reasonable. Having been directly connected with the project for four years prior to the slide, the writer is firmly convinced that the actual strength and condition of the rock in the slide area could have been determined only by large undisturbed samples. It is rather far fetched to assume, regardless of how carefully the geology of the region was studied, that the true character and strength of the rock at any given location could be determined by this procedure.

The writer fully agrees that "Had the true conditions been understood, the dam could have been designed to meet those conditions safely." This is merely another way, however, of saying that hindsight is better than foresight

It has not been the intent of the writer in this discussion to minimize the importance of thorough geological investigations. However, it is certainly false security to rely too much on a geological study of the region as proposed by Mr. Fahlquist and Mr. Crosby. In the future, under such circumstances, the writer will take large undisturbed samples of all weak rocks regardless of what the geologist might find in a study of the region.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 2145

ON THE METHOD OF COMPLEMENTARY ENERGY

AND ITS APPLICATION TO STRUCTURES STRESSED BEYOND THE PROPORTIONAL LIMIT, TO BUCKLING AND VIBRATIONS, AND TO SUSPENSION BRIDGES

BY H. M. WESTERGAARD,[1] M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. I. K. SILVERMAN, GEORGE R. RICH, R. V. SOUTHWELL, AND H. M. WESTERGAARD.

SYNOPSIS

The method of complementary energy is a general method of structural mechanics. The basic law was stated by F. Engesser in a paper in 1889. He extended Castigliano's law of least work to apply beyond the range of Hooke's law by replacing work by complementary work, which is an integral of distance times increment of force. Engesser's paper is little known.

The purpose of the present paper is to give proof and demonstration of the method. The proof goes back to fundamentals and includes a re-examination of the fundamentals; this is needed to remove doubts about the ranges of applicability. The demonstration consists of representative applications and may be interpreted as an exploration of the field.

HISTORICAL NOTES

The method of complementary energy is an extension of Castigliano's method of least work. Alberto Castigliano[2] published his method during the Seventies in papers and a treatise. His principle of least work applies to statically indeterminate structures stressed within the range of Hooke's law and subject to the restriction that all significant deformations must be linear homogeneous functions of the loads. Castigliano showed that among all the

NOTE.—Published in February, 1941, *Proceedings*.

[1] Dean, Graduate School of Eng., and Gordon McKay Prof. of Civ. Eng., Harvard Univ., Cambridge, Mass.

[2] Thesis to obtain diploma as engineer, Torino, 1873; two papers in *Atti della Reale Accademia delle Scienze di Torino*, Vol. 10, 1875, p. 380, and Vol. 11, 1876, p. 127; and "Théorie de l'équilibre des systèmes élastiques," Torino, 1879, 480 pp., translated into English by E. S. Andrews under the title "Elastic Stresses in Structures," Scott, Greenwood & Son, London, 1919.

statically possible states of stress in such a structure the correct one is that which makes the energy of the internal stresses a minimum. This state of stress satisfies automatically not only the requirement of equilibrium but also the requirement of geometrical continuity. L. F. Ménabréa[3] had stated this principle clearly for trusses in 1858, but his proof contained misunderstandings, and the method is credited justly to Castigliano.

Castigliano[4] himself gave the method its first extension; he stated a revised expression that must be made a minimum if imperfect fits of redundant members create initial stresses; and he applied this procedure to temperature stresses in a general discussion and in six examples.[5] H. Müller-Breslau[6] improved the procedure for temperature stresses and contributed much toward making Castigliano's method known. A useful and dependable critical account of the original works in the field was given by M. Grüning[7] in 1912. In the twentieth century Castigliano's method has become stock in trade; it holds a position today as one of several useful general procedures of structural mechanics. It is worthy of note that in a book published in 1936 R. V. Southwell[8] of Oxford University, Oxford, England, gave an attractive original derivation of Castigliano's principle, based on a discussion of self-strains.

The contribution that has the greatest interest for the present study was published by Fr. Engesser[9] in a paper in 1889. He derived the basic law of the method of complementary energy. It is a modification of Castigliano's law of least work in which work is replaced by complementary work or complementary energy. As work is an integral of force times increment of distance or of stress times increment of deformation, so is complementary work an integral of distance times increment of force or of deformation times increment of stress. Engesser's theory applies beyond the range of Hooke's law; it includes not only Castigliano's method but also Müller-Breslau's procedure for temperature stresses as special applications. In his review of the field in 1912 Grüning[10] quoted and discussed Engesser's contribution, but otherwise it has received little attention. A plausible explanation is that structural analysis has been concerned mainly with stresses below the proportional limit, and the applicability to buckling and vibrations had not been realized.

The method of complementary work or complementary energy is analogous to another method which has become important in structural statics; namely, the method based on the "principle of minimum of the potential energy by variation of the shape." It is advantageous to consider the two methods in

[3] "Nouveau principe sur la distribution des tensions dans les systèmes élastiques," by L. F. Ménabréa, *Comptes Rendus*, Paris, Vol. 46, 1858, pp. 1056–1060.

[4] "Théorie de l'équilibre des systèmes élastiques et ses applications," Torino, 1879, p. 39.

[5] *Loc. cit.*, pp. 39, 317, 324, 332, 347, 428, and 442.

[6] "Der Satz von der Abgeleiteten der ideellen Formänderungs-Arbeit," by H. Müller-Breslau, *Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover*, Vol. 30, 1884, columns 211–214; "Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktionen," Leipzig, 1886, 5th Ed., 1924, pp. 74–79; "Graphische Statik der Baukonstruktionen," Vol. 2, Subvolume 1, Leipzig, 1892, p. 49, 5th Ed., 1922, p. 47.

[7] "Theorie der Baukonstruktionen I: Allgemeine Theorie des Fachwerks und der vollwandigen Systeme," by M. Grüning, *Encyklopädie der mathematischen Wissenschaften*, Vol. 4, Subvolume 4, Leipzig, 1907–1914, pp. 419–534, especially pp. 437–454.

[8] "An Introduction to the Theory of Elasticity," by R. V. Southwell, Oxford Univ. Press, 1936, p. 91.

[9] "Ueber statisch unbestimmte Träger bei beliebigem Formänderungs-Gesetze und über den Satz von der kleinsten Ergänzungsarbeit," by Fr. Engesser, *Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover*, Vol. 35, 1889, columns 733–744, especially 738–744.

[10] *Loc. cit.*, p. 454.

conjunction. The basic principle of the second method is sometimes stated as a direct consequence of a general law of dynamics, but it can be derived from the simplest laws of statics. Daniel Bernouilli and Leonhard Euler[11] stated and used this principle in a special form in the first half of the eighteenth century. The method played a part in the nineteenth century. A mathematical paper published by W. Ritz[12] in 1908 gave new impetus to its use, which has been widespread since then. A number of applications are found in the writings of S. Timoshenko.[13]

The Laws of Least Energy and of Least Complementary Energy

Since it is desirable that no doubts shall remain about the ranges of applicability, all the steps of the derivations of the basic theorems will be shown.

The Structure.—Fig. 1 represents a structure of general type. The black parts are joints, which, by a definition adopted here, are rigid bodies. The shaded parts are deformable members, which are attached to the joints. External forces and reactions are assumed to act on joints only. Since the joints may be three-, two-, or one-dimensional or may be without extension, and any number of joints may be assumed to exist, it is difficult to conceive of a structure to which this picture could not be adapted.

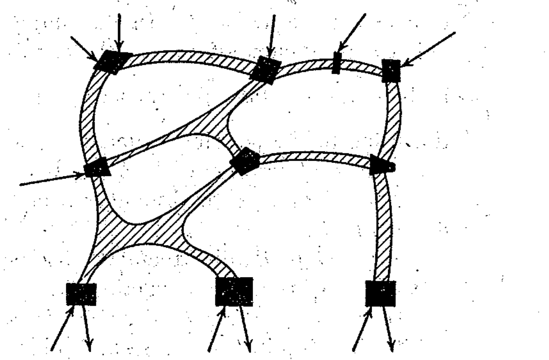


Fig. 1.—Structure of General Type

Notation.—The notation applied to the structure in Fig. 1 requires explanations, which follow:

P = external load. The load P may be a single force P, or it may consist of a group of forces $c_1 P$, $c_2 P$, $c_3 P$, $\cdots$, all proportional to P and varying together when P varies. This conception makes it possible to consider a whole dead load or a whole live load as "a load"; which is in accordance with common usage of the word "load."

p = path of the load P during the deformation from a starting shape of the structure to an assumed shape. If P is a single constant force, p is the displacement of the point of application in the direction of P, and Pp is the work of P. If the load P is a group of forces, then p is defined by the statement that with the value of P remaining constant during the assumed deformation, the work of P is Pp. For example, if P is a total load uniformly distributed over the length of a beam, p is the increase of the average deflection.

S = stress in a member. The stress in the sense adopted here is interpreted as a load exerted by joints on a member, and is defined in the same way

[11] "De curvis elasticis," by Leonhard Euler, 1744; annotated translation into English by W. A. Oldfather, C. A. Ellis, M. Am. Soc. C. E., and Donald M. Brown, in *Isis*, Vol. 20, 1, November, 1933, pp. 72–160, especially p. 78.

[12] *Crelles Journal*, Vol. 135, 1908.

[13] See especially "Theory of Elasticity," 1934, "Theory of Elastic Stability," 1936, and "Theory of Plates and Shells," 1940, by S. Timoshenko, McGraw-Hill Book Co., Inc.

as P except for the restriction that all the forces constituting a particular stress S must be in equilibrium. If the member is a simple tension member in a truss, S may be taken as the total tension, and the load on the member is the two equal and opposite pulls, each equal to S, at the ends. There can be more than one stress in a member. For example, in a beam flat joints may be assumed at two adjacent cross sections a distance dx apart. The member between these two joints has two stresses: One is the bending moment M, which consists of two equal and opposite couples exerted by the joints; the other is the transverse shear V, which consists of two equal and opposite shearing forces and a supplementary balancing couple $V dx$ at one of the joints. Instead one may in this case assume two members occupying the same space between the cross sections, one resisting M and the other V.

D = deformation in the direction of S; D is the path of S, defined in terms of S as p in terms of P. In the simple tension member with total tension S, D is the total elongation. In the member in the beam D in the direction of M is the relative rotation of the two joints, that is, $-\frac{d^2y}{dx^2}dx$ if y denotes the deflections due to the bending moments, y being measured from the starting shape; and D in the direction of V is the relative sliding of the joints.

R = reaction; a force or group of forces exerted by a support, otherwise defined as P.

r = path of R; displacement of a support in the direction or R; $-r$ is the settlement of the support against R.

Equation of Virtual Work in Infinitesimal Form.—The structure is in equilibrium if all members and all joints are in equilibrium. The members are loaded only by the stresses S, and are automatically in equilibrium by the definition of the stresses. A joint, by definition, is a rigid body. The equilibrium of a rigid body can be investigated by assuming an arbitrary infinitesimal movement of it while the forces remain constant. If during any such movement the sum of work of all the forces is zero except for an infinitesimal quantity of second or higher order, the rigid body is in equilibrium. Assume an arbitrary infinitesimal movement of each joint, whereby the deformations p, D, and r increase by the amounts δp, δD, and δr. The forces acting on all the joints are the forces constituting the loads P, the stresses reversed or $-S$, and the reactions R. The condition that the sum of work of all the forces on all the joints must be zero is expressed by the equation

$$\sum P\,\delta p - \sum S\,\delta D + \sum R\,\delta r = 0 \qquad (1)$$

This equation will be recognized as the equation of virtual work in infinitesimal form; it states a principle that has played a great part in mechanics. Eq. 1 is a general condition of equilibrium, unrestricted by any requirement of linearity of the relations between the deformations, stresses, and loads; that is, unrestricted by any law of superposition.

Initial Assumption.—Changes of temperature can be considered to have taken place in advance, before the forming of the starting state. Then it becomes feasible to assume, and it will be assumed, that in all operations that need be considered in the analysis each stress S is a definite continuous function of the deformations D of the member in which it belongs, and conversely each deformation is a definite continuous function of the stresses in the member.

Energy.—Now consider a series of changes of shape, a variation of the shape, during which the loads and the positions of the supports remain constant; that is, the values of P and r remain constant. Each joint is moved from the starting position into an arbitrarily assumed position; thereby the structure is changed from the starting shape to an assumed shape. The members remain attached to the joints, so that continuity is maintained, but the requirements of equilibrium are ignored in this operation; that is, the geometrical requirements are respected, while the statical requirements are abandoned temporarily. The stresses S will be the proper definite functions of the deformations D. Then, for each assumed shape the structure and the loads can be said to have a potential energy equal to

$$T = \sum \int_{D_0}^{D} S\,dD - \sum P\,p \qquad (2)$$

in which the lower limits D_0 are chosen arbitrarily and the summations include all stresses and loads. The phrase "potential energy" can be used here because the stresses are definite functions of the deformations in the operations of the analysis. An infinitesimal variation of the assumed shape causes an increment, a first variation, of the potential energy equal to

$$\delta T = \sum S\,\delta D - \sum P\,\delta p \qquad (3)$$

Let the requirements of equilibrium be imposed again. Then Eq. 1 must be satisfied. Since $\delta r = 0$ in the variation considered, it follows that

$$\delta T = 0 \qquad (4)$$

That is, if not only all the geometrical requirements but also all the statical requirements are to be satisfied, the first variation of the potential energy must be zero; that is, the potential energy must be a minimum or a maximum. Further arguments, omitted here, show that a stable equilibrium requires that T be a minimum. The statement

$$T = \min \qquad (5)$$

is interpreted then with the reservation that T is normally a minimum, but under some circumstances the minimum may be replaced by a maximum. Eqs. 5 and 2 together state the "law of minimum of the potential energy by variation of the shape." Eq. 5 like Eq. 1 is unrestricted by any law of superposition.

Assumption of Superposition of Deformations, and the Equation of Virtual Work in Finite Form.—It will now be assumed that the law of superposition (or linearity) applies in a limited form; namely to deformations only, without involving loads, stresses, and reactions: it is assumed that if p', D', r' and p'', D'', r'' are two geometrically possible sets of deformations p, D, r, then $p'+p''$, $D'+D''$, $r'+r''$ are a possible set, provided that all the deformations involved are within some range. If this law is assumed, δp, δD, and δr in Eq. 1 may be replaced by p', D', and r', so that

$$\sum P\,p' - \sum S\,D' + \sum R\,r' = 0 \qquad (6)$$

This is the equation of virtual work in finite form, which has been important in structural mechanics since the Seventies, when Otto Mohr applied it to trusses. It is noted that p', D', r' are not the deformations produced by the loads P.

Eq. 6 may be restated in the form

$$\sum P'\,p - \sum S'\,D + \sum R'\,r = 0 \qquad (7)$$

in which P', S', R' are a statically possible set of values of the loads, stresses, and reactions. A possible form of Eq. 7 is

$$\sum p\,\delta P - \sum D\,\delta S + \sum r\,\delta R = 0 \qquad (8)$$

in which δP, δS, δR are a statically possible set of increments of P, S, R.

Complementary Energy.—The preparations have now been made for the study of the second fundamental type of variation; namely, a variation of the state of stress, during which the statical requirements of equilibrium are the ones that remain satisfied, while the geometrical requirements of continuity are abandoned temporarily. As before, P and r are considered to remain constant, but now the deformations D are interpreted as functions of the stresses S, and values S_0 are arbitrarily chosen lower limits of the stresses. Then the complementary energy will be defined by the expression

$$U = \sum \int_{S_0}^{S} D\,dS - \sum r\,R \qquad (9)$$

An infinitesimal variation from any one of the statically possible states of stress to an adjacent statically possible state gives U an increment, a first variation, equal to

$$\delta U = \sum D\,\delta S - \sum r\,\delta R \qquad (10)$$

If the set of values of D and r happen to satisfy the geometrical requirements of continuity, then these values must satisfy Eq. 8 with $\delta P = 0$. It follows that

$$\delta U = 0 \qquad (11)$$

That is, the first variation of U vanishes; the complementary energy, U, is a minimum or a maximum, like the energy T ordinarily a minimum. The statement

$$U = \min \qquad (12)$$

is interpreted with the same reservation that was applied to Eq. 5: under abnormal circumstances the minimum may be changed into a maximum. Eq. 12, with U defined by Eq. 9, is the law of least complementary energy. This minimum is produced by a variation of the state of stress.

Analogy.—A comparison of Eqs. 1 to 5 with Eqs. 8 to 12, in order, shows that a complete analogy exists between the two laws of least energy and least

complementary energy, with a one to one correspondence of the quantities, as follows: in the shifting from one principle to the other the quantities P, S, R, T, U, p, D, r are replaced by the same quantities in the reversed order.

Castigliano's Principle as Special Case.—When Hooke's law applies and a stress-less state can be and is chosen as starting state, and the lower limits S_0 are chosen as zero, the part of the complementary energy contributed by the stresses, the first sum in Eq. 9, becomes

$$U_i = \tfrac{1}{2} \sum D\,S \qquad (13)$$

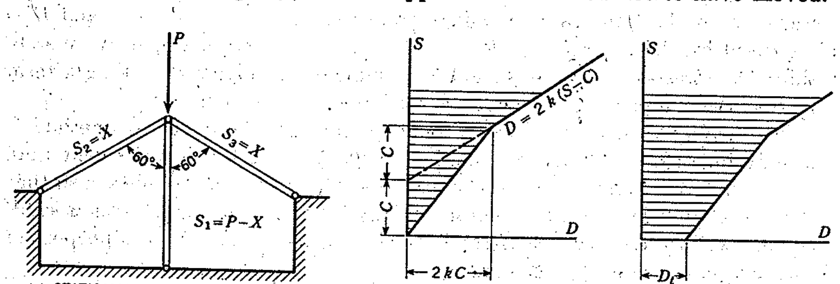
which is the same as the internal energy of stresses. If the supports have not moved, then $r = 0$. In this special yet general case the law of least complementary energy becomes

$$U_i = \min \qquad (14)$$

which is Castigliano's principle of least work, or, the principle of minimum of the internal energy by variation of the state of stress.

Application to a Simple Statically Indeterminate Structure Stressed Beyond the Proportional Limit

The simple truss in Fig. 2(*a*) will serve as an example. The stresses are total stresses, positive in compression. The state of stress is varied by varying X. The three members are assumed to be alike, with the stress-deformation diagram shown in Fig. 2(*b*). The supports are assumed not to have moved.



(*a*) STATICALLY POSSIBLE COMPRESSIVE STRESSES IN TRUSS (*b*) STRESS-DEFORMATION DIAGRAM (*c*) STRESS-DEFORMATION DIAGRAM AFTER COOLING

Fig. 2

When none of the stresses exceeds the proportional limit $2\,C$, the complementary energy is the same as the stress energy, which is

$$U = \tfrac{1}{2}\,k\,(P - X)^2 + k\,X^2 \qquad (15)$$

In Eq. 15, U must be a minimum by Castigliano's principle, which gives $X = \dfrac{P}{3}$.

When $P > 3\,C$, the stress in the vertical member will exceed the proportional limit $2\,C$. The complementary energy is determined by adding areas such as the shaded area in Fig. 2(*b*). If the proportional limit is not exceeded in the inclined members, the total complementary energy is

$$U = k\,C^2 + k\,(P - X - C)^2 + k\,X^2 \qquad (16)$$

In Eq. 16, U becomes a minimum when

$$X = \frac{P - C}{2} = S_2 = S_3 \qquad (17a)$$

and

$$S_1 = \frac{P + C}{2} \qquad (17b)$$

Eqs. 17 apply when $3\,C < P < 5\,C$. If $P > 5\,C$, the complementary energy is

$$U = 3\,k\,C^2 + k\,(P - X - C)^2 + 2\,k\,(X - C)^2 \qquad (18)$$

which becomes a minimum when

$$X = \frac{P + C}{3} = S_2 = S_3 \qquad (19a)$$

and

$$S_1 = \frac{2\,P - C}{3} \qquad (19b)$$

Computations such as these are simplified in less simple applications by expressing the derivatives that determine the minimum, without computing the value of the complementary energy itself.

If the support of the vertical member in Fig. 2(*a*) settles a distance c downward, the path of the corresponding upward reaction R is $r = -c$, and U is increased by the amount $-r\,R = c\,(P - X)$. Cooling of a member merely shifts the stress-deformation diagram as indicated in Fig. 2(*c*), adding a term $D_t\,S$ to the complementary energy contributed by that member.

In dealing with a structure of some complexity the choice of method is usually important, because one method is likely to be less inconvenient than others that are available; but the simple structure in Fig. 2 was chosen so that the two laws of least complementary energy and least energy lend themselves equally well to it. The use of the latter law will be shown for the purpose of comparison of the two procedures.

The shape is varied by moving the top joint downward a variable distance Y. Geometrical continuity is preserved when the deformations are $D_1 = Y$ and $D_2 = D_3 = \frac{1}{2}\,Y$ and $p = Y$. The strain energy of each member is measured by an area under the lines in Fig. 2(*b*). It is justifiable to call this quantity energy because the stress is a definite function of the deformation in the operations of the analysis; the quantity need not be energy in every physical sense; it is energy in the analysis. If the proportional limit has been exceeded in the vertical member only, the energy in Eq. 2 becomes

$$T = \left(\frac{Y^2}{4\,k} + C\,Y - k\,C^2\right) + \frac{1}{k}\left(\frac{Y}{2}\right)^2 - P\,Y \qquad (20)$$

In Eq. 20, T is a minimum when $Y = k\,(P - C)$, which reproduces the stresses in Eqs. 17.

Buckling of Column with Hinged Ends

When a column buckles under a critical load, the movement of the loaded end, the shortening of the chord, is proportional to the square of the lateral deflections; this is a departure from linearity and from the law of superposition, but the law of superposition of deformations will hold with good approximation in the significant applications if the starting shape is chosen close to the final shape.

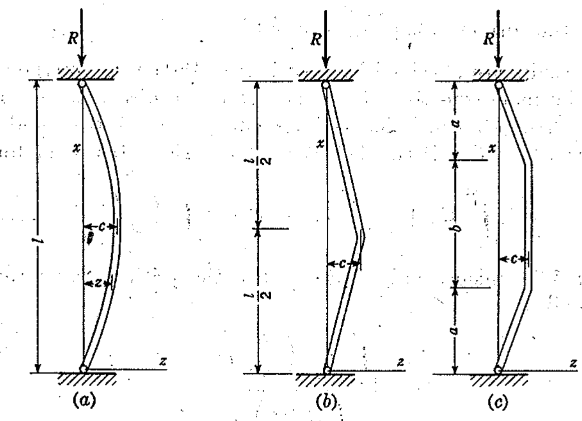


FIG. 3.—STARTING SHAPES OF COLUMN

Fig. 3(*a*) shows a plausible starting shape for a hinged-ended column, held between two fixed blocks. The blocks give the end pressure the character not of a load but of a reaction R.

Let: z = deflection in the starting shape; y = additional deflection from the starting shape to the final shape, making the total final deflection $y + z$; and $E\,I$ = modulus of elasticity times moment of inertia of the cross section.

In computing the variation of the complementary energy according to Eq. 10, S is taken as the bending moment $R\,z$ in the starting shape; this is approximately correct if the starting shape is close to the final shape. Then δS becomes $z\,\delta R$. The members are of infinitesimal length dx; therefore the summation is replaced by an integral. The deformation D becomes the relative change of slope

$$-\frac{d^2y}{dx^2}\,dx = \left(\frac{R\,z}{E\,I} + \frac{d^2z}{dx^2}\right)dx \qquad (21)$$

The displacement r is zero since the end blocks do not move. Thus one finds

$$\delta U = \int_0^l \left(\frac{R\,z}{E\,I} + \frac{d^2z}{dx^2}\right) dx\;(z\;\delta R) = 0 \qquad (22)$$

which gives the approximate formula

$$R = -\frac{\int_0^l z\,\frac{d^2z}{dx^2}\,dx}{\int_0^l \frac{z^2\,dx}{E\,I}} \qquad (23)$$

By choosing as a starting shape the parabola

$$z = 4\,c\,l^{-2}\,(l\,x - x^2) \qquad (24)$$

and assuming $E\,I$ constant, one finds by easy integrations

$$R = \frac{10\,E\,I}{l^2} \qquad (25)$$

which exceeds the correct value $\frac{\pi^2\,E\,I}{l^2}$ only by 1.3%.

The same results are obtained by a variant of the energy method to which reference will be made shortly. First, however, the direct use of the law of least energy will be shown. The end pressure is now interpreted not as a reaction but as a load P. If the deflections are z, Eq. 2 takes the form

$$T = \tfrac{1}{2}\int_0^l E\,I\left(\frac{d^2z}{dx^2}\right)^2 dx - P\cdot\tfrac{1}{2}\int_0^l\left(\frac{dz}{dx}\right)^2 dx \qquad (26)$$

The variation δT vanishes only when P has the critical value that makes $T = 0$. This gives

$$P = \frac{\int_0^l E\,I\left(\frac{d^2z}{dx^2}\right)^2 dx}{\int_0^l\left(\frac{dz}{dx}\right)^2 dx} \qquad (27)$$

With z as in Eq. 24 and $E\,I$ constant, Eq. 27 gives $P = 12\,E\,I\,l^{-2}$ which is 22% too great. The method of complementary energy gives the same result when z is taken as in Fig. 3(*b*); the numerator in Eq. 23 then being computed as c times the change of slope at the middle of the column. With the starting shape in Fig. 3(*c*) the method of complementary energy gives $R = \frac{6\,E\,I}{2\,a^2 + 3\,a\,b}$, which has its smallest and therefore most plausible value, $10.67\,E\,I\,l^{-2}$, when $a = \frac{3\,l}{8}$.

The comparison just made indicates superiority of the method of complementary energy over the direct application of the method of the potential energy to columns; but, as has been indicated, a variant of the energy method, applicable to columns, is available. S. Timoshenko[14] has described and used it. This improved variant can be explained by noting that the numerator in Eq. 27 is improved by replacing $\frac{d^2z}{dx^2}$ by $-\frac{P\,z}{E\,I}$. Then the formula becomes

$$P = \frac{\int_0^l\left(\frac{dz}{dx}\right)^2 dx}{\int_0^l \frac{z^2\,dx}{E\,I}} \qquad (28)$$

Eq. 28 can be rewritten in the form of Eq. 23. That is, Timoshenko's variant of the energy method for columns gives the same results as the method of complementary energy.

[14] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., 1936, p. 81.

Column with Initial Curvature

If the column that has been under discussion is slightly curved before loading, with the initial deflections z_0, and if z still denotes the total deflections in the starting shape, Eq. 22 will be changed to

$$\delta U = \int_0^l\left(\frac{R\,z}{E\,I} + \frac{d^2z}{dx^2} - \frac{d^2z_0}{dx^2}\right) dx\,(z\,\delta R) = 0 \qquad (29)$$

Assume that the shape before loading makes it plausible to choose z proportional to z_0. Denote by Q the critical value of R defined by Eq. 23 for $z_0 = 0$. Then Eq. 29 gives

$$R - \left(1 - \frac{z_0}{z}\right) Q = 0 \qquad (30)$$

or,

$$\frac{z}{z_0} = \frac{Q}{Q - R} \qquad (31)$$

Eq. 31 defines a magnification factor for the deflections. Such factors will be discussed again later.

Greatest Load on an Initially Curved Column Stressed Beyond the Proportional Limit

If the column in Fig. 3(*a*) is stressed beyond the proportional limit, it may be plausible to assume a law of deformations of the type,

$$\text{Change of curvature} = \frac{1}{E_r\,I}\,(M + F^{-2}\,M^3) \qquad (32)$$

in which E_r and F are functions of the end pressure R and dependent on the material and the shape of the cross section; E_r being a reduced modulus of elasticity, and F a coefficient measurable in inch-pounds. The theory developed by Engesser (from 1889), A. Considère (1891), F. S. Yasinsky (1895), and Theodor von Kármán, M. Am. Soc. C. E., (1910),[15] furnishes a method of computing E_r; for example, this method gives for a rectangular cross section

$$E_r = \frac{4\,E\,E_t}{(\sqrt{E} + \sqrt{E_t})^2} \qquad (33)$$

in which E_t is the tangent modulus of elasticity. The theory also will make it possible[16] to establish reasonable values of F. Moreover, it should be possible to determine both E_r and F by tests of short specimens under eccentric pressure.

Let z_0 denote, as before, the initial deflections before loading, and z the total starting deflections under the pressure R. With Eq. 32 accepted, Eq. 29

[15] For explanation of the basis of the theory and for references, see "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., 1936, pp. 157–159.

[16] See, for example, the paper "Strength of Steel Columns," by H. M. Westergaard and William R. Osgood, Members, Am. Soc. C. E., *Transactions*, A. S. M. E., Vol. 50, 1928, No. 17, p. 65.

changes into the following expression of the law of least complementary energy:

$$\frac{dU}{dR} = \int_0^l \left[\frac{1}{E_r I}(R z + F^{-2} R^3 z^3) + \frac{d^2 z}{dx^2} - \frac{d^2 z_0}{dx^2}\right] z\, dx = 0 \quad (34)$$

Let z_0 be assumed and z chosen as follows:

$$z_0 = c_0 \sin \frac{\pi x}{l} \quad (35a)$$

and

$$z = c \sin \frac{\pi x}{l} \quad (35b)$$

and denote,

$$Q = \frac{\pi^2 E_r I}{l^2} \quad (36)$$

Then Eq. 34 gives

$$R c + \frac{3}{4} \frac{(R c)^3}{F^2} = Q (c - c_0) \quad (37)$$

In the course of a test in a laboratory, as the distance between the end blocks is reduced, the maximum deflection c will increase gradually and may serve as a measure of the progress of the test. The end pressure R will be a function of c, dependent on the initial deflection c_0. It is desired to determine the maximum value of R as a function of c, for different definite values of c_0 and l.

It is noted that E_r in Eq. 36 is a function of R; therefore Q in Eqs. 36 and 37 is a function of R. With this in view Eq. 37 shows that in the special case of $c_0 = 0$, the maximum value of R occurs when $c = 0$, and is equal to Q, as it should be in accordance with the theory initiated by Engesser. With $Q = R$ and R chosen, Eq. 36 can be solved for l, thus serving as a formula for the idealized case of an initially straight column.

When the column is curved before loading (that is, when $c_0 > 0$), Q loses its significance as a maximum load and becomes merely a quantity introduced for convenience and defined by an equation. To make R a maximum the condition $\frac{dR}{dc} = 0$ is imposed. Since Q is a function of R, it follows that $\frac{dQ}{dc} = 0$. Under these conditions, by applying the operators $3 - c\frac{d}{dc}$ and $c\frac{d}{dc} - 1$ to Eq. 37, one finds

$$2 R c = Q (2 c - 3 c_0) \quad (38a)$$

and

$$\frac{3 (R c)^3}{2 F^2} = Q c_0 \quad (38b)$$

respectively. Elimination of c from Eqs. 38 gives

$$c_0 = \frac{4 F}{9 Q}\left(\frac{Q}{R} - 1\right)^{3/2} \quad (39)$$

or, with Q substituted from Eq. 36,

$$c_0 = \frac{4 F l^2}{9 \pi^2 E_r I}\left(\frac{\pi^2 E_r I}{R l^2} - 1\right)^{3/2} \quad (40)$$

Eq. 40 is not a design formula, but may lend itself to interpretations of tests. If E_r and F are numerically known functions of R, the values of R and l may be chosen, and the greatest initial deflection c_0 permitting these values may be computed by Eq. 40.

If Eq. 32 is replaced by the simpler but probably less plausible formula,

$$\text{Change of curvature} = \frac{1}{E_r I}(M + F^{-1} M^2) \quad (41)$$

the same procedure of analysis gives, instead of Eq. 40,

$$c_0 = \frac{3 F l^2}{32 \pi E_r I}\left(\frac{\pi^2 E_r I}{R l^2} - 1\right)^2 \quad (42)$$

Buckling of a Thin Circular Elastic Plate with Simply Supported Edge Under a Uniform Pressure at the Edge

A solution of this problem was indicated by G. H. Bryan[17] in 1891. The differential equation for the slope in the radial direction is solved by a Bessel function of order one. When Poisson's ratio is 0.3, this exact solution gives the following value of the critical pressure per unit of length:

$$R = \frac{4.62 E I}{a^2} \quad (43)$$

in which: I = moment of inertia of the cross section per unit of width; and a = radius of the plate.

The application of the method of complementary energy to this problem is not more convenient than the use of Bessel functions, but it will serve the purpose of an illustration. The procedure can be used in more complicated related problems, involving, for example, a varying thickness.

The following additional notation is used:

r = radial distance from the center of the plate.

m_r = bending moment in the radial direction per unit of width of the circumferential section on which it acts.

m_θ = bending moment in the circumferential direction per unit of width of the radial section on which it acts; both m_r and m_θ are measurable in inch-pounds per inch, or, in pounds.

z = starting deflection.

ζ = supplementary deflection, making the final deflection $z + \zeta$.

c = z at center.

$f = f(r)$ = stress function, measurable in inch-pounds.

X = constant to be determined.

μ = Poisson's ratio.

[17] *Proceedings*, London Mathematical Soc., Vol. 22, 1891, p. 54; also, "Theory of Elastic Stability," by S. Timoshenko, 1936, pp. 367–369.

It is permissible to assume that z, ζ, and f are functions of r only. When z is chosen so that the values of ζ will be insignificant relative to the values of z, the bending moments can be expressed by the formulas,

$$m_r = R\,z + \frac{f}{r} \qquad (44a)$$

and

$$m_\theta = R\,z + \frac{df}{dr} \qquad (44b)$$

A simple examination shows that no matter what differentiable function f is chosen, the moments in Eqs. 44 maintain equilibrium. The variation of the state of stress therefore is reduced to the variation of R and f. When the edge is simply supported, $z = 0$ and $f = 0$ at the edge.

As in the study of columns, R is interpreted as a reaction exerted by a strong support in a fixed position. The changes of curvature in the directions of m_r and m_θ, measured from the starting shape to the final shape, are $-\frac{d^2\zeta}{dr^2}$ and $-\frac{1}{r}\frac{d\zeta}{dr}$, respectively. Accordingly the law of least complementary energy takes the form

$$\delta U = \int_0^a 2\,\pi\,r\,dr\left[-\frac{d^2\zeta}{dr^2}\,\delta m_r - \frac{1}{r}\frac{d\zeta}{dr}\,\delta m_\theta\right] = 0 \qquad (45)$$

To obtain an approximate solution the following expressions are chosen:

$$z = c\left(1 - \frac{r^2}{a^2}\right) \qquad (46a)$$

and

$$f = c\,(X - R)\left(r - \frac{r^3}{a^2}\right) \qquad (46b)$$

By Eqs. 44 the corresponding moments are

$$m_r = c\,X\left(1 - \frac{r^2}{a^2}\right) \qquad (47a)$$

and

$$m_\theta = c\,X\left(1 - \frac{3\,r^2}{a^2}\right) + \frac{2\,c\,R\,r^2}{a^2} \qquad (47b)$$

The bending moments define the corresponding changes of curvature:

$$-\frac{d^2\zeta}{dr^2} = \frac{1}{E\,I}(m_r - \mu\,m_\theta) + \frac{d^2z}{dr^2}$$
$$= \frac{c}{E\,I}\left\{X\left[1 - \mu - (1 - 3\,\mu)\frac{r^2}{a^2}\right] - \frac{2\,\mu\,R\,r^2}{a^2}\right\} - \frac{2\,c}{a^2} \qquad (48)$$

and

$$-\frac{1}{r}\frac{d\zeta}{dr} = \frac{1}{E\,I}(m_\theta - \mu\,m_r) + \frac{1}{r}\frac{dz}{dr}$$
$$= \frac{c}{E\,I}\left\{X\left[1 - \mu - (3 - \mu)\frac{r^2}{a^2}\right] + \frac{2\,R\,r^2}{a^2}\right\} - \frac{2\,c}{a^2} \qquad (49)$$

By substituting from Eqs. 47, 48, and 49 in Eq. 45, and carrying out the integrations one finds

$$\frac{3\,E\,I}{\pi\,a^2\,c^2}\,\delta U = \left[4\,X - (3 + \mu)\,R\right]\delta X$$
$$+ \left[-(3 + \mu)\,X + 4\,R - \frac{6\,E\,I}{a^2}\right]\delta R = 0 \qquad (50)$$

Eq. 50 gives

$$X = \frac{3 + \mu}{4}\,R \qquad (51a)$$

and

$$R = \frac{24\,E\,I}{(7 + \mu)\,(1 - \mu)\,a^2} \qquad (51b)$$

With $\mu = 0.3$, Eqs. 51 give $R = 4.70\,\frac{E\,I}{a^2}$ which is only 1.7% greater than the correct value in Eq. 43.

By the choice,

$$z = c\cos\frac{\pi\,r}{2\,a} \qquad (52a)$$

and

$$f = c\,(X - R)\,r\cos\frac{\pi\,r}{2\,a} \qquad (52b)$$

instead of Eqs. 46, the slightly better value $R = 4.66\,\frac{E\,I}{a^2}$ was obtained with $\mu = 0.3$.

Application to Vibration of Beams

If an elastic beam vibrates freely in one of its modes, the deflections at the time t may be stated as

$$\eta = (z + y)\cos\omega\,t \qquad (53)$$

in which ω is a constant defining the period as $\frac{2\,\pi}{\omega}$, and z and y are functions of the distance x measured along the beam. The function z will be chosen so that the values of y are relatively small. Then z may be interpreted as a starting deflection and y as a supplementary deflection, both referring to the times when $\cos\omega\,t = 1$.

Let X denote the bending moment at any point due to an imagined static load defined as the product of z times the weight w per unit of length; w may be constant or a function of the distance x; X will be a function of x and will be measurable in lb-in.[2] When $\cos\omega\,t = 1$, the inertia force per unit of length is $\frac{\omega^2}{g}\,w\,(z + y)$, in which g is the acceleration due to gravity; $\frac{w}{g}$ is the mass per unit of length. When the contributions from y can be ignored, the bending moment due to the inertia forces becomes

$$M = \frac{\omega^2}{g}\,X \qquad (54)$$

The use of the inertia forces converts the problem into one of statics. The quantity $\frac{\omega^2}{g}$ now may be interpreted as an adjustable reaction, playing the same rôle as R in the application to columns. Then

$$\delta M = X\,\delta\left(\frac{\omega^2}{g}\right) \quad (55)$$

and the variation of the complementary energy becomes

$$\delta U = \int\left(\frac{M}{E\,I} + \frac{d^2z}{dx^2}\right)dx\,\delta M = \delta\left(\frac{\omega^2}{g}\right)\int\left(\frac{\omega^2 X}{E\,I\,g} + \frac{d^2z}{dx^2}\right)X\,dx = 0 \quad (56)$$

Eq. 56 gives

$$\omega^2 = -\frac{\int X\frac{d^2z}{dx^2}\,dx}{\int \frac{X^2\,dx}{E\,I\,g}} \quad (57)$$

from which the period may be computed as $\frac{2\,\pi}{\omega}$.

It is of interest to compare Eq. 57 with the next equation, which represents Rayleigh's[18] method. The left side of the equation is twice the kinetic energy when $\cos\omega\,t = 0$; the right side is twice the internal potential energy when $\cos\omega\,t = 1$; and either side is twice the total mechanical energy:

$$\omega^2\int\frac{w}{g}\,z^2\,dx = \int E\,I\left(\frac{d^2z}{dx^2}\right)^2 dx \quad (58)$$

When the deflection z is stated approximately, Eq. 58 gives an approximate value of ω^2. The example that follows will show that Eq. 58 requires a more accurate statement of z than does Eq. 57 for the same degree of approximation in the value of ω.

A cantilever of length l, with a fixed support at $x = 0$, and with w and $E\,I$ constant, is chosen as example. An exact solution[19] of the proper differential equation gives an expression for the extreme deflections $z + y$ in terms of two hyperbolic and two trigonometric functions. Thereby the value of ω in the first mode of vibration is found to be

$$\omega = 3.516\,\omega_0 \quad (59)$$

in which ω_0 denotes the quantity

$$\omega_0 = \sqrt{\frac{E\,I\,g}{w\,l^4}} \quad (60)$$

In an approximate solution by the method of complementary energy the starting curve may be taken as a parabola,

$$z = \frac{c\,x^2}{l^2} \quad (61)$$

[18] "Theory of Sound," by Lord Rayleigh, 1877, 2d Ed., MacMillan & Co., Vol. 1, 1894, pp. 109 and 257; see also "Vibration Problems in Engineering," by S. Timoshenko, Van Nostrand Co., 1928, pp. 55–60.

[19] See "Vibration Problems in Engineering," by S. Timoshenko, Van Nostrand Co., 1928, p. 234.

Then

$$X = -\frac{w\,l^2\,c}{12}\left[\left(\frac{x}{l}\right)^4 - \frac{4\,x}{l} + 3\right] \quad (62)$$

When these functions are substituted in Eq. 57, one finds

$$\omega = \omega_0\sqrt{\frac{162}{13}} = 3.530\,\omega_0 \quad (63)$$

which differs only by 0.4% from the correct value in Eq. 59.

Exactly the same result can be obtained by Rayleigh's method, with Eq. 58, but the function z must be chosen much closer to the true deflections; the function z equal to the deflection under a uniform load will serve the purpose.[20] If z were taken from Eq. 61, Eq. 58 would give $\omega = \omega_0\sqrt{20} = 4.47\,\omega_0$, which is 27% too great.

If a straight line, $z = \frac{c\,x}{l}$, is chosen as starting shape instead of the parabola in Eq. 61, the method of complementary energy gives $\omega = \omega_0\sqrt{\frac{140}{11}} = 3.57\,\omega_0$. Rayleigh's method gives the same result when z is chosen as the deflection due to a concentrated load at the free end.[21]

Stabilizing Loads

When a load producing buckling is reversed, it becomes a stabilizing load; but it can still be studied by the theory of buckling. The general theorems which are needed in subsequent applications of the method of complementary energy are derived most conveniently by considering the energy as a function of the shape. The derivations will be shown in brief form.[22]

It is assumed that any shape of the structure that is geometrically possible, when continuity is maintained, can be defined by assigning a definite set of values to a set of parameters $u_0, u_1, u_2, \cdots u_n, \cdots$. It is assumed that these parameters define any internal deformation by a linear equation of the form

$$D = D_0\,u_0 + D_1\,u_1 + D_2\,u_2 + \cdots + D_n\,U_n + \cdots = \sum_{0,1,2,\ldots}^{n} D_n\,u_n \quad (64)$$

Besides, it is assumed that each member obeys Hooke's law. Then each stress will be a linear function of the parameters, and the strain energy will be a quadratic function of the parameters.

Two loads, P and Q, are considered. It is assumed that the path p of P can be stated adequately as a linear function of the parameters,

$$p = \sum_{1,2,\ldots}^{n} p_n\,u_n \quad (65)$$

but that a quadratic function of the parameters is required for an adequate

[20] "Vibration Problems in Engineering," by S. Timoshenko, Van Nostrand Co., 1928, p. 59.

[21] *Loc. cit.*, p. 58.

[22] A fuller account of the theory represented by Eqs. 64 to 73 and 78 to 80 and explanations of the special terminology are given in the paper, "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. 85 (1922), p. 576, especially pp. 604–637.

statement of the path q of Q. It will be assumed that a special choice of parameters has made it possible to state the path q as

$$q = -q_0 u_0 - \tfrac{1}{2} \sum_{1,2,\ldots}^{n} q_n u_n^2 \quad \ldots\ldots (66)$$

and the strain energy as

$$V = \tfrac{1}{2} V_0 u_0^2 + \tfrac{1}{2} \sum_{1,2,\ldots}^{n} Q_n q_n u_n^2 \quad \ldots\ldots (67)$$

with all the coefficients q_0, q_n, V_0, and Q_n positive and all of the values Q_n different from one another. That this combination of relations is not only possible, but typical of buckling and stabilizing loads, will be brought out by the discussion that follows.

With Eqs. 65 to 67 accepted, the principle of minimum of the potential energy by variation of the shape takes the form,

$$T = V - Q\,q - P\,p = \tfrac{1}{2} V_0 u_0^2 + \tfrac{1}{2} \sum_{1,2,\ldots}^{n} Q_n q_n u_n^2$$

$$+ Q \left(q_0 u_0 + \tfrac{1}{2} \sum_{1,2,\ldots}^{n} q_n u_n^2 \right) - P \sum_{1,2,\ldots}^{n} p_n u_n \quad \ldots\ldots (68)$$

$$\delta T = 0, \quad \text{or,} \quad \frac{\partial T}{\partial u_n} = 0 \quad \text{for } n = 0, 1, 2, \cdots \quad \ldots\ldots (69)$$

Eqs. 68 and 69 will be applied to investigate a series of four actions, in which u_n will be denoted successively u_n, $\bar{u}_n$, $\bar{\bar{u}}_n$, and u_n. Table 1 will serve as a summary of the results.

TABLE 1.—SUMMARY OF THEORY OF BUCKLING AND STABILIZING LOADS

Action	Definition	Value of parameter u_n for $n = 1, 2, 3, \cdots$	Equation No.
Astatic	$P = 0, Q = -Q_n$	u_n = any value, others = 0	71
Orthostatic	$P \neq 0, Q = 0$	$\bar{u}_n = \frac{P\,p_n}{Q_n\,q_n} = \frac{Q}{Q_n}\bar{\bar{u}}_n$	73, 77
Relaxed	$P \neq 0, Q > 0$, V reduced	$\bar{\bar{u}}_n = \frac{P\,p_n}{Q\,q_n} = \frac{Q_n}{Q}\bar{u}_n$	77, 73
Heterostatic	$P \neq 0, Q \neq 0$, V restored	$u_n = \frac{Q_n}{Q_n + Q}\bar{u}_n = \frac{Q}{Q_n + Q}\bar{\bar{u}}_n$	80, 81

Astatic Action.—The first action is defined by

$$P = 0, \qquad Q = -Q_n \quad \ldots\ldots (70)$$

Inspection of Eq. 68 shows that $\delta T = 0$ when

$$u_0 = \frac{Q_n q_0}{V_0}, \quad u_n = \text{any value, all other parameters } u_m = 0 \quad \ldots\ldots (71)$$

Since the parameter u_n may pass through a continuous range of values, defining a continuous range of shapes of the structure, the equilibrium is called neutral, and the action is described as buckling under the critical load $-Q = Q_n$. Since all other parameters remain zero except u_0, this buckling is called pure buckling or astatic action. The load $-Q$ is called an astatic load; the parameters u_1, u_2, $\cdots u_n$, $\cdots$ are called astatic parameters; and u_0 is called the orthostatic parameter of the astatic action. If T is a maximum by variation of some of the parameters, the equilibrium is unstable, though neutral by variation of u_n. Further examination[23] of Eq. 68 shows that if the quadratic expressions for V and q are written with mixed terms included, such as $V_{mn} u_m u_n$ and $q_{mn} u_m u_n$, knowledge of the existence of two astatic actions, one with u_m = any value, $u_n = 0$ and the other with $u_m = 0$, u_n = any value, at two different values of the astatic load, leads to the conclusion that $V_{mn} = q_{mn} = 0$. It follows that if the states of pure buckling can be ascertained, q and V can be written in the relatively simple forms of Eqs. 66 and 67; and thereby the nature of the astatic parameters is also ascertained.

Orthostatic Action.—The second action to be examined is defined by

$$P \neq 0, \qquad Q = 0 \quad \ldots\ldots (72)$$

Eqs. 68 and 69 give $u_0 = 0$ and the following value of u_n for $n = 1, 2, \cdots$:

$$\bar{u}_n = \frac{P\,p_n}{Q_n\,q_n} \quad \ldots\ldots (73)$$

The parameters are proportional to the load P; the same applies to the deformations and stresses, which are linear functions of the parameters. No buckling is involved; therefore this is the usual action dealt with in structural mechanics. In the terminology of the theory of buckling, this action, in which the astatic load is absent, is called orthostatic action. The load P is called the orthostatic load. Eq. 73 defines the values of the astatic parameters in the orthostatic action.

Relaxed Action.—In the third action to be examined the structure is assumed to have been modified by a relaxation of stiffness. The strain energy V in Eq. 67 is reduced in the relaxed structure by removing all terms except the first, so that now

$$V = \tfrac{1}{2} V_0 u_0^2 \quad \ldots\ldots (74)$$

It will be assumed that

$$P \neq 0, \qquad Q > 0 \quad \ldots\ldots (75)$$

When the proper terms are omitted in Eq. 68 one finds

$$u_0 = -\frac{Q\,q_0}{V_0} \quad \ldots\ldots (76)$$

and the following value of u_n for $n = 1, 2, \cdots$:

$$\bar{\bar{u}}_n = \frac{P\,p_n}{Q\,q_n} \quad \ldots\ldots (77)$$

[23] "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. 85 (1922), p. 576, especially pp. 611–612.

Eq. 77 defines the values of the astatic parameters in the relaxed action. The equilibrium is stable, and Q is a stabilizing load.

Heterostatic Action.—In the fourth and last action to be considered the structure is assumed to have recovered its stiffness. Both the orthostatic and the astatic loads are present, that is,

$$P \neq 0, \qquad Q \neq 0 \qquad (78)$$

This general type of action is called heterostatic action; the combined load P, Q is a heterostatic load, with the orthostatic component P and the astatic component Q. By Eqs. 68 and 69 one finds u_0 as in Eq. 76, and for $n = 1, 2, 3, \ldots$:

$$u_n = \frac{P\,p_n}{(Q_n + Q)\,q_n} \qquad (79)$$

A comparison of Eq. 79 with Eqs. 73 and 77 shows that

$$\frac{u_n}{\bar{u}_n} = \frac{Q_n}{Q_n + Q} \qquad (80)$$

and

$$\frac{u_n}{\bar{\bar{u}}_n} = \frac{Q}{Q_n + Q} \qquad (81)$$

When Q is positive, Eq. 80 defines the reduction factor of the astatic parameter u_n by transition from the orthostatic action to the heterostatic action under the influence of the stabilizing load Q; and Eq. 81 defines the reduction factor of the same parameter by transition from the relaxed action to the heterostatic action by restoration of the stiffness. When Q is negative, the factors in Eqs. 80 and 81 become magnification factors. It may be noted in passing that $\frac{Q}{Q_n}$ is the transition factor $\frac{\bar{u}_n}{\bar{\bar{u}}_n}$ by change from the relaxed to the orthostatic action.

Example.—A slender horizontal tension member bending under its own weight will serve as illustration. The ends are assumed to be hinged, and the end load central. The total horizontal tension is the stabilizing load Q. The weight, w per unit of length, is the orthostatic load. Assume that the cross section is constant, with area A and moment of inertia I. Let E = modulus of elasticity, l = length, x = distance measured from one end. Then the nth critical value of the astatic load $-Q$ is defined by Euler's formula

$$Q_n = \frac{n^2 \pi^2 E I}{l^2} \qquad (82)$$

and the deflections maintained by it are

$$y = u_n \sin \frac{n \pi x}{l} \qquad (83)$$

in which u_n is a proper astatic parameter. As orthostatic parameter of the orthostatic action one may introduce the total shortening of the center line, $u_0 = -\frac{Q\,l}{E\,A}$, which makes $q_0 = 1$. The parameter u_0 presents no difficulties in this problem.

In the relaxed action the member is changed into a cable, which deflects according to the law

$$-\frac{d^2y}{dx^2} = \frac{w}{Q} \qquad (84)$$

Eq. 84 may be rewritten so that the right side expresses a sum of corresponding effects in astatic actions, as follows:[24]

$$-\frac{d^2y}{dx^2} = \frac{4\,w}{\pi\,Q} \sum_{1,3,\ldots}^{n} \frac{1}{n} \sin \frac{n \pi x}{l} \qquad (85)$$

When the stiffness is restored, each term in the summation in Eq. 85 is multiplied by a reduction factor defined by Eqs. 81 and 82. After multiplying by $-E\,I$, one obtains in this way the bending moments in the heterostatic action,

$$M = \frac{4\,w\,l^2}{\pi^3} \sum_{1,3,\ldots}^{n} \frac{Q_1}{n\,(Q_n + Q)} \sin \frac{n \pi x}{l} \qquad (86)$$

The series in Eq. 86 converges and defines the stresses.

Eq. 86 may also be derived by way of the orthostatic action. When $Q = 0$,

$$\frac{d^4y}{dx^4} = \frac{w}{E\,I} = \frac{4\,w}{\pi\,E\,I} \sum_{1,3,\ldots}^{n} \frac{1}{n} \sin \frac{n \pi x}{l} \qquad (87)$$

After two integrations one finds the bending moment in the orthostatic action

$$\bar{M} = \frac{4\,w\,l^2}{\pi^3} \sum_{1,3,\ldots}^{n} \frac{1}{n^3} \sin \frac{n \pi x}{l} \qquad (88)$$

When Q is introduced, and the reduction factors are applied according to Eq. 80, Eq. 86 is reproduced.

The general theory may be applied to the problem in other ways. For example, the theory will furnish the series needed first, such as those in Eqs. 85 and 87; this is important for the more complicated problems in which the corresponding series are not simple trigonometric series. The computations may be made as follows: The contribution of the deflections $y = u_n \sin \frac{n \pi x}{l}$ to the path of the distributed load w (which takes the place of P in Eq. 68) is

$$p = p_n\,u_n = \int_0^l y\,dx = \frac{2\,l\,u_n}{n\,\pi} \qquad (89)$$

if n is uneven, and zero if n is even; and the corresponding contribution to the path of Q is

$$q = -\tfrac{1}{2}\,q_n\,u_n^2 = -\tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx = -\frac{n^2 \pi^2 u_n^2}{4\,l} \qquad (90)$$

[24] See "A Short Table of Integrals," by B. O. Peirce, 2d Ed., Ginn & Co., 1910, formula 808.

The last two equations define p_n and q_n directly, and thereafter $\bar{u}_n$ through Eqs. 73 and 82. Thus one finds the deflections in the orthostatic action

$$y = \sum_{1,3,\ldots}^{n} \bar{u}_n \sin \frac{n \pi x}{l} = \frac{4 w l^4}{\pi^5 E I} \sum_{1,3,\ldots}^{n} n^{-5} \sin \frac{n \pi x}{l} \qquad (91)$$

The deflections in the heterostatic action are obtained by applying the reduction factor in Eq. 80 to each term in Eq. 91. Eq. 86 is reproduced easily by this method. Another route to the same result is to determine $\bar{u}_n$ by Eq. 77, and thereby the deflections in the relaxed action; from these one finds the deflections in the heterostatic action by applying the reduction factor in Eq. 81 to each term in the series.

The method of complementary energy is applicable to problems of stabilizing loads through its applicability to astatic actions.

Stiffening of Suspension Bridges

That a suspension bridge is stiffened by its own weight was brought out in an analysis published by Joseph Melan[25] in 1888. Leon S. Moisseiff, M. Am. Soc. C. E., developed the theory further for the design of the Manhattan Bridge (1909); and F. E. Turneaure,[26] Hon. M. Am. Soc. C. E., investigated this application and contributed to, and presented, the theory. Since then this principle of stiffening has become well appreciated in America; it has been utilized in the design of the great bridges, and has been discussed in a notable American technical literature on the subject.[25, 26, 27] The dead load has the character of a stabilizing load.

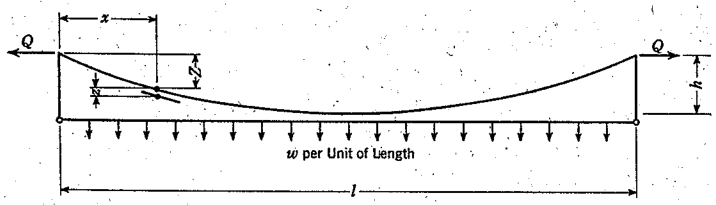


Fig. 4.—Main Span of Suspension Bridge

It is not intended here to present a full analysis of the stiffening of suspension bridges. It is desired only to show the applicability of the method of complementary energy to this problem through the theory of stabilizing loads and buckling.[28] The analysis therefore is limited to a simplified case, which is suggested in Fig. 4. The following actions are left out of account, as matters

[25] "Theorie der eisernen Bogenbrücken und Hängebrücken," by Joseph Melan, *Handbuch der Ingenieurwissenschaften*, Vol. 2, Subvolume 4, Leipzig, 1888, pp. 1–144, especially p. 38 (English translation of 3d Ed. (1906) entitled "Theory of Arches and Suspension Bridges," by D. B. Steinman, M. Am. Soc. C. E., The M. C. Clark Publishing Co., 1913).

[26] "Modern Framed Structures," by J. B. Johnson, C. W. Bryan, and F. E. Turneaure, John Wiley & Sons, Inc., Vol. 2, 9th Ed., 1911, pp. 276–318.

[27] "A Practical Treatise on Suspension Bridges," by D. B. Steinman, John Wiley & Sons Co., Inc., 1929, pp. 246–282; Leon S. Moisseiff, *Journal of the Franklin Institute*, October, 1925. *Transactions*, Am. Soc. C. E., Vol. 91 (1927), J. A. L. Waddell, pp. 884–910, especially pp. 893–895; Vol. 94 (1930), S. Timoshenko, pp. 377–391; Vol. 97 (1933), O. H. Ammann, pp. 1–65, especially pp. 39–44; Vol. 100 (1935), D. B. Steinman, pp. 1133–1170; and Vol. 104 (1939), Shortridge Hardesty and Harold E. Wessman, pp. 579–608.

[28] The application of the theory of buckling of "the suspension bridge upside-down" to suspension bridges was suggested by H. M. Westergaard in a discussion, *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 1021.

that can either be incorporated in a more extensive analysis by the same principles or be referred to independent supplementary analyses: elongations of the cable beyond the initial stretching; movements of the points of the cable at the tops of the towers; and elongations of the suspenders beyond the initial stretching. The suspenders are assumed to be spaced closely. The initial curve of the cable is assumed to be a parabola with a sag that is fairly small compared with the span. The following notation is used:

x = horizontal coordinate.

Z = initial vertical coordinate, positive downward, of any point of the cable.

z = addition to Z by the change from the initial shape to the starting shape; with the restrictions imposed, z is also the starting deflection of the stiffening truss.

y = addition to z by the change from the starting shape to the final shape.

Y = total deflection, addition to Z, in a relaxed action in which neither the cable nor the truss contributes stiffness against bending.

η = total deflection, addition to Z, in the heterostatic action.

l = span.

$h = Z_{max}$.

Q = horizontal component of the total tension in the cable; Q is interpreted as a measure of the stabilizing load.

Q_n = critical value of $-Q$ in an unstable astatic action.

w = uniformly distributed vertical load per unit of length; interpreted in this analysis as a reaction adjusting itself to Q as if it were a hydrostatic pressure.

P = orthostatic load.

K = force defined by Eq. 95.

M = sum of the bending moments in the stiffening truss and the cable at any value of x.

$E I$ = measure of the combined stiffness of the stiffening truss and the cable against bending.

k = ratio defined by Eq. 117.

The initial parabolic curve of the cable has the equation

$$Z = 4 h l^{-2} (l x - x^2) \qquad (92)$$

If the shape is changed by infinitesimal increments δZ, the potential energy of the uniformly distributed load w will be changed by the amount $- w \int_0^l \delta Z \, dx$. This amount is infinitesimal of second order because it is the total change of energy if the cable is unstiffened, and the parabola is the curve of equilibrium of the unstiffened cable. Therefore, when small quantities of second order, which will be without importance in the subsequent computations, are ignored, and under the simplifying restrictions that were imposed, and as long as the deflections z, y, Y, and η remain small, it can be asserted that

$$\int^l z \, dx = \int^l y \, dx = \int_0^l Y \, dx = \int_0^l \eta \, dx = 0 \qquad (93)$$

The first task is to investigate the astatic actions at critical negative values of the loads in Fig. 4. Under the loads in Fig. 4, when the deflections are z, the combined bending moment in the cable and the truss is

$$M = \tfrac{1}{2}\,w\,(l\,x - x^2) - 4\,Q\,h\,l^{-2}\,(l\,x - x^2) - Q\,z \qquad (94)$$

It is expedient to introduce a force K defined by the equation

$$w = 8\,(Q + K)\,h\,l^{-2} \qquad (95)$$

Then

$$M = -\,Q\,z + 4\,K\,h\,l^{-2}\,(l\,x - x^2) \qquad (96)$$

and the increment of M by variation of the state of stress is

$$\delta M = -\,z\,\delta Q + 4\,h\,l^{-2}\,(l\,x - x^2)\,\delta K \qquad (97)$$

When z is chosen so that the supplementary deflections y are small, the relation of deformations to stresses

$$-\frac{d^2y}{dx^2} = \frac{M}{E\,I} + \frac{d^2z}{dx^2} \qquad (98)$$

may be used with M as in Eq. 96.

The variation of the complementary energy may now be written:

$$\delta U = \int_0^l \left(\frac{M}{E\,I} + \frac{d^2z}{dx^2}\right) dx\,\delta M$$

$$= \frac{1}{E\,I}\int_0^l \left[-\,Q\,z + 4\,K\,h\,l^{-2}\,(l\,x - x^2) + E\,I\,\frac{d^2z}{dx^2}\right]\left[-\,z\,\delta Q + 4\,h\,l^{-2}\,(l\,x - x^2)\,\delta K\right] dx = 0 \qquad (99)$$

This equation is converted into the following two, in which all integrations are from 0 to l:

$$E\,I\,\frac{\partial U}{\partial Q} = Q\int z^2\,dx - 4\,K\,h\,l^{-2}\int (l\,x - x^2)\,z\,dx - E\,I\int z\,\frac{d^2z}{dx^2}\,dx = 0 \qquad (100)$$

$$E\,I\,\frac{\partial U}{\partial K} = -\,4\,Q\,h\,l^{-2}\int (l\,x - x^2)\,z\,dx + 16\,K\,h^2\,l^{-4}\int (l\,x - x^2)^2\,dx + 4\,E\,I\,h\,l^{-2}\int (l\,x - x^2)\,\frac{d^2z}{dx^2}\,dx = 0 \qquad (101)$$

Eqs. 100 and 101 are satisfied by the trivial solution $z = 0$, $K = 0$, $Q =$ any value. Non-trivial solutions exist, however. To find them, the following forms of z are suitable:

$$z = z_n \sin\frac{n\,\pi\,x}{l} \qquad \text{for } n = 2, 4, 6, \cdots \qquad (102)$$

and

$$z = z_n\left(\sin\frac{n\,\pi\,x}{l} - \frac{1}{n}\sin\frac{\pi\,x}{l}\right) \qquad \text{for } n = 3, 5, 7, \cdots \qquad (103)$$

Both satisfy the requirement in Eqs. 93.

With z as in Eq. 102 the solution is

$$K = 0, \qquad -\,Q = Q_n = n^2\,\pi^2\,\frac{E\,I}{l^2}, \qquad \text{for } n = 2, 4, \cdots \qquad (104)$$

This result could have been found more easily, without invoking the method of complementary energy, by noting that Eq. 102 is the solution of Eq. 98 with $y = 0$, and with K and Q as in Eq. 104.

It is when the diagram of deflections is symmetrical, as by Eq. 103, that the method of complementary energy shows merit; because Eq. 103 will be close, although not identical, to an exact solution of Eq. 98 with $y = 0$. With z as in Eq. 103, Eqs. 100 and 101 become, after multiplication by $\frac{2}{l}$:

$$(1 + n^{-2})\,z_n^2\,Q + 32\,(n^{-1} - n^{-3})\,\pi^{-3}\,h\,z_n\,K + (n^2 + n^{-2})\,\pi^2\,E\,I\,z_n^2\,l^{-2} = 0 \qquad (105)$$

and

$$32\,(n^{-1} - n^{-3})\,\pi^{-3}\,h\,z_n\,Q + \frac{16}{15}\,h^2\,K = 0 \qquad (106)$$

Eq. 106 shows that K is proportional to Q but also proportional to the small ratio $\frac{z_n}{h}$; therefore K is a small supplementary force, although its influence may not be small. Elimination of K gives

$$[1 + n^{-2} - 960\,(n^{-1} - n^{-3})^2\,\pi^{-6}]\,Q = -\,(n^2 + n^{-2})\,\pi^2\,E\,I\,l^{-2} \qquad (107)$$

Since π^6 is very close to 960, and n^{-6} will be insignificant, the left side of Eq. 107 can be restated as $(1 + 2\,n^{-4})\,Q$. A further simplification, representing no significant loss of accuracy, reduces the solution to the form

$$-\,Q = Q_n = (n^2 - n^{-2})\,\pi^2\,\frac{E\,I}{l^2}, \qquad \text{for } n = 3, 5, 7, \cdots \qquad (108)$$

Exact values of Q_n for $n = 3, 5, 7, \cdots$ can be found by solving Eq. 98 with $y = 0$ and M as in Eq. 94. It is found that these values are defined by the equations

$$\tan\alpha = \alpha + \frac{\alpha^3}{3} \qquad (109a)$$

and

$$Q_n = 4\,\alpha^2\,\frac{E\,I}{l^2} \qquad (109b)$$

Furthermore, it is found that Eq. 108 gives a very close approximation to these exact values.

This completes the study of the astatic actions, but the goal is the heterostatic action. This goal may be reached either by the route of the orthostatic action or by the route of the relaxed action, in which all stiffness identified with elasticity is removed. The latter route is chosen.

The stabilizing load Q consists of the two horizontal forces Q in Fig. 4 and the distributed load w which is proportional to Q. An added vertical load is an orthostatic load if it requires no change of Q in the relaxed action. Let M_0 denote the bending moments that such a load would produce in a simple beam replacing the bridge. Simple considerations of statics show that in the relaxed action this load produces the deflections

$$Y = \frac{M_0}{Q} \quad \ldots\ldots (110)$$

By referring to Eqs. 93 one obtains the condition

$$\int_0^l M_0 \, dx = 0 \quad \ldots\ldots (111)$$

The loads in Fig. 5 are ascertained as two examples of orthostatic loads by noting that the moments which they would produce in a simple beam satisfy

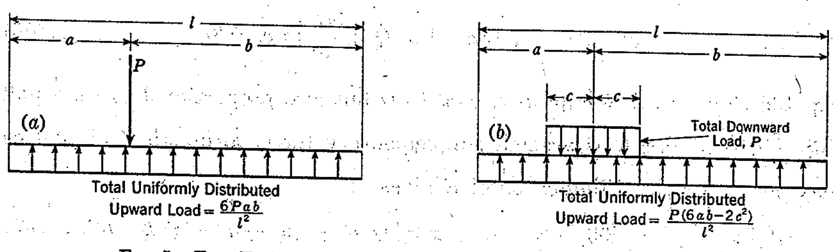


FIG. 5.—TWO EXAMPLES OF ORTHOSTATIC LOADS ON SUSPENSION BRIDGE

Eq. 111. It is observed further that any vertical load can be resolved into two component parts: one, an orthostatic load with moments satisfying Eq. 111; and the other a uniformly distributed load contributing to the astatic load Q. Therefore the function Y can be determined readily for any vertical load.

The next step is to express Y as a sum of functions representing deflections in astatic actions. The functions in Eqs. 102 and 103 are sufficiently close to the exact functions to serve this purpose. A suitable procedure is to expand the function Y into a Fourier series of the form

$$Y = \sum_{1,2,3,\ldots}^{n} Y_n \sin \frac{n \pi x}{l} \quad \ldots\ldots (112)$$

with the coefficients

$$Y_n = \frac{2}{l} \int_0^l Y \sin \frac{n \pi x}{l} \, dx \quad \ldots\ldots (113)$$

It may be convenient instead to begin by expressing the orthostatic load by a Fourier series, and then integrating twice to obtain M_0. In either case the series in Eq. 112 is obtained by a straight-forward and well-established process.

It is assured in advance that $\int_0^l Y \, dx = 0$. It follows that

$$Y_1 = - \sum_{3,5,\ldots}^{n} n^{-1} Y_n \quad \ldots\ldots (114)$$

Then Eq. 112 may be rewritten in the desired form

$$Y = \sum_{2,4,\ldots}^{n} Y_n \sin \frac{n \pi x}{l} + \sum_{3,5,\ldots}^{n} Y_n \left(\sin \frac{n \pi x}{l} - n^{-1} \sin \frac{\pi x}{l} \right) \quad \ldots (115)$$

The transition from the relaxed to the heterostatic action is made by applying the reduction factor in Eq. 81 to each term in Eq. 115, with Q_n defined by Eqs. 104 and 108. This reduction produces the deflections in the heterostatic action,

$$\eta = \sum_{2,4,\ldots}^{n} \frac{Q}{Q_n + Q} Y_n \sin \frac{n \pi x}{l} + \sum_{3,5,\ldots}^{n} \frac{Q}{Q_n + Q} Y_n \left(\sin \frac{n \pi x}{l} - \frac{1}{n} \sin \frac{\pi x}{l} \right) \quad \ldots\ldots (116)$$

By introducing the notation

$$k = \frac{Q l^2}{\pi^2 E I} \quad \ldots\ldots (117)$$

and referring to Eqs. 104 and 108, which define the critical values Q_n of $-Q$, the final deflection, η in Eq. 116, can be restated in either of the following two forms:

$$\eta = - \sin \frac{\pi x}{l} \sum_{3,5,\ldots}^{n} \frac{k}{n(n^2 - n^{-2} + k)} Y_n + \sum_{2,4,\ldots}^{n} \frac{k}{n^2 + k} Y_n \sin \frac{n \pi x}{l} + \sum_{3,5,\ldots}^{n} \frac{k}{n^2 - n^{-2} + k} Y_n \sin \frac{n \pi x}{l} \quad \ldots (118)$$

or,

$$\eta = Y + \sin \frac{\pi x}{l} \sum_{3,5,\ldots}^{n} \frac{n^2 - n^{-2}}{n(n^2 - n^{-2} + k)} Y_n - \sum_{2,4,\ldots}^{n} \frac{n^2}{n^2 + k} Y_n \sin \frac{n \pi x}{l} - \sum_{3,5,\ldots}^{n} \frac{n^2 - n^{-2}}{n^2 - n^{-2} + k} Y_n \sin \frac{n \pi x}{l} \quad \ldots (119)$$

After a certain value of n the series in Eq. 118 converge more rapidly than those in Eq. 119. For numerical computation the choice between the two formulas depends on the values of k and Y_n and on the accuracy that is desired.

Influence of Stabilizing Loads on Vibrations

A stabilizing load stiffens a structure. Therefore it will tend to reduce the periods of free vibration.

A mode of free vibration is characterized by the condition that all the significant deformations are proportional at each instant to a single parameter v which varies with the time t according to the law $v = u \cos \omega_n t$; u may have any value, is independent of t, and is a parameter corresponding to v but defining the extreme deformations.

It happens in some significant cases that the parameter u can be taken as any one of the astatic parameters u_n in Eq. 68. Then, by referring to Eq. 68, it will be seen that the part of the potential energy that varies during the vibration is $\frac{1}{2}(Q_n + Q)\, u_n^2 \cos^2 \omega_n t$. Since the velocities are proportional to $\omega_n u_n \sin \omega_n t$, the kinetic energy can be written as $\frac{1}{2} K_n \omega_n^2 u_n^2 \sin^2 \omega_n t$. Constancy of the combined potential and kinetic energy requires that

$$K_n \omega_n^2 = Q_n + Q \qquad (120)$$

If the stabilizing load Q is removed without changing the masses, ω_n and the corresponding period $t_n = \dfrac{2\pi}{\omega_n}$ will assume different values ω_{0n} and t_{0n}. Eq. 120 shows that the relations of the values are

$$\frac{t_n^2}{t_{0n}^2} = \frac{\omega_{0n}^2}{\omega_n^2} = \frac{Q_n}{Q_n + Q} \qquad (121)$$

That is, in the transition from a free vibration without the stabilizing load to one with the stabilizing load a reduction factor is applied to the square of the period equal to the reduction factor in Eq. 80, representing the transition from the orthostatic action to the heterostatic action under the stabilizing load Q.

Free Vibrations of a Suspension Bridge

While the astatic actions of a suspension bridge may be visualized by imagining gravity reversed or the bridge turned upside down, correspondingly, the free vibrations without stabilizing loads may be visualized by imagining gravity removed temporarily or the bridge merely keeled over so that cable and stiffening truss are placed in a horizontal plane.

The discussion will be limited again to the simplified case that has been under consideration. Furthermore, only vibrations in the plane of the cable and stiffening truss will be considered; and energies due to components of velocity in the direction of the span will be assumed to be relatively so insignificant that they can be ignored. Then the modes of vibration will be represented adequately by Eqs. 102 and 103. When the procedure leading to Eq. 57 is applied to Eq. 103, one finds that in the modes of $n = 3, 5, 7, \cdots$ the bending moments due to the load $w\,z$ are represented not exactly but with good approximation by the formula

$$X = \frac{w\, u_n\, l^2}{n^2 \pi^2}\left(\sin\frac{n\pi x}{l} - \frac{1}{n}\sin\frac{\pi x}{l}\right) \qquad (122)$$

By referring to Eq. 57, one finds, for $n = 3, 5, 7, \cdots$,

$$\omega_{0n}^2 = -\frac{\displaystyle\int X \frac{d^2 z}{dx^2}\,dx}{\displaystyle\int \frac{X^2\,dx}{E\,I\,g}}$$

$$= \frac{n^2\pi^4 E\,I\,g \displaystyle\int_0^l \left(\sin\frac{n\pi x}{l} - \frac{1}{n}\sin\frac{\pi x}{l}\right)\left(n^2 \sin\frac{n\pi x}{l} - \frac{1}{n}\sin\frac{\pi x}{l}\right)dx}{w\,l^4 \displaystyle\int_0^l \left(\sin\frac{n\pi x}{l} - \frac{1}{n}\sin\frac{\pi x}{l}\right)^2 dx}$$

$$= \frac{(n^6 + n^2)\,\pi^4\,E\,I\,g}{(n^2+1)\,w\,l^4} \qquad (123)$$

or, by substitution from Eq. 108 and with a further permissible approximation,

$$\omega_{0n}^2 = \frac{n^2\pi^2 g}{(1 + n^{-2} - 2\,n^{-4})\,w\,l^2}\,Q_n \qquad \text{for } n = 3, 5, 7, \cdots \qquad (124)$$

A similar but simpler computation based on Eq. 102 gives

$$\omega_{0n}^2 = \frac{n^2\pi^2 g}{w\,l^2}\,Q_n \qquad \text{for } n = 2, 4, 6, \cdots \qquad (125)$$

When the suspension bridge is again placed in its natural position and gravity contributes stiffness, the values of ω_{0n}^2 in Eqs. 124 and 125 will be increased to ω_n^2 in accordance with Eq. 121 by replacing the factors Q_n by $Q_n + Q$. Thereafter the periods of free vibration are obtained as $t_n = \dfrac{2\pi}{\omega_n}$. With the critical loads Q_n substituted from Eqs. 104 and 108, and the ratio k introduced from Eq. 117, the periods of free vibration become

$$t_n = \frac{2\,l^2}{n\pi}\sqrt{\frac{w}{(n^2+k)\,E\,I\,g}} \qquad \text{for } n = 2, 4, 6, \cdots \qquad (126)$$

and

$$t_n = \frac{2\,l^2}{n\pi}\sqrt{\frac{(1 + n^{-2} - 2\,n^{-4})\,w}{(n^2 + n^{-2} + k)\,E\,I\,g}} \qquad \text{for } n = 3, 5, 7, \cdots \qquad (127)$$

Conclusion

It has been shown that the method of complementary energy can be applied with advantage to a variety of problems in structural statics and dynamics.

DISCUSSION

I. K. SILVERMAN,[29] ASSOC. M. AM. SOC. C. E. (by letter).—The ideas presented in this paper throw a great deal of light on a concept that seems to have become axiomatic in many minds. This concept is one that deals with the economy of Nature, and it is expressed in textbooks dealing with the theory of structures by means of the "Principle of Least Work." Perhaps it is because the engineer, in dealing with forces, has come to have a respect for the way Nature behaves. The behavior of liquids, the action of falling bodies, the structure of organic materials such as bones and feathers,[30] the apparent minimal surfaces that may be obtained with thin films, cells of honeycombs, etc., all seem to be conclusive evidence of the economy of Nature.

It is a far cry from the Ptolemaic concept of the universe to the Principle of Least Work of modern structural analysis, but the conflict between the theological and mechanical concepts of the universe, which raged around the Ptolemaic theory, imparted a metaphysical twinge to the ideas of the philosophers who followed Galileo.

In their studies of the phenomena of Nature the investigators of the seventeenth and eighteenth centuries attempted to explain them from the inherent perfection of the Creator. It is not strange, therefore, to find the concepts of minimum arising from the efforts of the philosophers of this age to explain the universe. So great has been the influence of these precursors of modern science that their "minimum principles" have come down through the centuries to be included in modern textbooks with even the argument of the wisdom of the Creator in calling for economy in Nature. Such is the background of the principle of the minimum in the theory of structures.

Jules Henri Poincaré declared that any generalization which is based on experimental results has no right to be called a principle, and an examination of the assumptions which led to Castigliano's second theorem and thence to what is generally, but incorrectly, called the Principle of Least Work will bear this out. The basic assumptions are as follows, as applied to engineering structures:

1. The deformations produced are infinitesimal and thus it follows that the law of superposition holds; and
2. A linear relation exists between the forces and displacements.

From these two assumptions the energy stored in a body can be expressed as a quadratic in the forces or the displacements. Castigliano's first theorem follows: The partial derivative of the expression for energy containing the forces with respect to any force gives the displacement of the force in the direction it acts; and, for those forces which do not move because of their geometrical or kinematical restraints, the partial derivatives are zero. This latter result can be expressed as a theorem interpreting the vanishing of the derivative as a condition for a minimum. This theorem of "minimum" will not hold for the cases in which (*a*) the law of superposition is no longer true, and (*b*) the material does not follow Hooke's law.

[29] Care, Panama Canal, Canal Zone.

[30] "The Science of Mechanics," by E. Mach, The Open Court Publishing Co., Chicago, Ill., 1893, p. 452.

These conditions are purely experimental, and in the light of Poincaré's specifications the term "principle" is inadmissible. Most of the materials used in structural engineering may be considered to satisfy the conditions under which the theorem holds, and the results obtained in practise agree essentially with those predicted by the application of the theorem.

The author has introduced a new term, complementary energy, which is certainly not of anthropomorphic origin. It is nothing more than a mathematical expression involving stresses or forces which, upon differentiation with respect to the stress or force, gives the corresponding strain or displacement. When the forces involved are constrained kinematically or geometrically the ensuing derivative is set equal to zero, which is interpreted as a minimum. The author has shown that in the case of Hooke's law "complementary energy" coincides with "elastic energy."

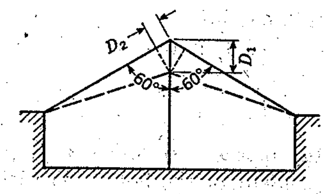


FIG. 6

FIG. 7

The only check on the results obtained by using either the "law of least complementary energy" or the "law of least elastic energy" is a geometrical one. In both cases the equations of equilibrium are first applied and when dealing with a statically indeterminate structure a "minimum principle" is used to determine the statically indeterminate quantities. In the last analysis the results obtained can be checked only by noting that the conditions of continuity are preserved. Thus the three solutions for S_1 of Fig. 2(*a*) can be shown to be correct as follows:

From Fig. 6 the deformation D_1 is connected with the deformations D_2 and D_3 by the following equation:

$$D_1 = \frac{D_2}{\cos 60°} = 2\,D_2 \qquad \text{(128a)}$$

For $P < 3\,C$,

$$D_1 = (P - X)\,k = 2\,k\,X = 2\,D_2; \quad \text{and} \quad X = \frac{P}{3} \qquad \text{(128b)}$$

For $3\,C < P < 5\,C$,

$$D_1 = 2\,k\,(P - X - C) = 2\,k\,X = 2\,D_2; \quad \text{and} \quad X = \frac{P - C}{2} \qquad \text{(128c)}$$

and, for $P > 5\,C$,

$$D_1 = 2\,k\,(P - X - C) = 4\,k\,(X - C) = 2\,D_2; \quad \text{and} \quad X = \frac{P + C}{3} \qquad \text{(128d)}$$

The fundamental character of a solution based on consistent deflections is apparent; but it must be admitted that a solution based on a "minimum" principle as presented by the author will prove more attractive to analysts because of its economy of thought.

Stability Problems.—The essential feature in the study of stability problems is that the structure in question is assumed to have a geometrical configuration entirely different from its original state. The equations of equilibrium are set up from this strained state rather than from the unstrained state as is done in all other problems of equilibrium. The question then becomes: What must be the value of the forces so that the strained state may be maintained? Two general methods are available that can be shown to be identical.

The first method involves the problem of the elastica—that is, the solution of the differential equation that describes the possible states of equilibrium. The second method, the energy method, is based on the fact that, for unstable equilibrium, the potential energy of a system is a maximum and any virtual variation from that state involves a decrease in potential energy. The criterion by which the conditions of unstable equilibrium are determined is that, in the transition from the actual state to one infinitely close to it, the variation of the potential energy is zero; that is,

$$\delta T = 0 \qquad (129)$$

in which T is the potential energy of the system. One important feature of this statement is that it is independent of the law of elasticity and holds for large displacements.

Instead of using the potential energy of the structure, the author has substituted for it a "complementary energy" and claims superiority of the method of "complementary energy" over the potential energy by showing that, by the use of Eq. 23, based on "complementary energy," a closer value to the buckling load can be obtained than by using Eq. 27, which is based on potential energy.

The reason for this apparent superiority lies in the fact that the author has satisfied all boundary conditions by using the expression $M = P\,z$ for the moment at any point whereas Eq. 27 utilizes the more general term $M = E\,I\,\dfrac{d^2z}{dx^2}$. When Eq. 27 is to be used the approximating function must satisfy all the boundary conditions and using Eq. 24 would naturally give a poor result. The Timoshenko variant takes this fact into account by writing

$$P\,z = M = E\,I\,\frac{d^2 z}{dx^2} \qquad (130)$$

If an expression that satisfies all boundary conditions is used, Eq. 27 furnishes a more accurate result than does Eq. 23 when Eq. 24 is used. For example, assume

$$z = c\,(l^3\,x - 2\,l\,x^3 + x^4) \qquad (131)$$

which satisfies all boundary conditions. From Eq. 27

$$P = \frac{168}{17}\,\frac{E\,I}{l^2} = 9.8824\,\frac{E\,I}{l^2}.$$

Substituting this expression in Eq. 23 or Eq. 28 will yield as a result

$$P = \frac{306}{31}\,\frac{E\,I}{l^2} = 9.8797\,\frac{E\,I}{l^2} \qquad (132)$$

As a matter of fact Eq. 27 has a marked superiority over Eq. 23 in that it is perfectly general and will hold under all boundary conditions, whereas Eq. 23 will have to be modified according to the boundary conditions. Consider the case shown in Fig. 7.

Assume a starting shape

$$z = c\,(3\,l\,x^2 - x^3) \qquad (133)$$

which satisfies all boundary conditions. The exact solution for this case is $P = \dfrac{\pi^2}{4}\,\dfrac{E\,I}{l^2}$. From Eq. 27, $P = 2.5\,\dfrac{E\,I}{l^2}$, whereas Eq. 23 gives the impossible value of $P = -\dfrac{14}{11}\,\dfrac{E\,I}{l^2}$. Evidently Eq. 23 must be modified to take the free end into account.

Eq. 27 was derived by equating Eq. 26 to zero, but an equivalent and (in the writer's opinion) more fundamental expression may be derived by the application of Eq. 129.

From Eq. 26

$$T = \frac{1}{2}\int_0^l \left[E\,I\left(\frac{d^2z}{dx^2}\right)^2 - P\left(\frac{dz}{dx}\right)^2\right] dx \qquad (134)$$

The variation of this integral is a problem in the Calculus of Variations, and on performing the variation the following result is obtained:

$$\delta T = \int_0^l \left[E\,I\,\frac{d^4z}{dx^4} + P\,\frac{d^2z}{dx^2}\right]\delta z\,dx - \left[\left(P\,\frac{dz}{dx} + E\,I\,\frac{d^3z}{dx^3}\right)\delta z\right]_0^l + E\,I\left[\frac{d^2z}{dx^2}\,\delta\left(\frac{dz}{dx}\right)\right]_0^l = 0 \qquad (135)$$

The terms in the brackets are called "the terms at the limits," and when they are zero Eq. 135 becomes

$$\int_0^l \left[E\,I\,\frac{d^4z}{dx^4} + P\,\frac{d^2z}{dx^2}\right]\delta z\,dx = 0 \qquad (136)$$

Since the variation δz is arbitrary, Eq. 136 is fulfilled when

$$E\,I\,\frac{d^4z}{dx^4} + P\,\frac{d^2z}{dx^2} = 0 \qquad (137a)$$

Eq. 137*a* may be integrated to

$$E\,I\,\frac{d^2z}{dx^2} + P\,z = 0 \qquad (137b)$$

which is nothing more than the differential equation given by Leonhard Euler. From this demonstration may be seen the connection between the problem of the elastica and the energy method.

The approximate solution of all problems in mechanics based on the variation of an integral can be simplified greatly, as follows: When the boundary conditions are such that the "terms at the limits" are zero, Eq. 135 becomes Eq. 136. In general, Eq. 136 may be written[31] as

$$\int_0^l [\text{Differential equation of equilibrium}] \times [\text{Virtual change of the function describing the state of equilibrium}]\, dx = 0 \quad \ldots\ldots (138)$$

When the "terms at the limits" are not zero an equation of the form of Eq. 135 must be dealt with. It must be remembered that Eq. 135 is the result of the variation of the integral of potential energy of a straight column subjected to end loads. Any other condition of loading will require a slightly different expression.

Consider the case shown by Fig. 3(*a*). The application of Eq. 135 to this case is as follows:

Since the ends are held, the terms at the limits vanish leaving Eq. 136 or 138. Assuming the value of z in Eq. 131, which satisfies all boundary conditions:

$$\frac{dz}{dx} = c\,(l^3 - 6\,l\,x^2 + 4\,x^3); \quad \frac{d^2z}{dx^2} = 12\,c\,(x^2 - l\,x) \quad \ldots\ldots (139a)$$

$$\frac{d^3z}{dx^3} = 12\,c\,(2\,x - l); \quad \frac{d^4z}{dx^4} = 24\,c \quad \ldots\ldots (139b)$$

and,

$$\delta z = \delta c\,(l^3\,x - 2\,l\,x^3 + x^4) \quad \ldots\ldots (139c)$$

Substituting in Eq. 136

$$\int_0^l [E\,I\,24\,c + 12\,P\,c\,(x^2 - l\,x)][l^3\,x - 2\,l\,x^3 + x^4]\,\delta c\,dx = 0 \quad \ldots (140)$$

Performing the integrations it will be found that

$$P = \frac{168}{17}\frac{E\,I}{l^2} \quad \ldots\ldots (141)$$

The differential equation of the strained column is given by Eq. 137*b*. From Eq. 138

$$\int_0^l \left[E\,I\,\frac{d^2z}{dx^2} + P\,z\right]\delta z\,dx = 0 \quad \ldots\ldots (142)$$

Substituting Eqs. 139 and performing the integrations will lead to

$$P = \frac{306}{31}\frac{E\,I}{l^2} \quad \ldots\ldots (143)$$

For the case shown by Fig. 7, Eq. 138 cannot be used and Eq. 135 including the "terms at the limits" must be used.

The same general method can be applied to vibrating systems.[32]

[31] "Eine Wichtige Vereinfachung der Methode von Ritz zur angenaherten Behandlung von Variationsproblemen," by H. Hencky, *Zeitschrift für angew. Mathematik und Mechanik*, Vol. 7, 1927, p. 80.

[32] "On Forced Pseudo-Harmonic Vibrations," by I. K. Silverman, *Journal of the Franklin Institute*, June, 1934, p. 743.

One point which should be emphasized in the foregoing approximations is that the critical loads and their analogous quantities in vibrating systems are always larger than the actual values and, therefore, are necessary but not sufficient magnitudes. Therefore, these approximate values are on the unsafe side and other methods[33] should be used to determine the lower limit.

Fundamentally there is no difference between the method presented by the author as expressed by Eq. 11 and that as expressed by Eq. 138. The writer believes that the differential equation or the energy expression of a given problem can be "set up" with greater ease than the deflections, displacements, and changes of curvature required by Eq. 10. Furthermore, all reference volumes on the theory of structures present the general theory of equilibrium via the differential equation and the investigator will find Eq. 138 quite easy to handle.

The method based on Eq. 138 also may be used in the approximate solution of problems of stable equilibrium (for example, in the determination of the deflections of beams, etc.). One important problem to which it has been applied is that dealing with the action of a suspension bridge subjected to lateral forces and to the determination of the natural frequencies of this type of structure when vibrating in a horizontal plane.

Under the action of a uniform wind load the deflection of a suspension bridge can be expressed as

$$\delta = \sum_{n=1,3,5\ldots} \frac{a_n\,l^4}{E\,I}\sin\frac{n\,\pi\,x}{l} \quad \ldots\ldots (144)$$

The moment and shear in the horizontal system may be determined from $M = E\,I\,\frac{d^2\delta}{dx^2}$; and shear $= E\,I\,\frac{d^3\delta}{dx^3}$. Eq. 138 may be used to determine the values of a_n. A set of simultaneous equations is obtained given by the general formula

$$a_n\,[(\pi\,n)^4\,A_1 + (\pi\,n)^2\,A_2 - A_3] - \sum_i a_i\,\frac{A_4\,(n\,i)^5}{(n-i)^2\,(n+i)^2} = \frac{A_5}{(n\,\pi)^3} - \frac{A_6}{n\,\pi}, \quad \text{for } n = 1, 3, 5, \cdots \quad \ldots\ldots (145)$$

in which i may have any odd integral value except n, depending upon the number of terms assumed in the series, Eq. 144; and A_1 to A_6 are constants of the structure and loading. Thus a direct solution is furnished by use of Eq. 138 to replace the tedious trial and error solution now available.[34]

George R. Rich,[35] M. Am. Soc. C. E. (by letter).—Dean Westergaard has prepared a valuable and interesting paper which, for maximum benefit, should be read as a sequel to his earlier paper on the "Buckling of Elastic Structures,"[36] from which the somewhat formidable terms "orthostatic action,"

[33] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., 1936, p. 84; also "Rayleigh's Principle," by G. F. J. Temple and W. G. Bickley, Oxford Press, 1933, p. 29 and p. 72.

[34] "Suspension Bridges Under the Action of Lateral Forces," by Leon S. Moisseiff and Frederick Lienhard, Members, Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1080.

[35] Chf. Design Engr., TVA, Knoxville, Tenn.

[36] "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 576.

"astatic action," and "heterostatic action" emerge as the more familiar expressions "simple stress or bending," "pure buckling," and "mixed buckling."

In determining critical buckling loads or natural periods of vibration, it appears from the examples adduced by the author that the method of complementary energy affords much greater latitude than the Rayleigh method in the selection of functions to represent the deflected elastic line. This is particularly striking in the author's use of parabolic loci in the cases of the buckling of hinged-end columns and the lateral vibration of prismatic beams. In the Rayleigh type of investigation, parabolic curves do not ordinarily yield dependable results for the reason that the resultant expression for curvature of the member, $\frac{d^2z}{dx^2}$, is a constant, whereas the curvature should vary with the bending moment. In problems similar to the analysis of vibrations in suspension bridges, in which the choice of functions naturally gravitates to certain standard trigonometric forms or Fourier expressions, the Rayleigh method appears to have the advantages of greater simplicity and directness without any considerable sacrifice of accuracy.

For example, Eqs. 125 and 126 may be verified exactly, Eq. 124 checked very closely, and Eq. 127 checked within 10% by direct substitution, in a single operation, of the functions

$$z = z_n \sin \frac{n \pi x}{l} \sin \omega t, \qquad \text{for } n = 2, 4, 6, \cdots \ldots \ldots \ldots \quad (146a)$$

and

$$z = z_n\left(\sin \frac{n \pi x}{l} - \frac{1}{n} \sin \frac{\pi x}{l}\right) \sin \omega t, \qquad \text{for } n = 3, 5, 7, \cdots \ldots \ldots \quad (146b)$$

in the standard Rayleigh form of the energy equation:

$$\int_0^l \frac{w}{2g}\left(\frac{dz}{dt}\right)^2 dx = \int_0^l \frac{EI}{2}\left(\frac{d^2z}{dx^2}\right)^2 dx + \int_0^l \frac{Q}{2}\left(\frac{d^2z}{dx^2}\right) z\, dx \ldots (147)$$

In fact, Eqs. 125 and 126 may be verified exactly by using the somewhat rougher form of Rayleigh expression:

$$\int_0^l \frac{w}{2g}\left(\frac{dz}{dt}\right)^2 dx = \int_0^l \frac{EI}{2}\left(\frac{d^2z}{dx^2}\right)^2 dx + \int_0^l \frac{Q}{2}\left(\frac{dz}{dx}\right)^2 dx \ldots (148)$$

in which the term $\int_0^l \frac{Q}{2}\left(\frac{dz}{dx}\right)^2 dx$ reflects the assumption that the increased length of cable due to the vibratory motion is equal to the increased length of elastic line of the deflecting stiffening truss. In using the Rayleigh method for this problem, the x-axis is taken as the undeflected axis of the stiffening truss, and all ordinates z are measured to the deflected elastic line of the truss. No ordinates are measured to the cable. The functions adopted for z simply represent the idea that the stiffening truss deflects during vibration into either an even number of alternating equal bay waves or an odd number of alternating unequal bay waves, the configurations in either case being such that the net departure of the cable during vibratory motion from the parabolic form in equilibrium under the dead load is a minimum. This action is implied by

Eqs. 93. In addition, these functions represent the familiar normal modes in which the oscillations of all elements of the elastic system are either in phase or in opposition, and the functions chosen have the convenient conjugate orthogonal property

$$\int \sin m x \sin n x \, dx = 0, \qquad \text{for } m \neq n \ldots \ldots \ldots \ldots (149)$$

In this connection, it is recognized that the author's primary purpose is to give examples of sufficient variety to demonstrate the power and range of the method of complementary energy. He is well aware, of course, that in certain instances alternative methods may prove equally convenient, as is shown by his statement that the problem of buckling of a thin circular disk supported at the edges and loaded by a normal edge load may be solved with equal ease by the standard method of attack using Bessel functions.

Dean Westergaard has rendered continued service to practicing engineers by demonstrating the great economy of time and labor, and the penetrating insight into structural action, that are afforded by the use of the more incisive methods of advanced analysis. Without attempting the mastery of existence theorems, reasonable facility in the use of the variational principles may be acquired for a surprisingly small expenditure of time; and, as remarked by Lord Rayleigh sixty years ago,[18] the difficulty in connection with Fourier series lies not in its practical physical applications, but only in a rigorous demonstration of what the professional mathematician means by the statement that the expansion represents the function "almost everywhere."

R. V. Southwell,[37] Esq. (by letter).—By this communication the writer wishes to record his appreciation of Dean Westergaard's very clear presentation of the "Method of Complementary Energy." The use of a quasi-energy function, defined as in Eq. 9, was propounded in one of the contributions to a symposium of engineering mechanics held at Ann Arbor, Mich., in the summer of 1935. The writer does not remember, however, that the notion was then attributed by any speaker to Engesser, and he was not aware that it had been propounded already. It must be of interest to all teachers who are concerned to understand the fundamentals of stress-strain theory.

Whether it will have equal appeal to the engineer whose first concern is to calculate correctly, only time can show. Use has so familiarized the concepts of kinetic and potential energy that as an investigator one is kept straight, so to speak, in one's working by a kind of physical intuition, and as a teacher one is likely to regard those concepts as natural and easy. It is salutary, perhaps, to find how groping are one's first attempts to use the complementary notion: One may thereby be made more sympathetic to the difficulties of students! Clearly, time must be found for a serious attempt to master the new technique, and to assess its sphere of usefulness.

Meanwhile, the writer can offer only one comment in regard to the comparison of methods given under the headings "Buckling of Column with Hinged Ends" and "Application to Vibration of Beams." Dean Westergaard has

[37] Prof. of Eng. Science, Oxford Univ., Oxford, England.

given two examples in which the error resulting from a use of Rayleigh's principle exceeds 20%. The writer suggests that these examples are not really fair to the principle, for the reason that modes such as Eqs. 24 and 61 are excluded (tacitly) in the argument by which it is derived. That argument is based on the concept of conservation of energy, and a formula such as Eq. 27 presumes that no energy can enter or leave the system. A mode in which both slope and curvature are non-zero at the ends of a strut, however, must entail there either input or output of energy. Neglecting this "leakage," it is not surprising that one should obtain seriously inaccurate results. Similar remarks apply to the second example.

The writer's reason for emphasizing this point is that an analogous condition for close accuracy should presumably be recognized when the complementary method is used. It would be worth while to examine whether this is so, and whether (for example) it excludes the use of a mode as shown in Fig. 3(*b*). In fact, when two methods have such close similarity, it is to be expected that every precaution that must be taken in using the first has its analogue in a precaution that must be taken in using the second. Provided that such precautions have been stated and are generally recognized, the method of complementary energy seems likely to play an important part in the study of plastic distortions, where the notion of elastic strain-energy has such limited utility.

H. M. Westergaard,[38] M. Am. Soc. C. E.—The writer is indebted to Mr. Rich and Professor Southwell for their comments, but he cannot accept the arguments presented by Mr. Silverman.

For example, Mr. Silverman notes that Eq. 23 does not apply well to a cantilever column with the axis of x along the original center line; but one would not write Eq. 23 for that case. The bending moment for the starting shape in this case is not $P\,z$ but $P(z - z_{max})$, which, with z as in Eq. 133, gives

$$P = \frac{42}{17}\frac{E\,I}{l^2} \qquad (150)$$

The value in Eq. 150 differs only by 0.13% from the true value,

$$P = \frac{\pi^2}{4}\frac{E\,I}{l^2} \qquad (151)$$

and is, in fact, a satisfactory result. Mr. Silverman uses the calculus of variations to derive Eq. 137*b*, but it should be observed that this equation does not apply to the cantilever column with the axis of x along the original center line. In Mr. Silverman's Eq. 138, "Differential equation of equilibrium" is a meaningless factor. If one guesses that "one side of a differential equation" is intended instead of "differential equation," the statement in Eq. 138 still would require a clear explanation of the relation of the mathematical forms to the actualities of structural mechanics. Mr. Silverman's Eq. 145 for vibrations of a suspension bridge contains undefined constants and is only a vague suggestion of a solution. His comment that Castigliano's laws are not

[38] Dean, Graduate School of Eng., and Gordon McKay Prof. of Civ. Eng., Harvard Univ., Cambridge, Mass.

principles can be dismissed as untenable. The entire discussion by Mr. Silverman can be dismissed; it throws no light on the method of complementary energy.

Referring to the understanding comments by Mr. Rich, the writer only wishes to state that he does not advocate the supplanting of other methods by the method of complementary energy; the writer does suggest the inclusion of the method of complementary energy among the tools of structural analysis.

The remarks by Professor Southwell on the importance of avoiding leakage in the account of energy and on the nature of analogies are much to the point. Illustrations surely can be found in which there is greater danger of leakage in the account of the complementary energy than in the corresponding account of the energy. As to analogies, it should be mentioned that they exist in different degrees. In the basic equations involving the energy T and the complementary energy U, in changing from the use of T to the use of U, the statical terms, load P, stress S, and reaction R, are replaced in order by the geometrical terms, path r of a reaction, deformation D, and path p of a load, and vice versa. This statement applies not only to the equations expressing the fact that T and U are minima when the quantities P and r are constant, but applies also to two further equations which can be derived from the principles of minimum and represent the extension of Castigliano's first and second laws. These two further equations apply when the quantities r are admitted as variables defining T, or the quantities P are admitted as variables defining U; and they are:

$$R = \frac{\partial T}{\partial r} \qquad (152)$$

and

$$p = \frac{\partial U}{\partial P} \qquad (153)$$

Thus an analogy is established in which there is a one-to-one correspondence of terms in two pairs of basic equations, but this analogy has not been extended to such a degree of completeness that, given a structure, one could always visualize an analogous structure in which the statical quantities of the original structure appear as geometrical quantities, and vice versa. Some of the well-known specific analogies possess this degree of completeness; for example, the soap-film analogy by which torsional stresses are represented by slopes of a film (which is a structure of a kind); or the analogy by which curvatures of a plate represent a two-dimensional state of stress in the plane of a slice. The comments made by Professor Southwell suggest the observation that the old field of structural mechanics is still open for profitable new exploration.