The Theorem of Least Work

The theorem of least work derives from what is known as Castigliano's second theorem. So, let's first state the two theorems of Carlo Alberto Castigliano (1847-1884) who was an Italian railroad engineer. In 1879, Castigliano published two theorems.

> Castigliano's first theorem
> *The first partial derivative of the total internal energy (strain energy) in a structure with respect to any particular deflection component at a point is equal to the force applied at that point and in the direction corresponding to that deflection component.*
>
> This first theorem is applicable to linearly or nonlinearly elastic structures in which the temperature is constant and the supports are unyielding.
>
> Castigliano's second theorem
> *The first partial derivative of the total internal energy in a structure with respect to the force applied at any point is equal to the deflection at the point of application of that force in the direction of its line of action.*
>
> The second theorem of Castigliano is applicable to linearly elastic (Hookean material) structures with constant temperature and unyielding supports.

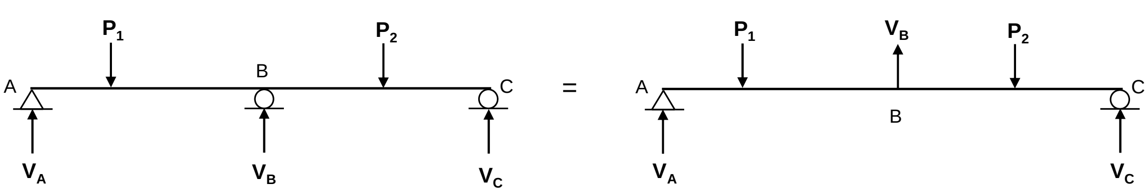
Note that in the above statements, *force* may mean point force or couple (moment) and *displacement* may mean translation or angular rotation. Proofs of Castigliano's theorems are given at the end of this document.

Without further due, here is the theorem of least work, a.k.a. Castigliano's theorem of least work:

> *The redundant reaction components of a statically indeterminate structure are such that they make the internal work (strain energy) a minimum.*

Please read the above statement again. It is a succinct statement of Nature's tendency to conserve energy. (Or it could be interpreted as Nature prefers to be lazy[1].)

We shall explain the proof of the theorem of least work and its application first by the use of a simple example shown below.



The beam shown on the left is statically indeterminate to the first degree. It is obvious that the simple determinate beam shown on the right is equivalent to the original beam on the left with a geometric condition (compatibility condition). That condition with which V_B can be determined is that the deflection at *B* of the equivalent beam should be zero. This deflection, by Castigliano's second theorem, is $\Delta_B = \dfrac{\partial U}{\partial V_B}$. But we know that support *B* has zero vertical deflection.

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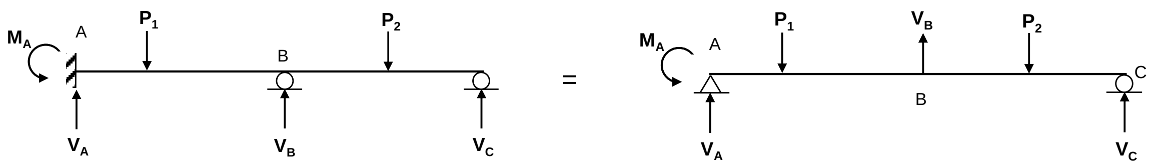
[1] It is said that it takes 43 muscles to frown and 17 muscles to smile (hence smiling ☺ is easier than frowning ☹), but none to do nothing.

Hence, the condition for determining V_B becomes $\frac{\partial U}{\partial V_B} = 0$, or V_B is such as to make the total internal work a minimum.

Note that when the first derivative of a function with respect to a variable and at a certain value of the variable is equal to zero, the function may be either a maximum or a minimum. Appealing to our sense of physics, we can eliminate the possibility that the total work can be a maximum. Hence the result: when Nature has its free choice, it will always tend to conserve energy.

To give an example with two redundants (i.e. statically indeterminate to second degree), we can consider the following system. Choosing moment at *A* and vertical force at *B* as the redundants, we can obtain the equivalent system on the right.



The conditions of geometry together with Castigliano's second theorem state that $\theta_A = \frac{\partial U}{\partial M_A} = 0$ and $\Delta_B = \frac{\partial U}{\partial V_B} = 0$ which simply means that the redundants M_A and V_B are such that they minimize the total internal strain energy *U*.

Proofs of Castigliano's Theorems

Castigliano referred to his theorems as 'theorem of the differential coefficients of the internal work, part I and part II'. Both theorems are related to statically indeterminate structures; the first one is to express equilibrium conditions while the second one is sometimes used to solve deflection problems in which case it is very similar to the method of virtual work.

Proof of the first theorem

Suppose a structure is in equilibrium under the action of forces $P_1, P_2, \cdots, P_n$. These forces have caused certain deflections $\delta_1, \delta_2, \cdots, \delta_n$ of the points where they were applied. They have done a certain amount of external work W_e and caused an equal amount of strain energy U to be stored in the structure.

If we vary the forces by infinitesimal amounts such that the deflection δ_n is changed a small amount $d\delta_n$ while all other deflections $\delta_1, \delta_2, \cdots, \delta_{n-1}$ are held constant, the total internal strain energy of the system will change to

$$U' = U + \frac{\partial U}{\partial \delta_n} d\delta_n$$

In the meantime, the external work done on the structure will increase and become $W_e' = W_e + P_n d\delta_n + \frac{1}{2} dP_n d\delta_n$ where $\frac{1}{2} dP_n d\delta_n$ is the second-order contribution due to the differential force dP_n going through the $d\delta_n$. If we neglect this second-order contribution, the total external work done becomes

$$W_e' = W_e + P_n d\delta_n$$

As U' must be equal to W_e', we have

$$\frac{\partial U}{\partial \delta_n} = P_n$$

Note that, to be able use this theorem we need to express the strain energy in terms of deflections $\delta_1, \delta_2, \cdots, \delta_n$ (as opposed to in terms of forces we have been doing so far).

Proof of the second theorem

Suppose a structure is in equilibrium under the action of forces $P_1, P_2, \cdots, P_n$. These forces would have caused certain deflections $\delta_1, \delta_2, \cdots, \delta_n$ of the points where they were applied. Accordingly, they have done a certain amount of external work W_e and caused an equal amount of strain energy U to be stored in the structure.

If we increase force P_n by a small amount dP_n, the total internal strain energy of the system will change to

$$U' = U + \frac{\partial U}{\partial P_n} dP_n$$

Now, let's reverse the order of the application of the forces. In other words, let's apply dP_n to the otherwise unloaded structure. Then, apply $P_1, P_2, \cdots, P_n$. Since the material in our structure follows Hooke's law, the final total internal strain will be same as U' given above.

The force dP_n applied first produces an infinitesimal displacement $d\delta_n$. The corresponding external work done during the application of dP_n is a small quantity of the second order and can be neglected. When $P_1, P_2, \cdots, P_n$ are applied, the external work done by them will not be modified by the presence of dP_n. However, dP_n will act through δ_n and provide an additional external work equal to $dP_n\delta_n$. Therefore, the total external work done by the entire system during this loading sequence will be equal to

$$W_e' = W_e + dP_n\delta_n$$

According to principle of the conservation of energy, $W_e' = U_e$, which means

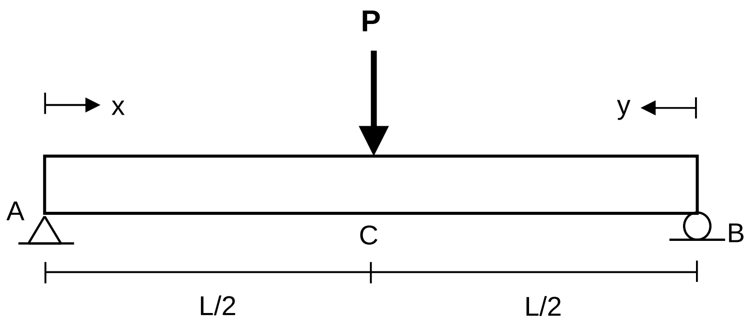
$$W_e + dP_n\delta_n = U + \frac{\partial U}{\partial P_n} dP_n$$

As $W_e = U$, we find that

$$\frac{\partial U}{\partial P_n} = \delta_n$$

which is the mathematical statement of Castigliano's second theorem.

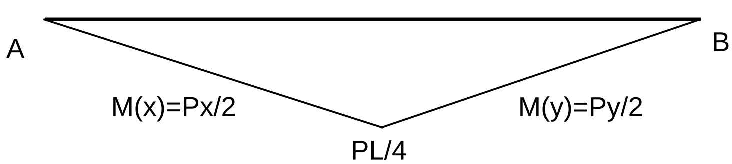
Ex. To illustrate Castigliano's second theorem, consider the simply supported cantilever beam below. Say we are interested in finding the vertical deflection at the point where the load P is applied.



Let's consider flexural behavior only. To find the total strain energy, we need to find the internal bending moment over the member and the corresponding curvature. To be able to make use of Castigliano's 2nd theorem, we need to express the internal strain energy in terms of P, the load.

Noting the symmetry in the system, we can use the coordinates x and y to find analytic expressions for the internal bending moment.

Bending moment:



Dividing the bending moment value at a location with EI will give the curvature at that location. Let's assume that EI is constant in this beam. The total internal strain energy is then

$$\begin{aligned}
U &= \frac{1}{2}\int_0^{L/2}\left(\frac{P}{2}x\right)\left(\frac{P}{2}x\frac{1}{EI}\right)dx + \frac{1}{2}\int_0^{L/2}\left(\frac{P}{2}y\right)\left(\frac{P}{2}y\frac{1}{EI}\right)dy \\
&= \frac{1}{2}\int_0^{L/2} P^2\frac{x^2}{4EI}dx + \frac{1}{2}\int_0^{L/2} P^2\frac{y^2}{4EI}dy \\
&= \frac{1}{2}\frac{P^2L^3}{48EI}
\end{aligned}$$

Applying Castigliano's 2nd theorem

$$\frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left(\frac{1}{2}\frac{P^2L^3}{48EI}\right) = \frac{PL^3}{48EI}$$

$$\Delta_C = \frac{PL^3}{48EI}$$

which we know to be the correct answer. You can verify it using virtual unit force approach or simply moment-area theorems.