Remembering The Column Analogy

Mete A. Sözen

It has been said that it took an age to understand Aristotle and another age to forget him. Hardy Cross may have been forgotten even before he was understood. If his name is mentioned at all today, he is remembered through his contribution to the solution of bending moments in structural frames and flow in networks. Sometimes he is even dismissed as a sleepwalker who stumbled on the relaxation method for solution of linear simultaneous equations. Actually, his close friends and students have said that Cross tended to advise strenuously against the use of the moment distribution method because he considered it to be too exact for inexact structures. Cross's influence on the profession of structural engineering is indelible and awesome. At the same time, it is subtle and easy to overlook. To get a flavor of his approach, consider his class notes for a course on indeterminate structures. Before he goes into explanations, he questions:

> What is theory? It is perhaps worthwhile to call attention to the double use of the word "theory" in scientific discussions. In some cases it is used to mean a body or group of facts the truth of which is not questioned, in others it means a hypothesis which has strong evidence in its favor though its truth is still open to some question. Thus the theory of elasticity is a group of geometrical relations which are not open to debate, but the idea that time yield of the concrete will delay failure from temperature stresses in a concrete arch is a theory in quite a different sense. Other debatable points in indeterminate structures are not theories at all, but merely convenient assumptions: thus no one holds any theory that the modulus of elasticity is constant throughout an arch ring, the only question being whether such variations as do occur produce an important effect on the results.

Theory, to Cross, is the axiom. He does not think that plane geometry needs to be proven for plane continua. But the student is cautioned against mixing "theory" and "theory in a quite different sense." All that is based on the observed is refutable. He expands on it.

> Much confusion of thought has come from misuse of this term [theory]. We may further cite: as groups of facts not open to experimentation or debate, the theory of the elastic arch, the theory of continuous girders, the theory of deflection; as hypotheses strongly supported but as yet not fully proved, the theories of fatigue failure, the theory of earth pressure, the theory that the strength of concrete in a structure is the same as that shown by a cylinder in a testing machine or that rate of application of load is a negligible factor in producing failure, and finally, as misuses of the word, the "theory" that the moment of inertia of a concrete beam varies as bd^3, that the tension rods do not slip in concrete beams, that there is no distortion due to shear. The first group of "theories" is not debatable, the second depend usually on experimental verification, while in the case of the third the important question is how significant is the error. The data often needed in the third group are elementary: when these are available, deductive processes furnish a definite answer as to the importance of the error.

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Taxonomy of the conceptual models was not his sole concern. His preoccupation was with engineering thinking and design in general. He never expressed it that way, but his constant quest was to determine whether or in which case an exact analysis of an approximate model was an approximate analysis of the exact model.*

He sought simplicity: "The analysis of a structure for continuity should be less complicated than the determination of anchorage and stirrup spacing…."** It is ironical that his wish came true, not because continuity analysis was simplified but because the determination of anchorage and stirrup spacing was made more difficult by illuminati who preferred the rigidity of rules to the flexibility of principles.

He revered statics: No indeterminate analysis – no structural analysis of any kind – is complete until the computer has satisfied himself

(1) that the forces balance, at least within the accuracy of the computation used.
(2) that he has not overlooked any forces.***

In our time, this wish was also fulfilled with the exception that tragically "himself" became "itself" and "he" became "it." The following paraphrase from an announcement by an institution that prides itself on being at the cutting edge of knowledge captures the intellectual fashion: "The advanced experimental capabilities will enable us to test and validate more complex and comprehensive analytical and computer numerical models to improve design and performance." In the complex and comprehensive environment envisioned, will the computer (it, he, or she) check simple equilibrium? Fat chance!

Traduttori traditori. It is unfair to Cross to pretend to synthesize his view with a few quotations misplaced in time. The reader is urged to read references 1 through 4. If he/she has already done so, he/she is urged to return to them. They will give him/her different insights, always valuable, at different times. In the text below, a conceptual invention of his is discussed primarily to illustrate Cross's creativity. How he arrived at his moment-distribution method can be understood, if with difficulty, in terms of deformations and the stiffness method. But his "Column Analogy" can only be classed as an artistic leap of imagination.

The Column Analogy

It is very interesting that Professor Cross started his lectures (Ref. 1) on indeterminate structures by referring to "three easily established principles:"

1. Column Analogy
2. Distribution of Moment
3. Virtual Work

Of the three principles he emphasized, the moment distribution survives, sometimes for the wrong reasons. Virtual work, being a theory and not developed but elegantly defined by

* All analyses are based on some assumptions which are not quite in accordance with the facts. From this, however, it does not follow that the conclusions of the analysis are not very close to the facts. (from Ref. 1, p. 2)
** Ref. 2, p. 2
*** Ref. 2, p. 3

Cross, has been a perennial. But the column analogy has been lost. It deserves recycling. The column analogy is essentially a theorem for finding indeterminate moments in a one-span restrained beam. What is important and useful about it is that it applies to straight and curved beams. It can be a very useful tool for determining flexural stiffness properties of nonprismatic beams.

To appreciate Cross's leap of imagination, let us examine the simplest column-analogy application.

Consider a prismatic beam with fixed ends over a span *L*. It is loaded at mid-span by a concentrated load *P*. What are the restraining moments at the ends?

To solve the problem, Cross takes us to an imagined world. In that world, the beam is represented by a section (section of an imagined or analogous column) with depth *L* and thickness *1/EI* where *E* is the Young's modulus for the material of the beam and *I* is its moment of inertia. This imaginary section responds linearly to an imagined load represented by the angle-change diagram, *M/EI*, distributed over the section just as the moment, *M*, is distributed over the span of a simply-supported beam.

The unit stresses at the ends, in the imagined world, are the moments sought:

$$M_{end} = \frac{P_{ana\log}}{A_{ana\log}} = \frac{PL^2}{8EI} * \frac{1}{L} * \frac{EI}{1} = \frac{PL}{8}$$

P_{analog} : Total load on the analogous section or (1/2)(PL/4)(L)

$A_{ana\log}$: Area of analogous section or L/EI

If the load on the beam is uniform,

$$M_{end} = \frac{2}{3}\frac{wL^3}{8EI} * \frac{EI}{L} = \frac{wL^2}{12}$$

The operation is so simple that the correctness of the results appears to be coincidence, but it is not. Plates I through V present examples.

Plate I contains the column-analogy solution for a concentrated load at any distance αL from one end of a prismatic beam with fixed ends. The extension of this model to a similar beam with uniform load is described in Plate II. It is to be noted that the stresses in the imagined world are calculated from the familiar expression

$$\alpha = \frac{P}{A} \pm \frac{Mc}{I}$$

hence the term "column analogy." The column analogy has its best use in determining fixed-end moments and stiffnesses for nonprismatic beams. An application is demonstrated in Plate III.

The solution, given in Plate IV, for a concentrated load on a prismatic beam with one end fixed and the other free, takes us down the rabbit-hole to the Queen of Hearts. Cross wants us to imagine a section which has an infinite width over an infinitesimal length. The only thing that the observer can say is that she has seen the unimaginable and it works. The application is also a witness to Cross's rare ability to associate images.

The example in Plate V is simply to show an application involving a frame. Though it is simple, its use cannot be recommended vis a vis other current methods. But it does demonstrate that Cross's Analogy can be used for "one cell" frames, arches, and curved beams.

Concluding Remarks

Cross's written works are replete with jewels of thought. Some, contained in references 1 and 2, are reasonably well known. But it is not a waste of print to revisit his judgment about the results of analysis rendered in relation to stresses computed in arches in reference 3 (paper #8).

> The investigations here recorded indicate:
> (1) that for a large part (over one-half) of the stresses in an arch there can be practically no uncertainty arising from assumptions involved in the method of analysis used.
> (2) that for the flexural stresses due to live load the true stresses cannot be predicted with absolute precision, because the stresses are a matter of chance.
> (3) that the departure of the stresses existing in any arch from the values given by the usual methods of analysis can scarcely be greater than the variations in the quality of the concrete, and will most probably be very much less.
>
> Beyond this it does not seem wise or profitable to draw conclusions, though others are apparently indicated by the data. The important fact is that any wide departure from the predicted values of the moments and thrusts in a concrete arch is not possible unless the variation in the properties of concrete is much greater than is commonly supposed. Within a narrow zone of uncertainty, then, the maximum moments and thrusts due to loads in a concrete arch are given without possible question by the "geometrical" (elastic) analysis. The zone of such uncertainty seems to have a width of about ±10 per cent; the zone of probable uncertainty seems to have a width of about ±5 per cent. The terms "true value" and "real value," however, are meaningless except as applied to a given arch under a given condition of loading and a given atmospheric condition; otherwise the reactions are a matter of chance.
>
> The geometrical theory of analysis for arch reactions appears more dependable than the theory of flexure used to compute the fiber stresses produced by these reactions, and much more dependable than the concrete itself. It is not exact or precise, but it is a safe and convenient guide in design.

Socrates is supposed to have told Antisthenes the Cynic, "Your pride shows through the holes of your rags." Is it intellectual arrogance, confidence, sensibility, or humility to have spent years in building, analyzing, testing concrete arches and then to arrive at such a modest set of conclusions? Whichever it is, Cross ought to be a model to current and future writers.

Acknowledgments

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HARDY CROSS
Brief Professional Record

1902	BA	Hampden-Sydney College, Virginia
1903	BS	Hampden-Sydney College, Virginia
1908	BS in CE	Massachusetts Institute of Technology, Cambridge
1911	MS in CE	Harvard University, Cambridge
1908-1910	Asst. Eng., Bridge Department, Missouri-Pacific Railway	
1912-1918	Asst. Prof. of Civil Eng., Brown University	
1918-1921	Practice in New York City and Boston	
1921-1937	Professor of Structural Engineering, Univ.of Illinois, Urbana	
1938-1951	Head, Department of Civil Eng., Yale University	

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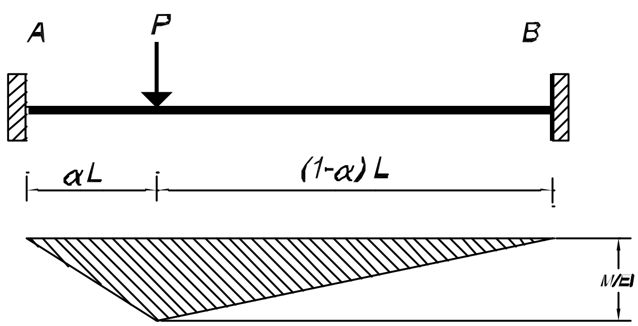
PRISMATIC BEAM WITH CONCENTRATED LOAD AND FIXED ENDS

The operation of determining the fixed-end moments in a prismatic beam subjected to a concentrated load is simple. It involves three steps.

Step 1. Determine the "load" on the analogous column.
Step 2. Determine the section properties of the analogous column.
Step 3. Determine the unit normal stresses in the analogous column.

Step 1: Load and Moment Acting on Section of Analogous Column

The entity defined by the area of the M/EI diagram for a simply supported beam is P_{analog}, the "load" acting on the section of the analogous column.



$$P_{analog} = \frac{P \cdot L^2}{2E \cdot I} \cdot \alpha \cdot (1 - \alpha)$$

The centroid of the load represented by the M/EI diagram is at

$$x = \frac{L}{3} \cdot (1 + \alpha)$$

from end A. The counterclockwise moment acting on the analogous column is

$$M_{analog} = P_{analog} \cdot \left(\frac{L}{2} - x \right)$$

substituting for P_{analog}

$$M_{analog} = \frac{1}{12} \cdot \frac{P \cdot L^3}{E \cdot I} \cdot \alpha \cdot (1 - \alpha) \cdot (1 - 2 \cdot \alpha)$$

Step 2: Section Properties of Analogous Column

$$A_{analog} = \frac{1}{E \cdot I} \cdot L$$

$$I_{analog} = \frac{1}{12} \cdot \frac{1}{E \cdot I} \cdot L^3$$

Step 3: Normal Stresses on Section of Analogous Column

The normal unit stress s_A at end A for the load on the analogous column section corresponds to the moment at in a beam with a concentrated load at aL from end A.

$$\sigma_A = \frac{P_{analog}}{A_{analog}} + \frac{M_{analog} \cdot c}{I_{analog}}$$

where c=L/2.

Substituting for P_{analog}, M_{analog}, A_{analog}, I_{analog}, and c,

$$\sigma_A = P \cdot L \cdot \alpha \cdot (\alpha - 1)^2$$

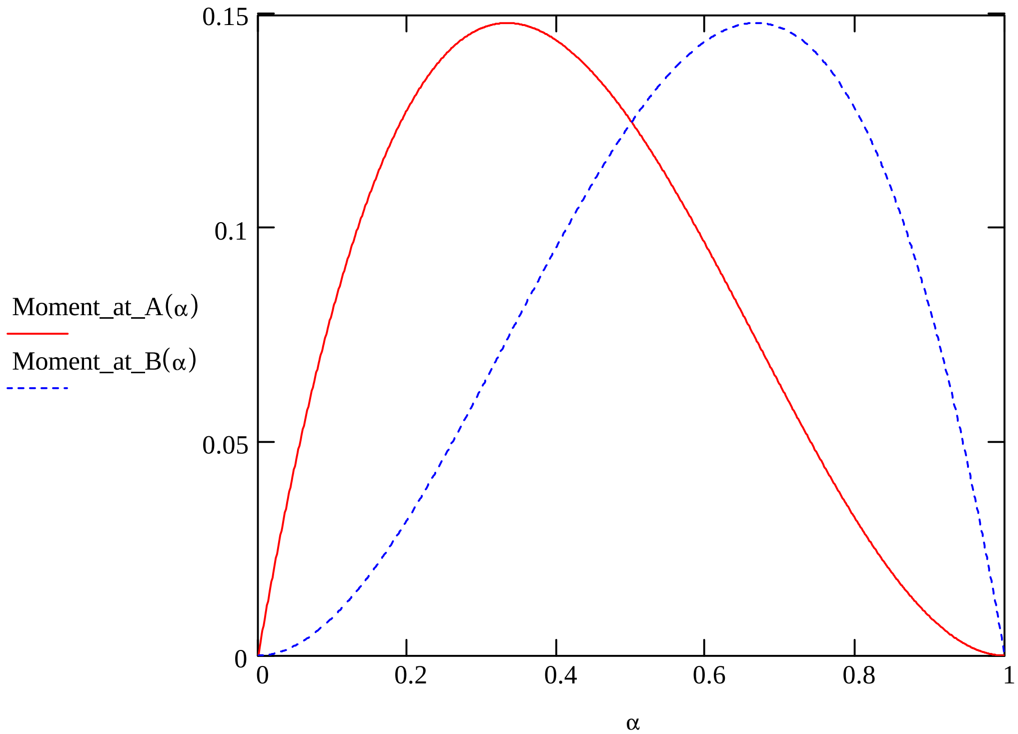
To obtain a solution for the variation of the coefficients for end moments at A and B , we set

$$P := 1 \qquad L := 1$$

$$\text{Moment\_at\_A}(\alpha) := P \cdot L \cdot \alpha \cdot (1 - \alpha)^2$$

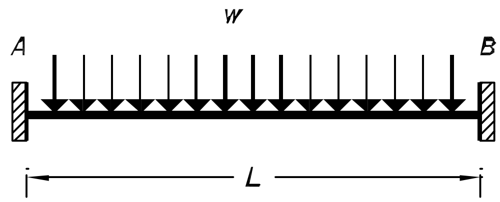
Similarly,

$$\text{Moment\_at\_B}(\alpha) := P \cdot L \cdot \alpha^2 \cdot (1 - \alpha)$$



Variation of Fixed-End Moment, Coefficient of PL, with **a**

FIXED-END MOMENT FOR UNIFORMLY LOADED PRISMATIC BEAM



Start with end moment at A for concentrated load P at distance **a** L from end A

$$\frac{M_A}{PL} = \alpha \cdot (\alpha - 1)^2$$

M_A = Moment at end A for a concentrated load at **a** L from end A
P = Concentrated load at **a** L from end A
L = Beam span
a = Ratio of distance to concentrated load from end A to beam span

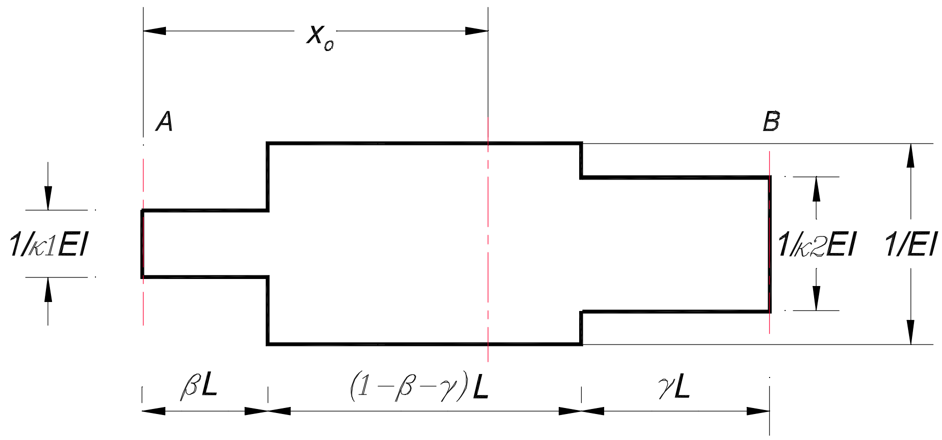
Assume unit load w = P/dL and integrate over beam span

$$M_A = w \cdot L^2 \int_0^1 \alpha \cdot (\alpha - 1)^2 \, d\alpha$$

$$M_A = \frac{1}{12} \cdot w \cdot L^2$$

FIXED -END MOMETS IN A NONPRISMATIC BEAM
Three Segments with Different EI's

Centroid of Analogous Column Section



Cross Section of Analogous Column

Definitions and Default Values

Young's Modulus	$E := 1$
Standard Beam Moment of Inertia (Segment 2)	$I := 1$
Beam Span	$L := 1$
First Segment, Ratio of Length of Segment to Beam Span	$\beta := 0.2$
Third Segment, Ratio of Length of Segment to Beam Span	$\gamma := 0.3$
First Segment, Ratio of Moment of Inertia to That of Second Segment	$\kappa 1 := 4$
Third Segment, Ratio of Moment of Inertia to That of Second Segment	$\kappa 2 := 2$
Unit Load	$w := 1$

Determination of Centroidal Distance from End A

Segment **b**

Area $A_\beta := \frac{1}{\kappa 1 \cdot E \cdot I} \cdot \beta \cdot L$

First Moment about End A $M_\beta := A_\beta \cdot \frac{\beta}{2} \cdot L$

Segment Standard

Area $A_S := \frac{1}{1 \cdot E \cdot I} \cdot (1 - \beta - \gamma) \cdot L$

First Moment about End A $M_S := A_S \cdot \left[(\beta) + \left(\frac{1 - \beta - \gamma}{2} \right) \right] \cdot L$

Segment **g**

Area $A_\gamma := \frac{1}{\kappa 2 \cdot E \cdot I} \cdot \gamma \cdot L$

First Moment about End A $M_\gamma := A_\gamma \cdot \left(1 - \frac{\gamma}{2} \right) \cdot L$

Centroid

$$x_o := \frac{(M_\beta + M_S + M_\gamma)}{A_\beta + A_S + A_\gamma} \qquad x_o = 0.51$$

Determination of Moment of Inertia

Segment 1

$$I_{\beta 1} := \frac{1}{12} \cdot \frac{1}{\kappa 1 \cdot E \cdot I} \cdot (\beta \cdot L)^3 \qquad I_{\beta 1} = 1.67 \times 10^{-4}$$

$$I_{\beta 2} := A_{\beta} \cdot \left(\frac{\beta \cdot L}{2} - x_o \right)^2 \qquad I_{\beta 2} = 8.43 \times 10^{-3}$$

Segment 2

$$I_{s1} := \frac{1}{12} \cdot \frac{1}{1 \cdot E \cdot I} \cdot [(1 - \beta - \gamma) \cdot L]^3 \qquad I_{s1} = 1.04 \times 10^{-2}$$

$$I_{s2} := A_s \cdot \left[\beta \cdot L + \frac{(1 - \beta - \gamma) \cdot L}{2} - x_o \right]^2 \qquad I_{s2} = 1.84 \times 10^{-3}$$

Segment 3

$$I_{\gamma 1} := \frac{1}{12} \cdot \frac{1}{\kappa 2 \cdot E \cdot I} \cdot (\gamma \cdot L)^3 \qquad I_{\gamma 1} = 1.12 \times 10^{-3}$$

$$I_{\gamma 2} := A_{\gamma} \cdot \left[\left(1 - \frac{\gamma}{2} \right) \cdot L - x_o \right]^2 \qquad I_{\gamma 2} = 1.73 \times 10^{-2}$$

Moment of Inertia and Area for Analogous Column Section

$$I_{analog} := I_{\beta 1} + I_{\beta 2} + I_{s1} + I_{s2} + I_{\gamma 1} + I_{\gamma 2}$$

$I_{analog} = 0.039$ 1/EI $x_o = 0.51$

$$A_{analog} := A_{\beta} + A_s + A_{\gamma}$$

$A_{analog} = 0.7$ 1/EI

"Load" on Analogous Column Section

$$P_a := \frac{1}{2\kappa 1}\cdot\left[\int_0^{\beta}\left(\alpha-\alpha^2\right)d\alpha\right]+\frac{1}{2}\left[\int_{\beta}^{1-\gamma}\left(\alpha-\alpha^2\right)d\alpha\right]+\frac{1}{2\kappa 2}\cdot\left[\int_{1-\gamma}^{1}\left(\alpha-\alpha^2\right)d\alpha\right] \qquad wL/EI$$

Moment about End A on Analogous Column Section

$$MA := \frac{1}{2\kappa 1}\cdot\int_0^{\beta}\alpha\cdot\left(\alpha-\alpha^2\right)d\alpha+\frac{1}{2}\left[\int_{\beta}^{1-\gamma}\alpha\cdot\left(\alpha-\alpha^2\right)d\alpha\right]+\frac{1}{2\kappa 2}\cdot\left[\int_{1-\gamma}^{1}\alpha\cdot\left(\alpha-\alpha^2\right)d\alpha\right] \qquad wL^2/EI$$

Clockwise Moment about Section Centroid

$$M_a := MA - P_a\cdot x_o$$

Fixed-End Moment at A

$$M_A := \frac{P_a}{A_{analog}} - \frac{M_a\cdot\frac{L}{2}}{I_{analog}}$$

$$M_A = 0.111 \quad wL^2$$

Fixed-End Moment at B

$$M_B := \frac{P_a}{A_{analog}} + \frac{M_a\cdot\frac{L}{2}}{I_{analog}}$$

$$M_B = 0.083 \quad wL^2$$

Stiffness, A to B

Apply rotation at A. Analogous action is to place concentrated unit load at A

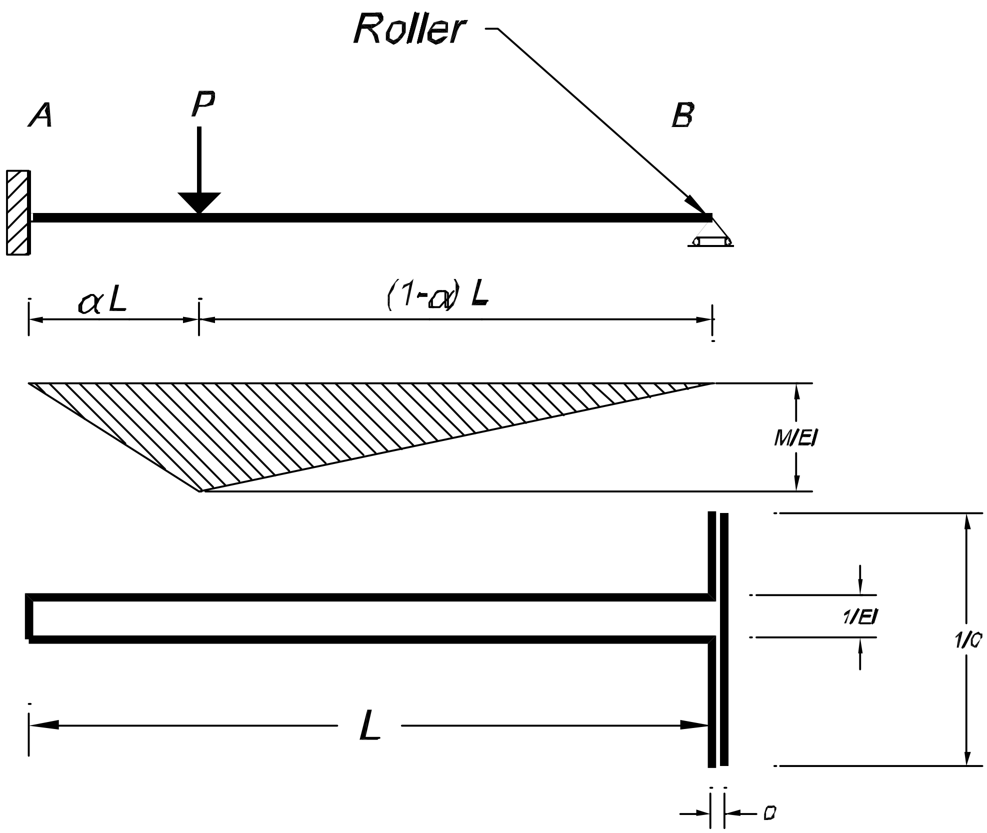
Moment of unit load is taken as counterclockwise. M_A is the moment at A for a unit rotation at A. M_B is the moment at B for a unit rotation at A.

$$M_A := \frac{1}{A_{analog}} + \frac{1\cdot\frac{L}{2}\cdot\frac{L}{2}}{I_{analog}} \qquad M_B := \frac{1}{A_{analog}} - \frac{1\cdot\frac{L}{2}\cdot\frac{L}{2}}{I_{analog}}$$

$$M_A = 7.8 \quad EI/L \qquad M_B = -4.9 \quad EI/L$$

Stiffness K_{AB} is $M_A = 7.8\ \frac{EI}{L}$ The "carry-over factor" is $\frac{M_B}{M_A} = -0.63$

PRISMATIC BEAM WITH FIXED AND SIMPLE SUPPORTS



Section of Analogous Column

The area of the M/EI diagram for a simply supported beam is P_{analog}, the "load" acting on the section of the analogous column.

$$P_{analog} = \frac{P \cdot L^2}{2E \cdot I} \cdot \alpha \cdot (1 - \alpha)$$

The centroid of the load represented by the M/EI diagram is at

$$x = \frac{L}{3} \cdot (1 + \alpha)$$

from end A. The counterclockwise moment acting on the analogous column is

$$M_{analog} = P_{analog} \cdot (L - x)$$

substituting for P_{analog}

$$M_{analog} = \frac{1}{6} \cdot \frac{PL^3}{E \cdot I} \cdot \alpha \cdot (1 - \alpha) \cdot (2 - \alpha)$$

Step 2: Section Properties of Analogous Column

$$A_{analog} = \infty$$

Because the centroid of the analogous-column section is at B

$$I_{analog} = \frac{1}{3} \cdot \frac{1}{E \cdot I} \cdot L^3$$

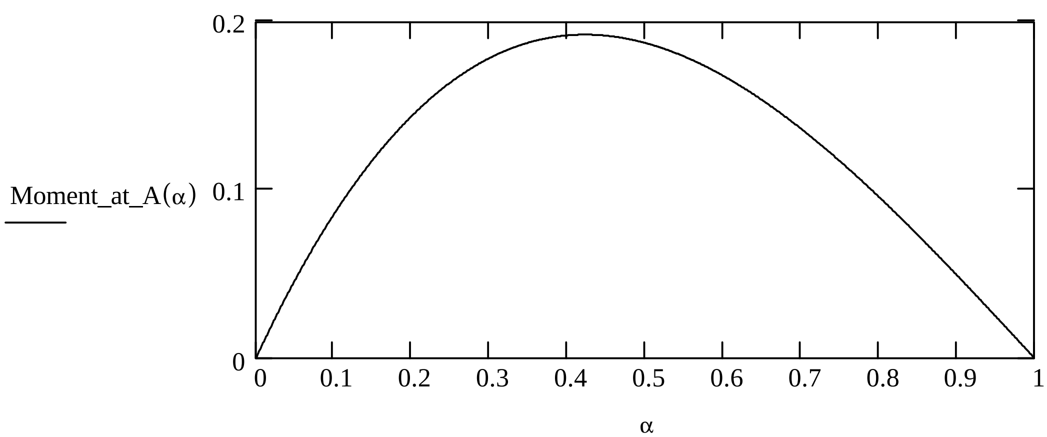
Step 3: Normal Stresses on Section of Analogous Column

Because $A_{analog} := \infty$ the normal-stress term disappears,

$$\sigma_A = \frac{M_{analog} \cdot L}{I_{analog}} \qquad \sigma_A = \frac{1}{2} \cdot P \cdot L \cdot \alpha \cdot (1 - \alpha) \cdot (2 - \alpha)$$

for $P := 1$ $L := 1$

$$\text{Moment\_at\_A}(\alpha) := \frac{1}{2} \cdot P \cdot L \cdot \alpha \cdot (1 - \alpha) \cdot (2 - \alpha)$$



Variation of M/ PL at A with **a**

Moment coefficient for end moment at A

Integrate a from 0 to 1:

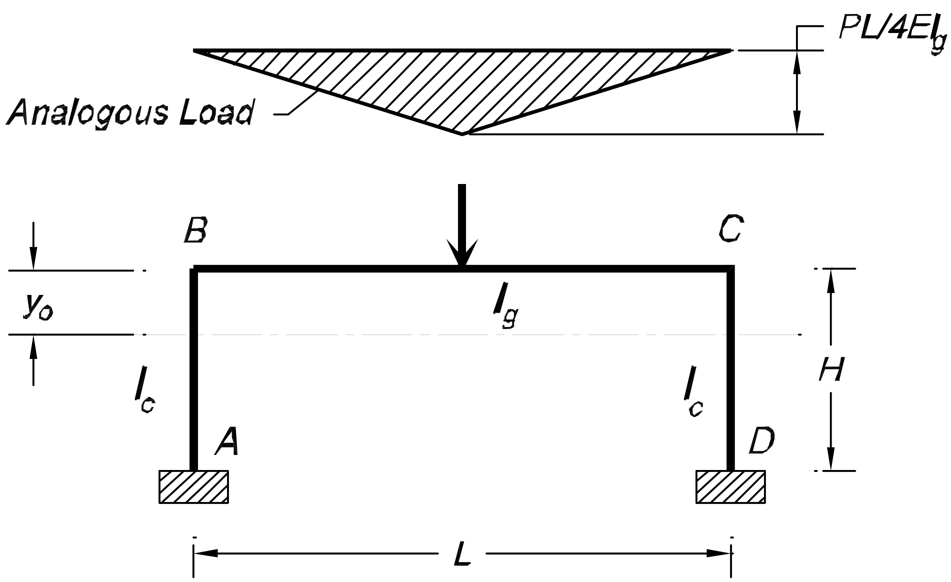
$$\int_{0}^{1} \frac{3}{2}\cdot\alpha\cdot(1-\alpha)\cdot\left[1-\frac{1}{3}\cdot(1+\alpha)\right]d\alpha$$

to obtain

$$\frac{M}{wL^{2}} = \frac{1}{8}$$

FRAME WITH CONCENTRATED LOAD

Prismatic Columns and Girder



Definitions and Default Values

Young's Modulus	$E := 1$
Moment of Inertia of Columns	$I_c := 1$
Moment of Inertia of Girder	$I_g := 1$
Span	$L := 2$
Height	$H := 1$
Load Applied at Center of Girder Span	$P := 1$

Properties of Analogous Section

Centroidal Distance from BC

$$y_o := \frac{2\cdot\frac{H}{E\cdot I_c}\cdot\frac{H}{2}}{2\cdot\frac{H}{E\cdot I_c}+\frac{L}{E\cdot I_g}} \qquad y_o = 0.25$$

Moment of Inertia of Analogous Column Section

$$I_{analog} := 2\cdot\frac{1}{12\cdot E\cdot I_c}\cdot H^3 + 2\cdot\frac{H}{E\cdot I_c}\cdot\left(\frac{H}{2}-y_o\right)^2 + \frac{L}{E\cdot I_g}\cdot y_o^{\,2} \qquad I_{analog} = 0.42$$

Cross-Sectional Area of Analogous Column

$$A_{analog} := \frac{2\cdot H}{E\cdot I_c}+\frac{L}{E\cdot I_g} \qquad A_{analog} = 4$$

"Load" on Analogous Section

$$P_{analog} := \frac{1}{8}\cdot\frac{P\cdot L}{E\cdot I_g} \qquad P_{analog} = 0.25$$

$$M_{analog} := \frac{1}{8}\cdot\frac{P\cdot L}{E\cdot I_g}\cdot y_o \qquad M_{analog} = 0.063$$

Moment at B $\qquad \sigma_B := \frac{P_{analog}}{A_{analog}} + \frac{M_{analog}\cdot y_o}{I_{analog}} \qquad \sigma_B = 0.1 \quad wL^2$

Moment at A $\qquad \sigma_A := \frac{P_{analog}}{A_{analog}} + \frac{M_{analog}\cdot(y_o - H)}{I_{analog}} \qquad \sigma_A = -0.05 \quad wL^2$