Approximating Lateral Stiffness of Stories in Elastic Frames

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Abstract: Despite advances in the computer analysis of frame buildings for lateral loads, there remains a need for simple models that provide accurate estimates of response. The concept of lateral stiffness is reviewed, and it is concluded that a single value can be used to represent the stiffness of a story in an elastic, rectangular frame with fixed base that is subjected to regular distributions of lateral load. Three existing expressions for approximating the lateral stiffness of stories are compared, but it is concluded that these are applicable only for uniform frames with girders that are flexurally stiffer than columns. An expression is proposed that includes three numerically derived factors that greatly improve the accuracy of stiffness estimates for regular and moderately irregular frames. The proposed expression simulates: (1) The effect of unequal heights for adjacent stories; (2) the influence of top and bottom boundaries; and (3) the stiffening effect of the fixed base in low-rise frames.

Introduction

Analysis of frame buildings subjected to lateral loads, such as those generated by earthquake motion and high wind, requires knowledge of lateral stiffness for calculation of lateral displacements in static analysis, and calculation of lateral displacements and dynamic properties (modal frequencies and shapes) in dynamic analysis. Modern computing equipment and current structural analysis techniques have greatly facilitated the development and use of complex computational models of building frames. Yet, there remains a need for a simple mathematical model that can be used to approximate, with a reasonable degree of accuracy, the response of building frames to lateral loads.

For a designer, approximate analysis may be used for obtaining estimates of building behavior during preliminary design, or for verifying the results of a more sophisticated computer analysis. For the researcher, the need is for an efficient mathematical model, and under certain conditions a tolerable degree of accuracy can be forsaken for computational expediency. This trade-off may be justified when numerical calculations are so extensive that an efficient mathematical model becomes a practical necessity. Probabilistic analyses, parametric studies, and system identification work provide ready examples of this situation.

The so-called shear building is often used to study the response of frame structures to lateral loads. It owes its popularity to the simplicity of the governing equilibrium equations and the computational ease with which these can be solved. The shear building is a lumped parameter model (Fig. 1) in which all mass at a story is placed at the corresponding lateral degrees of freedom. In the traditional shear building, joint rotations are assumed to be equal to zero, corresponding to girders that are rigid in relation to

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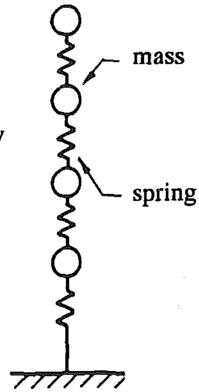

FIG. 1. Lumped-Parameter Model

columns. The lateral stiffness of a story is obtained by combining all columns into a single elastic spring that connects the lateral degrees of freedom at adjacent stories. The resulting mass and stiffness matrices are, respectively, diagonal (nonzero coefficients in principal diagonal only) and tridiagonal (nonzero coefficients in principal diagonal and adjacent minor diagonals only). In contrast, rigorous frame analysis yields stiffness matrices for frames that have nonzero coefficients outside the tridiagonal band. Solution of these equilibrium equations requires considerably more computational effort than solution of those for the shear building.

In most practical cases, the assumption of zero joint rotations introduces a substantial amount of error. Rubinstein and Hurty (1961) have indicated that neglecting the effect of joint rotations can lead to gross errors in computed dynamic properties. They demonstrated that the majority of this error can be eliminated with reasonable assumptions of joint behavior, such as equal rotations for exterior and interior joints in a floor of a frame and equal rotations for all joints in a given floor of a structure comprising multiple frames that are not identical. Goldberg (1972) successfully approximated the effect of joint flexibility by assuming an approximate average value for joint rotation at each floor of a multistory frame. He was using an iterative slope-deflection procedure to calculate drift. The works of these authors clearly demonstrate that if the stiffness of stories are modified to reflect girder flexibility in a realistic manner, the shear building becomes a viable mathematical model for approximating the response of laterally loaded elastic frames.

The purpose of this paper is to present explicit, closed-form expressions for approximating the lateral stiffnesses of stories in elastic frames. The expressions presented in this paper are limited to rectangular frames that are fixed at the base and for which only flexural deformations are important. Several existing expressions are reviewed and compared. An alternate formulation is presented that includes correction factors that enable the approximate stiffness expression to (1) Simulate the effect of variation in adjacent story heights; (2) more accurately represent the stiffnesses of boundary stories (first, second, and top); and (3) approximate the stiffening effect of a fixed base in low-rise frames. The approach taken herein achieves the same goal as static condensation of rotational degrees of freedom. However, this process is performed prior to formulation of equilibrium equations.

Consequently, the softening effect of joint rotations on story stiffness is only approximated. A simple example is included to illustrate the ease with which the proposed expression is applied.

Apparent Lateral Stiffness of a Story

Before presenting the approximate expressions, it is worthwhile to investigate the concept of lateral stiffness. The lateral stiffness K_s of a story is generally defined as the ratio of story shear to story drift. However, story drift, defined as the difference in the lateral displacements of floors bounding a story, is affected by vertical distribution of lateral loads, i.e., there is a unique displaced profile for each type of lateral load distribution. Consequently, the lateral stiffness of a story is not a stationary property, but an apparent one that depends on lateral load distribution. In the analysis of frame buildings subjected to wind or earthquake loads, it is generally assumed that lateral loads are distributed in a "regular" manner. Regular means that loads act in the same direction on all floors, and that lateral loads vary from floor to floor in a controlled manner. For frames subjected to regular lateral, load distributions, variations in the lateral stiffness of a given story for the several load cases are small enough to be neglected. Thus, a single value can be used to represent stiffness.

A series of nine-story, five-bay, elastic frames were analyzed to verify the concept of apparent lateral stiffness of a story. As indicated in Table 1, all stories above the first have the same height, H_s, and the first story is 33% taller. All bays have a span L equal to twice the nominal story height H_s. Moments of inertia for columns and girders are smaller at upper floors, as indicated in Table 1. This variation in stiffness is typical of actual building frames and introduces small or moderate irregularities in profile. Modulus of elasticity E is the same for all members of a frame. A relative stiffness parameter α is defined as the ratio of I_g/L to I_c/H_s, where I_g and I_c, respectively, are the nominal values of girder and column moments of inertia. The parameter α is used as a global indication of the relative flexural stiffnesses of girders to columns; its inverse β indicates column stiffness relative to girder stiffness. For each of the frames analyzed, α or β was assigned a value between 1 and 10. This value was used to specify girder moment of inertia on the basis of column moment of inertia, story height, and bay length, as indicated in Table 1.

Three distributions of lateral load were included in the analyses and are designated as constant, linear, and parabolic. The first of these has lateral loads of equal magnitudes acting on every floor of the frame. The linear and parabolic lateral load distributions, respectively, have lateral loads on each floor that are proportional to the height and to the square of the height of each floor from the base of the frame. A linear matrix analysis computer program was used to determine the drift response of the frames to each of lateral load distributions. No rigid zones were assumed in the joints of the frames, and shear and axial deformations were suppressed for all frame members.

Apparent stiffnesses (K_e) for the ninth, fifth, and first stories of the frames are summarized in Fig. 2 after being normalized by the stiffnesses obtained for these stories assuming rigid girders (K_∞). For the frames in question, this figure shows that apparent story stiffnesses are not affected much by the type of lateral load distribution, and that the concept of a single-valued story stiffness is quite accurate as long as the distribution is regular. It can

TABLE 1. Properties of Nine-Story, Five-Bay Frames

Floor (1)	Story (2)	Bay length[a] (3)	Story height (4)	Moment of Inertia: Girder[b] (5)	Moment of Inertia: Exterior column (6)	Moment of Inertia: Interior column (7)
9	—	L	—	$1/2I_g$	—	—
—	9	—	H_s	—	$1/3I_c$	$1/3I_c$
8	—	L	—	$1/2I_g$	—	—
—	8	—	H_s	—	$1/3I_c$	$1/2I_c$
7	—	L	—	$1/2I_g$	—	—
—	7	—	H_s	—	$1/2I_c$	$1/2I_c$
6	—	L	—	$3/4I_g$	—	—
—	6	—	H_s	—	$1/2I_c$	$1/2I_c$
5	—	L	—	$3/4I_g$	—	—
—	5	—	H_s	—	$1/2I_c$	$2/3I_c$
4	—	L	—	$3/4I_g$	—	—
—	4	—	H_s	—	$2/3I_c$	$2/3I_c$
3	—	L	—	I_g	—	—
—	3	—	H_s	—	$2/3I_c$	$2/3I_c$
2	—	L	—	I_g	—	—
—	2	—	H_s	—	$2/3I_c$	I_c
1	—	L	—	I_g	—	—
—	1	—	$4/3H_s$	—	I_c	I_c

[a] $L = 2H_s$.
[b] $I_g = (\alpha I_c)(L/H_s) = (I_c/\beta)(L/H_s)$.

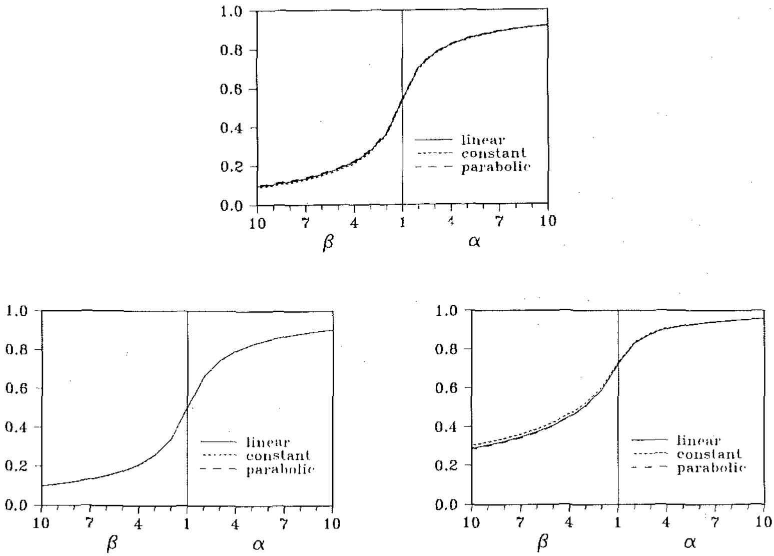

FIG. 2. Stiffness Ratios (K_e/K_∞) for Nine-Story, Five-Bay Frames: (*a*) Ninth Story; (*b*) Fifth Story; (*c*) First Story

further be seen that even for frames with girders that are nominally ten times as stiff as columns ($\alpha = 10$), apparent stiffness is not equal to that obtained using the rigid girder assumption. The region of largest changes in story stiffnesses is bounded by α and β equal to 4, and this region coincides with the range of relative flexural stiffnesses of members in typical building frames.

Literature Review

A review of the literature concerning analysis for lateral loads revealed many contributions, not mentioned here, on the subject of approximating drift. However, only three references were found that present explicit, closed-form expressions that can be used to approximate the lateral stiffness of stories in elastic frames.

Benjamin (1959)

In his text on indeterminate frame analysis, Benjamin (1959) outlines a method for estimating the stiffness of a story in a laterally loaded elastic frame. The slope deflection formulas are applied successfully to both ends of the four members bounding a typical panel. The effects of gravity loads are neglected, as well as axial deformations of the members. By appropriate manipulation, joint rotations are eliminated from the slope-deflection equations, yielding expressions for drift of the columns in the panel. Benjamin combines column drifts to obtain an average value for the story and indicates that this drift can be used to obtain story stiffness. With some rearrangement of terms, stiffness K_s can be expressed as

$$K_s = \frac{\left(\frac{24Vn}{\mathrm{H}}\right)}{\left[\sum\left(\frac{\Sigma M_{ec}}{k_{ec}}\right) + \sum\left(\frac{\Sigma M_{ic}}{k_{ic}}\right) - \sum\left(\frac{\Sigma M_{ga}}{k_{ga}}\right) - \sum\left(\frac{\Sigma M_{gb}}{k_{gb}}\right)\right]} \quad \ldots . (1)$$

when n, H, and ΣM = number of panels, the story height, and the sum of the two member end moments, respectively, for a story shear force V; flexural stiffness k of a member = EI/L, and the subscripts ec, ic, ga, and gb = respectively, exterior columns, interior columns, girders in the floor above, and girders in the floor below. Because the panel is indeterminate, Benjamin further recommends use of either the portal method, the cantilever method, or the factor method to approximate member end moments for the story shear V. While the factor method yields more accurate internal forces than the portal or cantilever methods (Wilbur et al. 1976), only the portal method provides general expressions for member end moments that are conducive to closed-form story stiffness expressions that are manageable. When the values for member end moments obtained from the portal method are introduced into (1), the apparent stiffness of the story becomes

$$K_s = \frac{\left(\frac{48n^2}{H^2}\right)}{\left[\sum\left(\frac{1}{k_{ec}}\right) + \sum\left(\frac{1}{k_{ic}}\right) + \sum\left(\frac{1}{k_{ga}}\right) + \sum\left(\frac{1}{k_{gb}}\right)\right]} \quad \ldots\ldots\ldots\ldots (2)$$

Blume et al. (1961)

Blume et al. (1961) present another procedure, whereby the method of moment distribution is used to determine how much the lateral stiffness of each column in a shear building is softened in proportion to girder flexibility. The apparent stiffness K_s of the story is obtained by adding the contributions of all columns ($K_s = \Sigma K_c$). Blume et al. assume that the typical column is in a regular frame and that the column end rotations are equal. Fixed-end moments for a column are calculated based on rigid girders and an arbitrarily selected story drift. Only a single cycle of moment distribution is needed because member stiffnesses are modified to reflect equal end rotations, for which case carryover movements are equal to zero. The resulting column moments are used to calculate the resisting shear force in the column, and from this shear force, apparent stiffness K_c of the column is approximated as

$$K_c = \left(\frac{12E_cI_c}{H^3}\right)\left[1 - \left(\frac{k_c}{\Sigma k_a}\right) - \left(\frac{k_c}{\Sigma k_b}\right)\right] \quad \text{(3)}$$

where k_c = the flexural stiffness of the column. The sums of the stiffnesses of all connecting members in the joints above and below the column are given by Σk_a and Σk_b, respectively.

Blume et al. recognized that this approximation breaks down at boundary stories, i.e., at the top and base of a frame. The disturbances introduced by abrupt termination of the frame and the fixed base are not consistent with the assumption of equal end rotations. To compensate for this shortcoming, Blume et al. recommend the use of charts that summarize multiplicative factors for modifying column end moments at boundary stories. In the present study, however, these charts are not used because explicit, closed-form expressions are sought.

Muto (1974)

In his treatise on seismic analysis of buildings, Muto (1974) approaches the problem of approximating lateral stiffnesses of columns in elastic stories by applying the slope-deflection equations to members in a panel of an idealized regular frame, as did Benjamin (1959). Muto, however, assumes that the frame is an infinite array of members, and that all columns at a story resist shear forces of equal magnitude. He further assumes that both ends of all members undergo equal end rotations. Using the slope-deflection formulas, expressions for member end moments are obtained. Muto uses these expressions in moment equilibrium equations for a typical beam-column joint, from which he extracts the following expression for stiffness K_c of the column

$$K_c = \left(\frac{12E_cI_c}{H^3}\right)\left(\frac{4k_g}{4k_c + 4k_g}\right) \quad \text{(4)}$$

To extend this equation to columns in real frames, Muto interprets the term $4k_g$ as the sum of the flexural stiffnesses of the two girders each framing into the joints at the top and the bottom of the column. Thus column stiffness can be rewritten as

$$K_c = \left(\frac{12E_cI_c}{H^3}\right)\left(\frac{\Sigma k_{ga} + \Sigma k_{gb}}{4k_c + \Sigma k_{ga} + \Sigma k_{gb}}\right) \quad \text{(5a)}$$

where Σk_{ga} and Σk_{gb} = respectively, the sum of the flexural stiffnesses of the girders framing into the joint above and the joint below the column. Story stiffness K_s is obtained by summing the stiffnesses of all columns at a story.

Muto recognizes that first-story stiffness is affected by the base fixity, and proposes a different expression for first-story columns. The derivation is similar to that (5*a*), except that a typical first-story column with a fixed base is assumed to have an inflection point located one-third of the column height from the top joint. Column stiffness for this case is given by

$$K_c = \left(\frac{12E_cI_c}{H^3}\right)\left(\frac{k_c + \Sigma k_{ga}}{4k_c + \Sigma k_{ga}}\right) \quad \text{(5b)}$$

ALTERNATE FORMULATION

An alternate model of the behavior of stories that are not adjacent to the base or the top of a laterally loaded frame is now presented. One story is isolated from the rest of the frame, as shown in Fig. 3(*a*). This story includes all columns as well as a portion of the girders at floor levels above and below. For a uniform frame comprising an infinite number of stories, it is assumed that total girder stiffness at a floor level is shared equally by adjacent stories. The story height factor η_a, which will be discussed later, is included to represent the effect on stiffness of adjacent stories having unequal heights. The representation of a story is further simplified by assuming that girders and columns act with points of inflection at midlength.

Based on these assumptions, a typical interior girder-column assemblage is isolated as shown in Fig. 3(*b*). Equations that satisfy equilibrium of the assemblage are written for column shear and moment at the joint. Because gravity effects are neglected, external moment on the joint is equal to zero and the rotation θ_a is eliminated by static condensation. The drift Δ_a cor-

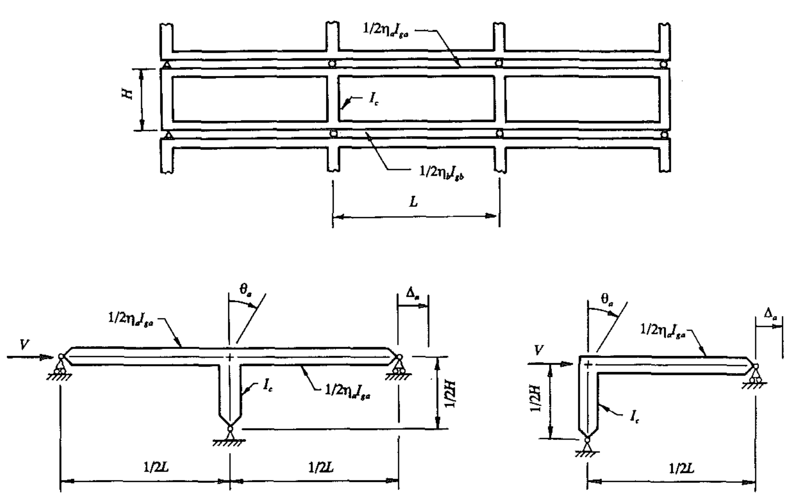

FIG. 3. Idealization of Story: (*a*) Isolated Story; (*b*) Interior Girder-Column Assemblage; (c) Exterior Girder-Column Assemblage

responding to the upper half of the column is obtained from the equilibrium equations as

$$\Delta_a = \left(\frac{VH^2}{24}\right)\left(\frac{k_c + \eta_a k_{ga}}{k_c \eta_a k_{ga}}\right) \quad (6)$$

By adding the drifts of both portions of the column and solving for the ratio of column shear to total drift, the apparent stiffness for an interior column becomes

$$K_{ic} = \left(\frac{24}{H^2}\right)\left(\frac{1}{\frac{2}{k_c} + \frac{1}{\eta_a k_{ga}} + \frac{1}{\eta_b k_{gb}}}\right) \quad (7a)$$

For a typical assemblage at an exterior joint, a similar expression for apparent stiffness is derived

$$K_{ec} = \left(\frac{24}{H^2}\right)\left(\frac{1}{\frac{2}{k_c} + \frac{2}{\eta_a k_{ga}} + \frac{2}{\eta_b k_{gb}}}\right) \quad (7b)$$

Story stiffness K_s is obtained by summing the contributions of all columns at that story.

If the factors η_a and η_b are taken equal to unity, (7) yields values for story stiffness that are practically identical to those obtained using Muto's intermediate-story expression [(5*a*)]. In both cases, stiffnesses must be obtained individually for the columns and then added to define story stiffness. This procedure is necessary because (5) and (7) recognize that interior and exterior columns differ in the number of connecting girders. However, if this difference is ignored, an analogy can be made between a typical story in a frame [Fig. 3(*a*)] and the column for which (7*a*) was derived [Fig. 3(*b*)]. The terms k_c, $n_a k_{ga}$, and $n_b k_{gb}$ in (7*a*) are substituted by the sums of relative flexural stiffnesses for columns (Σk_c), girders above ($n_a \Sigma k_{ga}$) and girders below ($n_b \Sigma k_{gb}$) the story, respectively, yielding the following expression for story stiffness

$$K_s = \left(\frac{24}{H^2}\right)\left(\frac{1}{\frac{2}{\Sigma k_c} + \frac{1}{\eta_a \Sigma k_{ga}} + \frac{1}{\eta_b \Sigma k_{gb}}}\right) \quad (8)$$

It is assumed that story height factors η_a and η_b are constant at a given story. For the first story of a frame with a fixed base Σk_{gb} is taken equal to infinity, thus eliminating the corresponding term in the denominator of (8).

Eq. (8) is the basis for the present study. If the story height factors η_a and η_b are taken equal to unity, as is the case for a uniform frame, the following expression is obtained

$$K_s = \left(\frac{24}{H^2}\right)\left(\frac{1}{\frac{2}{\Sigma k_c} + \frac{1}{\Sigma k_{ga}} + \frac{1}{\Sigma k_{gb}}}\right) \quad (9)$$

Eq. (9) or equivalent forms of that expression have been known and used in structural engineering practice in the United States in the recent past (M. A. Sozen, personal communication, 1985). However, the writer was unable to find published reference to this expression: it appears to have been largely forgotten.

Story Stiffnesses for Nine-Story, Five-Bay Frames

The expression derived from Benjamin's (1959) work [(2)] and those proposed by Blume et al. (1961) [(3)] and Muto (1974) [(5*a*) and (5*b*)], as well as (9), were used to approximate story stiffnesses for the nine-story frames defined earlier (Table 1). Exact apparent stiffnesses K_e were also evaluated using a matrix analysis program to compute frame response to a linear lateral load distribution. The results are summarized in for stories 9, 5, 2, and 1, where the approximate values K_s are normalized by the exact stiffness K_e. Results from (7*a*) and (7*b*) ($\eta_a = \eta_b = 1$) are not presented; they are practically identical to those obtained from Muto's intermediate-story expression [(5*a*)]. Stiffnesses from (8) are discussed after the story height factor η_i and a second factor for boundary stories (C_s) are introduced.

Stories in the middle of the frames most closely approximate the idealized conditions that were assumed in the deriving the approximate equations. However, only Muto's intermediate-story expression [(5*a*)] and (9) estimate the stiffnesses of the fifth story [Fig. 4(*a*)] with a reasonable degree of accuracy (7% error) over the entire range of member stiffnesses considered in this study, including frames for which columns are ten times as stiff as girders. Stiffnesses for stories 3, 4, 6, 7, and 8 are also approximated with a similar degree of accuracy by (5*a*) and (9) (not shown for brevity). Blume's expression [(3)] quickly loses accuracy for frames with columns stiffer than girders ($\alpha = 1/\beta < 1$). Benjamin's expression [(2)] has only a small range

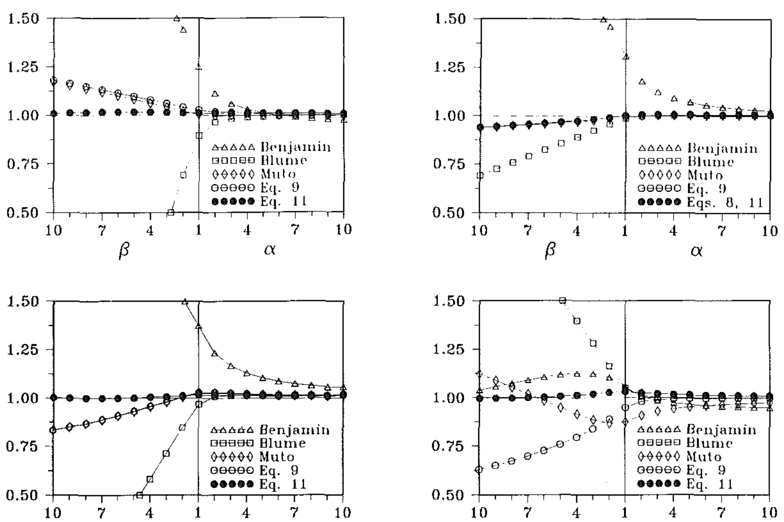

FIG. 4. Normalized Stiffnesses (K_s/K_e) for Uniform Nine-Story, Five-Bay Frames ($H_1 = 4/3H_s$, $H_2 - H_9 = H_s$: (*a*) Ninth Story; (*b*) Fifth Story; (*c*) Second Story; (*d*) First Story

of values for α over which it is accurate. In fact, Benjamin's expression does not always converge to the correct solution for frames with infinitely stiff girders. This shortcoming stems from the use of the portal method to evaluate member end moments.

At the boundaries, stories 9 [Fig. 4(*a*) and 1 [Fig. 4(*d*)], all four expressions diverge from the actual solution for frames with columns that are considerably stiffer than girders ($\alpha = 1/\beta << 1$). It is interesting that Muto's first-story stiffness [(5*b*)] does not fare much better than the others [Fig. 4(*d*)]. In general, (9) and Muto's expression [(5)] overestimate the stiffness of the ninth story and grossly underestimate the stiffness of the first story when $\alpha << 1$. Note that Muto's intermediate-story expression [(5*a*)] and (9) also underestimate the stiffness of the second story when $\alpha << 1$, thus the influence of the fixed base carries over to the second story. These equations do not accurately simulate the influence of the boundaries (i.e., the stiffening effect of the fixed base and the softening effect of the abrupt termination at the top) for frames with columns stiffer than girders ($\alpha < 1$). However, it should be noted that as α decreases, the normalization factor for the data in Fig. 4 becomes very small (Fig. 2), and the approximate expressions attempt to predict a quantity that is rapidly decreasing in magnitude relative to α.

The observed inaccuracy of (9) is due to a large extend on the assumption regarding points of inflection in the columns. Only in intermediate stories of uniform frames with many stories will these points be close to column midheight. The boundaries and any marked variation in the heights of adjacent stories will shift points of inflection away from column midheight. The introduction of appropriate correction factors in (9) can extend the range of relative member stiffness (α) for which this expression is accurate.

Variation in Heights of Adjacent Stories

In deriving the expressions for the approximate stiffnesses shown in Fig. 4, it was assumed that the frames are uniform, i.e., adjacent stories have the same height. Several series of frames, representing variations of the uniform frames defined in Table 1, were analyzed to illustrate the limitations of this assumption. For each series, one of either the ninth, fifth, or first stories were assigned heights that are shorter or taller than those in the corresponding stories of the uniform frames. Stories 9 and 5 were given heights equal to $3/4H_s$ and $4/3H_s$, where H_s is the nominal story height, while the first story was given heights of H_s and $5/3H_s$. Approximate stiffnesses are summarized in Fig. 5, for frames with short stories, and in Fig. 6, for frames with tall stories.

Figs. 5 and 6 indicate trends similar to those observed for the uniform frames (Fig. 4). In addition, it can be seen that all expressions lose accuracy when applied to the frames with short or tall stories. When the height of a story is smaller than that of adjacent stories, the restraining effect of the bounding stories is smaller than if all three stories had the same height. Thus, a story that is shorter than its bounding stories will have a smaller stiffness, in relation to K_∞ (calculated assuming rigid girders), than if it were of the same height as adjacent stories. Similarly, a story that is taller than its bounding stories will have a larger stiffness, in relation to K_∞, than if it were of the same height as adjacent stories. Neither Muto's intermediate-story expression [(5*a*)] nor (9) recognize this effect, and both overestimate the stiffness of short stories and underestimate the stiffness of tall stories. This can be observed clearly for the fifth story [Figs. 5(*b*) and 6(*b*)], and similar trends can be seen for the boundary stories (9, 2, and 1).

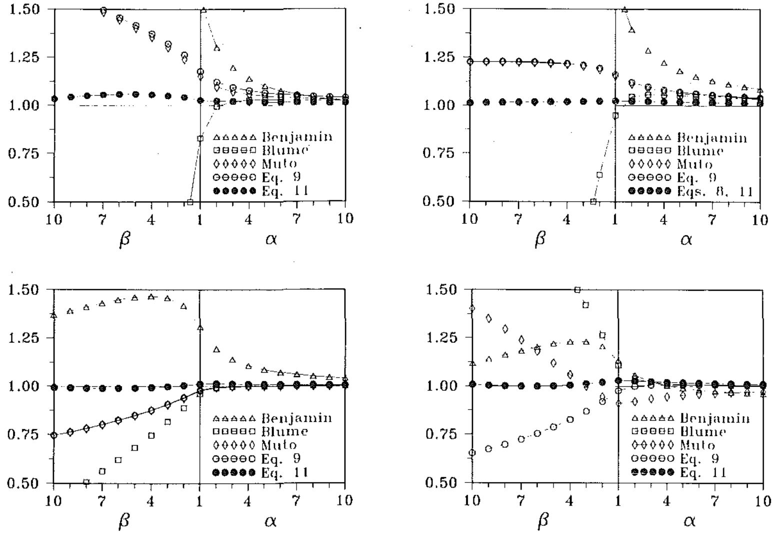

FIG. 5. Normalized Stiffnesses (K_s/K_e) for Nine-Story, Five-Bay Frames with Short Stories: (*a*) Ninth Story (H_9 = 3/4H_s); (*b*) Fifth Story (H_5 = 3/4H_s); (c) Second Story (H_1 = H_s); (*d*) First Story (H_1 = H_s)

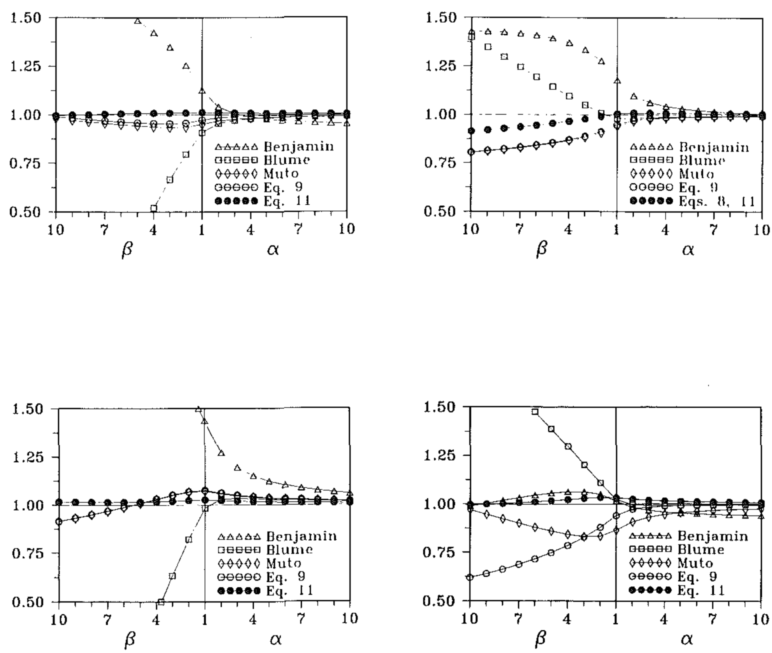

FIG. 6. Normalized Stiffnesses (K_s/K_e) for Nine-Story, Five-Bay Frames with Tall Stories: (*a*) Ninth Story (H_9 = 4/3H_s); (*b*) Fifth Story (H_5 = 4/3H_s); (c) Second Story (H_1 = 5/3H_s); (*d*) First Story (H_1 = 5/3H_s)

The story height factors η_a and η_b were introduced into (6) (7), and (8) to simulate the effect to adjacent stories with unequal heights on intermediate-story stiffness. Rather than deriving closed-form theoretical expressions, exact stiffnesses from the matrix analyses for the sixth, fifth, and fourth stories of the frames in Figs. 4, 5, and 6 were used in conjunction with (8) to obtain numerical values. If both adjacent stories have heights equal to the story considered, it is assumed that the height factor η_i for that story is equal to unity since no substantial correction is needed [Fig. 4(*b*)]. It is also assumed that η_a is equal to η_b for intermediate stories where the story above has a height H_a which is equal to the height H_b of the story below [Fig. 7(*b*)]. Additional values for η_a and η_b were generated by analyzing nine-story, five-bay frames similar to those considered earlier (Figs. 4, 5, and 6), but with fifth-story heights equal to 1/2, 2/3, 5/3, and 2 times the nominal story height H_s. For several values of α, computed fifth-story height factors η_i are shown in Fig. (7*b*) as a function of the ratio of fifth-story height to adjacent-story height (story-height ratio). Although there are differences in the magnitudes of the story-height factor η_i for different values of α, a single curve can be used to approximate η_i as a function of story-height ratio. The following relation for story-height factor was found to reproduce the primary trend in the computed values of η_i for the fifth story

$$\eta_i = \sqrt{\frac{H}{H_i}} \quad \text{.......... (10)}$$

Story-height factors η_a and η_b were also computed numerically from (8) for the fourth and sixth stories, respectively, of the nine-story frames. Only the fifth story was given a height different from H_s, thus, story-height

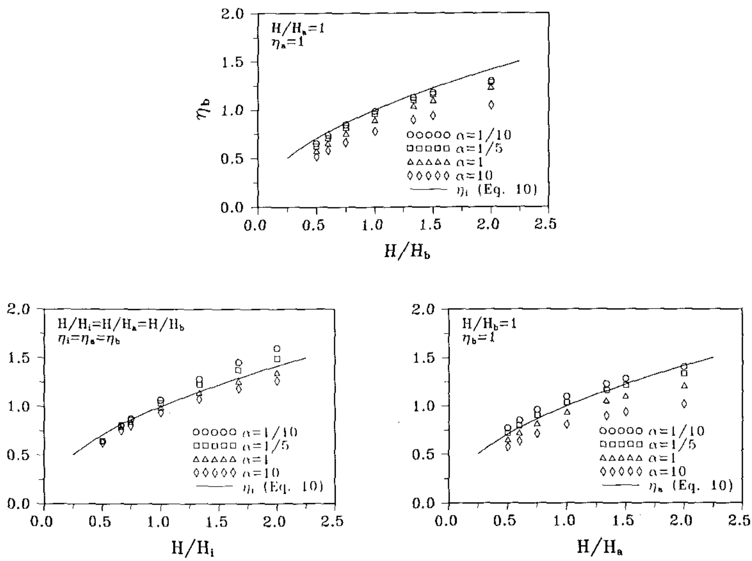

FIG. 7. Story-Height Factors (η_i) for Nine-Story, Five-Bay Frames: (*a*) Sixth Story (η_b); (*b*) Fifth Story ($\eta_a = \eta_b$); (*c*) Fourth Story (η_a)

fators η_a for the sixth story and η_b for the fourth story were assumed to be equal to unity. The computed values of η_b for the sixth story and η_a for the fourth story are shown in Figs. 7(*a*) and 7(*c*), respectively, as a function of story height ratio. The approximation given by (10) is not as accurate as for the fifth story, however, it is still judged sufficient.

Fifth-story normalized stiffnesses computed using (8) with story height factors given by (10) are shown in Figs. 5(*b*) and 6(*b*) for the nine-story frames. Comparison of these stiffnesses with those computed using (9) indicates a dramatic improvement in approximation accuracy. Similar improvements were observed for the fourth-story and sixth-story stiffnesses of these frames (not shown for brevity). For the frames in this study, approximation error does not exceed 7% for intermediate stories, even when relative column stiffness is ten times larger than relative girder stiffness ($\alpha = 1/\beta = 1/10$), and it generally does not exceed 5%.

Correction for Boundary Stories

It is evident from the analyses of the nine-story, five-bay frames (Figs. 4, 5, and 6) that (8) does not approximate the stiffnesses of boundary stories as well as stiffnesses of intermediate stories. Boundary stories include not only the first and top stories, but also the second story, because the stiffening effect of the fixed base propagates beyond the first story. From Figs. 4, 5, and 6 it can be seen that the error in the boundary stories is larger for frames that have columns that are flexurally stiffer than girders ($\alpha = 1/\beta < 1$). For boundary stories, (8) was modified by including a correction term C_s as follows

$$K_s = \left(\frac{24}{H^2}\right)\left(\frac{1 + C_s}{\frac{2}{\Sigma k_c} + \frac{1}{\eta_a \Sigma k_{ga}} + \frac{1}{\eta_b \Sigma k_{gb}}}\right) \qquad (11)$$

For intermediate stories, the correction term C_s is equal to zero, reducing (11) to (8).

For stories 9, 2, and 1, the correction terms C_s were evaluated numerically using the exact response taken from the matrix analyses of the nine-story, five-bay frames. These values for C_s are shown in Fig. 8. For all three boundary stories, the correction terms are approximately linear functions of the ratio of total columns stiffness at a story Σk_c to total girder stiffness Σk_g. The first-story correction term C_1 [Fig. 8(*a*)] is independent of the ratio of first-story height H_1 to second-story height H_2. For the second story [Fig. 8(*b*)] the correction term C_2 is proportional to the ratio of second-story to first-story heights H_2/H_1. For the top story [Fig. 8(*c*)], the corrections are negative because (8) cannot simulate the abrupt termination of the frame and overestimates stiffness. The correction term C_t for this story does not appear to linearize as well as those for the first and second stories, however, the magnitude of the corrections for the top story are much smaller than those for the other boundary stories. Based on these observations, the following equations

$$C_1 = \frac{\Sigma k_c}{22 \Sigma k_{ga}} \qquad (12a)$$

$$C_2 = \frac{\eta_b \Sigma k_c}{32 \Sigma k_{gb}} \qquad (12b)$$

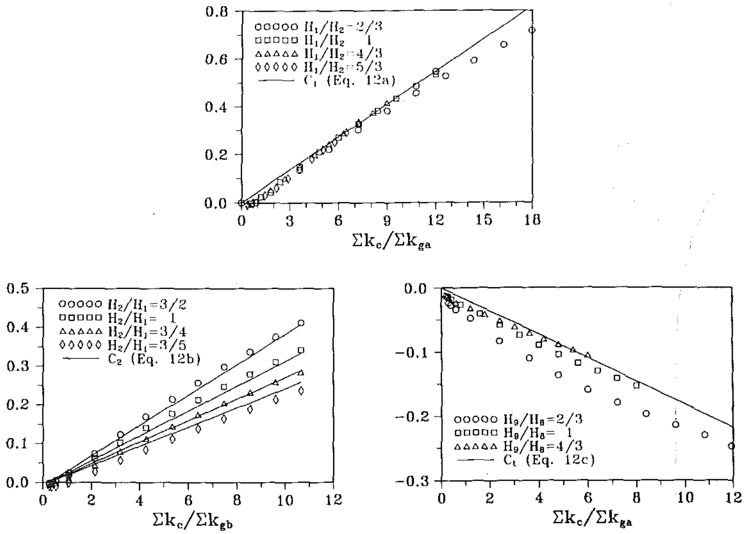

FIG. 8. Boundary Story Correction Terms (C_s) for Uniform Nine-Story, Five-Bay Frames: (*a*) First Story; (*b*) Second Story; (*c*) Ninth (Top) Story

$$C_t = \frac{-\Sigma k_c}{55\Sigma k_{ga}} \quad \text{(12c)}$$

for the first, second, and top stories, respectively, were found to represent the computed data for the correction terms reasonably well. These functions are indicated in Fig. 8 by solid lines. For the story correction term C_t [(12*c*)] it was assumed that η_a is equal to η_b.

Story stiffness obtained using (11) with story height factors η_i from (10) and boundary story correction terms C_t from (12) were used to evaluate stiffnesses for the nine-story, five-bay frames considered earlier (Figs. 4, 5, and 6). A dramatic improvement can be seen in the accuracy of the approximation. For the boundary stories, approximation error does not exceed 5%, even when column stiffness is ten times as large as girder stiffness ($\alpha = 1/\beta = 1/10$).

Top Displacements

Total lateral displacement of the top of the frames was chosen as a cumulative index for evaluating the performance of the approximate expression for story stiffness developed in this study [(11)]. Top displacements were evaluated for the uniform nine-story frames listed in Table 1, as well as other uniform multistory frames. These include 12-story, six-story, and three-story frames, which are summarized in Tables 2, 3, and 4, respectively. The parameter α, which was defined earlier as the ratio of nominal column stiffness (I_c/H_s) to nominal girder stiffness (I_g/L), and its inverse β, were used to defined moments of inertia of the girders as a fraction of moments of inertia of the columns. These parameters were assigned the same range of values considered earlier (1–10), so that a wide spectrum of frames could be studied.

TABLE 2. Properties of 12-Story, Six-Bay Frames

Floor (1)	Story (2)	Bay length[a] (3)	Story height (4)	Moment of Inertia: Girder[b] (5)	Exterior column (6)	Interior column (7)
12	—	L	—	$1/2I_g$	—	—
—	12	—	H_s	—	$1/4I_c$	$1/3I_c$
11	—	L	—	$1/2I_g$	—	—
—	11	—	H_s	—	$1/3I_c$	$1/3I_c$
10	—	L	—	$1/2I_g$	—	—
—	10	—	H_s	—	$1/3I_c$	$1/3I_c$
9	—	L	—	$1/2I_g$	—	—
—	9	—	H_s	—	$1/3I_c$	$1/2I_c$
8	—	L	—	$3/4I_g$	—	—
—	8	—	H_s	—	$1/2I_c$	$1/2I_c$
7	—	L	—	$3/4I_g$	—	—
—	7	—	H_s	—	$1/2I_c$	$1/2I_c$
6	—	L	—	$3/4I_g$	—	—
—	6	—	H_s	—	$1/2I_c$	$2/3I_c$
5	—	L	—	$3/4I_g$	—	—
—	5	—	H_s	—	$2/3I_c$	$2/3I_c$
4	—	L	—	I_g	—	—
—	4	—	H_s	—	$2/3I_c$	$2/3I_c$
3	—	L	—	I_g	—	—
—	3	—	H_s	—	$2/3I_c$	I_c
2	—	L	—	I_g	—	—
—	2	—	H_s	—	I_c	I_c
1	—	L	—	I_g	—	—
—	1	—	$4/3H_s$	—	I_c	I_c

[a] $L = 2H_s$.
[b] $I_g = (\alpha I_c)(L/H_s) = (I_c/\beta)(L/H_s)$.

Top displacements, which are reported in Fig. 9, were calculated using the approximate story stiffnesses given by (11) and are marked as being obtained using K_s. Approximate displacements Δ_a are reported as a fraction of the exact displacements Δ_e, which were computed using the matrix analysis program. The results include frame response to the three types of lateral load distribution defined earlier: constant, linear, and parabolic.

The most accurate results are those corresponding to the 12-story frames [Fig. 9(*a*)]; approximation error does not exceed 3%. As the number of stories decreases, however, the amount of error in computed top drift increases, with the three story frames [Fig. 9*d*)] having the largest amount of error (more than 20% when $\beta = 1/\alpha = 10$). It can also be seen that there is a small but measurable difference in the amount of error for each type of lateral load distribution. These differences also increase as the number of stories decreases.

Top displacements were also computed for a series of 15-story frames with setbacks [Fig. 10(*a*)] using the stiffnesses given by (11). Each bay of these frames has a different length, and stories 1, 5, and 9 have different heights than adjacent stories. At any given floor, all girders have the same

TABLE 3. Properties of Six-Story, Four-Bay Frames

Floor (1)	Story (2)	Bay length[a] (3)	Story height (4)	Moment of Inertia: Girder[b] (5)	Exterior column (6)	Interior column (7)
6	—	L	—	$1/2I_g$	—	—
—	6	—	H_s	—	$1/3I_c$	$1/2I_c$
5	—	L	—	$1/2I_g$	—	—
—	5	—	H_s	—	$1/2I_c$	$1/2I_c$
4	—	L	—	$3/4I_g$	—	—
—	4	—	H_s	—	$1/2I_c$	$2/3I_c$
3	—	L	—	$3/4I_g$	—	—
—	3	—	H_s	—	$2/3I_c$	$2/3I_c$
2	—	L	—	I_g	—	—
—	2	—	H_s	—	$2/3I_c$	I_c
1	—	L	—	I_g	—	—
—	1	—	$4/3H_s$	—	I_c	I_c

[a]$L = 2H_s$.
[b]$I_g = (\alpha I_c)(L/H_s) = (I_c/\beta)(L/H_s)$.

TABLE 4. Properties of Three-Story, Three-Bay Frames

Floor (1)	Story (2)	Bay length[a] (3)	Story height (4)	Moment of Inertia: Girder[b] (5)	Exterior column (6)	Interior column (7)
3	—	L	—	$1/2I_g$	—	—
—	3	—	H_s	—	$1/3I_c$	$1/3I_c$
2	—	L	—	$3/4I_g$	—	—
—	2	—	H_s	—	$2/3I_c$	$2/3I_c$
1	—	L	—	I_g	—	—
—	1	—	$4/3H_s$	—	I_c	I_c

[a]$L = 2H_s$.
[b]$I_g = (\alpha I_c)(L/H_s) = (I_c/\beta)(L/H_s)$.

moment of inertia, and at any given story all columns have the same moment of inertial (Table 5). Top displacements, for the three distributions of lateral load [Fig. 10(*b*)], indicate that even though the frames have a number of moderately large irregularities in profile, top displacements obtained using the stiffnesses from (11) are within 5% of the exact solution for the entire range of girder-to-column stiffnesses α.

Correction for Low-Rise Frames

The expression for story stiffness given by (14) does not accurately simulate the manner in which the stiffening effect of a fixed base dominates the behavior of low-rise frames. As the number of stories decreases, the stiffening effect increases and propagates beyond the second story. While

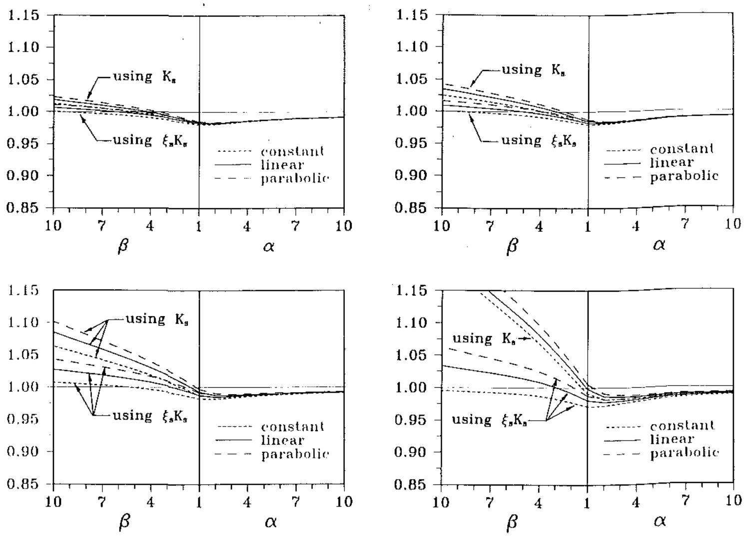

FIG. 9. Normalized Top Displacements (Δ_a/Δ_e) for Uniform Multistory Frames: (*a*) 12-Story, Six-Bay Frame; (*b*) Nine-Story, Five-Bay Frame; (*c*) Six-Story, Four-Bay Frame; (*d*) Three-Story, Three-Bay Frame

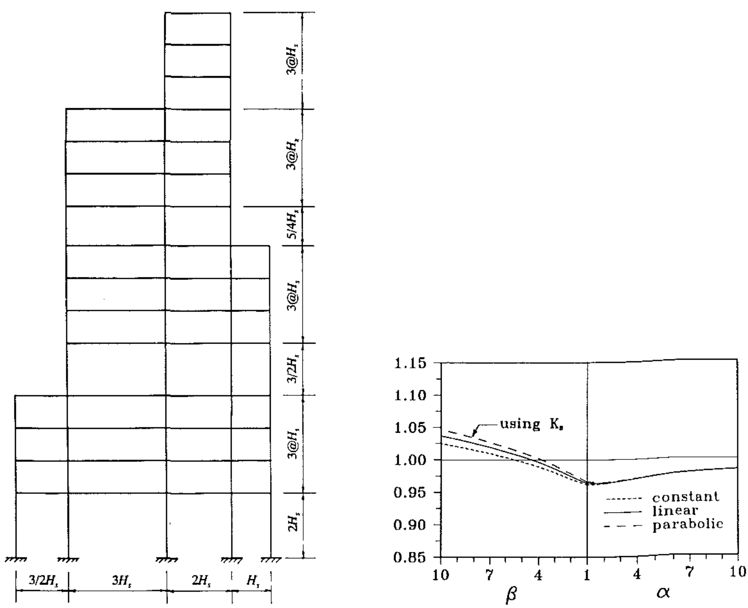

FIG. 10. Fifteen-Story Frame with Setbacks: (*a*) Elevation; (*b*) Normalized Top Displacements (Δ_a/Δ_e)

TABLE 5. Properties of 15-Story Frames with Setbacks

Floor (1)	Story (2)	Story height (3)	Moment of Inertia: Girder[a] (4)	Moment of Inertia: Column (5)
15	—	—	$1/2I_g$	—
—	15	H_s	—	$1/3I_c$
14	—	—	$1/2I_g$	—
—	14	H_s	—	$1/3I_c$
13	—	—	$1/2I_g$	—
—	13	H_s	—	$1/3I_c$
12	—	—	$2/3I_g$	—
—	12	H_s	—	$1/2I_c$
11	—	—	$2/3I_g$	—
—	11	H_s	—	$1/2I_c$
10	—	—	$2/3I_g$	—
—	10	H_s	—	$1/2I_c$
9	—	—	$2/3I_g$	—
—	9	$5/4H_s$	—	$1/2I_c$
8	—	—	$3/4I_g$	—
—	8	H_s	—	$2/3I_c$
7	—	—	$3/4I_g$	—
—	7	H_s	—	$2/3I_c$
6	—	—	$3/4I_g$	—
—	6	H_s	—	$2/3I_c$
5	—	—	$3/4I_g$	—
—	5	$3/2H_s$	—	$2/3I_c$
4	—	—	I_g	—
—	4	H_s	—	I_c
3	—	—	I_g	—
—	3	H_s	—	I_c
2	—	—	I_g	—
—	2	H_s	—	I_c
1	—	—	I_g	—
—	1	$2H_s$	—	I_c

[a] $I_g = (\alpha I_c)(L/H_s) = (I_c/\beta)(L/H_s)$.

this effect was not important for the nine-story and 12-story frames considered herein, it renders (11) practically useless for the three-story and six-story frames, which have flexurally stiffer columns than girders ($\alpha = 1/\beta < 1$). To gain more insight into the nature and scope of this effect, the exact stiffness K_e of every story was normalized by the corresponding approximate value K_s for each of the 12-story, nine-story, six-story, and three-story frames computed using (11). The mean value of the normalized stiffnesses (K_e/K_s) for all stories in a given frame was used as a global measure of the stiffening effect in low-rise frames, and it is presented in Fig. 11 for three values of α. The trends, with respect to number of stores n and α, in the data shown in Fig. 11 suggest a low-rise correction factor ξ_s which is approximated by

$$\xi_s = 1 + \frac{2\Sigma k_c}{5n^2(\Sigma k_{ga} + \Sigma k_{gb})} \quad \text{(13)}$$

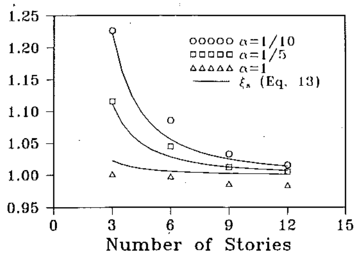

FIG. 11. Mean of Normalized Stiffnesses (K_e/K_s) for Uniform Multistory Frames

This multiplicative factor is calculated individually for every story; the corrected story stiffness is $\xi_s K_s$. For the first story, $2\Sigma k_{ga}$ replaces the sum of Σk_{ga} and Σk_{gb}.

Values for ξ_s obtained from (13) are also shown in Fig. 11 as solid lines. It can be seen that the low-rise correction factor is important for frames with few stories and with columns stiffer than girders; otherwise it can be neglected. Top displacements for the 12-story, nine-story, six-story, and three-story frames were recalculated using the amplified stiffnesses $\xi_s K_s$ and are shown in Fig. 9 along with the displacements previously obtained using K_s. The use of this factor considerably reduces the error in (14) for low-rise frames. Approximation error does not exceed 3% for all frames considered in Fig. 9, including the three-story frames. For the 15-story frames with setbacks, the difference in top displacements computed with and without the correction term ξ_s is very small, and the former are now shown in Fig. 10(*b*).

Illustrative Example

The analysis of a simple frame is presented to demonstrate the ease with which the proposed expression for story stiffness [(11)] is applied. The frame has four stories that are 20 ft tall and two 20-ft bays. Each floor level is subjected to a lateral force of 25 kips. All members have a modulus of elasticity equal to 3,000 k/-sq in. Column stiffnesses are held constant along frame height. Moments of inertia for the members are equal to 8,748, 5,461 and 2,531 in^4., respectively, for interior columns, exterior columns, and girders. These properties represent elastic idealization of a concrete frame for which member stiffnesses are based on gross cross-section dimensions. Interior and exterior columns, respectively, are assumed to have 18 in. and 16 in. square cross sections, and a 9 in. × 15 in. cross-section is assumed for all girders.

The calculations needed to evaluate story stiffnesses using (11) are summarized in Table 6. Because the stories have equal heights, all height ratios η_i are equal to unity. The low-rise correction factor ξ_s is the same for all stories since the quantities Σk_c, Σk_{ga}, and Σk_{gb} do not change from one story to the next. Story drifts are computed from story shears and approximate story stiffnesses $\xi_s K_s$, and approximate lateral displacements at the floor levels are obtained by accumulating story drifts. Lateral displacements, computed using a matrix analysis program, are listed in Table 6 for comparison.

It can be seen that the displacements calculated using the approximate stiffnesses are quite similar to those obtained from the matrix analysis pro-

TABLE 6. Illustrative Example

Floor (1)	Story (2)	Σk_c (kip-in.) (3)	Σk_g (kip-in.) (4)	C_s (5)	$\xi_s K_s$ (kip/in.) (6)	Story shear (kip) (7)	Story drift (in.) (8)	Approximate Displacement (in.) (9)	Exact Displacement (in.) (10)
4	—	—	63,275	—	—	—	—	5.380	5.404
—	4	409,790	—	−0.1178	30.25	25	0.826	—	—
3	—	—	63,275	—	—	—	—	4.554	4.491
—	3	409,790	—	0.0	34.29	50	1.458	—	—
2	—	—	63.275	—	—	—	—	3.096	3.020
—	2	409,790	—	0.2024	41.23	75	1.819	—	—
1	—	—	63,275	—	—	—	—	1.277	1.190
—	1	409,790	—	0.2944	78.29	100	1.277	—	—

Note: η_i = 1 and ξ_s = 1.081 for all stories.

gram. Also, the calculations needed to evaluate approximate stiffnesses and displacements for this example are simple enough to be readily performed on a hand-held calculator. For larger frames, the format summarized in Table 6 can be easily reproduced by a microcomputer-based spreadsheet program.

Summary and Conclusions

This paper was written in an effort to identify explicit, closed-form expressions for approximating the lateral stiffnesses of stories in elastic frames. Three expressions were found in the literature on approximate analysis of laterally loaded buildings, and a fourth expression was developed as part of this study. The approximate story stiffness formula developed in this study [(11)] includes correction factors that simulate the effects on story stiffness of (1) Adjacent stories with different heights; (2) boundaries; and (3) base fixity in low-rise frames. These expressions can be used in conjunction with the shear building model for analyzing frames subjected to lateral loads. They are limited to rectangular frames that are fixed at the base, and only flexural deformations are considered. The following observations and conclusions were made during the course of this study.

1. The apparent lateral stiffness of a story is not a stationary property; it can be accurately modeled by a single value for frames that resist lateral loads with regular distributions.

2. The approximate expression derived from Benjamin's work (1959), and that proposed by Blume et al. (1961) were found to be very inaccurate for frames with columns stiffer than girders ($\alpha < 1$), especially if there are large differences in the heights of adjacent stories.

3. Muto's expression (1971) for individual columns performs well for intermediate stories of frames with equal height stories.

4. The correction factors η_i, C_s, and ξ_s enable the proposed expression (11) to provide reasonably good estimates of story lateral stiffness, even for frames with columns that are as much as ten times stiffer than girders, and even when

story heights and member stiffnesses (I_c and I_g) differ by as much as 50% from one story to the next. For the frames considered, story stiffness estimates were usually 5% of the exact solution (always within 7%), and top displacement estimates did not exceed 3% error in most cases (never exceeded 5% error).

Appendix I. Conversion to SI Units

To convert	To	Multiply by
in.	m	0.0254
ft	m	0.3048
lb	N	4.448
kip	kN	4.448

Appendix II. References

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