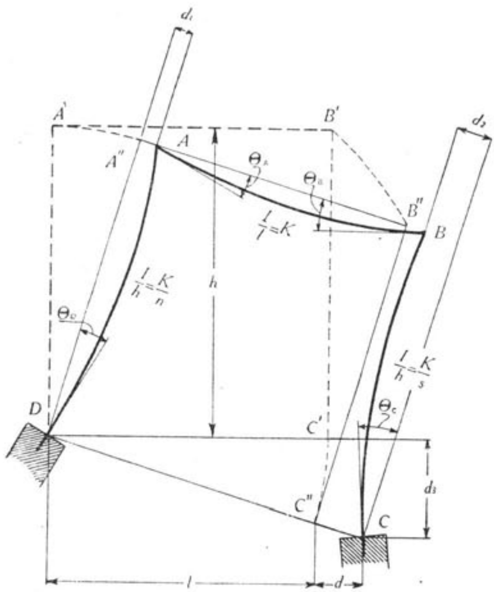


FIGURE 34

The foundations, moreover, have tipped so that whereas the tangents to the neutral axes of the columns at the bases were originally vertical now they are inclined. The motion of the bent can be considered as made up of two parts, as follows: first, without being strained the bent rotates about D until C' is at C'', a point on DC. At the same time the supports rotate so that the tangents to the neutral axes of the columns at the bases are normal to DC. These motions produce no stress in the bent; secondly, C'' moves to C, the support D rotates through the angle θ_D, and the support C rotates through the angle θ_C; that is, no matter what motion of C relative to D takes place, the stress in the bent depends only upon the change in the distance of C from D and upon the rotations of the supports at D and C *relative to the line DC* (not relative to DC').

Applying the equations of Table 1 gives

$$M_{AB} = -\frac{2EK}{n}\,(2\theta_A + \theta_D - 3R_1) \quad . \quad . \quad . \quad (193)$$

$$M_{AB} = 2EK(2\theta_A + \theta_B) \quad . \quad . \quad . \quad (194)$$

$$M_{BC} = -2EK(2\theta_B + \theta_A) \quad . \quad . \quad . \quad (195)$$

$$M_{BC} = \frac{2EK}{s}(2\theta_B + \theta_C - 3R_2) \quad . \quad . \quad . \quad (196)$$

$$M_{CB} = \frac{2EK}{s}\,(2\theta_C + \theta_B - 3R_2) \quad . \quad . \quad . \quad (197)$$

$$M_{DA} = \frac{2EK}{n}(2\theta_D + \theta_A - 3R_1) \quad . \quad . \quad . \quad (198)$$

For the columns to be in equilibrium

$$M_{DA} - M_{AB} + M_{BC} + M_{CB} = 0 \quad . \quad . \quad . \quad (199)$$

Combining these equations and solving for the moments gives

$$M_{AB} = \frac{2EK}{\Delta_o}\Big[3\,(6ns + 2n - s)\,\frac{d}{h} + (9ns + 8n - s)\,\theta_C$$

$$- (6ns + 2n - 7s - 3s^2)\,\theta_D\Big] \quad . \quad . \quad . \quad (200)$$

$$M_{BC} = \frac{2EK}{\Delta_o}\Big[\,3\,(6ns - n + 2s)\,\frac{d}{h} + (6ns - 7n + 2s - 3n^2)\,\theta_C$$

$$- (9ns - n + 8s)\,\theta_D\Big] \quad . \quad . \quad . \quad (201)$$

$$M_{CB} = \frac{2EK}{\Delta_o}\Big[3(6ns + 5n + 2s + 1)\frac{d}{h} + (12ns + 22n + 4s + 3 + 3n^2)\,\theta_C$$

$$- (9ns + 7n + 7s + 3)\,\theta_D\Big] \quad . \quad . \quad . \quad (202)$$

$$M_{DA} = -\frac{2EK}{\Delta_o}\Big[3(6ns + 2n + 5s + 1)\frac{d}{h} + (9ns + 7n + 7s + 3)\,\theta_C$$

$$- (12ns + 4n + 22s + 3 + 3s^2)\,\theta_D\Big] \quad . \quad . \quad . \quad (203)$$

in which

$$\Delta_o = 22ns + 2(s^2 + s + n^2 + n) + 6(n^2s + s^2n)$$

If the bent is symmetrical about a vertical center line $n=s$. Equations 200, 201, 202, and 203 then take the form

$$M_{AB}=EK\left[\frac{d}{h}\,\frac{3}{n+2}+\frac{\theta_C-\theta_D}{n+2}+\frac{3(\theta_C+\theta_D)}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad (204)$$

$$M_{BC}=EK\left[\frac{d}{h}\,\frac{3}{n+2}+\frac{\theta_C-\theta_D}{n+2}-\frac{3(\theta_C+\theta_D)}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad (205)$$

$$M_{CB}=EK\left[\frac{d}{h}\,\frac{3(n+1)}{n(n+2)}+\frac{2n+3}{n(n+2)}(\theta_C-\theta_D)+\frac{3(\theta_C+\theta_D)}{6n+1}\right] \quad . \quad (206)$$

$$M_{DA}=-EK\left[\frac{d}{h}\,\frac{3(n+1)}{n(n+2)}+\frac{2n+3}{n(n+2)}(\theta_C-\theta_D)-\frac{3(\theta_C+\theta_D)}{6n+1}\right] \quad (207)$$

If $n=s=1$

$$M_{AB}=\frac{EK}{21}\left[21\,\frac{d}{h}+16\theta_C+2\theta_D\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

$$M_{BC}=\frac{EK}{21}\left[21\frac{d}{h}-2\theta_C-16\theta_D\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

$$M_{CB}=\frac{EK}{21}\left[42\frac{d}{h}+44\theta_C-26\theta_D\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

$$M_{DA}=-\frac{EK}{21}\left[42\frac{d}{h}+26\theta_C-44\theta_D\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

In all these equations θ is measured from a line normal to the line CD.

If the foundations settle without tipping it is more convenient to measure θ from a line normal to the original position DC'. It is then necessary to consider the vertical settlement d_3.

Proceeding as before, if the tangents to the neutral axes of the columns at their bases remain vertical, if the bases are separated horizontally by an amount represented by d, and if C settles vertically by an amount represented by d_3, it can be proved that

$$M_{AB}=\frac{2EK}{\Delta_o}\left[3\,(6ns+2n-s)\frac{d}{h}-3\,(ns+2n+2s+s^2)\frac{d_3}{l}\right] \quad . \quad . \quad (212)$$

$$M_{BC}=\frac{2EK}{\Delta_o}\left[3\,(6ns-n+2s)\frac{d}{h}+3\,(ns+2n+2s+s^2)\frac{d_3}{l}\right] \quad . \quad . \quad (213)$$

$$M_{CB}=\frac{2EK}{\Delta_o}\left[3\,(6ns+5n+2s+1)\frac{d}{h}-3\,(ns+5n-s+n^2)\frac{d_3}{l}\right] \quad . \quad (214)$$

$$M_{DA}=-\frac{2EK}{\Delta_o}\left[3\,(6ns+2n+5s+1)\frac{d}{h}+3\,(ns-n+5s+s^2)\frac{d_3}{l}\right] \quad (215)$$

If the bent is symmetrical about a vertical center line, $n=s$. Equations 212, 213, 214, and 215 then reduce to the form

$$M_{AB}=EK\left[\frac{d}{h}\,\frac{3}{n+2}-\frac{d_3}{l}\,\frac{6}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

$$M_{BC}=EK\left[\frac{d}{h}\,\frac{3}{n+2}+\frac{d_3}{l}\,\frac{6}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

$$M_{CB}=EK\left[\frac{d}{h}\,\frac{3(n+1)}{n(n+2)}-\frac{d_3}{l}\,\frac{6}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

$$M_{DA}=-EK\left[\frac{d}{h}\,\frac{3(n+1)}{n(n+2)}+\frac{d_3}{l}\,\frac{6}{6n+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

If $n=s=1$

$$M_{AB}=EK\left[\frac{d}{h}-\frac{6}{7}\,\frac{d_3}{l}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

$$M_{BC}=EK\left[\frac{d}{h}+\frac{6}{7}\,\frac{d_3}{l}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221)$$

$$M_{CB}=EK\left[\frac{2d}{h}-\frac{6}{7}\,\frac{d_3}{l}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

$$M_{DA}=-EK\left[\frac{2d}{h}+\frac{6}{7}\,\frac{d_3}{l}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (223)$$

VII. Two-legged Rectangular Bent. One Leg Longer than the Other

32. *Two-legged Rectangular Bent. One Leg Longer than the Other. Concentrated Horizontal Load at Top of Bent—Legs Hinged at the Bases.*—Fig. 35 represents a two-legged rectangular bent having one leg longer than the other. The legs are hinged at C and D. Let q equal the ratio of the length AD to the length BC.

The horizontal deflections of A and B are equal and are represented by d. R_{AD} is $\frac{d}{h}$ and R_{BC} is $\frac{d}{h}q$. If R_{AD} is represented by R, $R_{BC}=q\,R$.

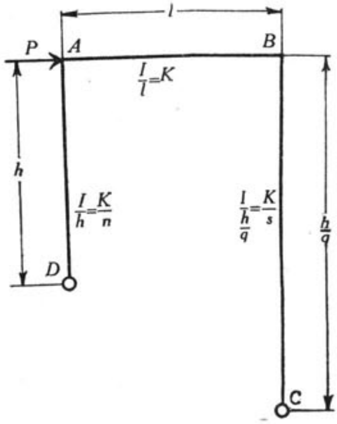


FIGURE 35

Applying the equations of Table 1 gives

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad . \quad . \quad . \quad . \quad . \quad . \quad (224)$$

$$M_{AD}=\frac{EK}{n}(3\theta_A-3R) \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

$$M_{BA}=2EK(2\theta_B+\theta_A) \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

$$M_{BC}=\frac{EK}{s}(3\theta_B-3\,q\,R) \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

$$M_{AD}+q\,M_{BC}+Ph=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

Substituting the value of θ_B from equation 224 in equation 226 and substituting $-M_{BC}$ for M_{BA} gives

$$\theta_A = \frac{1}{6EK}(M_{BC}+2M_{AB}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

Substituting the value of R from equation 225 and the value of θ_B from equation 224 in equation 227 and substituting $-M_{AB}$ for M_{AD} gives

$$\theta_A = \frac{1}{2EK(6+3q)}\Big[M_{AB}(3-2nq)-2s\,M_{BC}\Big]. \quad . \quad . \quad . \quad . \quad . \quad (230)$$

Equating the right-hand members of equations 229 and 230 gives

$$M_{BC}(2+q+2s)+M_{AB}(1+2nq+2q)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

Substituting $-M_{AB}$ for M_{AD} in equation 228 and eliminating M_{AB} from equations 228 and 231 gives

$$M_{BC}=-Ph\left[\frac{2nq+2q+1}{q(2nq+2q+1)+(2s+2+q)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

$$M_{AB}=Ph\left[\frac{2s+2+q}{q(2nq+2q+1)+(2s+2+q)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

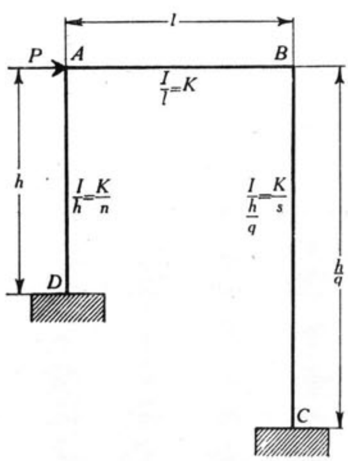


FIGURE 36

33. *Two-legged Rectangular Bent. One Leg Longer than the Other. Concentrated Horizontal Load at Top of Bent—Legs Fixed at the Bases.—*

Fig. 36 represents a two-legged rectangular bent having one leg longer than the other. The legs are fixed at C and D.

The horizontal deflections of A and B are equal and are represented by d. R_{AD} is $\frac{d}{h}$ and R_{BC} is $\frac{d}{h}\ q$. If R_{AD} is represented by R, $R_{BC}=qR$

Applying the equations of Table 1 gives

$$M_{AB}=2EK(2\theta_A+\theta_B) \qquad (234)$$

$$M_{AD}=\frac{2EK}{n}(2\theta_A-3R) \qquad (235)$$

$$M_{DA}=\frac{2EK}{n}(\theta_A-3R) \qquad (236)$$

$$M_{BA}=2EK(2\theta_B+\theta_A) \qquad (237)$$

$$M_{BC}=\frac{2EK}{s}(2\theta_B-3qR) \qquad (238)$$

$$M_{CB}=\frac{2EK}{s}(\theta_B-3qR) \qquad (239)$$

$$M_{AD}+M_{DA}+qM_{BC}+qM_{CB}+Ph=0 \qquad (240)$$

$$M_{AB}+M_{AD}=0 \qquad (241)$$

$$M_{BA}+M_{BC}=0 \qquad (242)$$

Substituting the values of M_{AB} and M_{AD} from equations 234 and 235 in equation 241 gives

$$2\left(\frac{\theta_A}{R}\right)(1+n)+n\left(\frac{\theta_B}{R}\right)=3 \qquad (243)$$

Substituting the values of M_{BA} and M_{BC} from equations 237 and 238 in equation 242 gives

$$s\left(\frac{\theta_A}{R}\right)+2\left(\frac{\theta_B}{R}\right)(s+1)=3q \qquad (244)$$

Substituting the values for the moments in equation 240 gives

$$s\left(\frac{\theta_A}{R}\right)+nq\left(\frac{\theta_B}{R}\right)=2(s+nq^2)-\frac{1}{R}\ \frac{ns\,Ph}{6EK} \qquad (245)$$

Solving equation 243 for $\frac{\theta_B}{R}$ gives

$$\frac{\theta_B}{R}=\frac{3}{n}-\frac{2\theta_A}{R}\left[\frac{1+n}{n}\right] \qquad (246)$$

Substituting the value of $\frac{\theta_B}{R}$ from equation 246 in equation 244 gives

$$\frac{\theta_A}{R}=\frac{3(2s+2-nq)}{3ns+4n+4s+4} \qquad (247)$$

Substituting the value of $\frac{\theta_A}{R}$ from equation 247 in equation 246 gives

$$\frac{\theta_B}{R}=\frac{3(2nq+2q-s)}{3ns+4n+4s+4} \qquad (248)$$

Substituting the values of $\frac{\theta_A}{R}$ and $\frac{\theta_B}{R}$ from equations 247 and 248 in equation 245 and solving for $\frac{1}{R}$ and letting Δ_o represent

$2(3ns^2+4ns+s^2+s+3nsq+3n^2sq^2+n^2q^2+nq^2+4nsq^2)$

$$\frac{1}{R}=\frac{6EK}{nsPh}\left(\frac{\Delta_o}{3ns+4n+4s+4}\right) \qquad (249)$$

Substituting the value of $\frac{1}{R}$ from equation 249 in equations 247 and 248 gives

$$\theta_A=\frac{nsPh}{6EK}\ \frac{3(2s+2-nq)}{\Delta_o} \qquad (250)$$

$$\theta_B=\frac{nsPh}{6EK}\ \frac{3(2nq+2q-s)}{\Delta_o} \qquad (251)$$

Substituting the values of θ_A and θ_B from equations 250 and 251 In equations 234 and 237 gives

$$M_{AB}=\frac{nsPh(3s+4+2q)}{\Delta_o} \qquad (252)$$

$$M_{BA}=\frac{n\,s\,Ph(3n\,q+4q+2)}{\Delta_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)$$

Substituting the values of θ_A and R in equations 236 and 239 gives

$$M_{DA}=-\frac{s\,Ph(3sn+4n+qn+2s+2)}{\Delta_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

$$M_{CB}=-\frac{n\,Ph(3nsq+4sq+2nq+s+2q)}{\Delta_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad (255)$$

34. *Two-legged Rectangular Bent. One Leg Longer than the Other. Any Horizontal Load on Vertical Leg—Legs Hinged at the Bases.*—Fig. 37 represents a two-legged bent having one leg longer than the other. P represents the resultant of any system of horizontal forces applied to the leg AD. The legs of the bent are hinged at D and C.

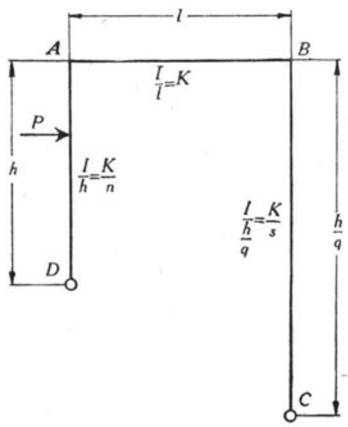


FIGURE 37

Applying the equations of Table 1 gives

$$n\,M_{AB}=-EK(3\theta_A-3R)-nH_{AD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (256)$$

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (258)$$

$$s\,M_{BC}=EK(3\theta_B-3q\,R) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (259)$$

The forces acting on the members AD and BC are shown in Fig. 38. M_D represents the moment of P about D.

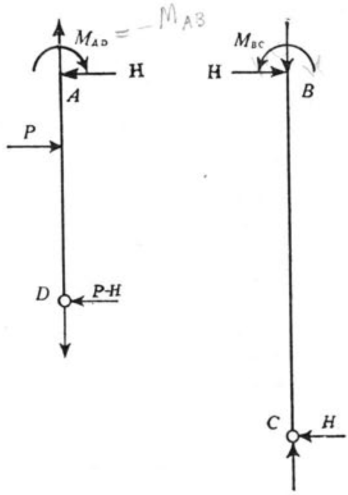


FIGURE 38

Equating to zero the moment of the external forces and reactions, about the point D, gives

$$M_D - M_{AB} + M_{BC} + H\left(\frac{h}{q} - h\right) = 0$$

Equating to zero the moments about the point B gives

$$M_{BC} + H\,\frac{h}{q} = 0$$

Combining these equations gives

$$M_{AB} - q\,M_{BC} = M_D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

Combining equations 256 to 259 as follows

$$-\frac{1}{3}\Big\{\Big[2(256) - 3(258) - 4(259)\Big]\Big[2+q\Big] - \Big[4(256) + 3($$

$$-2(259)\Big]\Big[1+2q\Big]\Big\}$$

(the numerals in the parentheses are the equation numbers) gives

$$M_{AB}(2qn + 2q + 1) + M_{BC}(2s + 2 + q) = -2qn\,H_{AD} \quad . \quad . \quad . \quad . \quad (261)$$

Substituting the value of M_{BC} from equation 260 in equation 261 gives

$$M_{AB}=\frac{1}{2}\ \frac{M_D(2s+2+q)-2q^2n\ H_{AD}}{q^2n+s+1+q+q^2} \quad \ldots \ldots \ldots \quad (262)$$

Substituting the value of M_{AB} in equation 261 gives

$$M_{BC}=-\frac{1}{2}\ \frac{M_D(2qn+2q+1)+2qn\ H_{AD}}{q^2n+s+1+q+q^2} \quad \ldots \ldots \quad (263)$$

If $n=s$, equations 262 and 263 take the form

$$M_{AB}=\frac{1}{2}\ \frac{M_D(2n+2+q)-2q^2n\ H_{AD}}{q^2n+n+1+q+q^2} \quad \ldots \ldots \ldots \quad (264)$$

$$M_{BC}=-\frac{1}{2}\ \frac{M_D(2qn+2q+1)+2qn\ H_{AD}}{q^2n+n+1+q+q^2} \quad \ldots \ldots \quad (265)$$

If $n=s=1$, equations 262 and 263 take the form

$$M_{AB}=\frac{1}{2}\ \frac{M_D(4+q)-2q^2\ H_{AD}}{2+q+2q^2} \quad \ldots \ldots \ldots \quad (266)$$

$$M_{BC}=-\frac{1}{2}\ \frac{M_D(1+4q)+2q\ H_{AD}}{2+q+2q^2} \quad \ldots \ldots \ldots \quad (267)$$

Values of H_{AD} for different systems of loadings to be used in equations 262 to 267 are given in Table 2.

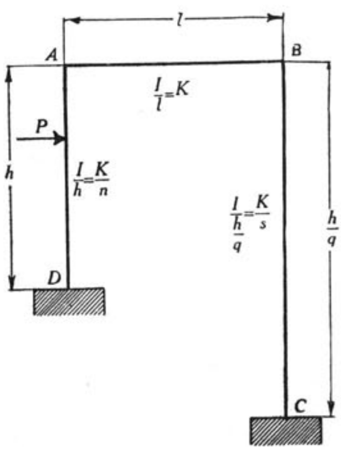


FIGURE 39

35. *Two-legged Rectangular Bent. One Leg Longer than the Other. Any Horizontal Load on Vertical Leg—Legs Fixed at the Bases.*—Fig. 39 represents a two-legged bent having legs of unequal length. P represents the resultant of any system of horizontal forces applied to the leg AD. The legs of the bent are fixed at D and C.

Applying the equations of Table 1 gives

$$n\,M_{AB} = -2EK(2\theta_A - 3R) - nC_{AD} \quad \ldots \quad (268)$$

$$M_{AB} = 2EK(2\theta_A + \theta_B) \quad \ldots \quad (269)$$

$$M_{BC} = -2EK(2\theta_B + \theta_A) \quad \ldots \quad (270)$$

$$s\,M_{BC} = 2EK(2\theta_B - 3qR) \quad \ldots \quad (271)$$

$$s\,M_{CB} = 2EK(\theta_B - 3qR) \quad \ldots \quad (272)$$

$$n\,M_{DA} = 2EK(\theta_A - 3R) - nC_{DA} \quad \ldots \quad (273)$$

The forces acting on the members AD and BC are shown in Fig. 40.

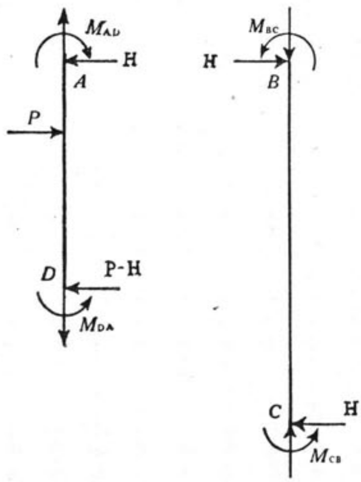


FIGURE 40

M_D represents the moment of P about D. Equating to zero the moment of the external forces and reactions about the point D gives

$$M_D - M_{AB} + M_{BC} - M_{CD} + M_{DA} + H\left(\frac{h}{q} - h\right) = 0$$

Also

$$M_{BC} - M_{CD} + H\,\frac{h}{q} = 0$$

TABLE 9

EQUATIONS FOR THE MOMENTS IN THE RECTANGULAR BENT REPRESENTED BY FIG. 39

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation	How Equation Was Obtained
	M_{AB}	M_{BC}	M_{CB}	M_{DA}		
274	-1	q	$+q$	1	$-M_D$	(274)
275	$1+2q-n(2+q)$	$2(s+1)\ (2+q)$	$-s(1+2q)$	$-2n(1+2q)$	$n\ [C_{AD}(2+q)+C_{DA}(2+4q)]$	$[2+q][2(270)-(268)+2(271)] +[1+2q][(269)-2(273) -(272)]$
276	1	$2+3s$	$-3s$	0	0	$(269)+2(270)+3(271) -3(272)$
277	$2(1+n)\ (1+2q)$	$(2+q)-s(1+2q)$	$2s(2+q)$	$n(2+q)$	$-n\ [C_{AD}(2+4q)+C_{DA}(2+q)]$	$[2+q]\ [(270)+2(272) +(273)]+[1+2q] [2(269)-(271)+2(268)]$

Combining these equations gives

$$M_{DA}-M_{AB}+q\,M_{BC}-q\,M_{CD}=-M_D \quad \text{. . . . . (274)}$$

Combining the equations as indicated gives the equations of Table 9.

Solving these equations simultaneously gives

$$M_{AB}=\frac{n}{\Delta_o}\Big\{(M_D-C_{DA})\,s\,(3s+4+2q)-C_{AD}(6q^2ns+3s^2+4s+qs+2q^2n)\Big\} \quad \text{. . . (278)}$$

$$M_{BC}=-\frac{n}{\Delta_o}\Big\{(M_D-C_{DA})\,s\,(3qn+2+4q)+C_{AD}(3qns-2s-2qs-q^2n)\Big\} \quad \text{. . . (279)}$$

$$M_{CB}=-\frac{n}{\Delta_o}\Big\{(M_D-C_{DA})(3qns+s+4qs+2qn+2q)+C_{AD}(3qns-s-2qs+2qn+q^2n-q)\Big\} \quad \text{. . . (280)}$$

$$M_{DA}=-\frac{1}{\Delta_o}\Big\{(M_D-C_{DA})\,s\,(3ns+4n+qn+2s+2)+C_{AD}n\,(3s^2+4s(1+q+q^2)+q^2)\Big\}-C_{DA} \quad \text{. . . (281)}$$

$\Delta_o=2\Big\{ns(4+3q+4q^2)+(s^2+s)+q^2(n^2+n)+3(q^2sn^2+s^2n)\Big\}$ for equations 278 to 281 inclusive.

If $n=s$, equations 278 to 281 take the form

$$M_{AB}=\frac{n^2}{\Delta_o}\Big\{(M_D-C_{DA})\;(3n+4+2q)-C_{AD}(6q^2n+3n+4+q+2q^2)\Big\} \quad \text{. . . (282)}$$

$$M_{BC}=-\frac{n^2}{\Delta_o}\Big\{(M_D-C_{DA})(3qn+2+4q)+C_{AD}(3qn-2-2q-q^2)\Big\} \quad \text{(283)}$$

$$M_{CB}=-\frac{n}{\Delta_o}\Big\{(M_D-C_{DA})(3qn^2+n+6qn+2q)+C_{AD}(3qn^2-n+q^2n-q)\Big\} \quad \text{. . . (284)}$$

$$M_{DA}=-\frac{n}{\Delta_o}\Big\{(M_D-C_{DA})\ (3n^2+6n+qn+2)+C_{AD}[3n^2+4n\ (1+q$$

$$+q^2)+q^2]\Big\}-C_{DA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (285)$$

$\Delta_o=2n\Big\{3n^2(1+q^2)+n(5+3q+5q^2)+1+q^2\Big\}$ for equations 282 to 285 inclusive.

If $n=s=1$, equations 282 to 285 take the form

$$M_{AB}=\frac{1}{\Delta_o}\Big\{(M_D-C_{DA})\ (7+2q)-C_{AD}(8q^2+q+7)\Big\} \quad . \quad . \quad . \quad (286)$$

$$M_{BC}=-\frac{1}{\Delta_o}\Big\{(M_D-C_{DA})\ (7q+2)+C_{AD}(q-2-q^2)\Big\} \quad . \quad . \quad (287)$$

$$M_{CB}=-\frac{1}{\Delta_o}\Big\{(M_D-C_{DA})\ (11q+1)+C_{AD}(2q-1+q^2)\Big\} \quad . \quad . \quad (288)$$

$$M_{DA}=-\frac{1}{\Delta_o}\Big\{(M_D-C_{DA})\ (11+q)+C_{AD}(7+4q+5q^2)\Big\}-C_{DA} \quad (289)$$

$\Delta_o=6(3+q+3q^2)$ for equations 286 to 289 inclusive.
Values of C_{AD} and C_{DA} are given in Table 2.

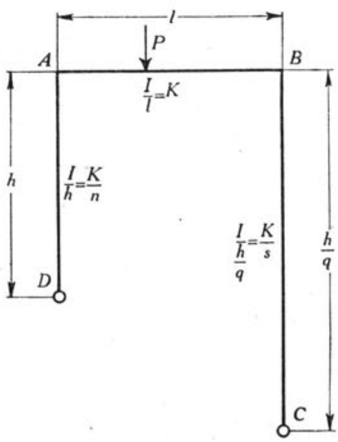


FIGURE 41

36. *Two-legged Rectangular Bent. One Leg Longer than the Other. Any System of Vertical Loads on Top of Bent—Legs Hinged at the Bases.*—Fig. 41 represents a two-legged bent having one leg longer

than the other. P represents the resultant of any system of vertical loads applied to AB. The legs of the bent are hinged at C and D.

Applying the equations of Table 1 gives

$$n\,M_{AB}=-EK(3\theta_A-3R) \quad (290)$$

$$M_{AB}=2EK(2\theta_A+\theta_B)-C_{AB} \quad (291)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A)-C_{BA} \quad (292)$$

$$s\,M_{BC}=EK(3\theta_B-3qR) \quad (293)$$

Since the sum of the shears in the two legs equals zero,

$$-M_{AB}+q\,M_{BC}=0 \quad (294)$$

Combining equations 290 to 293 as follows

$$-\left[\frac{2+q}{3}\right]\left[2(290)-3(292)-4(293)\right]+\left[\frac{1+2q}{3}\right]\left[4(290)+3(291)-2(293)\right]$$

(the numerals in the parentheses are the equation numbers) gives

$$M_{AB}(2qn+2q+1)+M_{BC}(2s+2+q)=$$
$$-[C_{BA}(2+q)+C_{AB}(1+2q)] \quad (295)$$

Substituting M_{BC} from equation 295 in equation 294 gives

$$M_{AB}=-\frac{q}{2}\frac{C_{BA}(2+q)+C_{AB}(1+2q)}{q^2n+s+1+q+q^2} \quad (296)$$

Substituting M_{AB} from equation 296 in equation 294 gives

$$M_{BC}=-\frac{1}{2}\frac{C_{BA}(2+q)+C_{AB}(1+2q)}{q^2n+s+1+q+q^2} \quad (297)$$

If the load is symmetrical about the center of $AB, C_{BA}=C_{AB}=\frac{F}{l}$ and equations 296 and 297 take the form

$$M_{AB}=-\frac{3}{2}q\,\frac{F}{l}\,\frac{1+q}{q^2n+s+1+q+q^2} \quad (298)$$

$$M_{BC}=-\frac{3}{2}\frac{F}{l}\frac{1+q}{q^2n+s+1+q+q^2} \quad (299)$$

Values of C_{AB}, C_{BA}, and $\frac{F}{l}$ are given in Tables 2 and 3.

37. *Two-legged Rectangular Bent. One Leg Longer than the Other. Any System of Vertical Loads on Top of Bent—Legs Fixed at the Bases.*—Fig. 42 represents a two-legged bent having legs of unequal length.

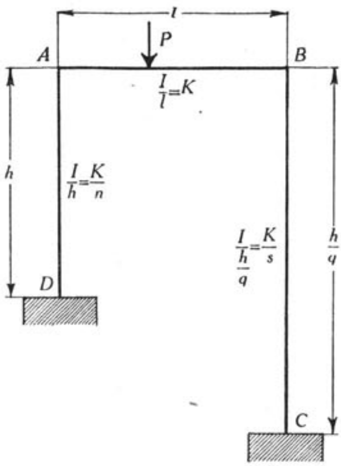


FIGURE 42

P represents the resultant of any system of vertical loads on AB. The legs are fixed at D and C.

Applying the equations of Table 1 gives

$$nM_{AB} = -2EK(2\theta_A - 3R) \qquad (300)$$

$$M_{AB} = 2EK(2\theta_A + \theta_B) - C_{AB} \qquad (301)$$

$$M_{BC} = -2EK(2\theta_B + \theta_A) - C_{BA} \qquad (302)$$

$$sM_{BC} = 2EK(2\theta_B - 3qR) \qquad (303)$$

$$sM_{CB} = 2EK(\theta_B - 3qR) \qquad (304)$$

$$nM_{DA} = 2EK(\theta_A - 3R) \qquad (305)$$

Since the sum of the shears in the two legs equals zero,

$$-M_{AB} + qM_{BC} + qM_{CB} + M_{DA} = 0 \qquad (306)$$

Combining equations 300 to 306 as indicated gives the equations of Table 10.

TABLE 10

EQUATIONS FOR THE MOMENTS IN THE RECTANGULAR BENT REPRESENTED BY FIG. 42

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation	How Equation Was Obtained
	M_{AB}	M_{BC}	M_{CB}	M_{DA}		
306	-1	q	q	1	0	(306)
307	$1+2q-n(2+q)$	$2(s+1)(2+q)$	$-s(1+2q)$	$-2n(1+2q)$	$-[2C_{BA}(2+q)+C_{AB}(1+2q)]$	$[2+q][2(302)-(300)+(2303)]$ $+[1+2q][(301)-2(305)$ $-(304)]$
308	1	$2+3s$	$-3s$	0	$-[2C_{BA}+C_{AB}]$	$(301)+3[(303)-(304)]$ $+2(302)$
309	$+2(1+n)(1+2q)$	$(2+q)-s(1+2q)$	$2s(2+q)$	$n(2+q)$	$-[C_{BA}(2+q)+2C_{AB}(1+2q)]$	$[2+q][(302)+2(304)+(305)]$ $+[1+2q][2(301)-(303)+$ $2(300)]$

Solving these equations simultaneously gives

$$M_{AB}=-\frac{1}{\Delta_o}\Big\{C_{BA}[2qns(3+2q)+s^2]+C_{AB}[qns(3+8q)+2s^2+2s+2q^2n]\Big\} \quad . \quad . \quad (310)$$

$$M_{BC}=-\frac{1}{\Delta_o}\Big\{C_{BA}[ns(8+3q)+2q^2n^2+2q^2n+2s]+C_{AB}[2ns(2+3q)+q^2n^2]\Big\} \quad . \quad . \quad (311)$$

$$M_{CB}=-\frac{1}{\Delta_o}\Big\{C_{BA}[ns(4+3q)-2q^2n^2-2q^2n+s]+C_{AB}[2ns(1+3q)-q^2n^2+3qn]\Big\} \quad . \quad . \quad (312)$$

$$M_{DA}=\frac{1}{\Delta_o}\Big\{C_{BA}[2qns(3+q)-s^2+3qs]+C_{AB}[qns(3+4q)-2s^2-2s+q^2n]\Big\} \quad . \quad . \quad (313)$$

in which $\Delta_o=2[ns(4+3q+4q^2)+q^2n(3ns+n+1)+s(3ns+s+1)]$

If the load is symmetrical about the center of $A\ B$, $C_{BA}=C_{AB}=\frac{F}{l}$, and equations 310 to 313 take the form

$$M_{AB}=-\frac{1}{\Delta_o}\frac{F}{l}\ [3qns(3+4q)+3s^2+2s+2q^2n] \quad . \quad . \quad (314)$$

$$M_{BC}=-\frac{1}{\Delta_o}\frac{F}{l}\ [3ns(4+3q)+3q^2n^2+2q^2n+2s] \quad . \quad . \quad (315)$$

$$M_{CB}=-\frac{1}{\Delta_o}\frac{F}{l}\ [3ns(2+3q)-3q^2n^2+qn(3-2q)+s] \quad . \quad . \quad (316)$$

$$M_{DA}=\frac{1}{\Delta_o}\frac{F}{l}\ [3qns(3+2q)-3s^2+s(3q-2)+q^2n] \quad . \quad . \quad (317)$$

If $n=s$ equations 310 to 313 take the form

$$M_{AB}=-\frac{n}{\Delta_o}\Big\{C_{BA}n(1+6q+4q^2)+C_{AB}[n(2+3q+8q^2)+2+2q^2]\Big\} \quad . \quad . \quad (318)$$

$$M_{BC}=-\frac{n}{\Delta_o}\left\{C_{BA}\,[n(8+3q+2q^2)+2+2q^2]\right.$$
$$\left.+C_{AB}n(4+6q+q^2)\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (319)$$

$$M_{CB}=-\frac{n}{\Delta_o}\left\{C_{BA}\,[n(4+3q-2q^2)+1-2q^2]\right.$$
$$\left.+C_{AB}[n(2+6q-q^2)+3q]\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (320)$$

$$M_{DA}=\frac{n}{\Delta_o}\left\{C_{BA}\,[n(-1+6q+2q^2)+3q]\right.$$
$$\left.+C_{AB}\,[n(-2+3q+4q^2)-2+q^2]\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (321)$$

in which $\Delta_o=2n[3n^2(1+q^2)+n(5+3q+5q^2)+(1+q^2)]$

If $n=s$ and $C_{BA}=C_{AB}=\frac{F}{l}$ equations 310 to 313 take the form

$$M_{AB}=-\frac{n}{\Delta_o}\frac{F}{l}\,[3n(1+3q+4q^2)+2+2q^2] \quad . \; . \; . \; . \; . \; . \quad (322)$$

$$M_{BC}=-\frac{n}{\Delta_o}\frac{F}{l}\,[3n(4+3q+q^2)+2+2q^2] \quad . \; . \; . \; . \; . \; . \quad (323)$$

$$M_{CB}=-\frac{n}{\Delta_o}\frac{F}{l}\,[3n(2+3q-q^2)+1+3q-2q^2] \quad . \; . \; . \; . \; . \quad (324)$$

$$M_{DA}=\frac{n}{\Delta_o}\frac{F}{l}\,[3n(-1+3q+2q^2)-2+3q+q^2] \quad . \; . \; . \; . \quad (325)$$

Values of C_{BA}, C_{AB}, and $\frac{F}{l}$ are given in Tables 2 and 3.

38. *Two-legged Rectangular Bent. One Leg Longer than the Other. External Moment at One Corner—Legs Hinged at the Bases.*— Fig. 43 represents a two-legged rectangular bent having one leg longer than the other. An external couple with moment M is applied at A. The legs are hinged at D and C.

Applying the equations of Table 1 gives

$$M_{AB}=-\frac{EK}{n}(3\theta_A-3R)+M \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (326)$$

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (327)$$

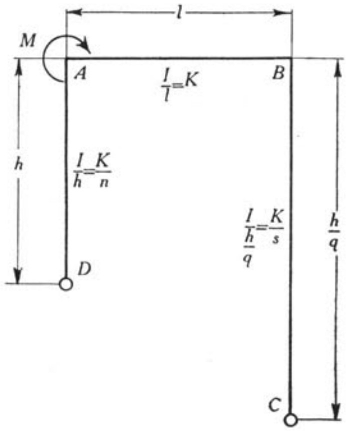


FIGURE 43

$$M_{BC}=-2EK(2\theta_B+\theta_A) \quad \text{(328)}$$

$$M_{BC}=\frac{EK}{s}(3\theta_B-3qR) \quad \text{(329)}$$

$$M_{AD}+qM_{BC}=0$$
$$M-M_{AB}+qM_{BC}=0$$
$$M_{AB}-qM_{BC}=M \quad \text{(330)}$$

Combining these equations and solving for the moments gives

$$M_{AB}=\frac{M}{2}\,\frac{2s+2+q+2q^2n}{q^2n+s+1+q+q^2} \quad \text{(331)}$$

$$M_{AD}=\frac{M}{2}\,\frac{q(1+2q)}{q^2n+s+1+q+q^2} \quad \text{(332)}$$

$$M_{BC}=-\frac{M}{2}\,\frac{1+2q}{q^2n+s+1+q+q^2} \quad \text{(333)}$$

39. *Two-legged Rectangular Bent. One Leg Longer than the Other. External Moment at One Corner—Legs Fixed at the Bases.*—Fig. 44 represents a two-legged rectangular bent having one leg longer than the other. An external couple with moment M is applied at A. The legs are fixed at C and D.

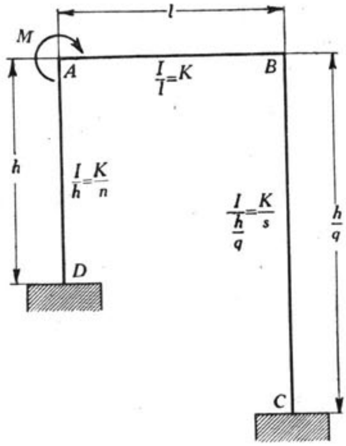


FIGURE 44

Applying the equations of Table 1 gives

$$M_{AB}=-\frac{2EK}{n}(2\theta_A-3R)+M \quad (334)$$

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad (335)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A) \quad (336)$$

$$M_{BC}=\frac{2EK}{s}(2\theta_B-3qR) \quad (337)$$

$$M_{CB}=\frac{2EK}{s}(\theta_B-3qR) \quad (338)$$

$$M_{DA}=\frac{2EK}{n}(\theta_A-3R) \quad (339)$$

For AD and BC to be in equilibrium

$$M_{DA}-M_{AB}+q\,M_{BC}+q\,M_{CB}=-M \quad (340)$$

Combining these equations and solving for the moments gives

$$M_{AB}=+\frac{Mn}{\Delta_o}(6s^2+8s+3qs+6q^2ns+2q^2n) \quad (341)$$

$$M_{BC}=-\frac{Mn}{\Delta_o}(4s+6qs+q^2n) \quad (342)$$

$$M_{CB}=-\frac{Mn}{\Delta_o}(2s+6qs+3q-q^2n) \quad (343)$$

$$M_{DA} = +\frac{M}{\Delta_o}(3qns - 2s^2 - 2s + 4q^2ns + q^2n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (344)$$

$$M_{AD} = +\frac{M}{\Delta_o}(3qns + 8q^2ns + 2s^2 + 2s + 2q^2n)$$

in which

$$\Delta_o = 2\left[ns(4 + 3q + 4q^2) + s^2 + s + q^2n(n+1) + 3ns(q^2n + s)\right]$$

40. *Two-legged Rectangular Bent. One Leg Longer than the Other. Settlement of Foundations—Legs Hinged at the Bases.*—Fig. 45 represents a two-legged rectangular bent having one leg longer than the other. The legs of the bent are hinged at D and C. The unstrained position of the bent is represented by the broken line $DA'B'C'$. Due

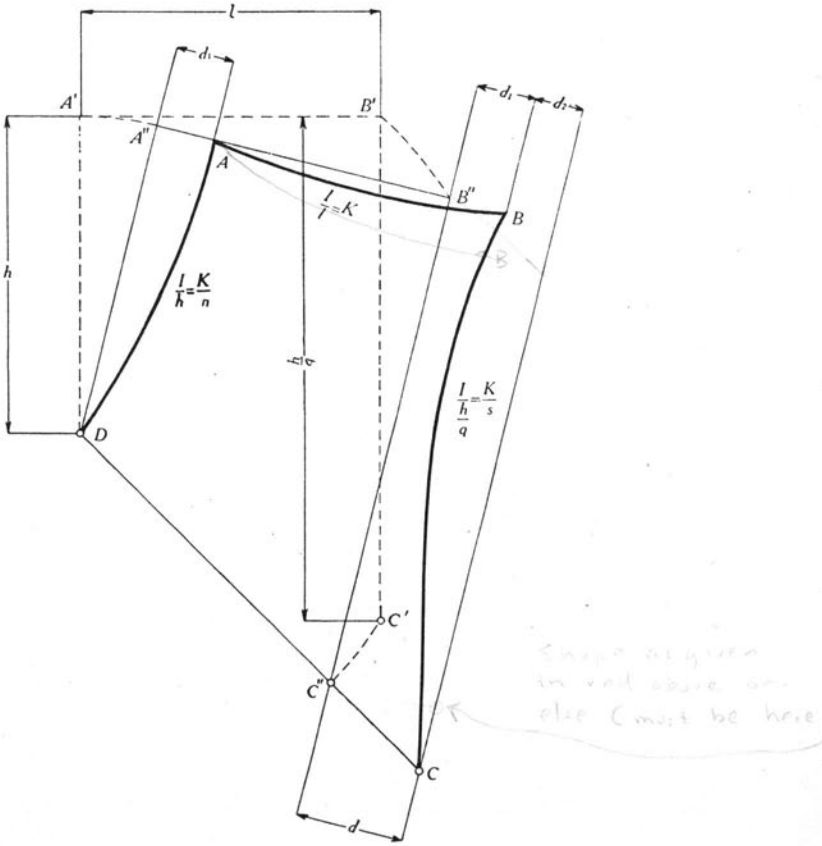


FIGURE 45

to settlement of the foundations the point C' has moved to C. The motion of C can be considered as made up of two parts, as follows: first, without being strained the bent rotates about D until C' is at C'', a point on DC; secondly, the bent is strained by applying a force at C'' which moves C'' to C. It is apparent that no matter what motion of C relative to D takes place, the stress in the bent depends only upon the change in the distance of C from D.

The change in the distance from D to C is proportional to d. d is made up of two parts, d_1 due to the deflection of AD, and d_2 due to the deflection of BC.

Applying the equations of Table 1 gives

$$M_{AB} = -\frac{EK}{n}(3\theta_A - 3R_1) \quad \ldots \quad (345)$$

$$M_{AB} = 2EK(2\theta_A + \theta_B) \quad \ldots \quad (346)$$

$$M_{BC} = -2EK(2\theta_B + \theta_A) \quad \ldots \quad (347)$$

$$M_{BC} = \frac{EK}{s}\left(3\theta_B - 3qR_1 + \frac{3qd}{h}\right) \quad \ldots \quad (348)$$

For the columns to be in equilibrium

$$M_{AB} - q\,M_{BC} = 0 \quad \ldots \quad (349)$$

Combining these equations and solving for the moments gives

$$M_{AB} = \frac{qd}{h}\,\frac{3qEK}{q^2n+s+1+q+q^2} \quad \ldots \quad (350)$$

$$M_{BC} = \frac{qd}{h}\,\frac{3EK}{q^2n+s+1+q+q^2} \quad \ldots \quad (351)$$

41. *Two-legged Rectangular Bent. One Leg Longer than the Other. Settlement of Foundations—Legs Fixed at the Bases.*—Fig. 46 represents a two-legged rectangular bent having one leg longer than the other. The legs of the bent are fixed at D and C. The unstrained position of the bent is represented by the broken line $DA'B'C'$. Due to settlement of the foundations the point C' has moved to C and the tangents to the elastic curves of the legs at their bases, originally vertical, have been rotated to the positions shown. This motion may be considered as made up of three parts, as follows: first, without being strained the

bent rotates about D until C' is at C'', a point on DC; secondly, the bent is strained by applying a force at C'' which moves C'' to C; thirdly, couples are applied at D and C, still further straining the bent so that the tangents to the elastic curves at the bases of the legs make angles of θ_D and θ_C respectively with lines normal to DC. It is apparent that,

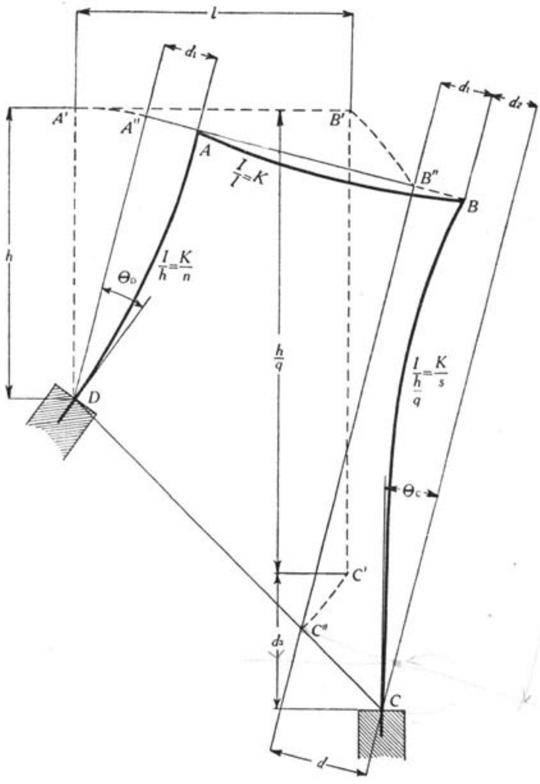


FIGURE 46

no matter what motion of C relative to D takes place, the stress in the bent depends only upon the change in the distance from C to D and upon the angles θ_D and θ_C.

The change in the distance from D to C is proportional to d. d is made up of two parts, d_1 due to the deflection of AD, and d_2 due to the deflection of BC.

Applying the equations of Table 1 gives

$$M_{AB}=-\frac{2EK}{n}\,(2\theta_A+\theta_D-3R_1) \qquad (352)$$

$$M_{AB}=2EK(2\theta_A+\theta_B) \qquad (353)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A) \qquad (354)$$

$$M_{BC}=\frac{2EK}{s}\,(2\theta_B+\theta_C-3R_2) \qquad (355)$$

$$M_{CB}=\frac{2EK}{s}\,(2\theta_C+\theta_B-3R_2) \qquad (356)$$

$$M_{DA}=\frac{2EK}{n}\,(2\theta_D+\theta_A-3R_1) \qquad (357)$$

For the legs of the bent to be in equilibrium

$$M_{DA}-M_{AB}+qM_{BC}+qM_{CB}=0 \qquad (358)$$

Combining these equations and solving for the moments gives

$$M_{AB}=\frac{2EK}{\Delta_o}\Big\{3\frac{qd}{h}\,[6qns+2qn-s]+\theta_C\,[9qns+2qn(3+q)-s] - \theta_D\,[6q^2ns+2q^2n-s(4+3q)-3s^2]\Big\} \qquad (359)$$

$$M_{BC}=\frac{2EK}{\Delta_o}\Big\{3\frac{qd}{h}\,[6ns-qn+2s]+\theta_C\,[6ns-qn\,(3+4q)+2s-3q^2n^2] - \theta_D\,[9qns-q^2n+2s(1+3q)]\Big\} \qquad (360)$$

$$M_{CB}=\frac{2EK}{\Delta_o}\Big\{3\frac{qd}{h}\,[6ns+n\,(4+q)+2s+1]+\theta_C\,[12ns+2n\,(6+3q+2q^2)+4s+3q^2n^2+3]-\theta_D\,[9qns+qn(6+q)+s(1+6q)+3q]\Big\} \qquad (361)$$

$$M_{DA}=-\frac{2EK}{\Delta_o}\Big\{3\frac{qd}{h}\,[6qns+2qn+s(1+4q)+q]+\theta_C\,[9qns+qn\,(6+q)+s(1+6q)+3q]-\theta_D\,[12q^2ns+4q^2n+2s\,(2+3q+6q^2)+3s^2+3q^2]\Big\} \qquad (362)$$

in which

$$\Delta_o = 2\ [ns(4+3q+4q^2)+q^2n(3ns+n+1)+s(3ns+s+1)]$$

It is to be noted that θ_C and θ_D are measured, not from the original direction of AD and BC, but from a line normal to AB.

If the tangents to the elastic curves of the legs at their bases remain vertical the moments can be expressed in terms of the vertical settlement of C relative to D. The θ's are then measured from the original direction of AD and BC and equal zero. The displacement d is measured in a horizontal direction. Proceeding as before the moments are found to be as follows:

$$M_{AB}=\frac{2EK}{\Delta_o}\left\{\frac{3\,qd}{h}\,[6qns+2qn-s]-\frac{3}{(4+5q)}\frac{d_3}{l}\,[q^2ns(13-4q)\right.$$
$$\left.+2qn(2+6q+q^2)+2s(9+q-q^2)+s^2(13-4q)]\right\} \quad . \quad . \quad . \quad (363)$$

$$M_{BC}=\frac{2EK}{\Delta_o}\left\{\frac{3\,qd}{h}\,[6ns-qn+2s]+\frac{3}{(4+5q)}\frac{d_3}{l}\,[ns\,(4+5q)\right.$$
$$\left.+\,2qn\,(1+5q+3q^2)+2s\,(5+6q-2q^2)+q^2n^2(4+5q)]\right\} . \quad . \quad (364)$$

$$M_{CB}=\frac{2EK}{\Delta_o}\left\{\frac{3\,qd}{h}\,[6ns+n(4+q)+2s+1]-\frac{3}{(4+5q)}\frac{d_3}{l}\,[ns\,(4+5q)\right.$$
$$+n\,(8+16q+15q^2+6q^3)-s(3+10q-4q^2)+q^2n^2\,(4+5q)$$
$$\left.-2(1-2q+q^2)]\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (365)$$

$$M_{DA}=-\frac{2EK}{\Delta_o}\left\{\frac{3\,qd}{h}\,[6qns+2qn+s(1+4q)+q]-\frac{3}{(4+5q)}\frac{d_3}{l}\,[q^2ns\right.$$
$$(13-4q)-qn(4+2q+3q^2)+s(18+15q+20q^2-8q^3)+s^2(13-4q)$$
$$\left.+2q(1-2q+q^2)]\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (366)$$

VIII. Two-legged Trapezoidal Bents. Bents and Loading Symmetrical about Vertical Center Line

42. *Two-legged Trapezoidal Bent. Bent and Load Symmetrical about Vertical Center Line. Vertical Load on Top—Legs Hinged at the Bases.*—Fig. 47 represents a two-legged trapezoidal bent. The

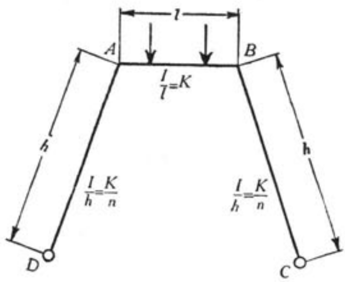


FIGURE 47

bent and the load are symmetrical about a vertical center line. The legs are hinged at C and D.

From symmetry $\theta_A = -\theta_B$. Since the deformation due to shearing and axial stresses may be neglected, the points A and B do not move, and R is zero for all members. By applying the equations of Table 1 five equations are obtained which are identical with equations 128 to 132 of section 26; hence it is seen that the moments in the members of this frame are independent of the angle of inclination of the legs, and this is true of any trapezoidal frame in which loading and frame are symmetrical about a vertical center line. The direct stress does vary with the angle of inclination, and may be found from the equations of statics when the moments are known.

The moments as found in section 26 when applied to this case in which bent and loading are symmetrical about a vertical center line are

$$M_{AD} = -M_{BC} = \left(\frac{3}{3+2n}\right)\frac{F}{l} \qquad (367)$$

If $n=1$

$$M_{AD} = -M_{BC} = \frac{3}{5}\frac{F}{l} \qquad (368)$$

Values of $\frac{F}{l}$ are given in Table 3.

43. *Two-legged Trapezoidal Bent. Bent and Load Symmetrical about Vertical Center Line. Vertical Load on Top—Legs Fixed at the Bases.*—Fig. 48 represents a two-legged trapezoidal bent. The bent

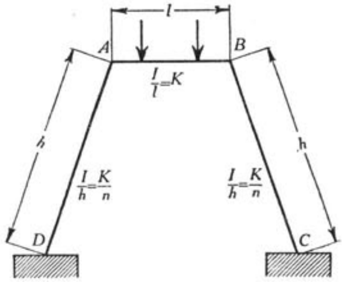


FIGURE 48

and loads are symmetrical about a vertical center line. The legs are fixed at C and D. From the preceding paragraph it is seen that equations 158 and 159 of section 27 apply to this case.

$$M_{AD}=-M_{BC}=\left(\frac{2}{n+2}\right)\frac{F}{l} \qquad (369)$$

$$M_{DA}=-M_{CB}=\left(\frac{1}{n+2}\right)\frac{F}{l} \qquad (370)$$

If $n=1$

$$M_{AD}=-M_{BC}=\frac{2}{3}\frac{F}{l} \qquad (371)$$

$$M_{DA}=-M_{CB}=\frac{1}{3}\frac{F}{l} \qquad (372)$$

Values of $\frac{F}{l}$ for different loads are given in Table 3.

44. *Two-legged Trapezoidal Bent. Bent and Loads Symmetrical about Vertical Center Line. Loads Normal to Legs of Bent—Legs Hinged at the Bases.*—Fig. 49 represents a trapezoidal bent having loads normal

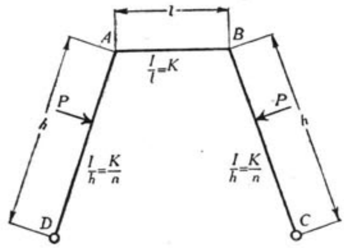


FIGURE 49

to the sides AD and BC. The bent and the loads are symmetrical about a vertical center line. P represents the resultant of the loads on AD, and likewise on BC. The legs are hinged at D and C. As in sections 42 and 43 the moments are independent of the angle of inclination of the legs. Hence equation 107 of section 24 applies.

$$M_{AD}=-M_{BC}=\left(\frac{2n}{2n+3}\right)H_{AD} \qquad (373)$$

If $n=1$

$$M_{AD}=\frac{2}{5}H_{AD} \qquad (374)$$

Values of H_{AD} are given in Table 2.

45. *Two-legged Trapezoidal Bent. Bent and Loads Symmetrical about Vertical Center Line. Loads Normal to Legs of Bent—Legs Fixed at the Bases.*—Fig. 50 represents a trapezoidal bent with loads and members similar to those of Fig. 49, except that the legs are fixed at D and C. Equations 126 and 127 of section 25 apply here.

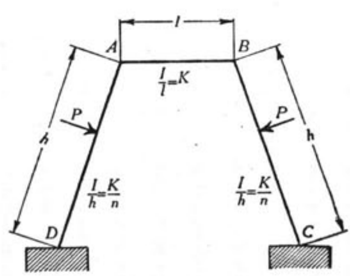


FIGURE 50

$$M_{AD} = -M_{BC} = \left(\frac{n}{n+2}\right) C_{AD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (375)$$

$$M_{DA} = -M_{CB} = -\left(\frac{C_{AD}}{n+2} + C_{DA}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (376)$$

If $n=1$

$$M_{AD} = -M_{BC} = \frac{1}{3} C_{AD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (377)$$

$$M_{DA} = -M_{CB} = -\left(\frac{1}{3} C_{AD} + C_{DA}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (378)$$

Values of C are given in Table 2.

46. *Two-legged Trapezoidal Bent. Bent and Loads Symmetrical about Vertical Center Line. External Moments at Upper Corners of Bent—Legs Hinged at the Bases.*—Fig. 51 represents a two-legged trapezoidal bent having couples acting at A and B. The bent and the couples are symmetrical about a vertical center line. The legs are hinged at C and D.

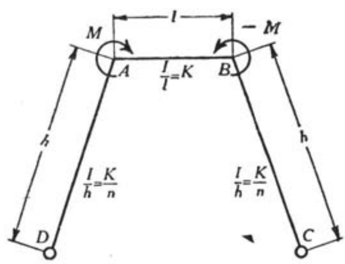


FIGURE 51

As in sections 42 and 43 because of the symmetry of loads and bent, the moments are independent of the angle of inclination of the legs. Equations 163 and 164 of section 28 are modified to apply here by the algebraic addition of moments due to the two couples.

$$M_{AD} = \left(\frac{3M}{2n+3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (379)$$

$$M_{AB} = \left(\frac{2nM}{2n+3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (380)$$

If $n=1$

$$M_{AD}=\frac{3}{5}M \qquad (381)$$

$$M_{AB}=\frac{2}{5}M \qquad (382)$$

47. *Two-legged Trapezoidal Bent. Bent and Loads Symmetrical about Vertical Center Line. External Moments at Upper Corners of Bent—Legs Fixed at the Bases.*—Fig. 52 represents a two-legged trapezoidal bent having couples acting at A and B. The bent and the couples are symmetrical about a vertical center line. The legs are fixed at C and D.

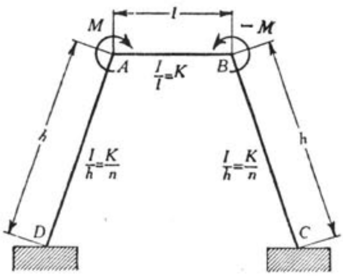


FIGURE 52

Since from symmetry of bent and loading, the moments are independent of the angle of inclination of the legs, equations 175 to 179 of section 29 apply here.

$$M_{AD}=-M_{BC}=\frac{2M}{n+2} \qquad (383)$$

$$M_{AB}=-M_{BA}=\frac{nM}{n+2} \qquad (384)$$

$$M_{DA}=-M_{CB}=\frac{M}{n+2} \qquad (385)$$

If $n=1$

$$M_{AD}=-M_{BC}=\frac{2}{3}M \qquad (386)$$

$$M_{AB}=-M_{BA}=\frac{1}{3}M \qquad (387)$$

$$M_{DA}=-M_{CB}=\frac{1}{3}M \qquad (388)$$

IX. Rectangular Frames

48. *Rectangular Frame. Horizontal Force at the Top.*—Fig. 53 represents a rectangular frame having a horizontal force P applied at

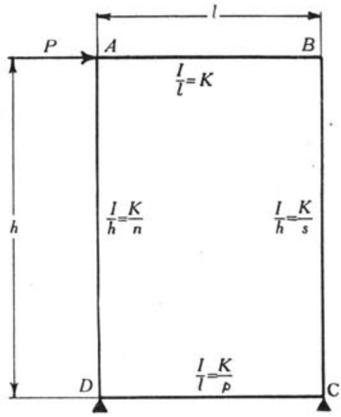


Figure 53

the top. The value of R for members AB and DC is zero, and R for AD equals R for BC.

Applying equation (A), Table 1, gives

$$nM_{DA}=2EK\ (2\theta_D+\theta_A-3R) \quad (389)$$

$$nM_{AB}=-2EK\ (2\theta_A+\theta_D-3R) \quad (390)$$

$$M_{AB}=2EK(2\theta_A+\theta_B) \quad (391)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A) \quad (392)$$

$$sM_{BC}=2EK\ (2\theta_B+\theta_C-3R) \quad (393)$$

$$sM_{CD}=-2EK\ (2\theta_C+\theta_B-3R) \quad (394)$$

$$pM_{CD}=2EK\ (2\theta_C+\theta_D) \quad (395)$$

$$pM_{DA}=-2EK(2\theta_D+\theta_C) \quad (396)$$

Considering AB and DC removed and equating the sum of the moments acting at the tops and bottoms of AD and BC to zero gives

$$M_{AD}+M_{DA}+M_{BC}+M_{CB}+Ph=0$$

Substituting $-M_{AB}$ for M_{AD} and $-M_{CD}$ for M_{CB} gives

$$-M_{AB}+M_{BC}-M_{CD}+M_{DA}=-Ph \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (397)$$

Combining equations 389 to 397 as indicated in Table 11 gives equations 397 to 400 of Table 11.

Solving these equations simultaneously and letting Δ represent the common denominator gives

$$\Delta=22(spn+sp+sn+np)+2(sp^2+s^2p+n^2p+p^2n+s^2+s+n^2+n)$$
$$+6(sn^2+s^2n+p^2+p) \text{ and}$$

$$M_{AB}=\frac{Ph}{\Delta}\left(3s^2n+5nps+2s^2p+2sp^2+6ns+6pn+5ps+3p^2 \right) \quad (401)$$

$$M_{BC}=-\frac{Ph}{\Delta}\left(3n^2s+5nps+2n^2p+2np^2+6ns+6ps+5pn+3p^2 \right)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (402)$$

$$M_{CD}=\frac{Ph}{\Delta}\left(3n^2s+6nps+5ns+6ps+5pn+2n+3p+2n^2\right) \quad . \quad (403)$$

$$M_{DA}=-\frac{Ph}{\Delta}\left(3ns^2+6nps+5ns+6pn+5ps+2s+3p+2s^2 \right) \quad (404)$$

If the frame is symmetrical about the vertical center line, that is, if AB and BC, Fig. 53, have the same section, $n=s$, and equations 401 to 404 take the form

$$M_{AB}=-M_{BC}=\frac{Ph}{2}\,\frac{3n+p}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (405)$$

$$M_{CD}=-M_{DA}=\frac{Ph}{2}\,\frac{3n+1}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (406)$$

in which $\beta=6n+p+1$.

TABLE 11

EQUATIONS FOR RECTANGULAR FRAME WITH HORIZONTAL FORCE AT THE TOP

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation	Method of Obtaining Equation
	M_{AB}	M_{BC}	M_{CD}	M_{DA}		
397	-1	$+1$	-1	$+1$	$-Ph$	Equation (397)
398	n	s	$2s+3p$	$2n+3p$	0	(390) + (393) + 2[(389) + (394)] + 3[(395) + (396)]
399	$-n$	$2s+1$	$2s+p$	$-n$	0	(392) + (395) + 2[(393) + (394)] − (389) − (390)
400	$n+1$	$s+1$	$s+p$	$n+p$	0	(389) + (390) + (391) + (392) + (393) + (394) + (395) + (396)

If the frame is symmetrical about the vertical center line and if the top and the bottom of the frame are alike, that is, if $n=s$ and $p=1$,

$$M_{AB}=-M_{BC}=\frac{Ph}{4} \qquad (407)$$

$$M_{CD}=-M_{DA}=\frac{Ph}{4} \qquad (408)$$

49. *Rectangular Frame. Any System of Horizontal Forces on One Vertical Side.*—Fig. 54 represents a rectangular frame subjected

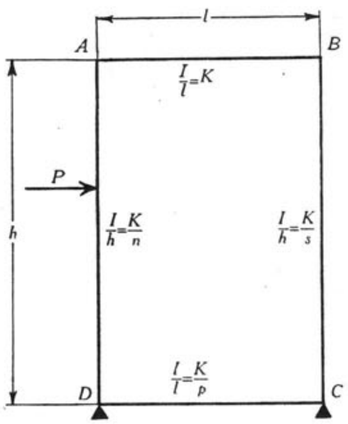


FIGURE 54

to any system of horizontal forces on the side AD.

Let M_D represent the moment of the external forces about D.

Applying the equations of Table 1 gives

$$nM_{DA}=2EK(2\theta_D+\theta_A-3R)-nC_{DA} \qquad (409)$$

$$nM_{AB}=-2EK(2\theta_A+\theta_D-3R)-nC_{AD} \qquad (410)$$

$$-M_{AB}+M_{BC}-M_{CD}+M_{DA}=-M_D \qquad (411)$$

Six other equations which are identical with equations 391 to 396 of section 48 may be written. The values of θ and R in these nine equations are identical with those in equations 389 to 397 of section 48, and hence the combination of equations to eliminate these two quantities is made in the manner indicated in the last column of Table 11. The equations thus obtained are given in Table 12.

TABLE 12

EQUATIONS FOR THE MOMENTS IN THE RECTANGULAR FRAME REPRESENTED BY FIG. 54

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation
	M_{AB}	M_{BC}	M_{CD}	M_{DA}	
411	-1	1	-1	1	$-M_D$
412	n	s	$2s+3p$	$2n+3p$	$-n(C_{AD}+2C_{DA})$
413	$-n$	$2s+1$	$2s+p$	$-n$	$n(C_{AD}+C_{DA})$
414	$n+1$	$s+1$	$s+p$	$n+p$	$-n(C_{AD}+C_{DA})$

Solving these equations simultaneously gives

$$M_{AB}=\frac{1}{\Delta}\Big\{M_D(3s^2n+5nps+2s^2p+2sp^2+6ns+6pn+5ps+3p^2)$$
$$-C_{AD}n(6sn+2pn+3s^2+17ps+2n+5s+11p+2p^2)$$
$$-C_{DA}n(3s^2+12ps+6s+10p+p^2)\Big\} \quad . \quad . \quad . \quad . \quad . \quad (415)$$

$$M_{BC}=\frac{1}{\Delta}\Big\{-M_D(3sn^2+2n^2p+5nps+2np^2+6ns+5pn+6ps+3p^2)$$
$$-C_{AD}n(3ns+2pn+5ps-n-7p-4s+2p^2)$$
$$+C_{DA}n(3ns+3pn-3ps+8p+6s-p^2)\Big\} \quad . \quad . \quad . \quad . \quad (416)$$

$$M_{CD}=\frac{1}{\Delta}\Big\{M_D(3sn^2+2n^2+6nps+5pn+5sn+6ps+2n+3p)$$
$$+C_{AD}n(3ns+6ps+3n+8p-3s-1)$$
$$-C_{DA}n(3ns-4ps+2n-7p-pn+5s+2)\Big\} \quad . \quad . \quad . \quad (417)$$

$$M_{DA}=\frac{1}{\Delta}\Big\{-M_D(5ns+6nps+3ns^2+2s^2+5ps+6pn+2s+3p)$$
$$-C_{AD}n(3s^2+6ps+12s+10p+1)$$
$$-C_{DA}n(3s^2+5ps+17s+11p+6ns+2pn+2n+2)\Big\} \quad . \quad (418)$$

in which

$$\Delta=22(spn+sp+sn+np)+2(sp^2+s^2p+n^2p+p^2n+s^2+s+n^2+n)$$
$$+6(sn^2+s^2n+p^2+p)$$

If $n=s$, equations 415 to 418 take the form

$$M_{AB}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{n+2p}{\alpha}+\frac{3}{\beta}\right]+C_{DA}n\left[\frac{p}{\alpha}+\frac{3}{\beta}\right]-M_D\frac{3n+p}{\beta}\right\} \quad (419)$$

$$M_{BC}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{n+2p}{\alpha}-\frac{3}{\beta}\right]+C_{DA}n\left[\frac{p}{\alpha}-\frac{3}{\beta}\right]+M_D\frac{3n+p}{\beta}\right\} \quad (420)$$

$$M_{CD}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{1}{\alpha}-\frac{3}{\beta}\right]+C_{DA}n\left[\frac{n+2}{\alpha}-\frac{3}{\beta}\right]-M_D\frac{3n+1}{\beta}\right\} \quad (421)$$

$$M_{DA}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{1}{\alpha}+\frac{3}{\beta}\right]+C_{DA}n\left[\frac{n+2}{\alpha}+\frac{3}{\beta}\right]+M_D\frac{3n+1}{\beta}\right\} \quad (422)$$

in which

$\alpha=n^2+2pn+2n+3p$
$\beta=6n+p+1$

If $n=s$ and $p=1$ equations 419 to 422 take the form

$$M_{AB}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{n+2}{\alpha}+\frac{3}{\beta}\right]+C_{DA}n\left[\frac{1}{\alpha}+\frac{3}{\beta}\right]-\frac{M_D}{2}\right\} \quad (423)$$

$$M_{BC}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{n+2}{\alpha}-\frac{3}{\beta}\right]+C_{DA}n\left[\frac{1}{\alpha}-\frac{3}{\beta}\right]+\frac{M_D}{2}\right\} \quad (424)$$

$$M_{CD}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{1}{\alpha}-\frac{3}{\beta}\right]+C_{DA}n\left[\frac{n+2}{\alpha}-\frac{3}{\beta}\right]-\frac{M_D}{2}\right\} \quad (425)$$

$$M_{DA}=-\frac{1}{2}\left\{C_{AD}n\left[\frac{1}{\alpha}+\frac{3}{\beta}\right]+C_{DA}n\left[\frac{n+2}{\alpha}+\frac{3}{\beta}\right]+\frac{M_D}{2}\right\} \quad (426)$$

in which

$\alpha=(n+3)\ (n+1)$
$\beta=2(3n+1)$

If $n=s=p=1$, equations 423 to 426 take the form

$$M_{AB}=-\frac{1}{8}\left\{3C_{AD}+2C_{DA}-2M_D\right\} \quad . \quad . \quad . \quad (427)$$

$$M_{BC}=-\frac{1}{8}\left\{-C_{DA}+2M_D\right\} \quad . \quad . \quad . \quad (428)$$

$$M_{CD}=\frac{1}{8}\left\{C_{AD}+2M_D\right\} \quad . \quad . \quad . \quad (429)$$

$$M_{DA}=-\frac{1}{8}\left\{2C_{AD}+3C_{DA}+2M_D\right\} \quad . \quad . \quad . \quad (430)$$

If both vertical sides are loaded and if the frame and the loads are symmetrical about a vertical center line

$$M_{AB}=M_{BC}=-\frac{n}{\alpha}\left[C_{AD}(n+2p)+C_{DA}p\right] \quad . \quad . \quad . \quad (431)$$

$$M_{CD}=M_{DA}=-\frac{n}{\alpha}\left[C_{AD}+C_{DA}(n+2)\right] \quad . \quad . \quad . \quad (432)$$

50. *Rectangular Frame. Any System of Vertical Forces on the Top.*—Fig. 55 represents a rectangular frame subjected to any system of vertical forces on the top member AB.

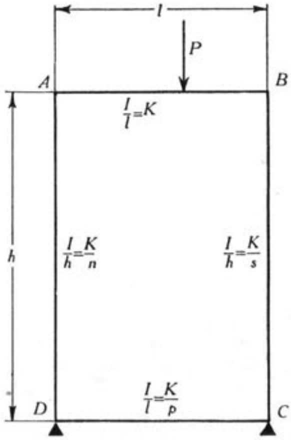


FIGURE 55

Applying the equations of Table 1 gives

$$M_{AB}=2EK(2\theta_A+\theta_B)-C_{AB} \quad \text{(433)}$$

$$M_{BC}=-2EK(2\theta_B+\theta_A)-C_{BA} \quad \text{(434)}$$

$$-M_{AB}+M_{DA}+M_{BC}-M_{CD}=0 \quad \text{(435)}$$

Six other equations which are identical with equations 389, 390, and 393 to 396 of section 48 may be written. Combining these nine equations to eliminate θ and R, as indicated in the last column of Table 11, gives the equations of Table 13.

TABLE 13

EQUATIONS FOR THE MOMENTS IN THE RECTANGULAR FRAME REPRESENTED BY FIG. 55

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation
	M_{AB}	M_{BC}	M_{CD}	M_{DA}	
435	-1	1	-1	1	0
436	n	s	$2s+3p$	$2n+3p$	0
437	$-n$	$2s+1$	$2s+p$	$-n$	$-C_{BA}$
438	$n+1$	$s+1$	$s+p$	$n+p$	$-(C_{BA}+C_{AB})$

Solving these equations simultaneously gives

$$M_{AB}=-\frac{1}{\Delta}\left\{C_{BA}(10ns+s^2+12ps+6pn+3p^2)\right.$$
$$\left.+C_{AB}(11ns+2s^2+2s+2n+17ps+5pn+3p^2+6p)\right\} \quad \text{(439)}$$

$$M_{BC}=-\frac{1}{\Delta}\left\{C_{BA}(11ns+2n^2+2n+2s+17pn+5ps+3p^2+6p)\right.$$
$$\left.+C_{AB}(10ns+n^2+12pn+6ps+3p^2)\right\} \quad \text{(440)}$$

$$M_{CD}=\frac{1}{\Delta}\left\{C_{BA}(7ns-2n^2-2n+s-5pn+4ps-3p)\right.$$
$$\left.+C_{AB}(8ns-n^2+3n-3pn+6ps+3p)\right\} \quad \text{(441)}$$

$$M_{DA}=\frac{1}{\Delta}\left\{C_{BA}(8ns-s^2+3s-3ps+6pn+3p)\right.$$
$$\left.+C_{AB}(7ns-2s^2-2s+n-5ps+4pn-3p)\right\} \quad . \quad . \quad . \quad . \quad (442)$$

in which

$$\Delta=22(spn+sp+sn+np)+2(sp^2+s^2p+n^2p+np^2+s^2+s+n^2+n)$$
$$+6(sn^2+s^2n+p^2+p)$$

If the load is symmetrical about the center of AB, that is, if $C_{AB}=C_{BA}$, equations 439 to 442 take the form

$$M_{AB}=-\frac{C_{AB}}{\Delta}\left[21ns+3s^2+2s+2n+29ps+11pn+6p+6p^2\right] \quad (443)$$

$$M_{BC}=-\frac{C_{AB}}{\Delta}\left[\ 21ns+3n^2+2n+2s+29pn+11ps+6p+6p^2\ \right] (444)$$

$$M_{CD}=\frac{C_{AB}}{\Delta}\left[\ 15ns-3n^2+n+s-8pn+10ps\ \right] . \quad . \quad . \quad . \quad . \quad (445)$$

$$M_{DA}=\frac{C_{AB}}{\Delta}\left[\ 15ns-3s^2+s+n-8ps+10pn\ \right] \quad . \quad . \quad . \quad . \quad (446)$$

If $n=s$ equations 439 to 442 take the form

$$M_{AB}=-\frac{1}{2}\left\{C_{BA}\left[\frac{2n+3p}{\alpha}-\frac{1}{\beta}\right]+C_{DA}\left[\frac{2n+3p}{\alpha}+\frac{1}{\beta}\right]\right\} \quad . \quad (447)$$

$$M_{BC}=-\frac{1}{2}\left\{C_{BA}\left[\frac{2n+3p}{\alpha}+\frac{1}{\beta}\right]+C_{DA}\left[\frac{2n+3p}{\alpha}-\frac{1}{\beta}\right]\right\} \quad . \quad (448)$$

$$M_{CD}=\frac{1}{2}\left\{C_{BA}\left[\frac{n}{\alpha}-\frac{1}{\beta}\right]+C_{DA}\left[\frac{n}{\alpha}+\frac{1}{\beta}\right]\right\} \quad . \quad . \quad . \quad . \quad (449)$$

$$M_{DA}=\frac{1}{2}\left\{C_{BA}\left[\frac{n}{\alpha}+\frac{1}{\beta}\right]+C_{DA}\left[\frac{n}{\alpha}-\frac{1}{\beta}\right]\right\} \quad . \quad . \quad . \quad . \quad (450)$$

in which $\alpha=n^2+2pn+2n+3p$

$\beta=6n+p+1$

If $n=s$ and $C_{AB}=C_{BA}$

$$M_{AB}=M_{BC}=-C_{AB}\frac{2n+3p}{\alpha} \quad \ldots \ldots \quad (451)$$

$$M_{CD}=M_{DA}=C_{AB}\frac{n}{\alpha} \quad \ldots \ldots \quad (452)$$

If $n=s$ and $p=1$

$$M_{AB}=-\frac{1}{2}\left\{C_{BA}\left[\frac{2n+3}{\alpha}-\frac{1}{\beta}\right]+C_{AB}\left[\frac{2n+3}{\alpha}+\frac{1}{\beta}\right]\right\}. \quad (453)$$

$$M_{BC}=-\frac{1}{2}\left\{C_{BA}\left[\frac{2n+3}{\alpha}+\frac{1}{\beta}\right]+C_{AB}\left[\frac{2n+3}{\alpha}-\frac{1}{\beta}\right]\right. \quad . \quad (454)$$

$$M_{CD}=\frac{1}{2}\left\{C_{BA}\left[\frac{n}{\alpha}-\frac{1}{\beta}\right]+C_{AB}\left[\frac{n}{\alpha}+\frac{1}{\beta}\right]\right\} \quad \ldots \quad (455)$$

$$M_{DA}=\frac{1}{2}\left\{C_{BA}\left[\frac{n}{\alpha}+\frac{1}{\beta}\right]+C_{AB}\left[\frac{n}{\alpha}-\frac{1}{\beta}\right]\right\} \quad \ldots \quad (456)$$

in which

$$\alpha=(n+1)\ (n+3)$$
$$\beta=2(3n+1)$$

If $n=s$, $p=1$, and $C_{AB}=C_{BA}$

$$M_{AB}=M_{BC}=-C_{AB}\frac{2n+3}{(n+1)\ (n+3)} \quad \ldots \ldots \quad (457)$$

$$M_{CD}=M_{DA}=C_{AB}\frac{n}{(n+1)\ (n+3)} \quad \ldots \ldots \quad (458)$$

If $n=s=p=1$

$$M_{AB}=-\frac{1}{8}\left[2C_{BA}+3C_{AB}\right] \quad \ldots \ldots \quad (459)$$

$$M_{BC}=-\frac{1}{8}\left[3C_{BA}+2C_{AB}\right] \quad \ldots \ldots \quad (460)$$

$$M_{CD}=\frac{1}{8}\,C_{AB} \quad \ldots \quad (461)$$

$$M_{DA}=\frac{1}{8}\,C_{BA} \quad \ldots \quad (462)$$

If $n=s=p=1$, and $C_{AB}=C_{BA}$

$$M_{AB}=M_{BC}=-\frac{5}{8}\,C_{AB} \quad \ldots \quad (463)$$

$$M_{CD}=M_{DA}=\frac{1}{8}\,C_{AB} \quad \ldots \quad (464)$$

Values of C_{AB} and C_{BA} to be used in equations 439 to 464 are given in Table 2.

51. *Rectangular Frame. External Moment at Upper Corner.*—Fig. 56 represents a rectangular frame subjected to an external moment M at the upper left-hand corner.

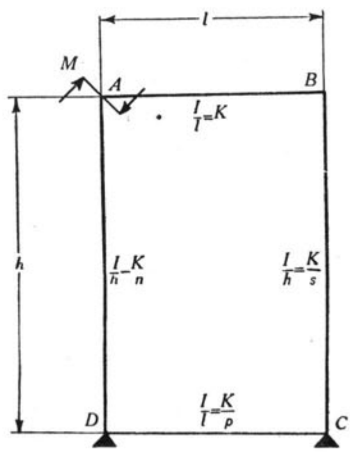


FIGURE 56

For equilibrium at A

$M-M_{AB}-M_{AD}=0$ or

$M_{AB}=M-M_{AD}$

Likewise

$$M_{BC} = -M_{BA}$$

$$M_{CD} = -M_{CB}$$

$$M_{DA} = -M_{DC}$$

$$nM_{AB} = -2EK(2\theta_A + \theta_D - 3R) + Mn \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (465)$$

$$-M_{AB} + M_{DA} + M_{BC} - M_{CD} = -M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (466)$$

Equations 389 and 391 to 396 of section 48 apply to this case. Eliminating values of θ and R, as indicated in Table 11, gives the equations of Table 14.

TABLE 14
EQUATIONS FOR THE MOMENTS IN THE RECTANGULAR FRAME REPRESENTED BY FIG. 56

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation
	M_{AB}	M_{BC}	M_{CD}	M_{DA}	
466	-1	1	-1	1	$-M$
467	n	s	$2s+3p$	$2n+3p$	$+Mn$
468	$-n$	$2s+1$	$2s+p$	$-n$	$-Mn$
469	$n+1$	$s+1$	$s+p$	$n+p$	$+Mn$

Solving these equations simultaneously gives

$$M_{AB} = -\frac{M}{\Delta}\left\{11ns + 2s^2 + 2s + 2n + 17ps + 5pn + 6p + 3p^2\right\} + M \quad (470)$$

$$M_{BC} = -\frac{M}{\Delta}\left\{n^2 + 10ns + 12pn + 6ps + 3p^2\right\} \quad . \quad . \quad . \quad . \quad . \quad (471)$$

$$M_{CD} = +\frac{M}{\Delta}\left\{3p - n^2 - 3pn + 8ns + 6ps + 3n\right\} \quad . \quad . \quad . \quad . \quad (472)$$

$$M_{DA} = -\frac{M}{\Delta}\left\{-7ns + 2s^2 + 5ps - 4pn + 2s + 3p - n\right\} \quad . \quad . \quad (473)$$

$$M_{AD} = +\frac{M}{\Delta}\left\{11ns + 2s^2 + 2s + 2n + 17ps + 5pn + 6p + 3p^2\right\} \quad . \quad (474)$$

in which

$$\Delta = 22(pns+sp+sn+np)+2(sp^2+s^2p+n^2p+p^2n+s^2+s+n^2+n)$$
$$+6(sn^2+s^2n+p^2+p)$$

If $n=s$, equations 470 to 474 take the form

$$M_{AB} = \frac{M}{2}\left\{1+\frac{n(n+2p)}{\alpha}-\frac{1}{\beta}\right\} \quad \ldots \ldots \quad (475)$$

$$M_{BC} = \frac{M}{2}\left\{\frac{n(n+2p)}{\alpha}-\frac{6n+p}{\beta}\right\} \quad \ldots \ldots \quad (476)$$

$$M_{CD} = \frac{M}{2}\left\{\frac{n}{\alpha}+\frac{1}{\beta}\right\} \quad \ldots \ldots \quad (477)$$

$$M_{DA} = \frac{M}{2}\left\{\frac{n}{\alpha}-\frac{1}{\beta}\right\} \quad \ldots \ldots \quad (478)$$

$$M_{AD} = \frac{M}{2}\left\{1-\frac{n(n+2p)}{\alpha}+\frac{1}{\beta}\right\} \quad \ldots \ldots \quad (479)$$

in which

$$\alpha = n^2+2pn+2n+3p$$
$$\beta = 6n+p+1$$

If $n=s$ and $p=1$, equations 475 to 479 take the form

$$M_{AB} = +\frac{M}{2}\left\{1+\frac{n(n+2)}{(n+1)\,(n+3)}-\frac{1}{6n+2}\right\} \quad \ldots \ldots \quad (480)$$

$$M_{AD} = +\frac{M}{2}\left\{1-\frac{n(n+2)}{(n+1)(n+3)}+\frac{1}{6n+2}\right\} \quad \ldots \ldots \quad (481)$$

$$M_{BC} = +\frac{M}{2}\left\{\frac{n(n+2)}{(n+1)(n+3)}-\frac{6n+1}{6n+2}\right\} \quad \ldots \ldots \quad (482)$$

$$M_{CD} = +\frac{M}{2}\left\{\frac{n}{(n+1)(n+3)}+\frac{1}{6n+2}\right\} \quad \ldots \ldots \quad (483)$$

$$M_{DA} = +\frac{M}{2}\left\{\frac{n}{(n+1)(n+3)} - \frac{1}{6n+2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (484)$$

If $n=s=p=1$, equations 480 to 484 take the form

$$M_{AB} = +\frac{5}{8}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (485)$$

$$M_{AD} = +\frac{3}{8}M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (486)$$

$$M_{BC} = -\frac{M}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (487)$$

$$M_{CD} = +\frac{M}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (488)$$

$$M_{DA} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (489)$$

If there is a couple at B as well as at A and if the frame and loading are symmetrical about a vertical center line, that is, if the couples are equal in magnitude and opposite in sense, and if $n=s$

$$M_{AB} = -M_{BA} = \frac{Mn}{\alpha}(n+2p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (490)$$

$$M_{AD} = -M_{BC} = \frac{M}{\alpha}(2n+3p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (491)$$

$$M_{CD} = M_{DA} = \frac{Mn}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (492)$$

X. Moments in Frames Composed of a Large Number of Rectangles as a Skeleton-Construction Building Frame

52. *Effect of Restraint at One End of a Member upon Moment at Other End.*—Fig. 57 represents any member in flexure. The end A is

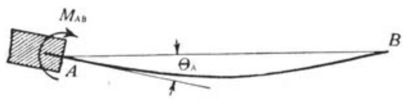


FIGURE 57

acted upon by a couple M_{AB} such that the tangent to the elastic curve at A makes an angle θ_A with AB. The magnitude of the moment M_{AB} depends not only upon the magnitude of θ_A, the moment of inertia of the section and the length of AB, but also upon the degree of restraint at B. This is illustrated by the following special problems.

Consider that AB is hinged at B. Applying equation (C) of Table 1 with R and H_{AB} equal to zero gives

$$M_{AB}=3EK\ \theta_A \quad (493)$$

Consider that AB is fixed at B. Applying equation (A) of Table 1 with θ_B, R, and C_{AB} equal to zero gives

$$M_{AB}=4EK\ \theta_A \quad (494)$$

Consider that $\theta_A=-\theta_B$. Applying equation (A) of Table 1 with R and C_{AB} equal to zero and with $\theta_A=-\theta_B$ gives

$$M_{AB}=2EK\ \theta_A \quad (495)$$

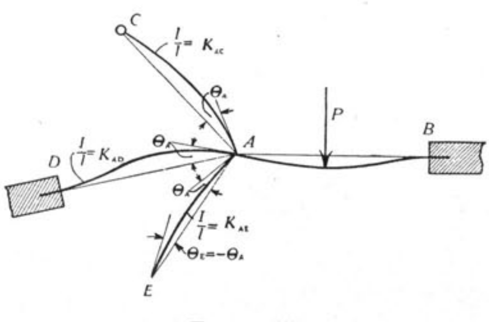


FIGURE 58

Fig. 58 represents a member AB having any degree of restraint at B and restrained at A by the members AC, AD, and AE. P represents any system of loads. AC is hinged at C, AD is fixed at D, and the restraint at E is such that $\theta_E = -\theta_A$. The moment at A in the member AB is taken as a measure of the restraint at A. Since A is in equilibrium, $M_{AB}+M_{AC}+M_{AD}+M_{AE}=0$. That is, the moment M_{AB} balances the three moments M_{AC}, M_{AD}, and M_{AE}. The moments M_{AC}, M_{AD}, and M_{AE} are therefore measures of the restraints which the members AC, AD, and AE exert on the member AB at A. From equations 493, 494, and 495

$$M_{AC} = 3EK_{AC}\,\theta_A$$
$$M_{AD} = 4EK_{AD}\,\theta_A$$
$$M_{AE} = 2EK_{AE}\,\theta_A$$

These equations have the general form

$$M = EK\,\theta\,N \qquad (496)$$

in which N depends upon the restraints at C, D, and E, and might be termed a "restraint factor;" that is, the restraint which a member can exert upon a joint at one end equals $EK\,\theta$ times a factor N whose value depends upon the degree of restraint at the other end of the member. As derived, if the far end is hinged, $N=3$; if the far end is fixed, $N=4$; and if the angular rotation at the two ends is equal in magnitude but opposite in sense $N=2$. In general, N depends upon the restraints at C, D, and E.

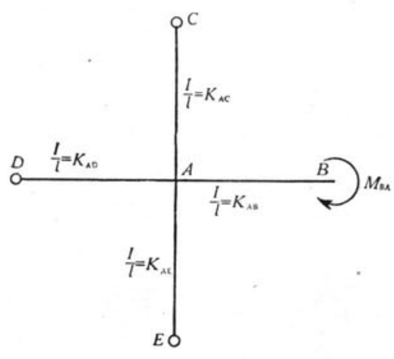


FIGURE 59

In Fig. 59, B is restrained by the couple M_{AB} and C, D, and E are hinged.

For equilibrium

$$M_{AB}+M_{AC}+M_{AD}+M_{AE}=0 \quad \ldots \quad (497)$$

From equation (A), Table 1

$$M_{AB}=2EK_{AB}(2\theta_A+\theta_B) \quad \ldots \quad (498)$$

$$M_{BA}=2EK_{AB}(2\theta_B+\theta_A) \quad \ldots \quad (499)$$

Substituting the values of M_{AC}, M_{AD}, and M_{AE} from equation 493 and the value of M_{AB} from equation 498 in equation 497 gives

$$\theta_A=-\theta_B\,\frac{2K_{AB}}{4K_{AB}+3K_{AC}+3K_{AD}+3K_{AE}} \quad \ldots \quad (500)$$

Substituting the value of θ_A from equation 500 in equation 499 gives

$$M_{BA}=EK_{AB}\;\theta_B\left[\frac{4(3K_{AB}+3K_{AC}+3K_{AD}+3K_{AE})}{4K_{AB}+3K_{AC}+3K_{AD}+3K_{AE}}\right] \quad \ldots \quad (501)$$

If C, D, and E of Fig. 59 are fixed, the values of M_{AC}, M_{AD}, and M_{AE} of equation 497 are given by equation 494. Proceeding as before gives

$$M_{BA}=EK_{AB}\;\theta_B\left[\frac{4(3K_{AB}+4K_{AC}+4K_{AD}+4K_{AE})}{4K_{AB}+4K_{AC}+4K_{AD}+4K_{AE}}\right] \quad \ldots \quad (502)$$

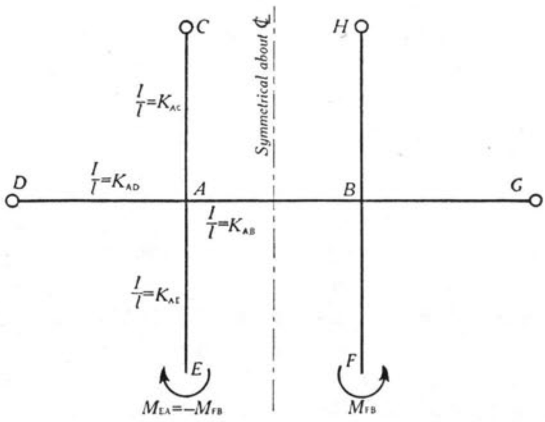


FIGURE 60

The frame represented by Fig. 60 is symmetrical about its vertical center line and is symmetrically loaded. θ_A therefore equals $-\theta_B$.

For the extremities of the members C, D, G, and H hinged, M_{AC} and M_{AD} are given by equation 493. M_{AB} is given by equation 495. From equation (A), Table 1

$$M_{AE}=2EK_{AE}(2\theta_A+\theta_E) \quad \text{. . . . .} \quad (503)$$

$$M_{EA}=2EK_{AE}(2\theta_E+\theta_A) \quad \text{. . . . .} \quad (504)$$

Substituting the values of M_{AB}, M_{AC}, M_{AD}, and M_{AE} in equation 497, solving for θ_A, and substituting the value of θ_A in equation 504 gives

$$M_{EA}=EK_{AE}\ \theta_E\left[\frac{4(2K_{AB}+3K_{AC}+3K_{AD}+3K_{AE})}{2K_{AB}+3K_{AC}+3K_{AD}+4K_{AE}}\right] \quad \text{. .} \quad (505)$$

If C and D are fixed

$$M_{EA}=EK_{AE}\ \theta_E\left[\frac{4(2K_{AB}+4K_{AC}+4K_{AD}+3K_{AE})}{2K_{AB}+4K_{AC}+4K_{AD}+4K_{AE}}\right] \quad \text{. .} \quad (506)$$

Equations 501, 502, 505, and 506 have the form $M=EK\ \theta\ N$ in which N corresponds to the quantity in the brackets.

It is to be noted that for the values of N in equations 501, 502, 505, and 506 the coefficient of K for the member in which the stress is to be determined is always 3 in the numerator and 4 in the denominator. For the members, furthermore, which restrain the member in which the moment is to be determined: if hinged at the far end the coefficient of K is 3; if fixed at the far end the coefficient of K is 4; and if the rotations of the two ends are equal in magnitude but opposite in sense the coefficient of the K is 2. These coefficients correspond to the coefficients of $EK\ \theta_A$ in the expressions for the moments M_{AC}, M_{AD}, and M_{AE} of Fig. 57.

53. *Moment in a Frame Composed of a Number of Rectangles Due to Vertical Loads.*—Fig. 61 represents a portion of a frame composed of a large number of rectangles. The portion considered is taken from the center of a frame symmetrical about a vertical line. The member AB carries any system of vertical loads symmetrical about the center line of AB. Under these conditions there is no horizontal deflection of the frame.

For equilibrium at A,

$$M_{AB}+M_{AD}+M_{AI}+M_{AH}=0 \quad \text{. . . . .} \quad (507)$$

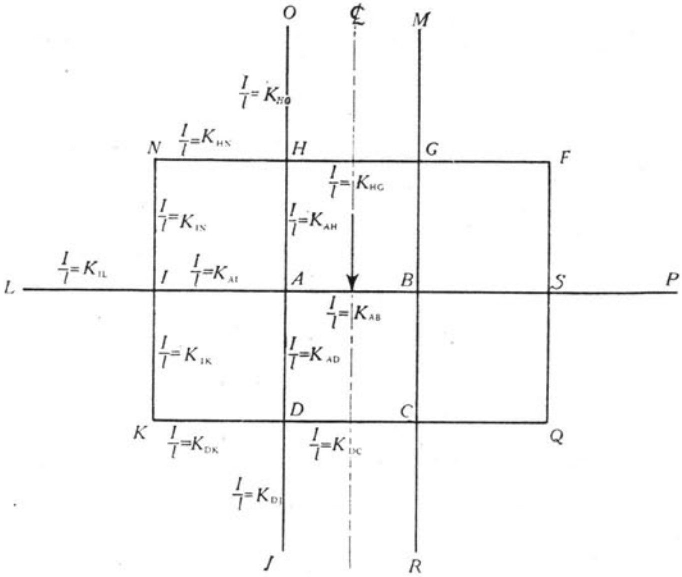


FIGURE 61

From equation 496, section 52

$$M_{AD}=EK_{AD}\,\theta_A\,N_{AD} \qquad (508)$$

$$M_{AI}=EK_{AI}\,\theta_A\,N_{AI} \qquad (509)$$

$$M_{AH}=EK_{AH}\,\theta_A\,N_{AH} \qquad (510)$$

Substituting the values of M_{AD}, M_{AI}, and M_{AH} from equations 508, 509, and 510 in equation 507 gives

$$M_{AB}=-E\,\theta_A\Big[K_{AD}\,N_{AD}+K_{AI}\,N_{AI}+K_{AH}\,N_{AH}\Big] \qquad (511)$$

Since the frame is symmetrical about a vertical center line, $\theta_A=-\theta_B$. From equation (C) of Table 1

$$M_{AB}=2EK_{AB}\,\theta_A-\frac{F}{l} \qquad (512)$$

Eliminating θ_A from equations 511 and 512 gives

$$M_{AB}=-\frac{F}{l}\left[\frac{K_{AD}\,N_{AD}+K_{AI}\,N_{AI}+K_{AH}\,N_{AH}}{K_{AD}\,N_{AD}+K_{AI}\,N_{AI}+K_{AH}\,N_{AH}+2K_{AB}}\right] \qquad (513)$$

The stress in the frame apparently depends upon the values of the N's in equation 513; that is, the stress depends upon the degrees of restraint of the extremities of the members, O, N, L, K, J, etc.

Although the degrees of restraint at these extremities are not known, it is known that the degree of restraint at each extremity is greater than if the extremity is hinged and less than if the extremity is fixed. If then the stresses are determined with the extremities hinged and again with the extremities fixed, although the true stresses will not be determined, they will be fixed between two limits.

54. *Extremities of Members Hinged.*—If the members are hinged at O, N, L, K, and J:

From equation 505, section 52,

$$N_{AD}=\frac{4(2K_{DC}+3K_{DJ}+3K_{DK}+3K_{AD})}{2K_{DC}+3K_{DJ}+3K_{DK}+4K_{AD}} \qquad (514)$$

$$N_{AH}=\frac{4(2K_{HG}+3K_{HN}+3K_{HO}+3K_{AH})}{2K_{HG}+3K_{HN}+3K_{HO}+4K_{AH}} \qquad (515)$$

From equation 501, section 52,

$$N_{AI}=\frac{4(3K_{AI}+3K_{IK}+3K_{IL}+3K_{IN})}{4K_{AI}+3K_{IK}+3K_{IL}+3K_{IN}} \qquad (516)$$

The restraint factor, N, in each of these equations has a value between 3 and 4.

Substituting the values of the N's from equations 514, 515, and 516 in equation 513 gives the value of M_{AB}.

The expression for M_{AB} in equation 513 is made up of three quantities, the three moments resisted by AD, AI, and AH. These moments are as follows:

$$M_{AD}=\frac{F}{l}\left[\frac{K_{AD}N_{AD}}{K_{AD}N_{AD}+K_{AI}N_{AI}+K_{AH}N_{AH}+2K_{AB}}\right] \qquad (517)$$

$$M_{AI}=\frac{F}{l}\left[\frac{K_{AI}N_{AI}}{K_{AD}N_{AD}+K_{AI}N_{AI}+K_{AH}N_{AH}+2K_{AB}}\right] \qquad (518)$$

$$M_{AH}=\frac{F}{l}\left[\frac{K_{AH}N_{AH}}{K_{AD}N_{AD}+K_{AI}N_{AI}+K_{AH}N_{AH}+2K_{AB}}\right] \qquad (519)$$

The values of the N's are given in equations 514, 515, and 516.

From the conditions for equilibrium at D and from the relation between M_{DA} and M_{AD} it can be proved that

$$M_{DA}=\frac{M_{AD}}{2}\left[\frac{3K_{DK}+3K_{DJ}+2K_{DC}}{3K_{DK}+3K_{DJ}+2K_{DC}+3K_{AD}}\right] \quad . \quad . \quad . \quad . \quad . \quad (520)$$

also

$$M_{IA}=\frac{M_{AI}}{2}\left[\frac{3K_{IK}+3K_{IN}+3K_{IL}}{3K_{IK}+3K_{IN}+3K_{IL}+3K_{AI}}\right] \quad . \quad . \quad . \quad . \quad . \quad (521)$$

and

$$M_{HA}=\frac{M_{AH}}{2}\left[\frac{3K_{HN}+3K_{HO}+2K_{HG}}{3K_{HN}+3K_{HO}+2K_{HG}+3K_{AH}}\right] \quad . \quad . \quad . \quad . \quad . \quad (522)$$

M_{DA} is made up of the three moments M_{DK}, M_{DJ}, and M_{DC}. These moments are as follows:

$$M_{DK}=-\frac{M_{AD}}{2}\left[\frac{3K_{DK}}{3K_{DK}+3K_{DJ}+2K_{DC}+3K_{AD}}\right] \quad . \quad . \quad . \quad . \quad (523)$$

$$M_{DJ}=-\frac{M_{AD}}{2}\left[\frac{3K_{DJ}}{3K_{DK}+3K_{DJ}+2K_{DC}+3K_{AD}}\right] \quad . \quad . \quad . \quad . \quad (524)$$

$$M_{DC}=-\frac{M_{AD}}{2}\left[\frac{2K_{DC}}{3K_{DK}+3K_{DJ}+2K_{DC}+3K_{AD}}\right] \quad . \quad . \quad . \quad . \quad (525)$$

In a similar manner M_{IA} can be divided into M_{IK}, M_{IN}, and M_{IL}, and M_{HA} can be divided into M_{HN}, M_{HO}, and M_{HG}.

55. *Extremities of Members Fixed.*—If the members are fixed at O, N, L, K, and J:

From equation 506, section 52,

$$N_{AD}=\frac{4(2K_{DC}+4K_{DJ}+4K_{DK}+3K_{AD})}{2K_{DC}+4K_{DJ}+4K_{DK}+4K_{AD}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (526)$$

$$N_{AH}=\frac{4(2K_{HG}+4K_{HN}+4K_{HO}+3K_{AH})}{(2K_{HG}+4K_{HN}+4K_{HO}+4K_{AH})} \quad . \quad . \quad . \quad . \quad . \quad . \quad (527)$$

From equation 502, section 52,

$$N_{AI}=\frac{3K_{AI}+4K_{IK}+4K_{IL}+4K_{IN}}{K_{AI}+K_{IK}+K_{IL}+K_{IN}} \qquad (528)$$

Substituting the values of the N's in equation 513 gives the value of M_{AB}.

Equations 517, 518, and 519 are applicable. By substituting the values of the N's, given in equations 526, 527, and 528, M_{AD}, M_{AI}, and M_{AH} can be determined.

Proceeding as in the case where the extremities of the members are hinged it can be proved that

$$M_{DA}=\frac{M_{AD}}{2}\left[\frac{4K_{DK}+4K_{DJ}+2K_{DC}}{4K_{DK}+4K_{DJ}+2K_{DC}+3K_{AD}}\right] \qquad (529)$$

$$M_{IA}=\frac{M_{AI}}{2}\left[\frac{4K_{IK}+4K_{IN}+4K_{IL}}{4K_{IK}+4K_{IN}+4K_{IL}+3K_{AI}}\right] \qquad (530)$$

$$M_{HA}=\frac{M_{AH}}{2}\left[\frac{4K_{HN}+4K_{HO}+2K_{HG}}{4K_{HN}+4K_{HO}+2K_{HG}+3K_{AH}}\right] \qquad (531)$$

M_{DA} is made up of three parts, one part corresponding to each of the moments M_{DK}, M_{DJ}, and M_{DC}. These latter moments are proportional respectively to the parts of the numerator of equation 529: $4K_{DK}$, $4K_{DJ}$, and $2K_{DC}$. Similarly, M_{IK}, M_{IL}, and M_{IN} can be determined from M_{IA}; and M_{HN}, M_{HO}, and M_{HG} can be determined from M_{HA}.

To determine the effect of the degree of restraint of the extremities O, N, L, K, and J upon the moments in the frame, and also to determine the effect of the magnitude of the K's upon the moments in the frame, moments have been determined for frames having fixed and hinged extremities, for frames having all K's equal, and for frames for which the K's of the columns equal ten times the K's for the girders. The values of the M's are given in Table 15.

From Table 15, it is apparent that only members directly connected to the member carrying the load are subjected to moments sufficiently large to be considered in the design of the structure. Furthermore, the moments in the members adjacent to the member carrying the load are practically independent of the degree of restraint at the extremities O, N, L, K, J, etc. Therefore, the moments in the frame of Fig. 61, due to the load on AB, are given with sufficient accuracy for purposes of design by equations 513, 517, 518, 519, 520, 521, and 522, based upon the assumption that the extremities O, N, L, K, J, etc., are hinged. The equations based upon the assumption that the extremities are fixed give almost exactly the same results and could also be used.

TABLE 15

MOMENTS IN FRAME REPRESENTED BY FIG. 61

Moments are expressed in terms of $\frac{F}{l}$

Moment	Extremities of Members Hinged at O, N, L, K, and J		Extremities of Members Fixed at O, N, L, K, and J	
	All K's Equal	K's of Columns Equal 10 Times K's of Girders	All K's Equal	K's of Columns Equal 10 Times K's of Girders
M_{AB}	$-.845$	$-.972$	$-.849$	$-.973$
M_{AD}	$+.281$	$+.462$	$+.282$	$+.462$
M_{AH}	$+.281$	$+.462$	$+.282$	$+.462$
M_{AI}	$+.283$	$+.048$	$+.285$	$+.049$
M_{DA}	$+.102$	$+.125$	$+.108$	$+.140$
M_{HA}	$+.102$	$+.125$	$+.108$	$+.140$
M_{IA}	$+.106$	$+.023$	$+.114$	$+.024$
M_{DK}	$-.038$	$-.011$	$-.043$	$-.012$
M_{DJ}	$-.038$	$-.107$	$-.043$	$-.122$
M_{DC}	$-.026$	$-.007$	$-.022$	$-.006$
M_{HN}	$-.038$	$-.011$	$-.043$	$-.012$
M_{HO}	$-.038$	$-.017$	$-.043$	$-.122$
M_{HG}	$-.026$	$-.007$	$-.022$	$-.006$
M_{IK}	$-.035$	$-.011$	$-.038$	$-.011$
M_{IN}	$-.035$	$-.011$	$-.038$	$-.011$
M_{IL}	$-.035$	$-.001$	$-.038$	$-.002$

56. *Distribution of Loads for Maximum Moments in a Frame Composed of a Large Number of Rectangles.*—Referring to Fig. 61, a load on AB produces a moment M_{KD} having the same sign as M_{AB}. That being the case, a load on GF produces a moment M_{AB} of the same sign as the M_{AB} produced by the load on AB; therefore if AB and GF are loaded simultaneously the moment at A in AB is greater than if either AB or GF is loaded alone. Reasoning in a similar manner, the members can be selected which, if loaded, produce a moment at A in

the member AB having the same sign as the moment at the same point due to a load on AB. If all these members are loaded simultaneously, the moment at A in the member AB is a maximum.

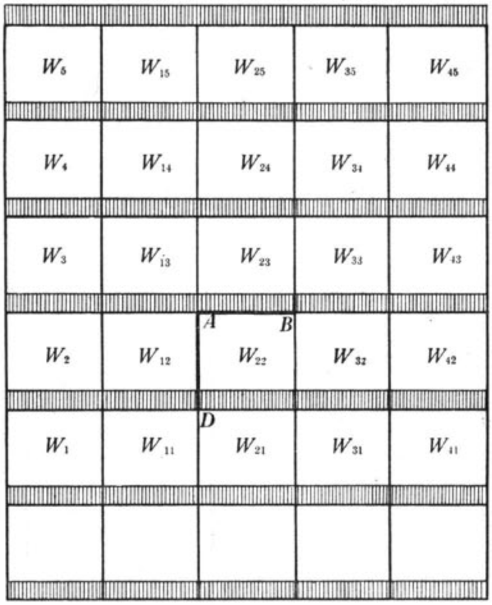


FIGURE 62

Fig. 62 represents a frame made up of similar rectangles. All girders are equally loaded. The moments M_{AB} and M_{AD}, due to these loads, as determined by the equations of section 53 are given in Table 16. For the frame of Table 16 the K's of all members are equal. The moments in similar frames for which the K's of all columns are equal and the K's of all girders are equal, but for which the K's of the columns do not equal the K's of the girders, have been determined. The relation between the maximum moments which it is possible to obtain in the girders, and the ratio of the K's of the columns to the K's of the girders, is presented graphically in Fig. 63. Similar data for the moment in the columns are given in Fig. 64.

Fig. 65 represents the loading which produces a maximum moment at A in the girder. Fig. 66 represents the loading which produces a maximum moment at A in the column.

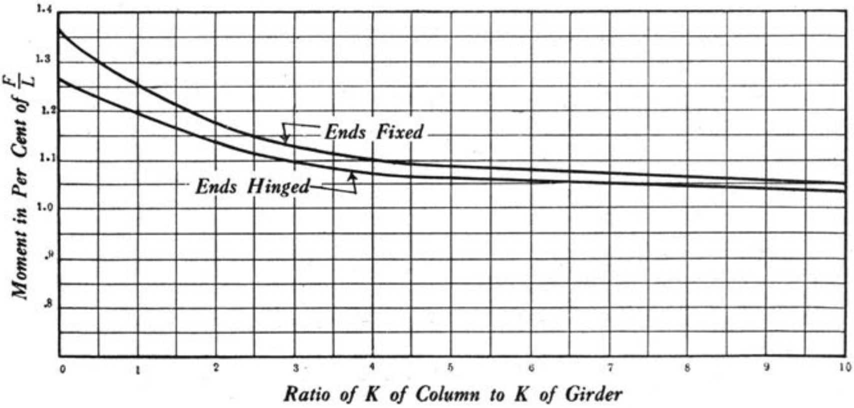


FIGURE 63

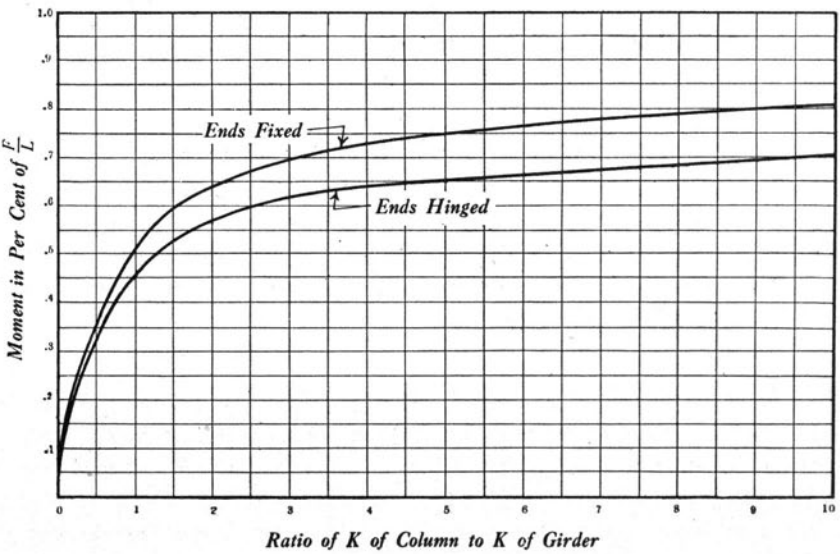


FIGURE 64

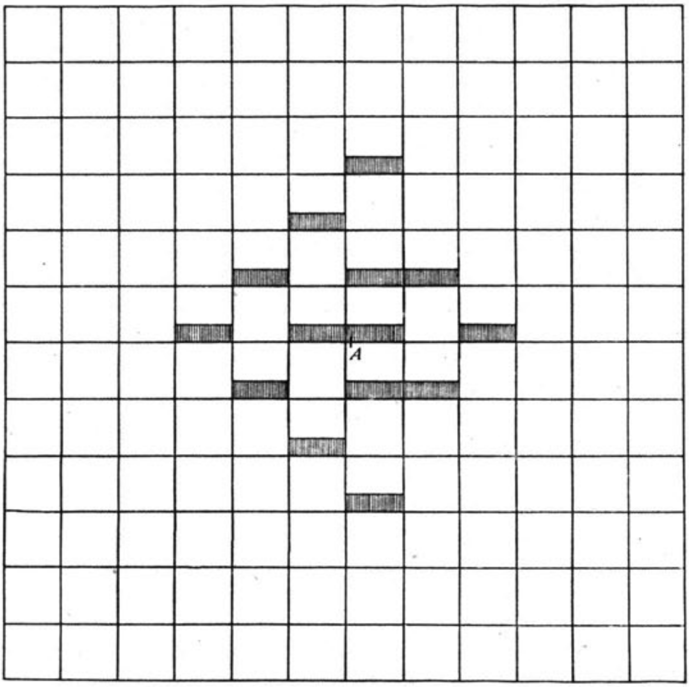


FIGURE 65

It is to be noted that for the frames of sections 52, 53, and 56 the horizontal deflection of one story of the frame relative to the other stories does not enter. If either the frame or the load is unsymmetrical there is a slight horizontal deflection. For the usual proportions of frames of engineering structures, the effect of this horizontal deflection is slightly to reduce the moments.

57. *Eccentric Load at Top of Exterior Column of a Frame. Connections of Girders to Columns Hinged.*—Fig. 67 represents a frame with eccentric loads at the tops of the exterior columns. The frame and the loading are symmetrical about the vertical center line of the frame. The connections of the girders to the columns are frictionless hinges The columns are continuous.

The moment in the column depends upon the restraint at 1. The degree of restraint at 1 is known to be between the restraint of a column hinged at 1 and a column fixed at 1. If, therefore, the moment is deter-

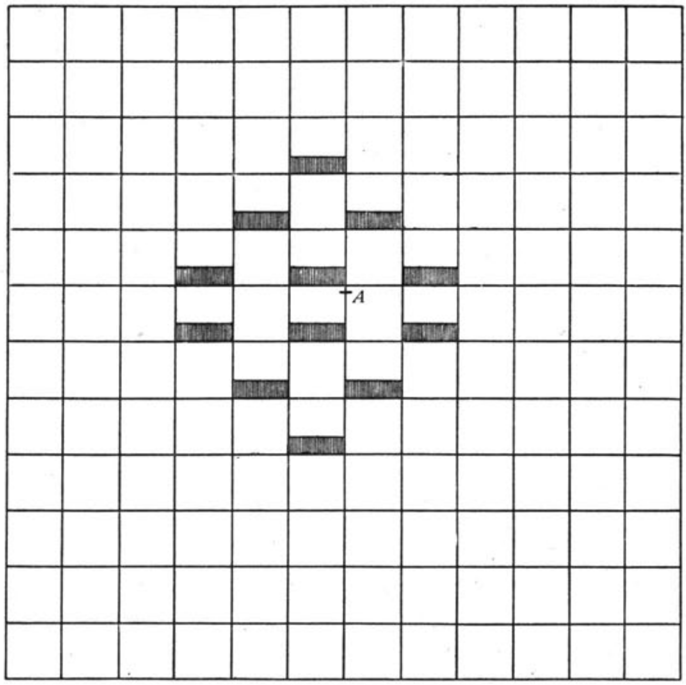


FIGURE 66

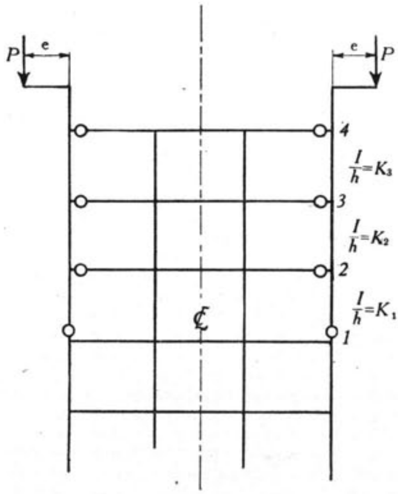


FIGURE 67

TABLE 16

MOMENTS IN FRAME REPRESENTED BY FIG. 62

K's of All Members Equal

Moment Produced by	M_{AB}		M_{AD}	
	Hinged	Fixed	Hinged	Fixed
$W2$			0	+.019
$W3$	+.035	+.038	+.035	+.038
$W11$			0	+.022
$W12$	+.038	+.043	−.102	−.108
$W13$	−.283	−.285	−.281	−.282
$W14$	+.038	+.043	+.038	+.043
$W21$			0	−.022
$W22$	−.026	−.022	+.102	+.108
$W23$	−.845	−.849	+.281	+.282
$W24$	−.026	−.022	−.038	−.043
$W32$	0	−.022	0	−.019
$W33$	+.106	+.114	−.035	−.038
$W34$	0	−.022		
$W43$	0	−.019		
Total	−.963	−1.003	0	0
Maximum	−1.180	−1.241	±.456	±.512

W's not given in table produce only very small moments.

mined for a column hinged at 1 and for a column fixed at 1 the true moment will be located between two limits.

Consider the column to be hinged at 1

$$M_{12}=0 \quad \ldots \quad (532)$$

$$M_{21}=3EK_1\theta_2 \quad \ldots \quad (533)$$

From equation 501, section 52,

$$M_{32}=4EK_2\theta_3\left(\frac{3K_2+3K_1}{4K_2+3K_1}\right) \quad . \quad . \quad . \quad (534)$$

$$M_{34}=-M_{32}=-4EK_2\theta_3\left(\frac{3K_2+3K_1}{4K_2+3K_1}\right)=2EK_3(2\theta_3+\theta_4) \quad . \quad . \quad (535)$$

$$M_{43}=2EK_3(2\theta_4+\theta_3) \quad . \quad . \quad . \quad (536)$$

Let $N_2=4\,\dfrac{3K_1+3K_2}{3K_1+4K_2}$

From equations 534 and 535

$$M_{43}=4EK_3\ \theta_4\left(\frac{N_2K_2+3K_3}{N_2K_2+4K_3}\right) \quad . \quad . \quad . \quad (537)$$

Also

$$M_{43}=Pe \quad . \quad . \quad . \quad (538)$$

From equations 537 and 538

$$\theta_4=\frac{Pe}{4EK_3}\,\frac{N_2K_2+4K_3}{N_2K_2+3K_3} \quad . \quad . \quad . \quad (539)$$

Eliminating θ_3 from equations 535 and 536 gives

$$M_{34}=-6EK_3\ \theta_4+2Pe \quad . \quad . \quad . \quad (540)$$

Substituting the value of θ_4 from equation 539 gives

$$M_{34}=\frac{Pe}{2}\ \frac{N_2K_2}{N_2K_2+3K_3} \quad . \quad . \quad . \quad (541)$$

$$M_{32}=-M_{34} \quad . \quad . \quad . \quad (542)$$

From the equations of Table 1

$$M_{32}=2EK_2(2\theta_3+\theta_2) \quad . \quad . \quad . \quad (543)$$

$$M_{23}=2EK_2(2\theta_2+\theta_3) \quad . \quad . \quad . \quad (544)$$

Eliminating θ_3 gives

$$2M_{23}-M_{32}=6EK_2\,\theta_2 \quad \ldots\ldots \quad (545)$$

From equations 533, 545, and 542, and since $M_{21}=-M_{23}$

$$M_{23}=-\frac{K_1}{2(K_1+K_2)}M_{34} \quad \ldots\ldots \quad (546)$$

If, therefore, the column is hinged at 1

$$M_{43}=Pe \quad \ldots\ldots \quad (538)$$

$$M_{34}=-M_{32}=\frac{Pe}{2}\left[\frac{N_2K_2}{N_2K_2+3K_3}\right] \quad \ldots\ldots \quad (541)$$

$$M_{21}=-M_{23}=\frac{K_1}{2(K_1+K_2)}M_{34} \quad \ldots\ldots \quad (546)$$

If the column is fixed at 1, letting N'_2 represent $4\,\dfrac{4K_1+3K_2}{4K_1+4K_2}$

$$M_{43}=Pe \quad \ldots\ldots \quad (547)$$

$$M_{34}=-M_{32}=\frac{Pe}{2}\;\frac{N'_2K_2}{N'_2K_2+3K_3} \quad \ldots\ldots \quad (548)$$

$$M_{21}=-M_{23}=\frac{2K_1}{4K_1+3K_2}M_{34} \quad \ldots\ldots \quad (549)$$

From a comparison of equations 538, 541, and 546 with 547, 548, and 549 it is apparent that the restraint at 1 does not materially affect the moments at 2 and 3. For purposes of design the moments as given by either equations 538, 541, and 546 or by equations 547, 548, and 549 are satisfactory. Moreover the average of the moments obtained by 538, 541, and 546, and 547, 548, and 549 approximate very closely the true moments.

If the K's are all equal, a condition often approximated in practice: For column hinged at 1

$$M_{43}=Pe \quad \ldots\ldots \quad (550)$$

$$M_{34} = -M_{32} = \frac{4}{15}\,Pe = 0.266\,Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (551)$$

$$M_{21} = -M_{23} = \frac{1}{15}\,Pe = 0.067\,Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (552)$$

For column fixed at 1

$$M_{43} = Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (553)$$

$$M_{34} = -M_{32} = \frac{7}{26}\,Pe = 0.269\,Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (554)$$

$$M_{21} = -M_{23} = \frac{2}{26}\,Pe = 0.077\,Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (555)$$

It is to be noted that the frame considered is symmetrical about a vertical center line and is symmetrically loaded. If there is a load on the right-hand column only, the moments in that column will be slightly smaller than the moments given by the equation, and the other columns will be subjected to a small moment. The error in the moment in the loaded column and the neglected moment in the other columns increase as the ratio of the stiffness of the loaded column to the combined stiffness of the other columns increase. Although they have not been able to establish this statement mathematically, it is the opinion of the writers that if the equations of this section are applied to a frame that is either unsymmetrical or unsymmetrically loaded, the error due to the horizontal deflection of the frame is negligible for purposes of design.

58. *Eccentric Load at Top of Exterior Column of a Frame. Connections of Girders to Columns Rigid.*—Fig. 68 represents a frame with eccentric loads at the tops of the exterior columns. The frame and the loading are symmetrical about the vertical center line of the frame. The connections of the girders to the columns are rigid.

An exact determination of the moments in the frame is practically impossible. From previous similar work, however, it is known that of the moments produced by P on the right-hand side of the frame, only the moments at A, B, C, and F are large enough to be considered in the design of the frame. Furthermore, from previous work it is known that the moments at A, B, C, and F are practically independent of the degree of restraint at J, K, G, H, and D.

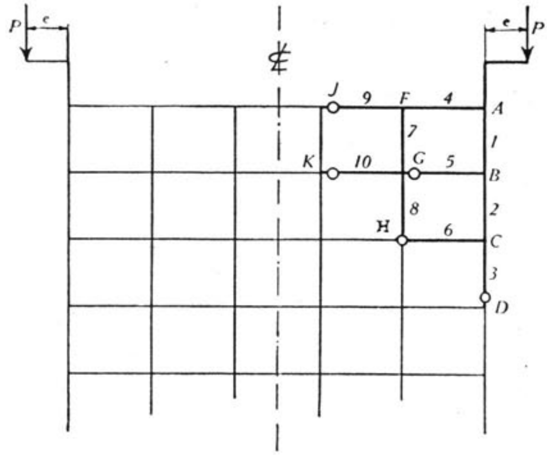


FIGURE 68

It is therefore considered that the girders are hinged at J, K, G, and H, and that the columns are hinged at H and D. Applying the same general method that was used in section 57, it can be proved that

$$M_{AB}=4EK_1\,\theta_A\left(\frac{3K_5+N_2K_2+3K_1}{3K_5+N_2K_2+4K_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (556)$$

$$M_{AF}=4EK_4\,\theta_A\left(\frac{3K_9+N_7K_7+3K_4}{3K_9+N_7K_7+4K_4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (557)$$

in which

$$N_2=4\left(\frac{3K_3+3K_6+3K_2}{3K_3+3K_6+4K_2}\right)$$

$$N_7=4\left(\frac{3K_8+3K_{10}+3K_7}{3K_8+3K_{10}+4K_7}\right)$$

For A to be in equilibrium

$$M_{AB}+M_{AF}-Pe=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (558)$$

From equations 556, 557, and 558 letting

$N_1=4\left(\frac{3K_5+N_2K_2+3K_1}{3K_5+N_2K_2+4K_1}\right)$ and $N_4=4\left(\frac{3K_9+N_7K_7+3K_4}{3K_9+N_7K_7+4K_4}\right)$ gives

$$M_{AB}=\frac{PeN_1K_1}{N_1K_1+N_4K_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (559)$$

$$M_{AF}=\frac{PeN_4K_4}{N_1K_1+N_4K_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (560)$$

Also

$$M_{BA}=\frac{M_{AB}}{2}\left(\frac{3K_5+N_2K_2}{3K_5+N_2K_2+3K_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (561)$$

$$M_{BG}=-\frac{M_{AB}}{2}\left(\frac{3K_5}{3K_5+N_2K_2+3K_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (562)$$

$$M_{BC}=-\frac{M_{AB}}{2}\left(\frac{N_2K_2}{3K_5+N_2K_2+3K_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (563)$$

$$M_{CB}=\frac{M_{BC}}{2}\left(\frac{K_3+K_6}{K_3+K_6+K_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (564)$$

$$M_{CH}=-\frac{M_{BC}}{2}\left(\frac{K_6}{K_3+K_6+K_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (565)$$

$$M_{CD}=-\frac{M_{BC}}{2}\left(\frac{K_3}{K_3+K_6+K_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (566)$$

$$M_{FA}=\frac{M_{AF}}{2}\left(\frac{3K_9+N_7K_7}{3K_9+N_7K_7+3K_4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (567)$$

$$M_{FG}=-\frac{M_{AF}}{2}\left(\frac{N_7K_7}{3K_9+N_7K_7+3K_4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (568)$$

$$M_{FJ}=-\frac{M_{AF}}{2}\left(\frac{3K_9}{3K_9+N_7K_7+3K_4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (569)$$

If the $\frac{I}{l}$'s of the girders are all equal and are represented by K, if the $\frac{I}{l}$'s of the columns are all equal and are represented by $\frac{K}{n}$, and if the l's are all equal, $N_2=N_7=N=4\left(\frac{3n+6}{3n+7}\right)$, and equations 559 to 569 reduce to the form

$$M_{AB}=Pe\left[\frac{\left(\frac{3n+3+N}{3n+4+N}\right)}{\left(\frac{3n+3+N}{3n+4+N}\right)+n\left(\frac{6n+N}{7n+N}\right)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (570)$$

$$M_{AF}=Pe\left[\frac{n\left(\frac{6n+N}{7n+N}\right)}{\left(\frac{3n+3+N}{3n+4+N}\right)+n\left(\frac{6n+N}{7n+N}\right)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (571)$$

$$M_{BA}=\frac{M_{AB}}{2}\left(\frac{3n+N}{3n+3+N}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (572)$$

$$M_{BG}=-\frac{M_{AB}}{2}\left(\frac{3n}{3n+3+N}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (573)$$

$$M_{BC}=-\frac{M_{AB}}{2}\left(\frac{N}{3n+3+N}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (574)$$

$$M_{CB}=\frac{M_{BC}}{2}\left(\frac{n+1}{n+2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (575)$$

$$M_{CH}=-\frac{M_{BC}}{2}\left(\frac{n}{n+2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (576)$$

$$M_{CD}=-\frac{M_{BC}}{2}\left(\frac{1}{n+2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (577)$$

$$M_{FA}=\frac{M_{AF}}{2}\left(\frac{3n+N}{6n+N}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (578)$$

$$M_{FG}=-\frac{M_{AF}}{2}\left(\frac{N}{6n+N}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (579)$$

$$M_{FJ}=-\frac{M_{AF}}{2}\left(\frac{3n}{6n+N}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (580)$$

If all the K's are equal

$$M_{AB} = .500\ Pe \quad (581)$$

$$M_{AF} = .500\ Pe \quad (582)$$

$$M_{BA} = .172\ Pe \quad (583)$$

$$M_{BG} = -.078\ Pe \quad (584)$$

$$M_{BC} = -.094\ Pe \quad (585)$$

$$M_{CB} = -.031\ Pe \quad (586)$$

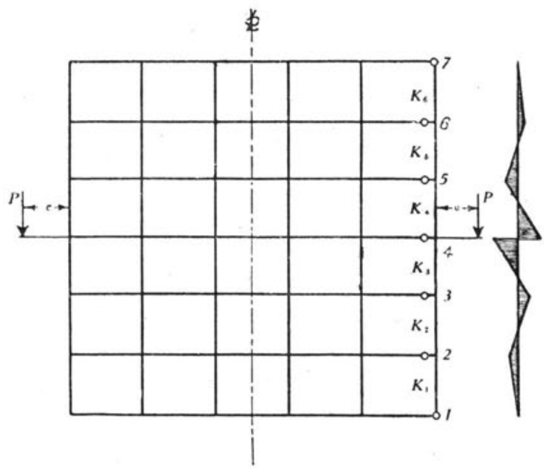
$$M_{CH} = .016\ Pe \quad (587)$$

$$M_{CD} = .016\ Pe \quad (588)$$

$$M_{FA} = .172\ Pe \quad (589)$$

$$M_{FG} = -.094\ Pe \quad (590)$$

$$M_{FJ} = -.078\ Pe \quad (591)$$



FIGURES 69 AND 70

59. *Eccentric Load at Middle Floor Level of Exterior Column of a Frame. Connections of Girders to Columns Hinged.*—Fig. 69 represents a frame with eccentric loads at the middle floor level of the exterior columns. The frame and the loading are symmetrical about the

vertical center line of the frame. The connections of the girders to the columns are hinged.

The moments are practically independent of the degree of restraint at 1 and 7. The column is therefore assumed to be hinged at 1 and 7.

$$-M_{43}-M_{45}+Pe=0 \qquad (592)$$

$$M_{43}=Pe\left[\frac{K_3\left(\frac{N_2K_2+3K_3}{N_2K_2+4K_3}\right)}{K_3\left(\frac{N_2K_2+3K_3}{N_2K_2+4K_3}\right)+K_4\left(\frac{N_5+3K_4}{N_5+4K_4}\right)}\right] \qquad (593)$$

$$M_{45}=Pe-M_{43} \qquad (594)$$

$$M_{34}=-M_{32}=\frac{M_{43}}{2}\left[\frac{N_2K_2}{N_2K_2+3K_3}\right] \qquad (595)$$

$$M_{54}=-M_{56}=\frac{M_{45}}{2}\left[\frac{N_5K_5}{N_5K_5+3K_4}\right] \qquad (596)$$

$$M_{21}=-M_{23}=M_{34}\left[\frac{K_1}{2(K_1+K_2)}\right] \qquad (597)$$

$$M_{67}=-M_{65}=M_{54}\left[\frac{K_6}{2(K_5+K_6)}\right] \qquad (598)$$

In these equations

$$N_2=4\left(\frac{3K_1+3K_2}{3K_1+4K_2}\right)$$

$$N_5=4\left(\frac{3K_6+3K_5}{3K_6+4K_5}\right)$$

When all K's are equal

$$M_{43}=M_{45}=\frac{1}{2}\,Pe=0.5000\,Pe \qquad (599)$$

$$M_{34}=M_{54}=\frac{2}{15}\,Pe=0.1333\,Pe \qquad (600)$$

$$M_{21}=M_{67}=\frac{1}{30}\,Pe=0.0333\,Pe \quad \ldots \quad (601)$$

If the ends of the column are fixed at 1 and 7, with all K's equal

$$M_{43}=\frac{1}{2}Pe=0.5000\,Pe \quad \ldots \quad (602)$$

$$M_{34}=\frac{7}{52}Pe=0.1346\,Pe \quad \ldots \quad (603)$$

$$M_{21}=\frac{2}{52}Pe=0.0385\,Pe \quad \ldots \quad (604)$$

If there is a number of loads, the moment due to all the loads is the sum of the moments due to each load considered separately.

With a load at each floor, if all K's are equal and all values of Pe are equal, $M=\frac{1}{2}\,Pe$.

The moment diagram for the right-hand exterior column is represented by Fig. 70. The loading which produces a maximum stress in the column just below 4 is represented by Fig. 71. If all K's are equal

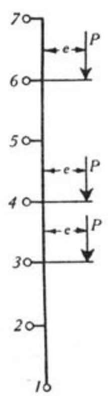


FIGURE 71

and all values of Pe are equal the maximum moment in a column that can be produced by eccentric loads is $\frac{Pe}{2}+\frac{2Pe}{15}+\frac{Pe}{30}=\frac{2Pe}{3}$

It is to be noted that the frame considered is symmetrical about a vertical center line and is symmetrically loaded. If there is a load on the right-hand column only, the moments in that column will be slightly smaller than the moments given by the equation, and the other columns will be subjected to a small moment. The error in the moment in the loaded column and the neglected moment in the other columns increase as the ratio of the stiffness of the loaded column to the combined stiffness of the other columns increases.

As in section 57, the error in the equations of this section due to the horizontal deflection of a frame under any vertical loading is undoubtedly negligible for purposes of design.

60. *Eccentric Load at Middle Floor Level of Exterior Column of a Frame. Connections of Girders to Columns Rigid.*—Fig. 72 represents

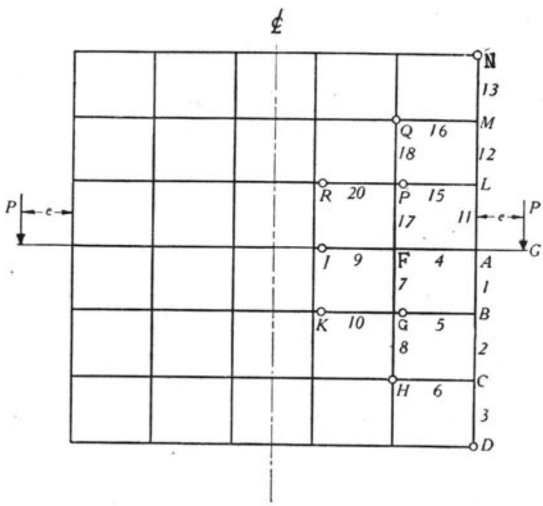


FIGURE 72

a frame with eccentric loads at the middle floor level of the exterior columns. The frame and the loading are symmetrical about the vertical center line of the frame. The connections of the girders to the columns are rigid.

From previous similar work it is known that of the moments produced by P on the right-hand side of the frame, only the moments at A, B, C, L, M, and F are large enough to be considered in the design of the frame. From previous work it is known, furthermore, that the moments at these points are practically independent of the degree of restraint at D, H, G, K, J, R, P, Q, and N. It will therefore be considered that the girders are hinged at H, G, K, J, R, P, and Q, and that the columns are hinged at D, H, Q, and N.

Applying the general method used in section 58, it can be proved that letting $N_1=4\left[\frac{3K_5+N_2K_2+3K_1}{3K_5+N_2K_2+4K_1}\right]$

$$N_4=4\left(\frac{3K_9+N_7K_7+N_{17}K_{17}+3K_4}{3K_9+N_7K_7+N_{17}K_{17}+4K_4}\right)$$

and $N_{11}=4\left(\frac{3K_{15}+N_{12}K_{12}+3K_{11}}{3K_{15}+N_{12}K_{12}+4K_{11}}\right)$ gives

$$M_{AB}=\frac{PeN_1K_1}{N_1K_1+N_4K_4+N_{11}K_{11}} \qquad (605)$$

$$M_{AF}=\frac{PeN_4K_4}{N_1K_1+N_4K_4+N_{11}K_{11}} \qquad (606)$$

$$M_{AL}=\frac{PeN_{11}K_{11}}{N_1K_1+N_4K_4+N_{11}K_{11}} \qquad (607)$$

$$M_{LA}=\frac{M_{AL}}{2}\left(\frac{3K_{15}+N_{12}K_{12}}{3K_{15}+N_{12}K_{12}+3K_{11}}\right) \qquad (608)$$

$$M_{LP}=-\frac{M_{AL}}{2}\left(\frac{3K_{15}}{3K_{15}+N_{12}K_{12}+3K_{11}}\right) \qquad (609)$$

$$M_{LM}=-\frac{M_{AL}}{2}\left(\frac{N_{12}K_{12}}{3K_{15}+N_{12}K_{12}+3K_{11}}\right) \qquad (610)$$

$$M_{ML}=\frac{M_{LM}}{2}\left(\frac{K_{13}+K_{16}}{K_{13}+K_{16}+K_{12}}\right) \qquad (611)$$

$$M_{MQ}=-\frac{M_{LM}}{2}\left(\frac{K_{16}}{K_{13}+K_{16}+K_{12}}\right) \qquad (612)$$

$$M_{MN}=-\frac{M_{LM}}{2}\left(\frac{K_{13}}{K_{13}+K_{16}+K_{12}}\right) \qquad (613)$$

$$M_{BA}=\frac{M_{AB}}{2}\left(\frac{3K_5+N_2K_2}{3K_5+N_2K_2+3K_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (614)$$

$$M_{BG}=-\frac{M_{AB}}{2}\left(\frac{3K_5}{3K_5+N_2K_2+3K_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (615)$$

$$M_{BC}=-\frac{M_{AB}}{2}\left(\frac{N_2K_2}{3K_5+N_2K_2+3K_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (616)$$

$$M_{CB}=\frac{M_{BC}}{2}\left(\frac{K_3+K_6}{K_3+K_6+K_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (617)$$

$$M_{CH}=-\frac{M_{BC}}{2}\left(\frac{K_6}{K_3+K_6+K_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (618)$$

$$M_{CD}=-\frac{M_{BC}}{2}\left(\frac{K_3}{K_3+K_6+K_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (619)$$

$$M_{FA}=\frac{1}{2}\,M_{AF}\left(\frac{3K_9+N_7K_7+N_{17}K_{17}}{3K_9+N_7K_7+N_{17}K_{17}+3K_4}\right) \quad . \quad . \quad . \quad . \quad (620)$$

$$M_{FG}=-\frac{1}{2}\,M_{AF}\left(\frac{N_7K_7}{3K_9+N_7K_7+N_{17}K_{17}+3K_4}\right) \quad . \quad . \quad . \quad . \quad (621)$$

$$M_{FJ}=-\frac{1}{2}\,M_{AF}\left(\frac{3K_9}{3K_9+N_7K_7+N_{17}K_{17}+3K_4}\right) \quad . \quad . \quad . \quad . \quad (622)$$

$$M_{FP}=-\frac{1}{2}\,M_{AF}\left(\frac{N_{17}K_{17}}{3K_9+N_7K_7+N_{17}K_{17}+3K_4}\right) \quad . \quad . \quad . \quad . \quad (623)$$

in which

$$N_2=4\left(\frac{3K_3+3K_6+3K_2}{3K_3+3K_6+4K_2}\right)$$

$$N_7=4\left(\frac{3K_8+3K_{10}+3K_7}{3K_8+3K_{10}+4K_7}\right)$$

$$N_{12}=4\left(\frac{3K_{13}+3K_{16}+3K_{12}}{3K_{13}+3K_{16}+4K_{12}}\right)$$

$$N_{17}=4\left(\frac{3K_{18}+3K_{20}+3K_{17}}{3K_{18}+3K_{20}+4K_{17}}\right)$$

If the $\frac{I}{l}$'s of all girders are equal and are represented by K, and if the $\frac{I}{l}$'s of all columns are equal and are represented by $\frac{K}{n}$,

$N_2 = N_7 = N_{12} = N_{17} = N = 4\left(\frac{3n+6}{3n+7}\right)$,

and equations 605 to 623 reduce to the form

$$M_{AB} = M_{AL} = \frac{Pe}{2}\left[\frac{\left(\frac{3n+3+N}{3n+4+N}\right)}{\left(\frac{3n+3+N}{3n+4+N}\right) + n\left(\frac{3n+N}{7n+2N}\right)}\right] \quad . \quad . \quad (624)$$

$$M_{AF} = \frac{nPe}{2}\left[\frac{\left(\frac{6n+2N}{7n+2N}\right)}{\left(\frac{3n+3+N}{3n+4+N}\right) + n\left(\frac{3n+N}{7n+2N}\right)}\right] \quad . \quad . \quad . \quad (625)$$

$$M_{BA} = M_{LA} = \frac{M_{AB}}{2}\left(\frac{3n+N}{3n+3+N}\right) \quad . \quad . \quad . \quad (626)$$

$$M_{BG} = M_{LP} = -\frac{M_{AB}}{2}\left(\frac{3n}{3n+3+N}\right) \quad . \quad . \quad . \quad (627)$$

$$M_{BC} = M_{LM} = -\frac{M_{AB}}{2}\left(\frac{N}{3n+3+N}\right) \quad . \quad . \quad . \quad (628)$$

$$M_{CB} = M_{ML} = \frac{M_{BC}}{2}\left(\frac{n+1}{n+2}\right) \quad . \quad . \quad . \quad (629)$$

$$M_{CH} = M_{MQ} = -\frac{M_{BC}}{2}\left(\frac{n}{n+2}\right) \quad . \quad . \quad . \quad (630)$$

$$M_{CD} = M_{MN} = -\frac{M_{BC}}{2}\left(\frac{1}{n+2}\right) \quad . \quad . \quad . \quad (631)$$

$$M_{FA} = \frac{M_{AF}}{2}\left(\frac{3n+2N}{6n+2N}\right) \quad . \quad . \quad . \quad (632)$$

$$M_{FJ} = -\frac{M_{AF}}{2}\left(\frac{3n}{6n+2N}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (633)$$

$$M_{FG} = M_{FP} = -\frac{M_{AF}}{2}\frac{N}{6n+2N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (634)$$

If all the K's are equal, equations 605 to 623 reduce to the form

$$M_{AB} = M_{AL} = 0.331\ Pe \quad . \quad . \quad . \quad (635)$$

$$M_{BA} = M_{LA} = 0.113\ Pe \quad . \quad . \quad . \quad (636)$$

$$M_{BG} = M_{LP} = -0.052\ Pe \quad . \quad . \quad . \quad (637)$$

$$M_{BC} = M_{LM} = -0.062\ Pe \quad . \quad . \quad . \quad (638)$$

$$M_{CB} = M_{ML} = -0.021\ Pe \quad . \quad . \quad . \quad (639)$$

$$M_{CH} = M_{MQ} = 0.010\ Pe \quad . \quad . \quad . \quad (640)$$

$$M_{CD} = M_{MN} = 0.010\ Pe \quad . \quad . \quad . \quad (641)$$

$$M_{AF} = 0.338\ Pe \quad . \quad . \quad . \quad (642)$$

$$M_{FA} = 0.131\ Pe \quad . \quad . \quad . \quad (643)$$

$$M_{FJ} = -0.038\ Pe \quad . \quad . \quad . \quad (644)$$

$$M_{FG} = M_{FP} = -0.046\ Pe \quad . \quad . \quad . \quad (645)$$

If there are loads at each of the points M, L, A, B, and C, the moment at any point, as A, can be determined by taking the sum of the moments due to each load separately. If the values of Pe are equal for all the points, if the $\frac{I}{l}$'s of all girders are equal and are represented by K, if the $\frac{I}{l}$'s of all columns are equal and are represented by $\frac{K}{n}$, and all values of $N = 4\left(\frac{3n+6}{3n+7}\right)$, the moments are

$$M_{AB} = M_{AL} = \frac{Pe}{2}\frac{\left[\dfrac{4.5n+3+N\left(\dfrac{3n+8}{4n+8}\right)}{3n+4+N}\right]}{\left(\dfrac{3n+3+N}{3n+4+N}\right)+n\left(\dfrac{3n+N}{7n+2N}\right)} \quad . \quad . \quad . \quad . \quad . \quad (646)$$

$$M_{AF}=\frac{nPe}{2}\frac{\left(\frac{6n+2N}{7n+2N}\right)+\left[\frac{\frac{N}{2n+4}-3}{3n+4+N}\right]}{\left(\frac{3n+3+N}{3n+4+N}\right)+n\left(\frac{3n+N}{7n+2N}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (647)$$

If the K's of the columns equal the K's of the girders

$$M_{AB}=M_{AL}=1\tfrac{1}{8} \quad (.331 \; Pe)=0.372 \; Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (648)$$

$$M_{AF}=0.254 \; Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (649)$$

The moments M_{AL} and M_{AB} are a maximum when the frame is loaded as shown in Fig. 73. If the $\frac{I}{l}$'s of the girders all equal K, and if the $\frac{I}{l}$'s of the columns all equal $\frac{K}{n}$, and if all values of Pe are equal, the maximum moments are

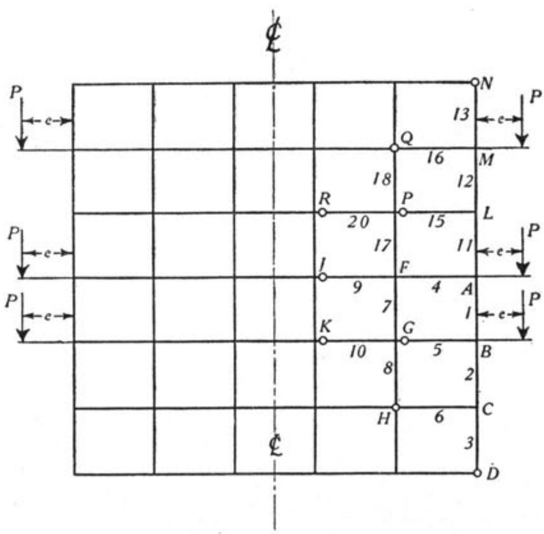


FIGURE 73

$$M_{AL}=M_{AB}=\frac{Pe}{2}\;\frac{\left[\dfrac{4.5n+3+N\left(\dfrac{6n+13}{4n+8}\right)}{3n+4+N}\right]}{\left(\dfrac{3n+3+N}{3n+4+N}\right)+n\left(\dfrac{3n+N}{7n+2N}\right)} \quad . \quad . \quad . \quad . \quad (650)$$

M_{AF} is maximum when alternate floors are loaded. For such oading

$$M_{AF}=n\,\frac{Pe}{2}\;\frac{\left(\dfrac{6n+2N}{7n+2N}\right)+\left[\dfrac{\left(\dfrac{N}{2n+4}\right)}{3n+4+N}\right]}{\left(\dfrac{3n+3+N}{3n+4+N}\right)+n\left(\dfrac{3n+N}{7n+2N}\right)} \quad . \quad . \quad . \quad . \quad (651)$$

If the K's are all equal and the values of Pe are all equal, the maximum moments are

$$M_{AB}=M_{AL}=0.454\ Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (652)$$

$$M_{AF}=0.358\ Pe \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (653)$$

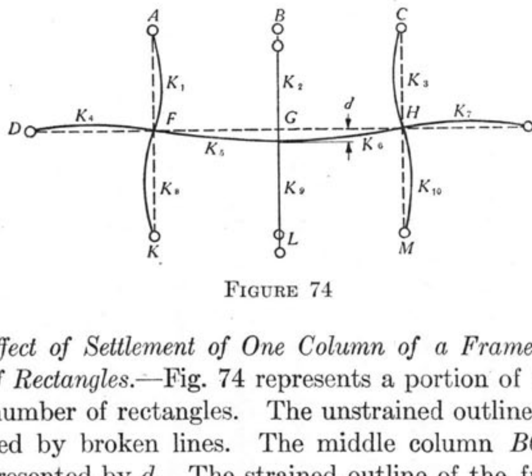


FIGURE 74

61. *Effect of Settlement of One Column of a Frame Composed of a Number of Rectangles.*—Fig. 74 represents a portion of a frame composed of a number of rectangles. The unstrained outline of the frame is represented by broken lines. The middle column BGL settles an amount represented by d. The strained outline of the frame is represented by the full lines. The points D, A, C, J, M, K, F, and H remain stationary.

For F to be in equilibrium

$$M_{FK}+M_{FD}+M_{FA}+M_{FG}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (654)$$

Substituting the values of the M's as given by the equations of Table 1 gives

$$3E\theta_F\left(K_8+K_4+K_1\right)+2EK_5\left(2\theta_F+\theta_G-3\frac{d}{l_5}\right)=0 \quad . \quad . \quad . \quad (655)$$

or

$$\theta_F=\frac{6K_5\frac{d}{l_5}-2K_5\,\theta_G}{3K_4+3K_1+3K_8+4K_5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (656)$$

$$M_{GF}=2EK_5(2\theta_G+\theta_F-3\frac{d}{l_5}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (657)$$

Substituting the value of θ_F from equation 656 in equation 657 gives

$$M_{GF}=EK_5\left[4\theta_G\left(\frac{3K_4+3K_1+3K_8+3K_5}{3K_4+3K_1+3K_8+4K_5}\right)-\right.$$

$$\left.\frac{6d}{l_5}\left(\frac{3K_4+3K_1+3K_8+2K_5}{3K_4+3K_1+3K_8+4K_5}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (658)$$

Similarly for the right-hand portion of the structure

$$M_{GH}=EK_6\left[4\theta_G\left(\frac{3K_7+3K_3+3K_{10}+3K_6}{3K_7+3K_3+3K_{10}+4K_6}\right)+\right.$$

$$\left.\frac{6d}{l_6}\left(\frac{3K_7+3K_3+3K_{10}+2K_6}{3K_7+3K_3+3K_{10}+4K_6}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (659)$$

$$M_{GB}=3EK_2\theta_G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (660)$$

$$M_{GL}=3EK_9\theta_G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (661)$$

For G to be in equilibrium

$$M_{GF}+M_{GH}+M_{GB}+M_{GL}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (662)$$

Substituting the value of the M's in equation 662 gives

$$\theta_G=\frac{6d\left[\frac{K_5}{l_5}\left(\frac{3K_4+3K_1+3K_8+2K_5}{3K_4+3K_1+3K_8+4K_5}\right)-\frac{K_6}{l_6}\left(\frac{3K_7+3K_3+3K_{10}+2K_6}{3K_7+3K_3+3K_{10}+4K_6}\right)\right]}{4K_5\left[\frac{3K_4+3K_1+3K_8+3K_5}{3K_4+3K_1+3K_8+4K_5}\right]+4K_6\left[\frac{3K_7+3K_3+3K_{10}+3K_6}{3K_7+3K_3+3K_{10}+4K_6}\right]+3K_2+3K_9} \quad (663)$$

From equation 656

$$\theta_F=\frac{6d\frac{K_5}{l_5}-2K_5\theta_G}{3K_4+3K_1+3K_8+4K_5} \quad (664)$$

Also

$$\theta_H=-\frac{6d\frac{K_6}{l_6}+2K_6\theta_G}{3K_7+3K_3+3K_{10}+4K_6} \quad (665)$$

If the portion of the frame DJ is symmetrical about G, $\theta_G=0$, and $\theta_F=-\theta_H$.

With θ_G, θ_F, and θ_H known the moments can be determined from the equations:

$$M_{FA}=3EK_1\theta_F \quad (666)$$

$$M_{FD}=3EK_4\theta_F \quad (667)$$

$$M_{FK}=3EK_8\theta_F \quad (668)$$

$$M_{FG}=2EK_5\left(2\theta_F+\theta_G-3\frac{d}{l_5}\right) \quad (669)$$

$$M_{GF}=2EK_5\left(2\theta_G+\theta_F-3\frac{d}{l_5}\right) \quad (670)$$

$$M_{GB}=3EK_2\theta_G \quad (671)$$

$$M_{GL}=3EK_9\theta_G \quad (672)$$

$$M_{GH}=2EK_6\left(2\theta_G+\theta_H+3\frac{d}{l_6}\right) \quad (673)$$

$$M_{HG}=2EK_6\left(2\theta_H+\theta_G+3\,\frac{d}{l_6}\right) \qquad (674)$$

$$M_{HC}=3EK_3\theta_H \qquad (675)$$

$$M_{HJ}=3EK_7\theta_H \qquad (676)$$

$$M_{HM}=3EK_{10}\theta_H \qquad (677)$$

If the members, instead of being hinged, are fixed at D, A, B, C, J, M, L, and K

$$\theta_G=\frac{6d\left[\frac{K_5}{l_5}\left(\frac{4K_4+4K_1+4K_8+2K_5}{4K_4+4K_1+4K_8+4K_5}\right)-\frac{K_6}{l_6}\left(\frac{4K_7+4K_3+4K_{10}+2K_6}{4K_7+4K_3+4K_{10}+4K_6}\right)\right]}{4K_5\left(\frac{4K_4+4K_1+4K_8+3K_5}{4K_4+4K_1+4K_8+4K_1}\right)+4K_6\left(\frac{4K_7+4K_3+4K_{10}+3K_6}{4K_7+4K_3+4K_{10}+4K_6}\right)+4K_2+4K_9} \qquad (678)$$

$$\theta_F=\frac{6d\,\frac{K_5}{l_5}-2K_5\,\theta_G}{4K_4+4K_1+4K_8+4K_5} \qquad (679)$$

$$\theta_H=-\frac{6d\,\frac{K_6}{l_6}+2K_6\,\theta_G}{4K_7+4K_3+4K_{10}+4K_6} \qquad (680)$$

$$M_{FA}=4EK_1\theta_F \qquad (681)$$

$$M_{FD}=4EK_4\theta_F \qquad (682)$$

$$M_{FK}=4EK_8\theta_F \qquad (683)$$

$$M_{FG}=2EK_5\left(2\theta_F+\theta_G-3\,\frac{d}{l_5}\right) \qquad (684)$$

$$M_{GF}=2EK_5\left(2\theta_G+\theta_F-3\,\frac{d}{l_5}\right) \qquad (685)$$

$$M_{GB}=4EK_2\theta_G \qquad (686)$$

$$M_{GL}=4EK_9\theta_G \quad . \quad . \quad . \quad (687)$$

$$M_{GH}=2EK_6\left(2\,\theta_G+\theta_H+\frac{3d}{l_6}\right) \quad . \quad . \quad . \quad (688)$$

$$M_{HG}=2EK_6\left(2\,\theta_H+\theta_G+\frac{3d}{l_6}\right) \quad . \quad . \quad . \quad (689)$$

$$M_{HC}=4EK_3\theta_H \quad . \quad . \quad . \quad (690)$$

$$M_{HJ}=4EK_7\theta_H \quad . \quad . \quad . \quad (691)$$

$$M_{HM}=4EK_{10}\theta_H \quad . \quad . \quad . \quad (692)$$

If the portion of the frame DJ is symmetrical about G, $\theta_G=0$, and $\theta_F=-\theta_H$.

The moments due to the settlement of one column for frames having the K's of all members equal and the lengths of all girders equal, and also for frames having the K's of all columns equal, the K's of all girders equal and the lengths of all girders equal are given in Table 17.

It is to be noted that the work in this section is based upon the assumption that there is no horizontal motion in the frame.

TABLE 17

MOMENTS IN A FRAME COMPOSED OF A NUMBER OF RECTANGLES, DUE TO SETTLEMENT OF ONE COLUMN

Moment	K's of All Members Equal Lengths of All Girders Equal		$\frac{I}{h}$ for All Columns Equals K; $\frac{I}{l}$ for All Girders Equals nK; Lengths of All Girders Equal	
	Ends Hinged	Ends Fixed	Ends Hinged	Ends Fixed
M_{GB} and M_{GL}	0	0	0	0
M_{FA}, M_{FK}, $-M_{HC}$, and $-M_{HM}$	$\frac{18}{13}\,\frac{KEd}{l}$	$\frac{3}{2}\,\frac{KEd}{l}$	$\frac{18n}{6+7n}\,\frac{KEd}{l}$	$\frac{3n}{1+n}\,\frac{KEd}{l}$
M_{FD} and $-M_{HJ}$	$\frac{18}{13}\,\frac{KEd}{l}$	$\frac{3}{2}\,\frac{KEd}{l}$	$\frac{18n^2}{6+7n}\,\frac{KEd}{l}$	$\frac{3n^2}{1+n}\,\frac{KEd}{l}$
M_{FG} and $-M_{HG}$	$-\frac{54}{13}\,\frac{KEd}{l}$	$-\frac{9}{2}\,\frac{KEd}{l}$	$\frac{-18n(2+n)}{6+7n}\,\frac{KEd}{l}$	$-3n\left(\frac{2+n}{1+n}\right)\frac{KEd}{l}$
M_{GF} and $-M_{GH}$	$-\frac{66}{13}\,\frac{KEd}{l}$	$-\frac{21}{4}\,\frac{KEd}{l}$	$\frac{-6n(6+5n)}{6+7n}\,\frac{KEd}{l}$	$\frac{-3n}{2}\left(\frac{4+3n}{1+n}\right)\frac{KEd}{l}$

XI. Three-legged Bent

62. *Three-legged Bent. Lengths of All Legs Different. Vertical Load on Left-hand Span—Legs Hinged at the Bases.*—Fig. 75 represents a three-legged bent. The lengths of all legs are different P represents the resultant of any system of loads on AB. The legs of the bent are hinged at D, C, and E.

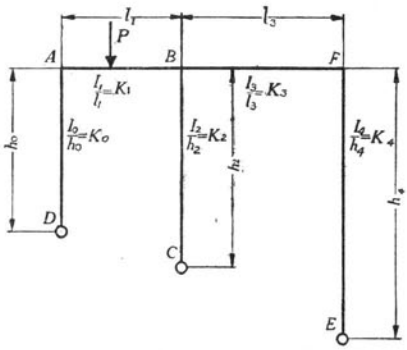


Figure 75

Since axial strains are neglected the vertical deflections of B relative to A and F are zero. Likewise the horizontal deflections of A, B, and F are equal. This deflection is represented by d.

Applying the equations of Table 1 gives

$$M_{AD}=3EK_o\left(\theta_A-\frac{d}{h_o}\right) \quad (693)$$

$$M_{AB}=2EK_1(2\theta_A+\theta_B)-C_{AB} \quad (694)$$

$$M_{BA}=2EK_1(2\theta_B+\theta_A)+C_{BA} \quad (695)$$

$$M_{BC}=3EK_2\left(\theta_B-\frac{d}{h_2}\right) \quad (696)$$

$$M_{BF}=2EK_3(2\theta_B+\theta_F) \quad (697)$$

$$M_{FB}=2EK_3(2\theta_F+\theta_B) \quad (698)$$

$$M_{FE}=3EK_4\left(\theta_F-\frac{d}{h_4}\right) \quad (699)$$

For equilibrium at A, B, and F

$$M_{AD}+M_{AB}=0 \qquad (700)$$

$$M_{BA}+M_{BC}+M_{BF}=0 \qquad (701)$$

$$M_{FB}+M_{FE}=0 \qquad (702)$$

For the bent as a whole to be in equilibrium

$$\frac{M_{AD}}{h_o}+\frac{M_{BC}}{h_2}+\frac{M_{FE}}{h_4}=0 \qquad (703)$$

Substituting the values of the moments from equations 693 to 699 inclusive in equations 700, 701, 702, and 703 gives

$$3EK_o\theta_A-3EK_o\frac{d}{h_o}+4EK_1\theta_A+2EK_1\theta_B=C_{AB} \qquad (704)$$

$$4EK_1\theta_B+2EK_1\theta_A+3EK_2\theta_B-3EK_2\frac{d}{h_2}+4EK_3\theta_B+2EK_3\theta_F = -C_{BA} \qquad (705)$$

$$4EK_3\theta_F+2EK_3\theta_B+3EK_4\theta_F-3EK_4\frac{d}{h_4}=0 \qquad (706)$$

$$\frac{3EK_o}{h_o}\theta_A-3EK_o\frac{d}{h_o^2}+\frac{3EK_2}{h_2}\theta_B-3EK_2\frac{d}{h_2^2}+\frac{3EK_4}{h_4}\theta_F-3EK_4\frac{d}{h_4^2} =0 \qquad (707)$$

These four equations contain only four unknowns, and it is therefore possible to combine the equations and solve for the unknowns. The resulting expressions, however, are so complex that it is more practicable to substitute numerical values for E, the K's and the h's and then solve for the unknowns, d and the θ's, by a process of elimination. Knowing d and the θ's, the moments can be determined from equations 693 to 699 inclusive.

For convenience in eliminating the unknowns, equations 704, 705, 706, and 707 are reproduced in Table 18. In this table the unknowns are at the tops of the columns and the coefficients of the unknowns are in the lines below. Equations A to D of Table 18 are identical with equations 704 to 707.

TABLE 18

EQUATIONS FOR THREE-LEGGED BENT

Lengths of All Legs Different. Vertical Load on Left-hand Span. Legs Hinged at the Bases.

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation
	θ_A	θ_B	θ_F	d	
A	$3K_o+4K_1$	$2K_1$	0	$-\dfrac{3K_o}{h_o}$	$+\dfrac{C_{AB}}{E}$
B	$2K_1$	$4K_1+3K_2+4K_3$	$2K_3$	$-\dfrac{3K_2}{h_2}$	$-\dfrac{C_{BA}}{E}$
C	0	$2K_3$	$4K_3+3K_4$	$-\dfrac{3K_4}{h_4}$	0
D	$-\dfrac{3K_o}{h_o}$	$-\dfrac{3K_2}{h_2}$	$-\dfrac{3K_4}{h_4}$	$+3\left[\dfrac{K_o}{h_o^2}+\dfrac{K_2}{h_2^2}+\dfrac{K_4}{h_4^2}\right]$	0

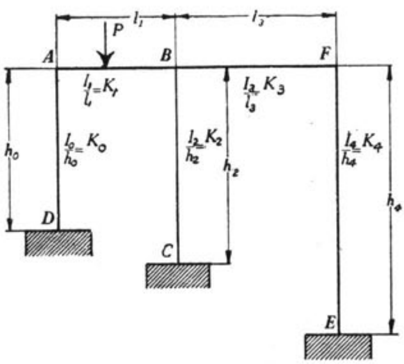


FIGURE 76

63. *Three-legged Bent. Lengths of All Legs Different. Vertical Load on Left-hand Span—Legs Fixed at the Bases.*—Fig. 76 represents a three-legged bent. The lengths of all legs are different. P represents the resultant of any system of loads on AB. The legs of the bent are fixed at D, C, and E.

From the equations of Table 1

$$M_{AD}=2EK_o\left(2\theta_A-3\frac{d}{h_o}\right) \quad . \quad . \quad . \quad (708)$$

$$M_{AB}=2EK_1(2\theta_A+\theta_B)-C_{AB} \quad . \quad . \quad . \quad (709)$$

$$M_{BA}=2EK_1(2\theta_B+\theta_A)+C_{BA} \quad . \quad . \quad . \quad (710)$$

$$M_{BC}=2EK_2\left(2\theta_B-3\frac{d}{h_2}\right) \quad . \quad . \quad . \quad (711)$$

$$M_{BF}=2EK_3(2\theta_B+\theta_F) \quad . \quad . \quad . \quad (712)$$

$$M_{FB}=2EK_3(2\theta_F+\theta_B) \quad . \quad . \quad . \quad (713)$$

$$M_{FE}=2EK_4\left(2\theta_F-3\frac{d}{h_4}\right) \quad . \quad . \quad . \quad (714)$$

$$M_{DA}=2EK_o\left(\theta_A-3\frac{d}{h_o}\right) \quad . \quad . \quad . \quad (715)$$

$$M_{CB}=2EK_2\left(\theta_B-3\frac{d}{h_2}\right) \quad . \quad . \quad . \quad (716)$$

$$M_{EF}=2EK_4\left(\theta_F-3\frac{d}{h_4}\right) \quad . \quad . \quad . \quad (717)$$

For equilibrium at A, B, and F, equations 700, 701, and 702 are applicable.

For the bent as a whole to be in equilibrium

$$\frac{M_{AD}+M_{DA}}{h_o}+\frac{M_{BC}+M_{CB}}{h_2}+\frac{M_{FE}+M_{EF}}{h_4}=0 \quad . \quad . \quad . \quad (718)$$

Substituting the values of the moments from equations 708 to 717 in equations 700, 701, 702, and 718 gives the equations of Table 19.

64. *Three-legged Bent. Lengths of All Legs Different. Any System of Loads.*—No matter what system of loads is applied to the three-legged bents of Figs. 75 and 76, equations similar to equations 693 to 699, 708 to 718 and 700 to 703 can be written. These equations will

contain no unknown quantities not found in the equations of section 62, and therefore they can be combined to obtain equations similar to the equations of Tables 18 and 19. The left-hand members of the equations for all bents with legs hinged at the bases will be identical with the left-hand members of the equations of Table 18, and the left-hand members of the equations for all bents with legs fixed at the bases will be identical with the left-hand members of the equations of Table 19.

The equations of Table 1 as applied to a three-legged bent having legs hinged at the bases and carrying a number of systems of loads are given in Table 20. Similar equations for a three-legged bent having legs fixed at the bases are given in Table 21. Four equations containing four unknowns, derived from the equations of Table 20 are given in Table 22, and four equations containing four unknowns based upon the equations of Table 21 are given in Table 23. The equations of Table 22 can be used to determine the stresses in a bent having legs hinged at the bases, and the equations of Table 23 can be used to determine the stresses in a bent having legs fixed at the bases. A numerical problem illustrating the use of the equation in Table 23 is presented in section 76.

TABLE 19

EQUATIONS FOR THREE-LEGGED BENT

Lengths of All Legs Different. Vertical Load on Left-hand Span. Legs Fixed at the Bases.

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation
	θ_A	θ_B	θ_F	d	
A	$4K_o+4K_1$	$2K_1$	0	$-6\dfrac{K_o}{h_o}$	$+\dfrac{C_{AB}}{E}$
B	$2K_1$	$4K_1+4K_2+4K_3$	$2K_3$	$-6\dfrac{K_2}{h_2}$	$-\dfrac{C_{BA}}{E}$
C	0	$2K_3$	$4K_3+4K_4$	$-6\dfrac{K_4}{h_4}$	0
D	$-6\dfrac{K_o}{h_o}$	$-6\dfrac{K_2}{h_2}$	$-6\dfrac{K_4}{h_4}$	$+12\left[\dfrac{K_o}{h_o^2}+\dfrac{K_2}{h_2^2}+\dfrac{K_4}{h_4^2}\right]$	0

The 's and d can be determined from the equations of Table 19 by a process of elimination. Knowing the θ's and d, the moments can be determined from equations 708 to 717.

TABLE 20

FUNDAMENTAL EQUATIONS FOR THREE-LEGGED BENTS

Lengths of All Legs Different. Any System of Loads. Legs Hinged at the Bases.

No. of Equation	This part of the equation is independent of the loads	Additional term of the right-hand member of the equation, different for each system of loads, as follows: Case I Vertical Load on Left-hand Span See Fig. 77	Case II Vertical Load on Right-hand Span See Fig. 78	Case III Horizontal Load to the Right on Left-hand Leg See Fig. 79	Case IV Horizontal Load to the Left on Right-hand Leg See Fig. 80	Case V Settlement and Sliding of Foundations See Fig. 81.	Case VI External Couple at Upper Left-hand Corner See Fig. 82
1	$M_{AD}=3EK_o\left(\theta_A-\frac{d}{h_o}\right)$	0	0	$+H_{AD}$	0	0	0
2	$M_{AB}=2EK_1(2\theta_A+\theta_B)$	$-C_{AB}$	0	0	0	$-6EK_1\frac{d_1}{l_1}$	0
3	$M_{BA}=2EK_1(2\theta_B+\theta_A)$	$+C_{BA}$	0	0	0	$-6EK_1\frac{d_1}{l_1}$	0
4	$M_{BC}=3EK_2\left(\theta_B-\frac{d}{h_2}\right)$	0	0	0	0	$+3EK_2\frac{d_2}{h_2}$	0

TABLE 20—CONTINUED

5	$M_{BF}=2EK_3(2\theta_B+\theta_F)$	0	$-C_{BF}$	0	0	$-6EK_3\frac{(d_3-d_1)}{l_3}$	0
6	$M_{FB}=2EK_3(2\theta_F+\theta_B)$	0	$+C_{FB}$	0	0	$-6EK_3\frac{(d_3-d_1)}{l_3}$	0
7	$M_{FE}=3EK_4\left(\theta_F-\frac{d}{h_4}\right)$	0	0	0	$-H_{FE}$	$+3EK_4\frac{d_4}{h_4}$	0
8	$0=M_{AD}+M_{AB}$	0	0	0	0	0	$-M$
9	$0=M_{BA}+M_{BF}+M_{BC}$	0	0	0	0	0	0
10	$0=M_{FB}+M_{FE}$	0	0	0	0	0	0
11	$0=\frac{M_{AD}}{h_o}+\frac{M_{BC}}{h_2}+\frac{M_{FE}}{h_4}$	0	0	$+\frac{M_D}{h_o}$	$+\frac{M_E}{h_4}$	0	0

M_D is the statical moment about point D of the horizontal loads on left-hand leg; M_E is the statical moment about point E of the horizontal loads on right-hand leg. The statical moment is positive when the indicated rotation is clockwise.

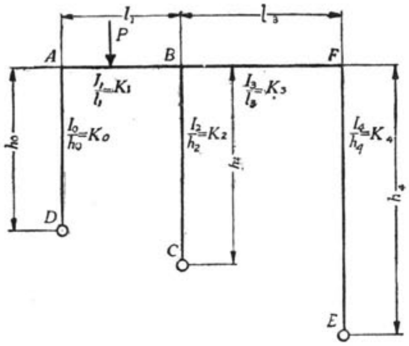


FIGURE 77

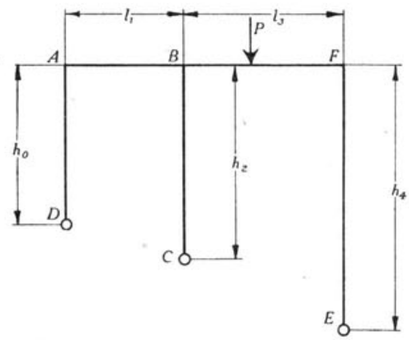


FIGURE 78

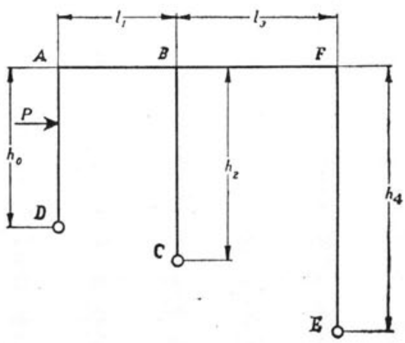


FIGURE 79

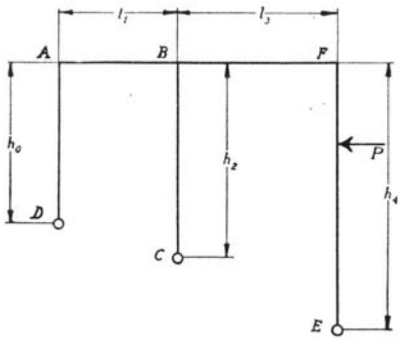


FIGURE 80

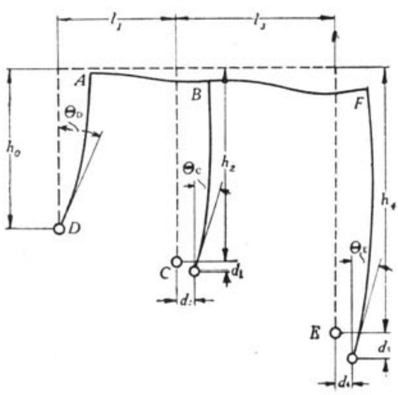


FIGURE 81

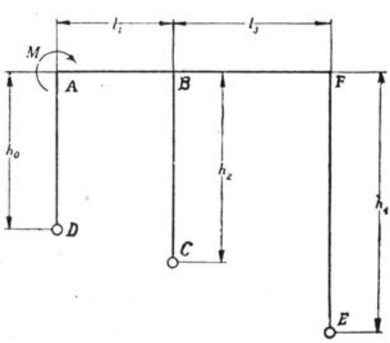


FIGURE 82

TABLE 21

FUNDAMENTAL EQUATIONS FOR THREE-LEGGED BENTS

Lengths of All Legs Different. Any System of Loads. Legs Fixed at the Bases.

No. of Equation	This part of the equation is independent of the loads	Additional term of the right-hand member of the equation, different for each system of loads, as follows: Case VII Vertical Load on Left-hand Span See Fig. 83	Case VIII Vertical Load on Right-hand Span See Fig. 84	Case IX Horizontal Load to the Right on Left-hand Leg See Fig. 85	Case X Horizontal Load to the Left on Right-hand Leg See Fig. 86	Case XI Settlement and Rotation of Foundations See Fig. 87	Case XII External Couple at Upper Left-hand Corner See Fig. 88
1	$M_{AD}=2EK_o\left(2\theta_A-3\frac{d}{h_o}\right)$	0	0	$+C_{AD}$	0	$+2EK_o\theta_D$	0
2	$M_{AB}=2EK_1(2\theta_A+\theta_B)$	$-C_{AB}$	0	0	0	$-6EK_1\frac{d_1}{l_1}$	0
3	$M_{BA}=2EK_1(2\theta_B+\theta_A)$	$+C_{BA}$	0	0	0	$-6EK_1\frac{d_1}{l_1}$	0
4	$M_{BC}=2EK_2\left(2\theta_B-3\frac{d}{h_2}\right)$	0	0	0	0	$+6EK_2\frac{d_2}{h_2}+2EK_2\theta_C$	0

TABLE 21—CONTINUED

5	$M_{BF}=2EK_3(2\theta_B+\theta_F)$	0	$-C_{BF}$	0	0	$-6EK_3\dfrac{d_3-d_1}{l_3}$	0
6	$M_{FB}=2EK_3(2\theta_F+\theta_B)$	0	$+C_{FB}$	0	0	$-6EK_3\dfrac{d_3-d_1}{l_3}$	0
7	$M_{FE}=2EK_4\left(2\theta_F-3\dfrac{d}{h_4}\right)$	0	0	0	$-C_{FE}$	$+6EK_4\dfrac{d_4}{h_4}+2EK_4\theta_E$	0
8	$M_{DA}=2EK_o\left(\theta_A-3\dfrac{d}{h_o}\right)$	0	0	$-C_{DA}$	0	$+4EK_o\theta_D$	0
9	$M_{CB}=2EK_2\left(\theta_B-3\dfrac{d}{h_2}\right)$	0	0	0	0	$+6EK_2\dfrac{d_2}{h_2}+4EK_2\theta_C$	0
10	$M_{EF}=2EK_4\left(\theta_F-3\dfrac{d}{h_4}\right)$	0	0	0	$+C_{EF}$	$+6EK_4\dfrac{d_4}{h_4}+4EK_4\theta_E$	0
11	$0=M_{AD}+M_{AB}$	0	0	0	0	0	$-M$
12	$0=M_{BA}+M_{BF}+M_{BC}$	0	0	0	0	0	0
13	$0=M_{FB}+M_{FE}$	0	0	0	0	0	0
14	$0=\dfrac{M_{AD}+M_{DA}}{h_o}+\dfrac{M_{BC}+M_{CB}}{h_2}+\dfrac{M_{FE}+M_{EF}}{h_4}$	0	0	$+\dfrac{M_D}{h_o}$	$+\dfrac{M_E}{h_4}$	0	0

For definition of M_D and M_E see Table 20.

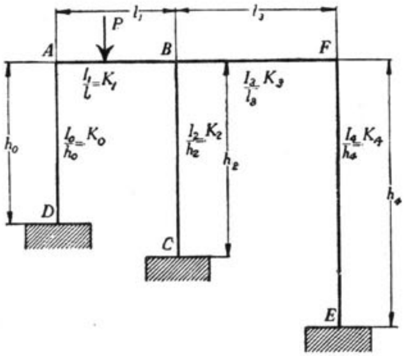


FIGURE 83

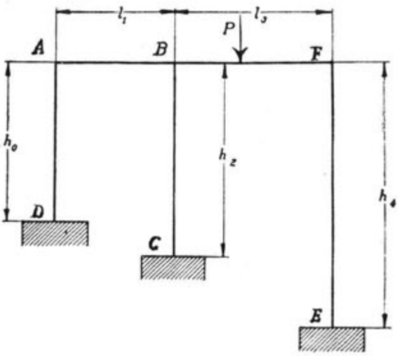


FIGURE 84

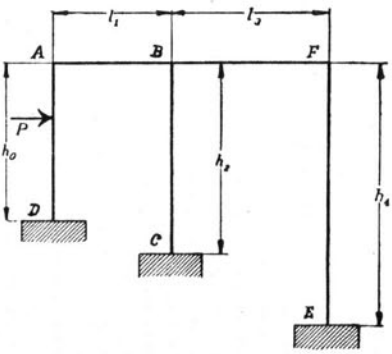


FIGURE 85

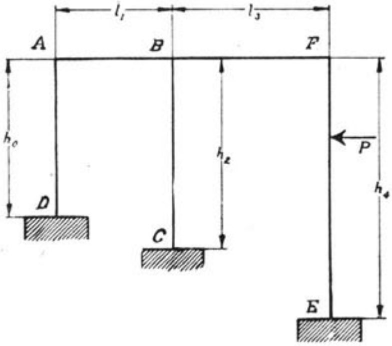


FIGURE 86

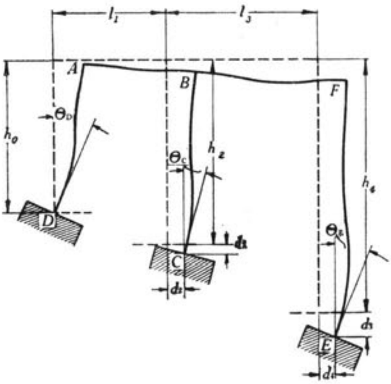


FIGURE 87

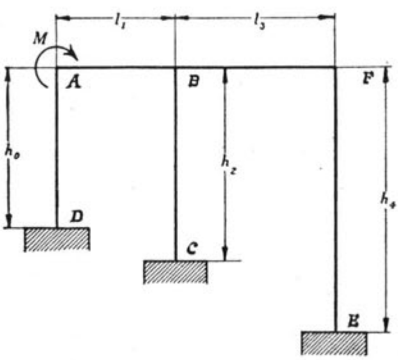


FIGURE 88

TABLE 22

EQUATIONS FOR THREE-LEGGED BENT

Lengths of All Legs Different. Any System of Loads. Legs Hinged at the Bases.

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation					
	θ_A	θ_B	θ_F	d	Case I Vertical Load on Left-hand Span See Fig. 77	Case II Vertical Load on Right-hand Span See Fig. 78	Case III Horizontal Load to the Right on Left-hand Leg See Fig. 79	Case IV Horizontal Load to the Left on Right-hand Leg See Fig. 80	Case V Settlement and Sliding of Foundations See Fig. 81	Case VI External Couple at Upper Left-hand Corner See Fig. 82
A	$3K_o+4K_1$	$2K_1$	0	$-3\frac{K_o}{h_o}$	$\frac{C_{AB}}{E}$	0	$-\frac{H_{AD}}{E}$	0	$6K_1\frac{d_1}{l_1}$	$\frac{M}{E}$
B	$2K_1$	$4K_1+3K_2+4K_3$	$2K_3$	$-3\frac{K_2}{h_2}$	$-\frac{C_{BA}}{E}$	$\frac{C_{BF}}{E}$	0	0	$6K_1\frac{d_1}{l_1}+6K_3\frac{d_3-d_1}{l_3}-3K_2\frac{d_2}{h_2}$	0
C	0	$2K_3$	$4K_3+3K_4$	$-3\frac{K_4}{h_4}$	0	$-\frac{C_{FB}}{E}$	0	$\frac{H_{FE}}{E}$	$6K_3\frac{(d_3-d_1)}{l_3}-3K_4\frac{d_4}{h_4}$	0
D	$\frac{K_o}{h_o}$	$\frac{K_2}{h_2}$	$\frac{K_4}{h_4}$	$-\left[\frac{K_o}{h_o^2}+\frac{K_2}{h_2^2}+\frac{K_4}{h_4^2}\right]$	0	0	$-\frac{1}{3Eh_o}\left[M_D+H_{AD}\right]$	$-\frac{1}{3Eh_4}\left[M_E-H_{FE}\right]$	$-K_2\frac{d_2}{h_2^2}-K_4\frac{d_4}{h_4^2}$	0

For definition of M_D and M_E see Table 20.

TABLE 23

EQUATIONS FOR THREE-LEGGED BENT

Lengths of All Legs Different. Any System of Loads. Legs Fixed at the Bases.

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation					
	θ_A	θ_B	θ_F	d	Case VII Vertical Load on Left-hand Span See Fig. 83	Case VIII Vertical Load on Right-hand Span See Fig. 84	Case IX Horizontal Load to the Right on Left-hand Leg See Fig. 85	Case X Horizontal Load to the Left on Right-hand Leg See Fig. 86	Case XI Settlement and Rotation of Foundations See Fig. 87	Case XII External Couple at Upper Left-hand Corner See Fig. 88
A	$4K_o+4K_1$	$2K_1$	0	$-6\frac{K_o}{h_o}$	$+\frac{C_{AB}}{E}$	0	$-\frac{C_{AD}}{E}$	0	$6K_1\frac{d_1}{l_1}-2K_o\theta_D$	$+\frac{E}{M}$
B	$2K_1$	$4K_1+4K_2+4K_3$	$2K_3$	$-6\frac{K_2}{h_2}$	$-\frac{C_{BA}}{E}$	$+\frac{C_{BF}}{E}$	0	0	$6K_1\frac{d_1}{l_1}-6K_2\frac{d_2}{h_2}+6K_3\frac{d_3-d_1}{l_3}-2K_2\theta_C$	0
C	0	$2K_3$	$4K_3+4K_4$	$-6\frac{K_4}{h_4}$	0	$-\frac{C_{FB}}{E}$	0	$+\frac{C_{FE}}{E}$	$6K_3\frac{d_3-d_1}{l_3}-6K_4\frac{d_4}{h_4}-2K_4\theta_E$	0
D	$\frac{K_o}{h_o}$	$\frac{K_2}{h_2}$	$\frac{K_4}{h_4}$	$-2\left[\frac{K_o}{h_o^2}+\frac{K_2}{h_2^2}+\frac{K_4}{h_4^2}\right]$	0	0	$-\frac{1}{6Eh_o}\left[M_D+C_{AD}-C_{DA}\right]$	$-\frac{1}{6Eh_4}\left[M_E+C_{EF}-C_{FE}\right]$	$-2K_2\frac{d_2}{h_2^2}-2K_4\frac{d_4}{h_4^2}-\frac{K_o\theta_D}{h_o}-\frac{K_2\theta_C}{h_2}-\frac{K_4\theta_E}{h_4}$	0

For definition of M and M_E see Table 20.

65. *Three-legged Bent. Lengths and Sections of All Legs Equal. Lengths and Sections of Top Members Equal. Any System of Loads. Let* $\frac{I}{l}$ *of a Top Member be* n *times* $\frac{I}{h}$ *of a Leg. Legs Hinged at the Bases.*—Fig. 89 represents a three-legged bent. The lengths of the legs are equal, and the lengths of the top members are equal. The $\frac{I}{l}$ of a top member is designated as K, and the $\frac{I}{h}$ of a leg is designated as $\frac{K}{n}$. The legs of the bent are hinged at D, C, and E.

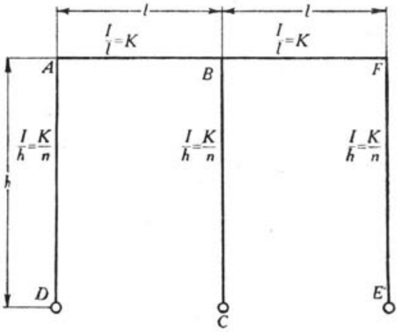


FIGURE 89

The equations of Table 1 as applied to the three-legged bent represented in Fig. 89 are given in Table 24 for a number of systems of loads. Four equations containing four unknowns, derived from the equations

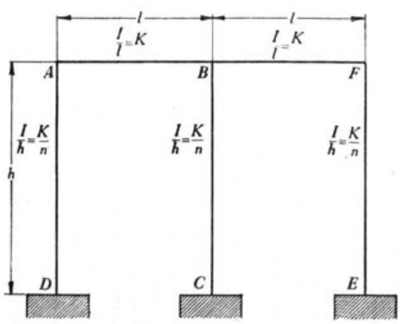


FIGURE 90

of Table 24, are given in Table 25, and the moments at the ends of the members as found from the equations of Tables 24 and 25 are given in Table 26.

66. *Three-legged Bent. Lengths and Sections of All Legs Equal. Lengths and Sections of Top Members Equal. Any System of Loads—Legs Fixed at the Bases.*—Fig. 90 represents a three-legged bent. The lengths of the legs are equal, and the lengths of the top members are equal. The $\frac{I}{l}$ of a top member is designated as K, and the $\frac{I}{h}$ is designated as $\frac{K}{n}$. The legs of the bent are fixed at D, C, and E.

The equations of Table 1 as applied to the three-legged bent represented in Fig. 90 are given in Table 27 for a number of systems of loads. Four equations containing four unknowns, derived from the equations of Table 27, are given in Table 28, and the moments at the ends of the members as found from the equations of Tables 27 and 28 are given in Table 29.

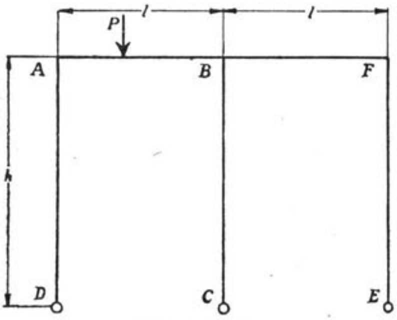


FIGURE 91

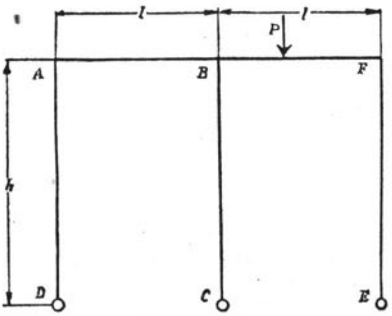


FIGURE 92

TABLE 24

FUNDAMENTAL EQUATIONS FOR THREE-LEGGED BENT

Lengths and Sections of Top Members Equal. $\left(\frac{I}{l} = K\right)$

Lengths and Sections of All Legs Equal. $\left(\frac{I}{h} = \frac{K}{n}\right)$

Any System of Loads. Legs Hinged at the Bases.

No. of Equation	This part of the equation is independent of the loads	Additional terms of the right-hand member of the equation, different for each system of loads, as follows:					
		Case I Vertical Load on Left-hand Span See Fig. 91	Case II Vertical Load on Right-hand Span See Fig. 92	Case III Horizontal Load to the Right on Left-hand Leg See Fig. 93	Case IV Horizontal Load to the Left on Right-hand Leg See Fig. 94	Case V Settlement and Sliding of Foundations See Fig. 95	Case VI External Couple at Upper Left-hand Corner See Fig. 96
1	$M_{AD} = 3E\frac{K}{n}\left(\theta_A - R\right)$	0	0	$+H_{AD}$	0	0	0
2	$M_{AB} = 2EK(2\theta_A + \theta_B)$	$-C_{AB}$	0	0	0	$-6EK\frac{d_1}{l}$	0
3	$M_{BA} = 2EK(2\theta_B + \theta_A)$	$+C_{BA}$	0	0	0	$-6EK\frac{d_1}{l}$	0

TABLE 24—CONTINUED

4	$M_{BC}=3E\ \frac{K}{n}\left(\theta_B-R\right)$	0	0	0	0	$+3E\ \frac{K}{n}\ \frac{d_2}{h}$	0
5	$M_{BF}=2EK(2\theta_B+\theta_F)$	0	$-C_{BF}$	0	0	$-6EK\ \frac{d_3-d_1}{l}$	0
6	$M_{FB}=2EK(2\theta_F+\theta_B)$	0	$+C_{FB}$	0	0	$-6EK\ \frac{d_3-d_1}{l}$	0
7	$M_{FE}=3E\ \frac{K}{n}\left(\theta_F-R\right)$	0	0	0	$-H_{FE}$	$+3E\ \frac{K}{n}\ \frac{d_4}{h}$	0
8	$0=M_{AD}+M_{AB}$	0	0	0	0	0	$-M$
9	$0=M_{BA}+M_{BF}+M_{BC}$	0	0	0	0	0	0
10	$0=M_{FB}+M_{FE}$	0	0	0	0	0	0
11	$0=M_{AD}+M_{BC}+M_{FE}$	0	0	$+M_D$	$+M_E$	0	0

M_D is the statical moment about point D of the horizontal loads on left-hand leg.
M_E is the statical moment about point E of the horizontal loads on right-hand leg.
The statical moment is positive when the indicated rotation is clockwise.

TABLE 25

EQUATIONS FOR THREE-LEGGED BENT

Lengths and Sections of Top Member Equal. $\left(\frac{I}{l}=K.\right)$ Lengths and Sections of All Legs Equal. $\left(\frac{I}{h}=\frac{K}{n}\right)$

Any System of Loads. Legs Hinged at the Bases.

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation					
	θ_A	θ_B	θ_F	$3R$	Case I Vertical Load on Left-hand Span See Fig. 91	Case II Vertical Load on Right-hand Span See Fig. 92	Case III Horizontal Load to the Right on Left-hand Leg See Fig. 93	Case IV Horizontal Load to the Left on Right-hand Leg See Fig. 94	Case V Settlement and Sliding of Foundations See Fig. 95	Case VI External Couple at Upper Left-hand Corner See Fig. 96
A	$4n+3$	$2n$	0	-1	$\frac{n.C_{AB}}{EK}$	0	$\frac{-n.H_{AD}}{EK}$	0	$6n\ \frac{d_1}{l}$	$\frac{Mn}{EK}$
B	$2n$	$8n+3$	$2n$	-1	$\frac{-n.C_{BA}}{EK}$	$\frac{n.C_{BF}}{EK}$	0	0	$6n.\frac{d_3}{l}-\frac{3d_2}{h}$	0
C	0	$2n$	$4n+3$	-1	0	$\frac{-n.C_{FB}}{EK}$	0	$\frac{n.H_{FE}}{EK}$	$\frac{6n(d_3-d_1)}{l}-\frac{3d_4}{h}$	0
D	1	1	1	-1	0	0	$\frac{-n}{3EK}\left[M_D+H_{AD}\right]$	$\frac{-n}{3EK}\left[M_E-H_{FE}\right]$	$-\frac{d_2+d_4}{h}$	0

For definition of M_D and M_E see Table 24.

Table 26

Moments Due to Various Systems of Loads on a Three-legged Bent

Lengths and Sections of Top Members Equal. $\left(\frac{I}{l} = K\right)$

Lengths and Sections of All Legs Equal. $\left(\frac{I}{h} = \frac{K}{n}\right)$

Legs Hinged at the Bases.

No. of Equation	Moment	Case I Vertical Load on Left-hand Span See Fig. 91	Case II Vertical Load on Right-hand Span See Fig. 92	Case III Horizontal Load to the Right on Left-hand Leg See Fig. 93	Case IV Horizontal Load to the Left on Right-hand Leg See Fig. 94	Case V Settlement and Sliding of Foundation See Fig. 95	Case VI External Couple at Upper Left-hand Corner See Fig. 96
1	$M_{AD}=$	$+\frac{C_{AB}(10n+9)+C_{BA}(4n+3)}{4(n+1)\ (4n+3)}$	$-\frac{C_{BF}(4n+3)-C_{FB}(2n+3)}{4(n+1)\ (4n+3)}$	$+\frac{2nH_{AD}(16n+15)-M_D(4n+3)^2}{12(n+1)\ (4n+3)}$	$+\frac{2nH_{FE}(8n+9)-M_E(4n+3)^2}{12(n+1)\ (4n+3)}$	$+\frac{EK}{2hl}\quad\frac{18h(n+1)\ (2d_1-d_3)-l\,[d_2(8n+6)+d_4(8n+9)]}{(n+1)\ (4n+3)}$	$+\frac{M(10n+9)}{4(n+1)\ (4n+3)}$
2	$M_{AB}=$	$-\frac{C_{AB}(10n+9)+C_{BA}(4n+3)}{4(n+1)\ (4n+3)}$	$+\frac{C_{BF}(4n+3)-C_{FB}(2n+3)}{4(n+1)\ (4n+3)}$	$-\frac{2nH_{AD}(16n+15)-M_D(4n+3)^2}{12(n+1)\ (4n+3)}$	$-\frac{2nH_{FE}(8n+9)-M_E(4n+3)^2}{12(n+1)\ (4n+3)}$	$-\frac{EK}{2hl}\cdot\frac{18h(n+1)\ (2d_1-d_3)-l\,[d_2(8n+6)+d_4(8n+9)]}{(n+1)\ (4n+3)}$	$+\frac{M(16n^2+18n+3)}{4(n+1)\ (4n+3)}$
3	$M_{BA}=$	$+\frac{(2n+3)\ [C_{AB}(2n+1)+C_{BA}(4n+3)]}{4(n+1)\ (4n+3)}$	$+\frac{C_{BF}(4n+3)\ (2n+1)+C_{FB}(4n^2-3)}{4(n+1)\ (4n+3)}$	$-\frac{(2n+3)\ [2nH_{AD}-M_D(4n+3)]}{12(n+1)\ (4n+3)}$	$-\frac{2nH_{FE}(10n+9)-M_E(4n+3)\ (2n+3)}{12(n+1)\ (4n+3)}$	$-\frac{EK}{2hl}\cdot\frac{6h(n+1)\ (2n+3)\ (2d_1-d_3)+l\,[d_2(8n+6)-d_4(10n+9)]}{(n+1)\ (4n+3)}$	$+\frac{M(2n+1)\ (2n+3)}{4(n+1)\ (4n+3)}$
4	$M_{BC}=$	$-\frac{C_{AB}+C_{BA}}{2(n+1)}$	$+\frac{C_{BF}+C_{FB}}{2(n+1)}$	$-\frac{2nH_{AD}+M_D(2n+3)}{6(n+1)}$	$+\frac{2nH_{FE}-M_E(2n+3)}{6(n+1)}$	$+\frac{EK}{h}\cdot\frac{2d_2-d_4}{n+1}$	$-\frac{M}{2(n+1)}$
5	$M_{BF}=$	$-\frac{C_{AB}(4n^2-3)+C_{BA}(4n+3)\ (2n+1)}{4(n+1)\ (4n+3)}$	$-\frac{(2n+3)\ [C_{BF}(4n+3)+C_{FB}(2n+1)]}{4(n+1)\ (4n+3)}$	$+\frac{2nH_{AD}(10n+9)+M_D(4n+3)\ (2n+3)}{12(n+1)\ (4n+3)}$	$+\frac{(2n+3)\ [2nH_{FE}+M_E(4n+3)]}{12(n+1)\ (4n+3)}$	$+\frac{EK}{2hl}\cdot\frac{6h(n+1)\ (2n+3)\ (2d_1-d_3)-l\,[d_2(8n+6)+d_4(2n+3)]}{(n+1)\ (4n+3)}$	$-\frac{M(4n^2-3)}{4(n+1)\ (4n+3)}$
6	$M_{FB}=$	$+\frac{C_{AB}(2n+3)-C_{BA}(4n+3)}{4(n+1)\ (4n+3)}$	$+\frac{C_{BF}(4n+3)+C_{FB}(10n+9)}{4(n+1)\ (4n+3)}$	$+\frac{2nH_{AD}(8n+9)+M_D(4n+3)^2}{12(n+1)\ (4n+3)}$	$+\frac{2nH_{FE}(16n+15)+M_E(4n+3)^2}{12(n+1)\ (4n+3)}$	$+\frac{EK}{2hl}\cdot\frac{18h(n+1)\ (2d_1-d_3)+l\,[d_2(8n+6)-d_4(16n+15)]}{(n+1)\ (4n+3)}$	$+\frac{M(2n+3)}{4(n+1)\ (4n+3)}$
7	$M_{FE}=$	$-\frac{C_{AB}(2n+3)-C_{BA}(4n+3)}{4(n+1)\ (4n+3)}$	$-\frac{C_{BF}(4n+3)+C_{FB}(10n+9)}{4(n+1)\ (4n+3)}$	$-\frac{2nH_{AD}(8n+9)+M_D(4n+3)^2}{12(n+1)\ (4n+3)}$	$-\frac{2nH_{FE}(16n+15)+M_E(4n+3)^2}{12(n+1)\ (4n+3)}$	$-\frac{EK}{2hl}\cdot\frac{18h(n+1)\ (2d_1-d_3)+l\,[d_2(8n+6)-d_4(16n+15)]}{(n+1)\ (4n+3)}$	$-\frac{M(2n+3)}{4(n+1)\ (4n+3)}$

For definition of M_D and M_E see Table 24.

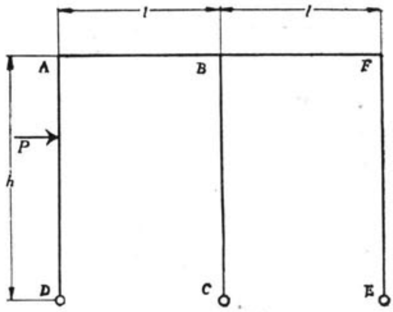


FIGURE 93

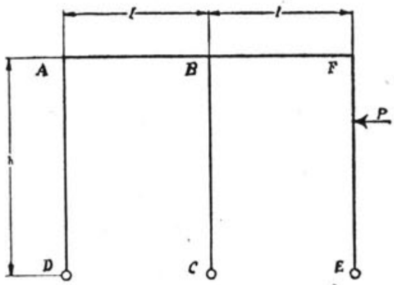


FIGURE 94

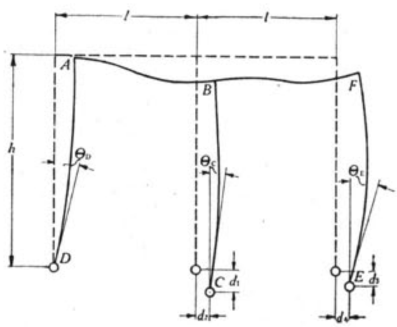


FIGURE 95

TABLE 27

FUNDAMENTAL EQUATIONS FOR THREE-LEGGED BENT

Lengths and Sections of Top Members Equal. $\left(\frac{I}{l}=K\right)$ Lengths and Sections of All Legs Equal. $\left(\frac{I}{h}=\frac{K}{n}\right)$

Any System of Loads. Legs Fixed at the Bases.

No. of Equation	This part of the equation is independent of the loads	Additional term of the right-hand member of the equation, different for each system of loads, as follows:				
		Case VII Vertical Load on Left-hand Span See Fig. 97	Case VIII Vertical Load on Right-hand Span See Fig. 98	Case IX Horizontal Load to the Right on Left-hand Leg See Fig. 99	Case X Horizontal Load to the Left on Right-hand Leg See Fig. 100	Case XII External Couple at Upper Left-hand Corner See Fig. 101
1	$M_{AD}=2E\frac{K}{n}(2\theta_A-3R)$	0	0	$+C_{AD}$	0	0
2	$M_{AB}=2EK(2\theta_A+\theta_B)$	$-C_{AB}$	0	0	0	0
3	$M_{BA}=2EK(2\theta_B+\theta_A)$	$+C_{BA}$	0	0	0	0
4	$M_{BC}=2E\frac{K}{n}(2\theta_B-3R)$	0	0	0	0	0
5	$M_{BF}=2EK(2\theta_B+\theta_F)$	0	$-C_{BF}$	0	0	0

TABLE 27—CONTINUED

6	$M_{FB}=2EK(2\theta_F+\theta_B)$	0	$+C_{FB}$	0	0	0
7	$M_{FE}=2E\frac{K}{n}(2\theta_F-3R)$	0	0	0	$-C_{FE}$	0
8	$M_{DA}=2E\frac{K}{n}(\theta_A-3R)$	0	0	$-C_{DA}$	0	0
9	$M_{CB}=2E\frac{K}{n}(\theta_B-3R)$	0	0	0	0	0
10	$M_{EF}=2E\frac{K}{n}(\theta_F-3R)$	0	0	0	$+C_{EF}$	0
11	$0=M_{AD}+M_{AB}$	0	0	0	0	$-M$
12	$0=M_{BA}+M_{BF}+M_{BC}$	0	0	0	0	0
13	$0=M_{FB}+M_{FE}$	0	0	0	0	0
14	$0=M_{AD}+M_{DA}+M_{BC}+M_{CB}+M_{FE}+M_{EF}$	0	0	$+M_D$	$+M_E$	0

M_D is the statical moment about point D of the horizontal loads on left-hand leg.
M_E is the statical moment about point E of the horizontal loads on right-hand leg.
The statical moment is positive when the indicated rotation is clockwise.

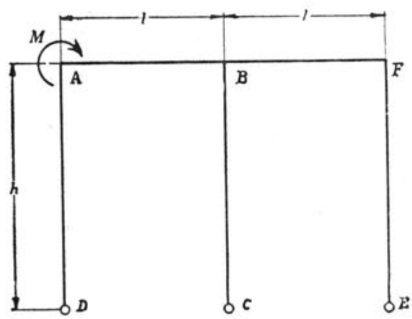


FIGURE 96

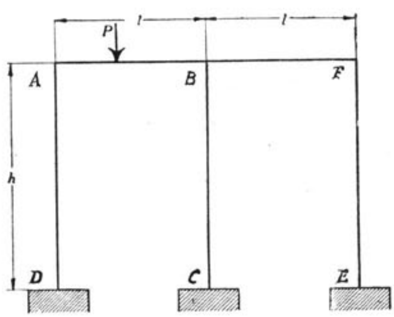


FIGURE 97

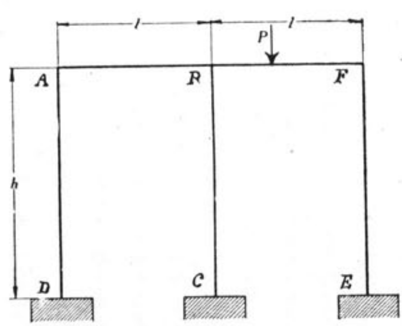


FIGURE 98

TABLE 28

EQUATIONS FOR THREE-LEGGED BENT

Lengths and Sections of Top Members Equal. $\left(\frac{I}{l}=K\right)$

Lengths and Sections of All Legs Equal. $\left(\frac{I}{h}=\frac{K}{n}\right)$

Any System of Loads. Legs Fixed at the Bases.

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation				
	θ_A	θ_B	θ_F	$3R$	Case VII Vertical Load on Left-hand Span See Fig. 97	Case VIII Vertical Load on Right-hand Span See Fig. 98	Case IX Horizontal Load to the Right on Left-hand Leg See Fig. 99	Case X Horizontal Load to the Left on Right-hand Leg See Fig. 100	Case XII External Couple at Upper Left-hand Corner See Fig. 101
A	$2n+2$	n	0	-1	$+\frac{nC_{AB}}{2EK}$	0	$-\frac{nC_{AD}}{2EK}$	0	$+\frac{Mn}{2EK}$
B	n	$4n+2$	n	-1	$-\frac{nC_{BA}}{2EK}$	$\frac{nC_{BF}}{2EK}$	0	0	0
C	0	n	$2n+2$	-1	0	$-\frac{nC_{FB}}{2EK}$	0	$+\frac{nC_{FE}}{2EK}$	0
D	1	1	1	-2	0	0	$-\frac{n(M_D+C_{AD}-C_{DA})}{6EK}$	$-\frac{n(M_E+C_{EF}-C_{FE})}{6EK}$	0

For definition of M_D and M_E see Table 27.

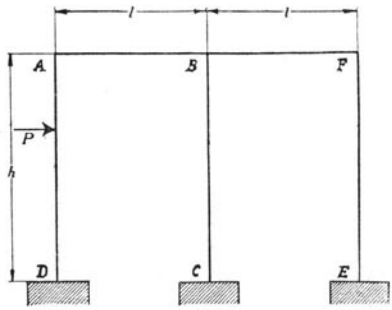


FIGURE 99

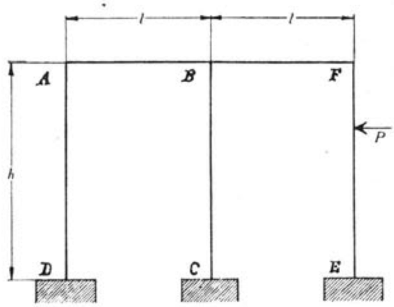


FIGURE 100

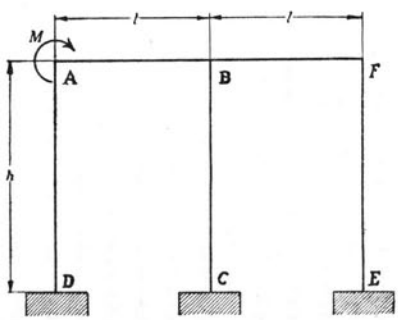


FIGURE 101

Table 29

Moments Due to Various Systems of Loads on a Three-legged Bent

Lengths and Sections of Top Members Equal. $\left(\frac{I}{l}=K\right)$

Lengths and Sections of All Legs Equal. $\left(\frac{I}{h}=\frac{K}{n}\right)$

Legs Fixed at the Bases.

No. of Equa- tion	Mo- ment	Case VII Vertical Load on Left-hand Span See Fig. 97	Case VIII Vertical Load on Right-hand Span See Fig. 98	Case IX Horizontal Load to the Right on Left-hand Leg See Fig. 99	Case X Horizontal Load to the Left on Right-hand Leg See Fig. 100	Case XII External Couple at Upper Left-hand Corner See Fig. 101
1	$M_{AD}=$	$+\frac{C_{AB}(11n^2+15n+2)+4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)}$	$-\frac{4nC_{BF}(n+1)-nC_{FB}(n+3)}{2(n+1)(6n^2+9n+1)}$	$+\frac{nC_{AD}(10n^2+15n+3)-2n(M_D-C_{DA})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$+\frac{nC_{FE}(2n^2+3n-1)-2n(M_E+C_{EF})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$+\frac{M(11n^2+15n+2)}{2(n+1)(6n^2+9n+1)}$
2	$M_{AB}=$	$-\frac{C_{AB}(11n^2+15n+2)+4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)}$	$+\frac{4nC_{BF}(n+1)-nC_{FB}(n+3)}{2(n+1)(6n^2+9n+1)}$	$-\frac{nC_{AD}(10n^2+15n+3)-2n(M_D-C_{DA})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$-\frac{nC_{FE}(2n^2+3n-1)-2n(M_E+C_{EF})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$+\frac{Mn(3n+1)(4n+5)}{2(n+1)(6n^2+9n+1)}$
3	$M_{BA}=$	$+\frac{nC_{AB}(3n^2+8n+4)+C_{BA}(n+1)(6n^2+13n+2)}{2(n+1)(6n^2+9n+1)}$	$+\frac{nC_{BF}(n+1)(6n+5)+nC_{FB}(3n^2+n-3)}{2(n+1)(6n^2+9n+1)}$	$-\frac{n(n+2)[C_{AD}(2n+1)-(M_D-C_{DA})(n+1)]}{2(n+1)(6n^2+9n+1)}$	$-\frac{nC_{FE}(4n^2+4n-1)-n(M_E+C_{EF})(n+1)(n+2)}{2(n+1)(6n^2+9n+1)}$	$+\frac{Mn(n+2)(3n+2)}{2(n+1)(6n^2+9n+1)}$
4	$M_{BC}=$	$-\frac{7nC_{AB}+2C_{BA}(4n+1)}{2(6n^2+9n+1)}$	$+\frac{2C_{BF}(4n+1)+7nC_{FB}}{2(6n^2+9n+1)}$	$-\frac{nC_{AD}(2n-3)+2n(M_D-C_{DA})(n+2)}{2(6n^2+9n+1)}$	$+\frac{nC_{FE}(2n-3)-2n(M_E+C_{EF})(n+2)}{2(6n^2+9n+1)}$	$-\frac{7Mn}{2(6n^2+9n+1)}$
5	$M_{BF}=$	$-\frac{nC_{AB}(3n^2+n-3)+nC_{BA}(n+1)(6n+5)}{2(n+1)(6n^2+9n+1)}$	$-\frac{C_{BF}(n+1)(6n^2+13n+2)+nC_{FB}(3n^2+8n+4)}{2(n+1)(6n^2+9n+1)}$	$+\frac{nC_{AD}(4n^2+4n-1)+n(M_D-C_{DA})(n+1)(n+2)}{2(n+1)(6n^2+9n+1)}$	$+\frac{n(n+2)[C_{FE}(2n+1)+(M_E+C_{EF})(n+1)]}{2(n+1)(6n^2+9n+1)}$	$-\frac{Mn(3n^2+n-3)}{2(n+1)(6n^2+9n+1)}$
6	$M_{FB}=$	$+\frac{nC_{AB}(n+3)-4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)}$	$+\frac{4nC_{BF}(n+1)+C_{FB}(11n^2+15n+2)}{2(n+1)(6n^2+9n+1)}$	$+\frac{nC_{AD}(2n^2+3n-1)+2n(M_D-C_{DA})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$+\frac{nC_{FE}(10n^2+15n+3)+2n(M_E+C_{EF})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$+\frac{Mn(n+3)}{2(n+1)(6n^2+9n+1)}$
7	$M_{FE}=$	$-\frac{nC_{AB}(n+3)-4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)}$	$-\frac{4nC_{BF}(n+1)+C_{FB}(11n^2+15n+2)}{2(n+1)(6n^2+9n+1)}$	$-\frac{nC_{AD}(2n^2+3n-1)+2n(M_D-C_{DA})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$-\frac{nC_{FE}(10n^2+15n+3)+2n(M_E+C_{EF})(n+1)^2}{2(n+1)(6n^2+9n+1)}$	$-\frac{Mn(n+3)}{2(n+1)(6n^2+9n+1)}$
8	$M_{DA}=$	$+\frac{nC_{AB}(4n+5)+C_{BA}(n+1)(2n+1)}{2(n+1)(6n^2+9n+1)}$	$-\frac{C_{BF}(n+1)(2n+1)-C_{FB}(2n^2+4n+1)}{2(n+1)(6n^2+9n+1)}$	$-\frac{C_{AD}(6n^3+27n^2+26n+2)}{6(n+1)(6n^2+9n+1)}-\frac{M_D(6n^2+9n+2)+C_{DA}(30n^2+45n+4)}{6(6n^2+9n+1)}$	$+\frac{C_{FE}(6n^3+9n^2-n-1)-(M_E+C_{EF})(n+1)(6n^2+9n+2)}{6(n+1)(6n^2+9n+1)}$	$+\frac{Mn(4n+5)}{2(n+1)(6n^2+9n+1)}$
9	$M_{CB}=$	$-\frac{C_{AB}(5n+1)+4nC_{BA}}{2(6n^2+9n+1)}$	$+\frac{4nC_{BF}+C_{FB}(5n+1)}{2(6n^2+9n+1)}$	$-\frac{C_{AD}(6n^2-3n-1)+2(M_D-C_{DA})(3n^2+6n+1)}{6(6n^2+9n+1)}$	$+\frac{C_{FE}(6n^2-3n-1)-2(M_E+C_{EF})(3n^2+6n+1)}{6(6n^2+9n+1)}$	$-\frac{M(5n+1)}{2(6n^2+9n+1)}$
10	$M_{EF}=$	$-\frac{C_{AB}(2n^2+4n+1)-C_{BA}(n+1)(2n+1)}{2(n+1)(6n^2+9n+1)}$	$-\frac{C_{BF}(n+1)(2n+1)+nC_{FB}(4n+5)}{2(n+1)(6n^2+9n+1)}$	$-\frac{C_{AD}(6n^3+9n^2-n-1)+(M_D-C_{DA})(n+1)(6n^2+9n+2)}{6(n+1)(6n^2+9n+1)}$	$+\frac{C_{FE}(6n^3+27n^2+26n+2)}{6(n+1)(6n^2+9n+1)}-\frac{M_E(6n^2+9n+2)-C_{EF}(30n^2+45n+4)}{6(6n^2+9n+1)}$	$-\frac{M(2n^2+4n+1)}{2(n+1)(6n^2+9n+1)}$

For definitions of M and M_E see Table 27.

XII. Effect of Error in Assumptions

67. *Error Due to Assumption that Axial Deformation is Zero.*— In determining the stresses in frames it has been assumed that the members of the frame do not change in length, but since the members are subjected to axial stress there must be a corresponding axial deformation. It remains to determine the effect of the axial deformation upon the stresses in the frames.

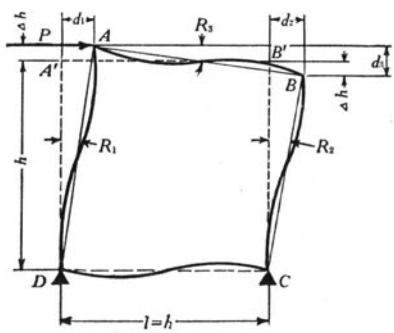


Figure 102

Fig. 102 represents a square frame with all sides identical in section. The frame is subjected to a single horizontal force P at the top, and is in equilibrium under the action of P and the reactions at D and C. Since there are no loads except at the ends of the members, H and C of Table 2 are zero for all members.

Case 1. Assume horizontal reactions at C and D equal. From equations 407 and 408 of section 48

$$M_{AB}=\frac{Ph}{4}$$

$$M_{BC}=-\frac{Ph}{4}$$

$$M_{CD}=\frac{Ph}{4}$$

$$M_{DA}=-\frac{Ph}{4}$$

$$\text{Shear in } AD = \frac{M_{AD}+M_{DA}}{h} = \frac{1}{2}P$$

Likewise

$$\text{Shear in } BC = \frac{1}{2}P$$

$$\text{Shear in } AB = \frac{1}{2}P$$

$$\text{Shear in } DC = \frac{1}{2}P$$

$$\text{Stress in } AB = \frac{1}{2}P \text{ compression}$$

$$\text{Stress in } CD = 0$$

$$\text{Stress in } AD = \frac{1}{2}P \text{ tension}$$

$$\text{Stress in } BC = \frac{1}{2}P \text{ compression}$$

Since AB, AD, and BC have the same sections, the same lengths, and are subjected to axial stresses of the same magnitudes, $\Delta\ h$, representing the change in length due to axial stress, is the same for all members.

Referring to Fig. 102

$$R_1 - R_2 = \frac{\Delta h}{h} = \frac{P}{2AE}$$

$$R_3 = \frac{2\Delta h}{l} = \frac{P}{AE}$$

Applying the equations of Table 1 gives

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R_3) \quad \ldots \quad (719)$$

$$M_{AB} = -2EK(2\theta_A + \theta_D - 3R_1) \quad \ldots \quad (720)$$

$$M_{BC} = 2EK(2\theta_B + \theta_C - 3R_2) \quad \ldots \quad (721)$$

$$M_{BC} = -2EK(2\theta_B + \theta_A - 3R_3) \qquad (722)$$

$$M_{CD} = 2EK(2\theta_C + \theta_D) \qquad (723)$$

$$M_{CD} = -2EK(2\theta_C + \theta_B - 3R_2) \qquad (724)$$

$$M_{DA} = 2EK(2\theta_D + \theta_A - 3R_1) \qquad (725)$$

$$M_{DA} = -2EK(2\theta_D + \theta_C) \qquad (726)$$

For AD and BC to be in equilibrium

$$-M_{AB} + M_{BC} - M_{CD} + M_{DA} + Pl = 0 \qquad (727)$$

Combining equations 719 to 727, substituting values for R_1, R_2, and R_3, and solving for the moments gives

$$M_{AB} = \frac{Ph}{4}\left[1 - \frac{3K}{2Ah}\right] \qquad (728)$$

$$M_{BC} = -\frac{Ph}{4}\left[1 - \frac{9K}{2Ah}\right] \qquad (729)$$

$$M_{CD} = \frac{Ph}{4}\left[1 + \frac{3K}{2Ah}\right] \qquad (730)$$

$$M_{DA} = -\frac{Ph}{4}\left[1 + \frac{9K}{2Ah}\right] \qquad (731)$$

In these equations $\frac{9K}{2Ah}$ represents the largest error due to neglecting the axial deformation in the members, and the ratio of $\frac{9K}{2Ah}$ to 1 represents the relative error.

Case 2. Assume horizontal reaction at C only. If the horizontal reaction is all taken at C

Stress in $AB = \frac{1}{2}P$ compression

Stress in $CD = \frac{1}{2}P$ compression

$$\text{Stress in } AD = \frac{1}{2}P \text{ tension}$$

$$\text{Stress in } BC = \frac{1}{2}P \text{ compression}$$

Since AB, CD, AD, and BC have the same sections, the same lengths, and are subjected to axial stresses of the same magnitude, Δh, representing the change in length due to axial stress, is the same for all members. The magnitude of Δh is given by the expression

$$\Delta h = \frac{Ph}{2AE}$$

As Δh for AB equals Δh for DC, $R_1 = R_2$

Since AD is in tension and BC in compression

$$R_3 = \frac{2\Delta h}{l} = \frac{P}{AE}$$

Equations 719 to 727 are applicable. Combining the equations, substituting the values of the R's and solving for the moments gives

$$M_{AB} = \frac{Ph}{4}\left(1 - \frac{6K}{2Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (732)$$

$$M_{BC} = -\frac{Ph}{4}\left(1 - \frac{6K}{2Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (733)$$

$$M_{CD} = \frac{Ph}{4}\left(1 + \frac{6K}{2Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (734)$$

$$M_{DA} = -\frac{Ph}{4}\left(1 + \frac{6K}{2Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (735)$$

In these equations $\frac{6K}{2Ah}$ represents the error due to neglecting the axial deformation in the members, and the ratio of $\frac{6K}{2Ah}$ to 1 represents the relative error.

Comparing Case 1 with Case 2, it is apparent that the error due to neglecting the axial deformations is smaller if the horizontal reaction is all at one corner than if half of it comes at each of the lower corners. The maximum relative error, for Case 1, is $\frac{9K}{2Ah}\cdot$ Substituting for K its value $\frac{I}{l}$, the expression $\frac{9K}{2Ah}$ becomes $\frac{9I}{2Ah^2}\cdot$ That is, the error due to neglecting the axial deformation in a square frame with all sides identical, subjected to a single horizontal force at the top, varies directly with the moments of inertia of the section of the members, inversely with the area of the members, and inversely with the square of the length.

If for Case 1 the frame, instead of being square as shown in Fig. 102, is a rectangle twice as wide as it is high, and if the sections of all members are identical, the moments in the frame due to a single horizontal force P at the top are

$$M_{AB}=\frac{Ph}{4}\left(1+\frac{9}{4}\,\frac{K}{Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad (736)$$

$$M_{BC}=-\frac{Ph}{4}\left(1-\frac{15}{4}\,\frac{K}{Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad (737)$$

$$M_{CD}=\frac{Ph}{4}\left(1-\frac{9}{4}\,\frac{K}{Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad (738)$$

$$M_{DA}=-\frac{Ph}{4}\left(1+\frac{15}{4}\,\frac{K}{Ah}\right) \quad . \quad . \quad . \quad . \quad . \quad (739)$$

in which h and K are for the vertical members.

The maximum relative error in this case is $\frac{15}{4}\,\frac{K}{Ah}$ divided by 1.

It is to be noted that $\frac{15}{4}$, the coefficient of $\frac{K}{Ah}$ in equation 739, is less than $\frac{9}{2}$, the coefficient of $\frac{K}{Ah}$, in equation 731. Hence for two frames, one a rectangle twice as wide as it is high with sections of all members identical, and the other a square having all sides identical with the vertical sides of the rectangle, and both frames having a single horizontal force applied at the top, the error due to axial deformation is less for the rectangular frame than it is for the square frame.

If for Case 1 the frame, instead of being square as shown in Fig. 102, is a rectangle twice as high as it is wide, and if the sections of all members are identical, the moments in the frame due to a single horizontal force P at the top are

$$M_{AB}=\frac{Ph}{4}\left(1-\frac{51}{8}\frac{K}{Ah}\right) \qquad (740)$$

$$M_{BC}=-\frac{Ph}{4}\left(1-\frac{51}{8}\frac{K}{Ah}\right) \qquad (741)$$

$$M_{CD}=\frac{Ph}{4}\left(1+\frac{51}{8}\frac{K}{Ah}\right) \qquad (742)$$

$$M_{DA}=-\frac{Ph}{4}\left(1+\frac{51}{8}\frac{K}{Ah}\right) \qquad (743)$$

in which h and K are for the vertical members.

The maximum relative error in this case is $\frac{51}{8}\frac{K}{Ah}$ divided by 1.

It is to be noted that $\frac{51}{8}$, the coefficient of $\frac{K}{Ah}$ in equation 743, is more than $\frac{9}{2}$, the coefficient of $\frac{K}{Ah}$ in equation 731. Hence for two frames, one a rectangle twice as high as it is wide with sections of all members identical, and the other a square having all sides identical with the vertical sides of the rectangle, and both frames having a single horizontal force at the top, the error due to axial deformation is greater for the rectangular frame than it is for the square frame.

Table 30 gives the errors due to axial deformations in a number of rectangular frames. The largest error in the table is 3.70 per cent for a frame 10 feet square composed of 27 in.-83 lb.-I-beams. Although this frame is composed of members having sectional areas so large compared with their length that it would be impracticable to provide connections strong enough to develop the strength of the members, yet the error, 3.7 per cent, is well within the range of permissible error. For steel frames of the proportions common in engineering structures the error is well under 2 per cent. For concrete frames the ratio $\frac{I}{A}$ is much less than for steel frames, and the error due to neglecting the axial deformations is correspondingly less for concrete frames than for steel frames.

Table 30

Error in Stresses in Rectangular Frames Due to Neglecting Axial Deformation

Frame is subjected to a single horizontal force at the top. The horizontal reaction at the bottom is equally divided between the two lower corners. The sections of all members of a frame are identical.

Section	Square Frame		Frame Having Width Twice Height			Frame Having Height Twice Width		
	Height and Width feet	Error per cent	Height feet	Width feet	Error per cent	Height feet	Width feet	Error per cent
27″—I—83 lb.	20 15 10	.92 1.64 3.70	20 15 10	40 30 20	.77 1.37 3.08	40 30 20	20 15 10	.33 .58 1.31
24″—I—80 lb.	20 15 10	.70 1.24 2.80	20 15 10	40 30 20	.58 1.03 2.33	40 30 20	20 15 10	.25 .43 .99
20″—I—65 lb.	20 15 10	.48 .85 1.92	20 15 10	40 30 20	.40 .71 1.60	40 30 20	20 15 10	.17 .30 .68
15″—I—42 lb.	20 15 10	.28 .49 1.10	20 15 10	40 30 20	.23 .41 .92	40 30 20	20 15 10	.10 .17 .39
12″—I—31.5 lb.	20 15 10	.18 .32 .73	20 15 10	40 30 20	.15 .27 .61	40 30 20	20 15 10	.06 .11 .26

68. *Error Due to Assumption that Shearing Deformation is Zero.*—In the derivation of the fundamental propositions upon which the analyses are based, deflection due to shear was not considered. That being the case, R in the equations of Table 1 depends upon the bending deflection only. In the analyses of the rectangular frames the R's of the two vertical members are taken equal. This is true of the bending

deflections only when the shearing deformations in the two vertical members are equal. (Axial deformation is here neglected.) Likewise, in considering that the deflection due to bending for the top of the frame is zero, the shear deflection in the top of the frame is neglected.

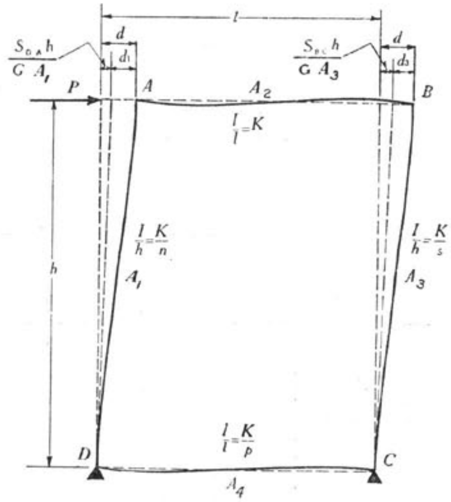


FIGURE 103

Fig. 103 represents a rectangular frame having a horizontal force applied at the top. The members of the frame are subjected to shear as follows:

$$\text{Shear in } DA,\ S_{DA} = \frac{M_{DA} - M_{AB}}{h} = -\frac{Ph}{h}\,\frac{\Delta_{DA} + \Delta_{AB}}{\Delta} \quad . \quad . \quad (744)$$

$$\text{Shear in } AB,\ S_{AB} = \frac{M_{AB} - M_{BC}}{l} = \frac{Ph}{l}\,\frac{\Delta_{AB} + \Delta_{BC}}{\Delta} \quad . \quad . \quad (745)$$

$$\text{Shear in } BC,\ S_{BC} = \frac{M_{BC} - M_{CD}}{h} = \frac{Ph}{h}\,\frac{\Delta_{BC} + \Delta_{CD}}{\Delta} \quad . \quad . \quad (746)$$

$$\text{Shear in } CD,\ S_{CD} = \frac{M_{CD} - M_{DA}}{l} = \frac{Ph}{l}\,\frac{\Delta_{CD} + \Delta_{DA}}{\Delta} \quad . \quad . \quad (747)$$

In these equations Δ_{AB}, Δ_{BC}, Δ_{CD}, and Δ_{DA} represent the quantities in the parentheses of equations 401, 402, 403, and 404, of section 48, respectively, and Δ represents the common denominator of the same equations.

The points D and C are considered to remain stationary, and axial strain is neglected.

The total horizontal deflection of A equals the total horizontal deflection of B. These deflections are represented in Fig. 103 by d. The shear deflection of DA is $\frac{S_{DA}h}{GA_1}$ and the shear deflection of BC is $\frac{S_{BC}h}{GA_3}$. The bending deflection of DA, d_1, is therefore given by the equation

$$d_1 = d - \frac{S_{DA}h}{GA_1} \qquad (748)$$

and the bending deflection of BC, d_3, is given by the equation

$$d_3 = d - \frac{S_{BC}h}{GA_3} \qquad (749)$$

The point C does not move vertically relatively to D, but there is a shear S_{CD} which produces a shear deflection in CD of $-\frac{S_{CD}l}{GA_4}$; therefore there must be an equal and opposite moment deflection of $\frac{S_{CD}l}{GA_4}$. Similarly, there is a bending deflection in AB of $\frac{S_{AB}l}{GA_2}$.

From the preceding equations

$$R_1 = \frac{d_1}{h} = \frac{d}{h} - \frac{S_{DA}}{GA_1} \qquad (750)$$

$$R_2 = \frac{d_2}{l} = \frac{S_{AB}}{GA_2} \qquad (751)$$

$$R_3 = \frac{d_3}{h} = \frac{d}{h} - \frac{S_{BC}}{GA_3} \qquad (752)$$

$$R_4 = \frac{d_4}{l} = \frac{S_{CD}}{GA_4} \qquad (753)$$

Applying the equations of Table 1 gives

$$M_{DA}=\frac{2EK}{n}(2\theta_D+\theta_A-3R_1) \quad \ldots \quad (754)$$

$$M_{AB}=-\frac{2EK}{n}(2\theta_A+\theta_D-3R_1) \quad \ldots \quad (755)$$

$$M_{AB}=2EK(2\theta_A+\theta_B-3R_2) \quad \ldots \quad (756)$$

$$M_{BC}=-2EK(2\theta_B+\theta_A-3R_2) \quad \ldots \quad (757)$$

$$M_{BC}=\frac{2EK}{s}(2\theta_B+\theta_C-3R_3) \quad \ldots \quad (758)$$

$$M_{CD}=-\frac{2EK}{s}(2\theta_C+\theta_B-3R_3) \quad \ldots \quad (759)$$

$$M_{CD}=\frac{2EK}{p}(2\theta_C+\theta_D-3R_4) \quad \ldots \quad (760)$$

$$M_{DA}=-\frac{2EK}{p}(2\theta_D+\theta_C-3R_4) \quad \ldots \quad (761)$$

For AD and BC to be in equilibrium

$$M_{DA}-M_{AB}+M_{BC}-M_{CD}+Ph=0 \quad \ldots \quad (762)$$

Substituting the values of the shears from equations 744, 745, 746, and 747 in equations 750, 751, 752, and 753, substituting the resulting values of the R's in equations 754 to 761, combining the resulting equations and 762, and solving for the moments gives

$$M_{AB}=\begin{bmatrix}M_{AB} \text{ when shearing strain is}\\ \text{neglected, a positive quantity}\\ \text{(see equation 401, section 48).}\end{bmatrix}+\left[\frac{6PK}{\Delta^2}\frac{E}{G}\right]\left[B_{AB}D+O_{AB}Q\right] \quad \ldots \quad (763)$$

$$M_{BC}=\begin{bmatrix}M_{BC} \text{ when shearing strain is}\\ \text{neglected, a negative quantity}\\ \text{(see equation 402, section 48).}\end{bmatrix}+\left[\frac{6PK}{\Delta^2}\frac{E}{G}\right]\left[B_{BC}D-O_{BC}Q\right] \quad \ldots \quad (764)$$

$$M_{CD}=\begin{bmatrix}M_{CD}\text{ when shearing strain is}\\ \text{neglected, a positive quantity}\\ \text{(see equation 403, section 48).}\end{bmatrix}-\left[\frac{6PK}{\Delta^2}\,\frac{E}{G}\right]\left[B_{CD}D+O_{CD}Q\right] \quad (765)$$

$$M_{DA}=\begin{bmatrix}M_{DA}\text{ when shearing strain is}\\ \text{neglected, a negative quantity}\\ \text{(see equation 404, section 48).}\end{bmatrix}-\left[\frac{6PK}{\Delta^2}\,\frac{E}{G}\right]\left[B_{DA}D-O_{DA}Q\right] \quad (766)$$

in which

$$B_{AB}=6ns+5ps+2pn+p^2+2n+p-s$$

$$B_{BC}=6ns+5pn+2ps+p^2-n+2s+p$$

$$B_{CD}=6ns+5n+2ps+p-pn+2s+1$$

$$B_{DA}=6ns+5s-ps+p+2pn+2n+1$$

$$O_{AB}=\frac{h}{l}(ns-pn+s^2+5ps+2n+2s+6p)$$

$$O_{BC}=\frac{h}{l}(ns+5pn-ps+n^2+2n+6p+2s)$$

$$O_{CD}=\frac{h}{l}(ns+2ps+5n+6p+n^2+2pn-s)$$

$$O_{DA}=\frac{h}{l}(s^2+2ps+5s+6p+ns+2pn-n)$$

$$D=\frac{11ns+11nps+6ns^2+2s^2+10ps+12pn+2s+3p+2s^2p+2sp^2+3p^2}{A_1}$$

$$-\frac{6sn^2+2n^2+11nps+10pn+11sn+12ps+2n+3p+2n^2p+2np^2+3p^2}{A_3}$$

$$Q=\frac{10ns+12nps+3ns^2+2s^2+11ps+11pn+2s+6p+3sn^2+2n^2+2n}{A_4}$$

$$-\frac{12ns+10nps+3ns^2+11ps+11pn+2s^2p+2sp^2+6p^2+3sn^2+2n^2p+2np^2}{A_2}$$

$$\Delta=22(pns+ps+ns+np)+2(p^2s+ps^2+n^2p+p^2n+s^2+s+n^2+n)$$
$$+6(n^2s+ns^2+p^2+p)$$

If the members AB and CD are identical, equations 763, 764, 765, and 766 reduce to the form

$M_{AB} = M_{AB}$ when shearing deformation is neglected (positive quantity) $+k$ . . . . (767)

$M_{BC} = M_{BC}$ when shearing deformation is neglected (negative quantity) $+k$ . . . . (768)

$M_{CD} = M_{CD}$ when shearing deformation is neglected (positive quantity) $-k$ . . . . (769)

$M_{DA} = M_{DA}$ when shearing deformation is neglected (negative quantity) $-k$ . . . . (770)

in which

$$k = 6\,\frac{E}{G}\,\frac{PK}{(n+s+6)^2}\left(\frac{s+3}{A_1} - \frac{n+3}{A_3}\right)$$

If the members AB and CD are identical, and if DA and BC are also identical but not necessarily identical with AB and CD, k of equations 767 to 770 is zero, showing that for a rectangular frame with opposite sides identical the shearing deformation does not affect the moment in the frame.

The error due to shearing deformation in a number of frames is given in Table 31.

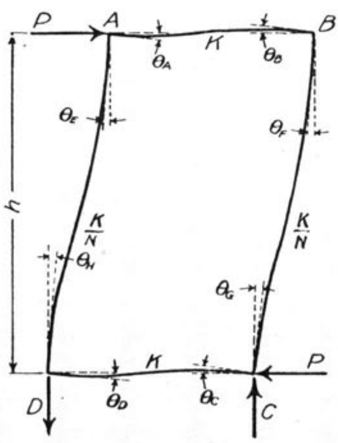


FIGURE 104

TABLE 31

ERROR IN STRESSES IN RECTANGULAR FRAMES DUE TO NEGLECTING SHEARING DEFORMATIONS

All frames are subjected to a single horizontal force at the top.

	Description of Frame						Error Due to Shearing Strain Per cent			
	Height in feet	Width in feet	Member	Area of Section in.^{2}	I in.^{4}	K in.^{3}	M_{AB}	M_{BC}	M_{CD}	M_{DA}
No Two Members Alike	30	50	AB BC CD DA	20 20 40 100	2500 300 5000 6000	4.16 .83 8.33 16.67	.07	.03	.09	.02
	24	40	AB BC CD DA	20 20 40 100	2500 300 5000 6000	5.21 1.04 10.41 20.81	.115	.05	.14	.03
	15	25	AB BC CD DA	20 20 40 100	2500 300 5000 6000	8.33 1.67 16.67 33.33	.30	.14	.35	.07
AB and CD identical	30	50	AB and CD BC DA	20 20 100	2500 300 6000	4.16 0.83 16.67	.032	.078	.078	.032
	24	40	AB and CD BC DA	20 20 100	2500 300 6000	5.21 1.04 20.81	.050	.122	.122	.050
	15	25	AB and CD BC DA	20 20 100	2500 300 6000	8.33 1.67 33.33	.127	.313	.313	.127
	50	30	AB and CD BC DA	20 20 100	2500 300 6000	6.95 0.50 10.00	.016	.072	.072	.016
	40	24	AB and CD BC DA	20 20 100	2500 300 6000	8.69 0.62 12.50	.025	.113	.113	.025
	25	15	AB and CD BC DA	20 20 100	2500 300 6000	13.90 1.00 20.00	.063	.290	.290	.063

69. *Effect of Slip in the Connections upon the Moments in a Rectangular Frame Having a Single Horizontal Force at the Top.*—Fig. 104 represents a rectangular frame having a single horizontal force applied at the top. The sections of opposite sides of the frame are identical. The connections at the corners of the frame slip.

The changes in the slopes at the ends of the member AB are represented by θ_A at A and θ_B at B; likewise for the member CD the change in slope at C is represented by θ_C and at D by θ_D; for the member AD the change in slope at A is represented by θ_E and at D by θ_H; and for the member BC the change in slope at B is represented by θ_F and at C by θ_G. That is: slip at A equals $\theta_E - \theta_A$; slip at B equals $\theta_F - \theta_B$; slip at C equals θ_G-θ_C; and slip at D equals θ_H-θ_D.

Represent $\frac{I}{l}$ for AB and CD by K, and $\frac{I}{l}$ for AD and BC by $\frac{K}{n}$. Applying equation A, Table 1, and equating the sum of the moments at each of the points A, B, C, and D to zero and also equating the sums of the moments at the ends of the two members AD and BC to $-Ph$ gives

$$2EK(2\theta_A + \theta_B) + \frac{2EK}{n}(2\theta_E + \theta_H - 3R) = 0 \quad \ldots \quad (771)$$

$$2EK(2\theta_B + \theta_A) + \frac{2EK}{n}(2\theta_F + \theta_G - 3R) = 0 \quad \ldots \quad (772)$$

$$2EK(2\theta_C + \theta_D) + \frac{2EK}{n}(2\theta_G + \theta_F - 3R) = 0 \quad \ldots \quad (773)$$

$$2EK(2\theta_D + \theta_C) + \frac{2EK}{n}(2\theta_H + \theta_E - 3R) = 0 \quad \ldots \quad (774)$$

$$\frac{2EK}{n}(2\theta_E + \theta_H - 3R + 2\theta_H + \theta_E - 3R + 2\theta_F + \theta_G - 3R + 2\theta_G + \theta_F - 3R) + Ph = 0 \quad \ldots \quad (775)$$

Letting A represent the quantity, slip at A divided by R, likewise letting B, C, and D represent the corresponding quantities at the points B, C, and D, respectively, gives

$$A = \frac{\theta_E}{R} - \frac{\theta_A}{R}$$

$$B=\frac{\theta_F}{R}-\frac{\theta_B}{R}$$

$$C=\frac{\theta_G}{R}-\frac{\theta_C}{R}$$

$$D=\frac{\theta_H}{R}-\frac{\theta_D}{R}$$

Substituting the values of θ_E, θ_F, θ_G, and θ_H from these equations in the preceding equations gives

$$\frac{2\theta_A}{nR}\left(1+n\right)+\frac{\theta_B}{R}+\frac{\theta_D}{nR}=\frac{1}{n}\left(3-2A-D\right) \quad . \quad . \quad . \quad . \quad . \quad (776)$$

$$\frac{2\theta_B}{nR}\left(1+n\right)+\frac{\theta_A}{R}+\frac{\theta_C}{nR}=\frac{1}{n}\left(3-2B-C\right) \quad . \quad . \quad . \quad . \quad . \quad (777)$$

$$\frac{2\theta_C}{nR}\left(1+n\right)+\frac{\theta_D}{R}+\frac{\theta_B}{nR}=\frac{1}{n}\left(3-2C-B\right) \quad . \quad . \quad . \quad . \quad . \quad (778)$$

$$\frac{2\theta_D}{nR}\left(1+n\right)+\frac{\theta_C}{R}+\frac{\theta_A}{nR}=\frac{1}{n}\left(3-2D-A\right) \quad . \quad . \quad . \quad . \quad . \quad (779)$$

$$2EKR=\frac{\frac{1}{3}\,Phn}{-\left[\frac{\theta_A}{R}+\frac{\theta_B}{R}+\frac{\theta_C}{R}+\frac{\theta_D}{R}\right]+4-(A+B+C+D)} \quad . \quad . \quad (780)$$

These five equations contain four unknown angles and one unknown deflection. Solving these equations and substituting the values for the θ's and R in the expressions for the moments gives

$$M_{AD}=\frac{1}{3}\;\frac{Ph}{4-(A+B+C+D)}\cdot$$

$$\left[\frac{(6A+3B-9)+n(16A+5B+4C+5D-30)+n^2(6A+3D-9)}{(n+3)\;(3n+1)}\right]$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (781)$$

$$M_{DA}=\frac{1}{3}\;\frac{Ph}{4-(A+B+C+D)}$$
$$\left[\frac{(6D+3C-9)+n\,(16D+5C+4B+5A-30)+n^2(6D+3A-9)}{(n+3)\;(3n+1)}\right] \quad \ldots \quad (782)$$

$$M_{BC}=\frac{1}{3}\;\frac{Ph}{4-(A+B+C+D)}$$
$$\left[\frac{(6B+3A-9)+n(16B+5A+4D+5C-30)+n^2(6B+3C-9)}{(n+3)\;(3n+1)}\right] \quad \ldots \quad (783)$$

$$M_{CB}=\frac{1}{3}\;\frac{Ph}{4-(A+B+C+D)}$$
$$\left[\frac{(6C+3D-9)+n(16C+5D+4A+5B-30)+n^2(6C+3B-9)}{(n+3)\;(3n+1)}\right] \quad \ldots \quad (784)$$

The values of $\frac{\theta_A}{R}$, $\frac{\theta_B}{R}$, $\frac{\theta_C}{R}$, and $\frac{\theta_D}{R}$ determined from equations 776 to 779 substituted in equation 780 gives

$$R=\frac{1}{4}\left[(RA+RB+RC+RD)+\frac{1}{6}\,\frac{Ph(n+1)}{EK}\right] \quad \ldots \quad (785)$$

in which RA, RB, RC, and RD represent the slips in the connections at A, B, C, and D respectively. If the slips are measured, R may be computed from equation 786. Knowing R and the slips, the values of A, B, C, and D may be computed. Substituting the values of A, B, C, and D in equations 781 to 784, the moments in a frame having connections which are not rigid may be determined.

If there is no slip in the connections the moment at each corner of the frame of Fig. 104 is $-\frac{1}{4}Ph$. The differences between $-\frac{1}{4}Ph$ and the moments given by equations 781, 782, 783, and 784 represent the effect of the slip in the connections.

If A, B, C, and D, of equations 781 to 784 inclusive, are equal to each other; that is, if the slips in all the connections of the rectangular frame represented by Fig. 104 are equal, equations 781 to 784 reduce to the form

$$M_{AD}=-\frac{1}{4}Ph \quad \ldots \quad (781a)$$

$$M_{DA}=-\frac{1}{4}Ph \quad \ldots \quad (782a)$$

$$M_{BC}=-\frac{1}{4}Ph \quad \ldots \quad (783a)$$

$$M_{CB}=-\frac{1}{4}Ph \quad \ldots \quad (784a)$$

That is, if the slips in all the connections of the rectangular frame shown in Fig. 104* are equal, the stresses in the frame are the same as they are in a similar frame having connections which are perfectly rigid.

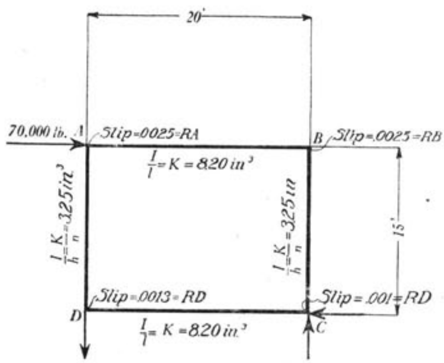


FIGURE 105

To illustrate the magnitude of the effect of slip in the connections upon the stresses in a rectangular frame consider the frame represented by Fig. 105. Equations 781, 782, 783, 784, and 785 are applicable. Substituting the values of the quantities given in Fig. 105 in equations 781 to 785 gives

*Wilson, W. M., and Moore, H. F. "Tests to Determine the Rigidity of Riveted Joints of Steel Structures," Univ. of Ill. Eng. Exp. Sta., Bul. 104, p. 28, 1917.

$M_{AD} = 3{,}030{,}000$ in. lb.
$M_{DA} = 3{,}240{,}000$ in. lb.
$M_{BC} = 3{,}050{,}000$ in. lb.
$M_{CB} = 3{,}300{,}000$ in. lb.

If there is no slip in the connections, each of these moments is 3,150,000 in. lb. The errors in the moments due to neglecting the slip in the connections are as follows:

M_{AD}, error $= -3.8$ per cent
M_{DA}, error $= +3.0$ per cent
M_{BC}, error $= -3.0$ per cent
M_{CB}, error $= +4.6$ per cent

For a given slip, the error due to slip is greater for a frame having short stiff members than it is for a frame having long flexible members.

XIII. Numerical Problems

The following numerical problems illustrate the use of the equations which have been derived.

70. *Girder Restrained at the Ends. Supports on Different Levels. Any System of Vertical Loads.*—Fig. 106 represents a 12 in.–31.5 lb. I-beam embedded in masonry at both ends. Because of upheaval by frost, or other causes, the beam which was originally horizontal now has one end higher than the other and the tangents to the elastic curve of the beam at the ends are inclined to the horizontal. The beam carries a concentrated load and a uniform load as shown. It is required to find the bending moments in the beam.

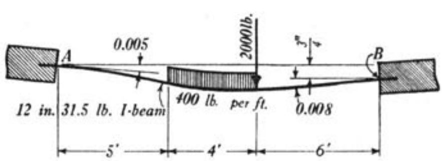


Figure 106

Equations 25 and 26, section 13, are applicable. E for steel is 30,000,000 lb. per sq. in.

$d_B = .75$ inch $\qquad\qquad \theta_A = +.005$

$I = 215.8$ in.^{4} $\qquad\qquad \theta_B = -.008$

$l = 180$ in. $\qquad\qquad R = \dfrac{.75}{180} = .00416$

$$K = \frac{215.8}{180} = 1.2 \text{ in.}^3$$

From Table 2, C_{AB} for a single concentrated load is

$$\frac{Pab^2}{l^2} \text{ in which}$$

$$a = 108 \text{ in.}$$

$$b = 72 \text{ in.}$$

$$P = 2000 \text{ lb.}$$

$$\frac{Pab^2}{l^2} = 34560 \text{ in. lb.}$$

Also from Table 2, C_{AB} for a uniform load over a portion of the span is $\frac{w}{12l^2}\left[d^3(4l-3d)-b^3(4l-3b)\right]$ in which

$d=120$ in.

$b=\ 72$ in.

$w=400$ lb. per ft. $=\frac{400}{12}$ lb. per in.

$$\frac{w}{12l^2}\left[d^3(4l-3d)-b^3(4l-3b)\right]=37{,}200 \text{ in. lb.}$$

Total $C_{AB}=37{,}200+34{,}560=71{,}760$ in. lb.

Substituting the values of the constants in equation 25, section 13, gives

$M_{AB}=-826{,}300$ in. lb.

From Table 2, C_{BA} for the concentrated load is 52,000 in. lb., and C_{BA} for the uniform load is 32,700 in. lb.

Total $C_{BA}=52{,}000+32{,}700=84{,}700$ in. lb.

Substituting the values of the constants in equation 26, section 13, gives

$M_{BA}=-1{,}606{,}300$ in. lb.

It is to be noted in the solution of this problem, in solving for both M_{AB} and M_{BA}, that θ_A and R are both positive whereas θ_B is negative. The minus sign, moreover, is used before C_{AB} and the plus sign is used before C_{BA}. The signs are in accordance with the conventional method of fixing signs given at the bottom of Table 1.

If the supports had been on the same level and if the tangents to the elastic curves at the ends of the beams had been horizontal, the moments would have been given by the equations of section 11. Using the same values of C_{AB} and C_{BA} as before

$M_{AB}=-C_{AB}=-71{,}760$ in. lb.
$M_{BA}=+C_{BA}=+84{,}700$ in. lb.

71. *Girder Continuous over Four Supports. Supports on Different Levels. Any System of Vertical Loads.*—Fig. 107 represents a 20 in.-80 lb. I-beam supported on four supports and having hinged ends. The supports are all on different levels. It is required to find the moments in the girder.

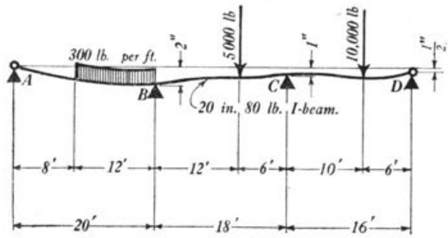


FIGURE 107

Equations (a), Table 6, are applicable.

$I_o = I_1 = I_2 = 1466$ in.^{4}

$l_o = 240$ in.

$l_1 = 216$ in.

$l_2 = 192$ in.

$$K_o = \frac{1466}{240} = 6.10 \text{ in.}^3$$

$$K_1 = \frac{1466}{216} = 6.79 \text{ in.}^3$$

$$K_2 = \frac{1466}{192} = 7.64 \text{ in.}^3$$

$$n_o = \frac{6.79}{6.10} = 1.11$$

$$n_1 = \frac{7.64}{6.79} = 1.125$$

$E = 30{,}000{,}000$ lb. per sq. in.

$d_A = 0$

$d_B = 2$ in.

$d_C = 1$ in.

$d_D = .5$ in.

From Table 2

$$H_{BA} = \frac{3600 \times 144}{8 \times 240 \times 240}\left(2 \times 240 - 144\right)^2 = 127{,}000 \text{ in. lb.}$$

$$H_{BC} = \frac{5000 \times 144 \times 72(216+72)}{2 \times 216 \times 216} = 160{,}000 \text{ in. lb.}$$

$$H_{CB} = \frac{5000 \times 144 \times 72(216+144)}{2 \times 216 \times 216} = 200{,}000 \text{ in. lb.}$$

$$H_{CD} = \frac{10000 \times 120 \times 72}{2 \times 192 \times 192}\left(192 + 72\right) = 310{,}000 \text{ in. lb.}$$

Substituting these values in equations (a) of Table 6 gives

$M_{BC} = 4{,}071{,}500$ in. lb.

$M_{CD} = -1{,}987{,}500$ in. lb.

If the supports had all been on the same level, the moments would have been

$M_{BC} = -88{,}500$ in. lb.

$M_{CD} = -227{,}500$ in. lb.

72. *Girder Continuous over Five Supports. Supports on Different Levels. Any System of Vertical Loads. Ends of Girder Restrained.*—Fig. 108 represents a girder continuous over five supports. All the supports are on different levels and the girder is restrained at the ends. The slopes of the elastic curve at the ends are known. It is required to determine the moments in the girder in terms of I, the moment of inertia of the girder.

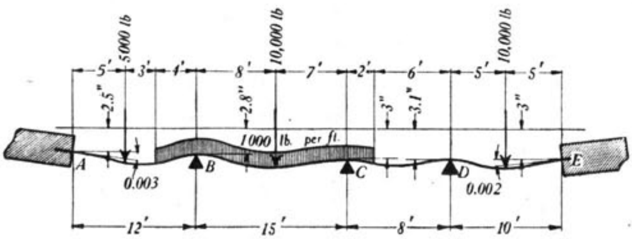


FIGURE 108

The solution of the problem involves writing the equations of three moments and solving these equations for the moments.

Cases (a), (d), and (e), Table 4, and the equations of Table 2 are applicable.

$$K_o = \frac{I}{12\times12} = \frac{I}{144}\text{ in.}^3 \quad n_1 = \frac{K_1}{K_0} = \frac{4}{5} = .800 \quad l_o = 12\times12 = 144\text{ in.}$$

$$K_1 = \frac{I}{12\times15} = \frac{I}{180}\text{ in.}^3 \quad n_2 = \frac{K_2}{K_1} = \frac{15}{8} = 1.875 \quad l_1 = 12\times15 = 180\text{ in.}$$

$$K_2 = \frac{I}{12\times8} = \frac{I}{96}\text{ in.}^3 \quad n_3 = \frac{K_3}{K_2} = \frac{4}{5} = .800 \quad l_2 = 12\times8 = 96\text{ in.}$$

$$K_3 = \frac{I}{12\times10} = \frac{I}{120}\text{ in.}^3 \qquad l_3 = 12\times10 = 120\text{ in.}$$

$d_A = 2.5$ in. $\quad d_B - d_A = 2.8\text{-}2.5 = +0.3$ in. $\quad \theta_A = +.003$

$d_B = 2.8$ in. $\quad d_C - d_B = 3.0\text{-}2.8 = +0.2$ in. $\quad \theta_E = -.002$

$d_C = 3.0$ in. $\quad d_D - d_C = 3.1\text{-}3.0 = +0.1$ in. $\quad E = 30{,}000{,}000$ lb. per sq. in.

$d_D = 3.1$ in. $\quad d_E - d_D = 3.0\text{-}3.1 = -0.1$ in.

$d_E = 3.0$ in.

$$H_{AB} = \frac{5000\times5\times7\times19\times12}{2\times144} + \frac{12000(16)\times(2\times144-16)}{8\times144}$$
$$= -183{,}900 \text{ in. lb.}$$

$$H_{BA} = \frac{5000\times5\times7\times17\times12}{2\times144} + \frac{12000}{8\times144}\left(144-64\right)\left(2\times144-144-64\right)$$
$$= 190{,}700 \text{ in. lb.}$$

$$H_{BC} = \frac{10000\times8\times7\times22\times12}{2\times225} + \frac{12000}{8}\times225 = 665{,}500 \text{ in. lb.}$$

$$H_{CB} = \frac{10000\times8\times7\times23\times12}{2\times225} + \frac{12000}{8}\times225 = 681{,}500 \text{ in. lb.}$$

$$H_{CD} = \frac{12000}{8\times64}\times\left(64-36\right)\left(2\times64-36-64\right) = 18{,}400 \text{ in. lb.}$$

$$H_{DC} = \frac{12000}{8\times64}\left(4\right)\left(2\times64-4\right) = 11{,}600 \text{ in. lb.}$$

$$H_{DE} = \frac{10000\times5\times5(15)\times12}{2\times100} = 225{,}000 \text{ in. lb.}$$

$$H_{ED} = \frac{10000\times5\times5(15)\times12}{200} = 225{,}000 \text{ in. lb.}$$

Substituting the values of these quantities in the equations of Table 4 gives

From equation (d)

$$6.4\ M_{BC} + 2.0\ M_{CD} = 1030\ I - 2{,}978{,}000 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (786)$$

From equation (a)

$$1.875\,M_{BC}+5.75\,M_{CD}+M_{DE}=130\,I-2{,}593{,}000 \quad . \quad . \quad . \quad (787)$$

From equation (e)

$$1.6\,M_{CD}+6.2\,M_{DE}=3900\,I-487{,}100 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (788)$$

Fı ı equations 786, 787, and 788

$$M_{DE}=(672\,I+9{,}500) \text{ in. lb.}$$

$$M_{CD}=-(166\,I+341{,}000) \text{ in. lb.}$$

$$M_{BC}=(213\,I-358{,}000) \text{ in. lb.}$$

Substituting the values of M_{BC} and M_{CD} in equation (a) of Table 4 gives

$$M_{AB}=(467\,I+18{,}750) \text{ in. lb.}$$

Also from equation (a) of Table 4

$$M_{ED}=-(530I-193{,}400) \text{ in. lb.}$$

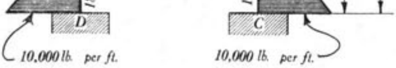


FIGURE 109

73. *Two-legged Rectangular Bent with Legs Fixed at the Bases. Bent and Loading Symmetrical about a Vertical Center Line.*—Fig. 109 represents a two-legged rectangular bent with legs fixed at the bases. Both legs are subjected to hydrostatic pressure. The bent and loading are symmetrical about a vertical center line. It is required to determine the moments in the frame.

Equations 126 and 127, section 25, are applicable.

$$n=\frac{K \text{ for } AB}{K \text{ for } AD}=\frac{\dfrac{441.8}{240}}{\dfrac{841.8}{192}}=.42$$

From Table 2

$$C_{AD}=\frac{Wa^2}{30l^2}(5l-3a) \text{ in which}$$

$$C_{DA}=\frac{Wa}{30l^2}(3a^2-10al+10l^2)$$

$$W=\frac{10{,}000\times 10}{2}=50{,}000 \text{ lb}$$

$a=120$ inches

$l=192$ inches

$C_{AD}=391{,}000$ in. lb.

$C_{DA}=984{,}400$ in. lb.

$$M_{AB}=M_{BC}=-\frac{0.42}{0.42+2}\times 391{,}000=-67{,}900 \text{ in. lb.}$$

$$M_{CB}=-M_{DA}=\frac{1}{0.42+2}\times 391{,}000+984{,}400=1{,}145{,}900 \text{ in. lb.}$$

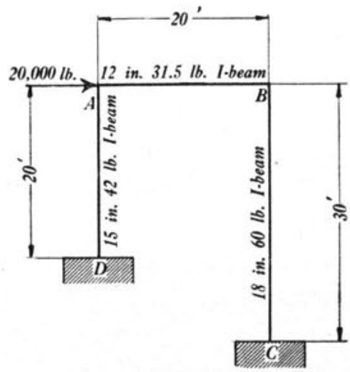


FIGURE 110

74. *Two-legged Rectangular Bent. One Leg Longer than the Other. Concentrated Horizontal Load at the Top. Legs Fixed at the Bases.—*

Fig. 110 represents a rectangular bent having one leg longer than the other. There is a horizontal force of 20,000 lb. applied at the top. The bases of the legs are fixed. It is required to determine the moments in the bent.

Equations 252, 253, 254, and 255, section 33, are applicable.

$P = 20{,}000$ lb.

$l = 240$ inches

$h = 240$ inches

$$q = \frac{240}{360} = 0.667$$

$$K \text{ for } AB = \frac{215.8}{240} = 0.90 \text{ in.}^3$$

$$K \text{ for } AD = \frac{441.8}{240} = 1.84 \text{ in.}^3$$

$$K \text{ for } BC = \frac{841.8}{360} = 2.34 \text{ in.}^3$$

$$n = \frac{K \text{ for } AB}{K \text{ for } AD} = 0.49$$

$$s = \frac{K \text{ for } AB}{K \text{ for } BC} = 0.385$$

$$\begin{aligned}\Delta_o = 2(&3\times.49\times.385\times.385+4\times.49\times.385+.385\times.385+.385+3\\ &\times.49\times.385\times.667+3\times.49\times.49\times.385\times.667\times.667+.49\\ &\times.49\times.667\times.667+.49\times.667\times.667+4\times.49\times.385\times.667\\ &\times.667) = 5.334\end{aligned}$$

Substituting the values of the quantities in equations 252, 253, 254, and 255, of section 33, gives

$M_{AB} = 1{,}100{,}000$ in. lb.

$M_{BA} = 958{,}000$ in. lb.

$M_{DA} = -1{,}950{,}000$ in. lb.

$M_{CB} = -1{,}664{,}000$ in. lb.

75. *Two-legged Rectangular Bent. Settlement of Foundation.*—Fig. 111 represents a rectangular portal. Due to upheaval by frost or other causes the foundation originally at C' moves 2 inches to the right, settles 3 inches, and turns in a negative direction an angle of .01 radians. It is required to determine the moments in the portal.

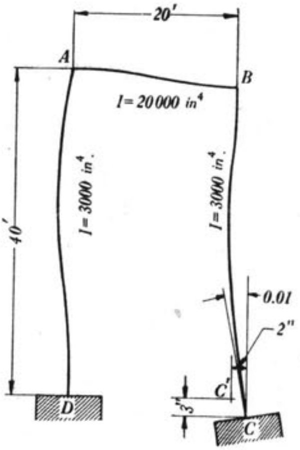


FIGURE 111

Equations 204, 205, 206, and 207, section 31, are applicable.

$l = 240$ inches

$h = 480$ inches

$$K = \frac{20{,}000}{240} = 83.4 \text{ in.}^3$$

$$n = \frac{K \text{ for } AB}{K \text{ for } AD} = \frac{83.4}{6.25} = 13.3$$

$$\theta_C = -.01 - \frac{3}{242} = -.0224$$

$\theta_D = 0$

$d = 2$ inches

$E = 30{,}000{,}000$ lb. per sq. in

$$M_{AB}=2{,}502{,}000{,}000 \quad [0.000816-0.001461-0.000831]$$
$$=-3{,}700{,}000 \text{ in. lb.}$$

$$M_{BC}=2{,}502{,}000{,}000 \quad [0.000816-0.001461+0.000831]$$
$$=+470{,}000 \text{ in. lb.}$$

$$M_{CB}=2{,}502{,}000{,}000 \quad [0.000877-0.003255-0.000831]$$
$$=-8{,}040{,}000 \text{ in. lb.}$$

$$M_{DA}=-2{,}502{,}000{,}000 \quad [0.000877-0.003255+0.000831]$$
$$=+3{,}875{,}000 \text{ in. lb.}$$

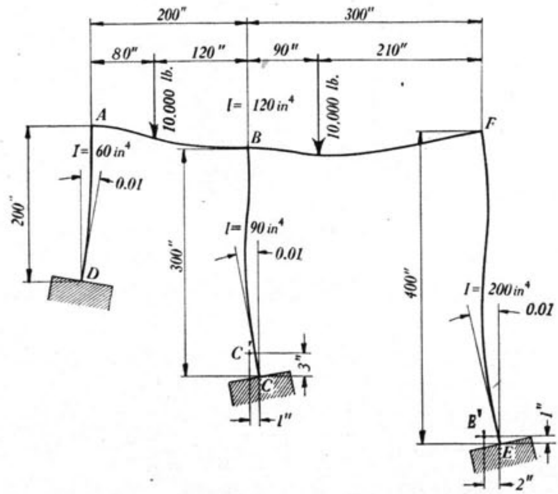


FIGURE 112

76. *Three-legged Bent. Lengths of All Legs Different. Loads on Top and Settlement of Foundations.*—Fig. 112 represents a three-legged bent having vertical loads on top. The legs are restrained at the bases. Due to upheaval by frost or other causes the foundation at D has rotated in a positive direction through an angle of 0.01 radians. Likewise the foundations at C and E have rotated in a negative direction through an angle 0.01. The foundation, originally at C', has moved, moreover, to the right 1 inch and has settled 3 inches. Likewise the foundation, originally at E', has moved to the right 2 inches and has settled 1 inch. It is required to determine the moments in the frame.

The equations of Table 21 are applicable. Since there are no horizontal loads on the legs nor external couples at the top, Cases IX, X, and XII will not enter.

The total right-hand members of the equations of Table 21 will therefore be the algebraic sum of the right-hand members due to Case VII, Case VIII, and Case XI.

$h_o = 200$ in. $\qquad I_o = 60$ in.^{4}

$h_2 = 300$ in. $\qquad I_1 = 120$ in.^{4}

$h_4 = 400$ in. $\qquad I_2 = 90$ in.^{4}

$l_1 = 200$ in. $\qquad I_3 = 120$ in.^{4}

$l_3 = 300$ in. $\qquad I_4 = 200$ in.^{4}

$K_o = 0.30$ in.^{3} $\qquad C_{AB} = 2{,}880{,}000$ in. lb.

$K_1 = 0.60$ in.^{3} $\qquad C_{BA} = 1{,}920{,}000$ in. lb.

$K_2 = 0.30$ in.^{3} $\qquad C_{BF} = 4{,}410{,}000$ in. lb.

$K_3 = 0.40$ in.^{3} $\qquad C_{FB} = 1{,}890{,}000$ in. lb.

$K_4 = 0.50$ in.^{3} $\qquad E = 30{,}000{,}000$ lb. per sq. in.

$$\frac{6d_1K_1}{l_1} = \frac{6\times3\times0.6}{200} = 0.054 \text{ in.}^3 \qquad K_o\theta_D = 0.003 \text{ in.}^3$$

$$K_2\theta_C = -0.003 \text{ in.}^3$$

$$\frac{6d_2K_2}{h_2} = \frac{6\times1\times0.3}{300} = 0.006 \text{ in.}^3 \qquad K_4\theta_E = -0.005 \text{ in.}^3$$

$$\frac{6(d_3-d_1)K_3}{l_3} = \frac{6\times(-2)\times0.4}{300}$$
$$= -0.016 \text{ in.}^3$$

$$\frac{6d_4K_4}{h_4} = \frac{6\times2\times0.5}{400} = 0.015 \text{ in.}^3$$

Substituting the values of these quantities in the equations of Table 23 gives equations 1, 2, 3, and 4 of Table 32. It is to be noted in the equations of Table 32 that the quantities under the headings Case VII and Case VIII are coefficients of $\frac{100}{E}\cdot$ Solving equations 1, 2, 3, and 4 by the method of elimination, as given in Table 32, gives

$$d = \frac{1}{E}\left[49{,}373{,}300 - 19{,}722{,}100\right] + 1.19314$$

$$\theta_F = \frac{1}{E}\left[238{,}155 - 797{,}034\right] - 0.0047062$$

$$\theta_B = \frac{1}{E}\left[-608{,}835 + 1{,}039{,}263\right] + 0.0061134$$

$$\theta_A = \frac{1}{E}\left[1{,}126{,}370 - 395{,}727\right] + 0.014278$$

Substituting these values of the θ's and d in the equations of Table 21 gives the moments in the frame. The moments are itemized and presented in Table 33.

TABLE 32

EQUATIONS FOR THE THREE-LEGGED BENT OF FIG. 112

No. of Equation	Left-hand Member of Equation				Right-hand Member of Equation		
	θ_A	θ_B	θ_F	d	Case VII Coefficients of $\frac{100}{E}$	Case VIII Coefficients of $\frac{100}{E}$	Case XI
1	3.6	1.2	0	−.0090	+28800	0	.04800
2	1.2	5.2	.80	−.0060	−19200	+44100	.03800
3	0	.8	3.60	−.0075	0	−18900	−.02100
4×1000	1.5	1.0	1.25	−.02792	0	0	−.01167
1	+1.0	+ .3333	0	−.002500	+ 8000	0	+.013333
2	+1.0	+4.3333	+ .6667	−.005000	−16000	+36750	+.031667
3	0						
4	+1.0	+ .6667	+ .8333	−.018611	0	0	−.007777
$(2-1)=a$	0	+4.0000	+ .6667	−.002500	−24000	+36750	+.018333
$(2-4)=b$	0	+3.6667	− .1667	+.013611	−16000	+36750	+.039444
a		+1.0	+.166667	−.000625	−6000	+ 9187.5	+.004583
b		+1.0	−.045454	+.003712	−4363.63	+10022.7	+.010757
3		+1.0	+4.500000	−.009375	0	−23625.0	−.026250
$(3-a)=c$		0	+4.333333	−.008750	+6000	−32812.5	−.030833
$(3-b)=d$		0	+4.545454	−.013087	+4363.63	−33647.7	−.037007
c			+1.0	−.0020192	+1384.61	−7572.11	−.0071154
d			+1.0	−.0028792	+ 960.00	−7402.50	−.0081415
$(c-d)=e$			0	+.0008600	+424.61	−169.61	+.0010261
e				+1.0	+493733	−197221	+1.19314

TABLE 33

MOMENTS IN THREE-LEGGED BENT OF FIG. 112

Moments are expressed in inch-pounds.

Moment	Due to Load on AB	Due to Load on BF	Due to Settlement of Foundations	Total
M_{DA}	$+231500$	-59900	$+294900$	$+466500$
M_{AB}	-907300	$+297400$	-371800	-981700
M_{BA}	$+1810400$	$+2019400$	-665800	$+3164000$
M_{BC}	-1026800	$+1365400$	$+5300$	$+343900$
M_{CB}	-661500	$+741900$	-284700	-204300
M_{BF}	-783600	-3384800	$+660500$	-3507900
M_{FB}	-106000	$+1446200$	$+400800$	$+1741000$
M_{EF}	-132100	-649100	-559600	-1340800

XIV. Conclusions

Some outstanding features of the use of the slope-deflection equations, as brought out by the analysis in Part II, may well be emphasized.

Two general methods of using the equations have been illustrated. In one case, after the equations have been written for each member of a frame, by equating the sum of the moments at each joint to zero and employing one equation of statics, a number of equations is obtained which contain values of θ and R as the only unknowns. From these equations can be found values of θ and R, which, when substituted in the original slope-deflection equations, give values of the various moments. This method applies especially well to a frame in which a large number of members meet at each joint. Such a problem is generally best solved in numerical terms. Examples of this method are found in sections 23, 61, 64, and 65.

The procedure in the other case is more direct. The slope-deflection equations for each member may be combined to eliminate values of θ and R, leaving equations involving the unknown moments, the properties of the members, and the given loading of the frame. These equations may be solved directly for the moments. Examples of this method are found in sections 24, 35, and 49.

Special attention is called to the form of the equations, which are independent of the magnitude and location of the individual loads, except as the magnitude and location of such loads influence the numerical values of the quantities C and H of the equations. The quantities C and H are determinate and their numerical values may be readily found for any known system of loads.

It is well to note also that the treatment of continuous girders for which the supports are not on the same level is comparatively simple. This part of the work is of considerable value inasmuch as it permits the determination of the effect of the settlement of supports. The treatment of the settlement of foundations for two-legged bents is of equal importance.

Advantages of the slope-deflection equations which are worthy of appreciation are:

(1) The general form of the fundamental equation is easily memorized, and the equations may be written for all members of a structure with little effort. The value of the quantities C or H

for loaded members may be calculated by reference to Tables 2 and 3. It is frequently possible to simplify the equations through noting where values of θ and R must be equal to zero from the conditions of the problem.

(2) No integrations need be performed except possibly to find values of C or H, and there is little danger of the omission of the effect of a single indeterminate quantity, as there is in methods involving the work of internal forces or moments.

(3) The physical conception of a problem is easier than in the case where differentiation or integration is performed. When the slopes and deflections are determined, it is easy to visualize the approximate shape of the elastic curve of a member, whereas an expression involving the work of an indeterminate force or moment may have little physical meaning. Neither does the method of cutting a member and equating expressions for the linear and angular movement of the adjoining ends give so clear an idea of the actual deformation. To one unfamiliar with such a method, the determination of the sign of the movement of the ends of the member cut is also more or less difficult.

(4) It is shown in sections 67, 68, and 69 that the effect of axial and shearing deformations and of slip of joints may be calculated by the use of the slope-deflection equations. This makes possible a complete treatment of any problem, though it is shown that it is seldom necessary to make use of such refinements in an analysis.

(5) The use of the quantity K for $\frac{I}{l}$ of a member, and also of quantities n, s, and p, as ratios of K's for different members, is of great help in writing equations in a workable form. The restraint factor N is also useful in simplifying both analyses and final equations.

(6) Although the fact has not been brought out in this bulletin, these equations may be applied to many structures not composed of rectangular units. The determination of secondary stresses in bridge trusses is an example of such use which has been in print for some time. With trapezoidal and triangular frames, care must be taken in the use of the term R for adjoining members.

(7) While the method is readily applicable to all the problems solved in this bulletin, its advantage over other methods is seen when applied to structures which are statically indeterminate to a high degree and in which a number of members meets at each joint.

The use of statically indeterminate structures in recent years has grown rapidly and many new types of structures have been evolved. With the use of riveted connections in steel frames and the development of monolithic reinforced concrete structures of all sorts, it often happens that statically indeterminate stresses cannot be avoided. On the other hand, structures are frequently made of an indeterminate type for the purpose of securing economy of material. Rational methods of design will do much to inspire confidence in the reliability and economy of such structures, thus insuring their more widespread use.

It is felt that the treatment of statically indeterminate structures given in this bulletin will be helpful in giving information regarding such structures. The method has been explained in sufficient detail to enable the designing engineer to use it in the solution of his particular problems. It is believed that the fundamental principles can be quickly coördinated with the ordinary principles of mechanics so that the more complex problems and even the simpler ones may be studied from a new viewpoint.

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