From "Structural Analysis" by A. Kassimali, 1999.

In this appendix, some basic concepts of matrix algebra necessary for formulating the computerized analysis of structures are briefly reviewed. A more comprehensive and mathematically rigorous treatment of these concepts can be found in any textbook on matrix algebra, such as [11] and [26].

B.1 DEFINITION OF A MATRIX

A matrix is a rectangular array of quantities arranged in rows and columns. A matrix containing m rows and n columns can be expressed as:

$$\mathbf{A} = [A] = \begin{bmatrix} A_{11} & A_{12} & \cdots & & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & & \cdots & A_{2n} \\ \cdots & & \cdots & A_{ij} & \cdots & \\ A_{m1} & A_{m2} & \cdots & | & \cdots & A_{mn} \end{bmatrix} \begin{matrix} \\ \\ i\text{th row} \\ \\ \end{matrix} \qquad \textbf{(B.1)}$$

$$j\text{th column} \qquad m \times n$$

As Eq. (B.1) indicates, matrices are usually denoted either by *boldface letters* (e.g., **A**) or by *italic letters* enclosed within brackets (e.g., $[A]$). The quantities that form a matrix are referred to as the *elements* of the matrix, and each element is represented by a double-subscripted letter, with the first subscript identifying the row and the second subscript identifying the column in which the element is located. Thus in Eq. (B.1), A_{12} represents the element located in the first row and the second column of the matrix **A**, and A_{21} represents the element in the second row and the first column of **A**. In general, an element located in the ith row and the jth column of matrix **A** is designated as A_{ij}. It is common practice to enclose the entire array of elements between brackets, as shown in Eq. (B.1).

The *size* of a matrix is measured by its *order,* which refers to the number of rows and columns of the matrix. Thus the matrix **A** in Eq. (B.1), which consists of m rows and n columns, is considered to be of order $m \times n$ (m by n). As an example, consider a matrix **B** given by

$$\mathbf{B} = \begin{bmatrix} 5 & 21 & 3 & -7 \\ 40 & -6 & 19 & 23 \\ -8 & 12 & 50 & 22 \end{bmatrix}$$

The order of this matrix is 3×4, and its elements can be symbolically represented by B_{ij}, with $i = 1$ to 3 and $j = 1$ to 4; for example, $B_{23} = 19$, $B_{31} = -8$, $B_{34} = 22$, etc.

B.2 TYPES OF MATRICES

Row Matrix

If all the elements of a matrix are arranged in a single row (i.e., $m = 1$), then the matrix is called a *row matrix.* An example of a row matrix is

$$\mathbf{C} = [50 \quad -3 \quad -27 \quad 35]$$

Column Matrix

A matrix with only one column of elements (i.e., $n = 1$) is called a *column matrix*. For example,

$$\mathbf{D} = \{D\} = \begin{bmatrix} -10 \\ 33 \\ -6 \\ 15 \end{bmatrix}$$

Column matrices are also referred to as *vectors* and are sometimes denoted by *italic* letters enclosed within braces (e.g., $\{D\}$).

Square Matrix

A matrix with the same number of rows and columns ($m = n$) is called a *square matrix*. An example of a 3×3 square matrix is

$$\mathbf{A} = \begin{bmatrix} 5 & 21 & 3 \\ 40 & -6 & 19 \\ -8 & 12 & 50 \end{bmatrix} \tag{B.2}$$

Main diagonal

The elements with the same subscripts—that is, $A_{11}, A_{22}, \ldots, A_{nn}$—form the *main diagonal* of the square matrix **A**. These elements are referred to as the *diagonal elements*. As shown in Eq. (B.2), the main diagonal extends from the upper left corner to the lower right corner of the square matrix. The remaining elements of the matrix (i.e., A_{ij} with $i \neq j$) that are not along the main diagonal are termed the *off-diagonal elements*.

Symmetric Matrix

If the elements of a square matrix are symmetric about its main diagonal (i.e., $A_{ij} = A_{ji}$), the matrix is called a *symmetric matrix*. An example of a 4×4 symmetric matrix is

$$\mathbf{A} = \begin{bmatrix} -12 & -6 & 13 & 5 \\ -6 & 7 & -28 & 31 \\ 13 & -28 & 10 & -9 \\ 5 & 31 & -9 & -2 \end{bmatrix}$$

Diagonal Matrix

If all the off-diagonal elements of a square matrix are zero (i.e., $A_{ij} = 0$ for $i \neq j$), the matrix is referred to as a *diagonal matrix*. For example,

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

Unit or Identity Matrix

A diagonal matrix with all its diagonal elements equal to 1 (i.e., $I_{ii} = 1$ and $I_{ij} = 0$ for $i \neq j$) is called a *unit*, or *identity, matrix*. Unit matrices are usually denoted by **I** or $[I]$. An example of a 4 × 4 unit matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Null Matrix

When all the elements of a matrix are zero (i.e., $O_{ij} = 0$), the matrix is called a *null matrix*. Null matrices are commonly denoted by **O** or $[O]$. For example,

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

B.3 MATRIX OPERATIONS

Equality

Two matrices **A** and **B** are equal if they are of the same order and if their corresponding elements are identical (i.e., $A_{ij} = B_{ij}$). Consider, for example, the matrices

$$\mathbf{A} = \begin{bmatrix} -3 & 5 & 6 \\ 4 & 7 & 9 \\ 12 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -3 & 5 & 6 \\ 4 & 7 & 9 \\ 12 & 0 & 1 \end{bmatrix}$$

Since both **A** and **B** are of order 3 × 3 and since each element of **A** is equal to the corresponding element of **B**, the matrices are considered to be equal to each other; that is, $\mathbf{A} = \mathbf{B}$.

Addition and Subtraction

The addition (or subtraction) of two matrices **A** and **B**, which must be of the same order, is carried out by adding (or subtracting) the corresponding elements of the two matrices. Thus if $\mathbf{A} + \mathbf{B} = \mathbf{C}$, then $C_{ij} = A_{ij} + B_{ij}$; and if $\mathbf{A} - \mathbf{B} = \mathbf{D}$, then $D_{ij} = A_{ij} - B_{ij}$. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 0 \\ 8 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 10 & 4 \\ 6 & 7 \\ 9 & 2 \end{bmatrix}$$

then

$$\mathbf{A} + \mathbf{B} = \mathbf{C} = \begin{bmatrix} 12 & 9 \\ 9 & 7 \\ 17 & 3 \end{bmatrix}$$

and

$$\mathbf{A} - \mathbf{B} = \mathbf{D} = \begin{bmatrix} -8 & 1 \\ -3 & -7 \\ -1 & -1 \end{bmatrix}$$

Note that matrices **C** and **D** have the same order as matrices **A** and **B**.

Multiplication by a Scalar

To obtain the product of a scalar and a matrix, each element of the matrix must be multiplied by the scalar. Thus if

$$\mathbf{B} = \begin{bmatrix} 7 & 3 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad c = -3$$

then

$$c\mathbf{B} = \begin{bmatrix} -21 & -9 \\ 3 & -12 \end{bmatrix}$$

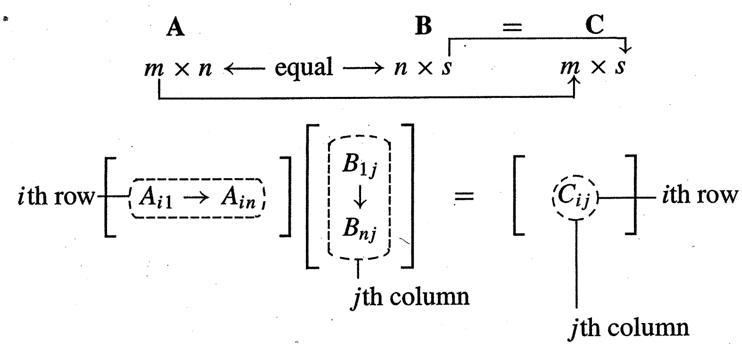
Multiplication of Matrices

The multiplication of two matrices can be carried out only if the number of columns of the first matrix equals the number of rows of the second matrix. Such matrices are referred to as being *conformable* for multiplication. Consider, for example, the matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 5 \\ 7 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 3 & -6 \\ 4 & -8 & 9 \end{bmatrix} \tag{B.3}$$

in which **A** is of order 2 × 2 and **B** is of order 2 × 3. Note that the product **AB** of these matrices is defined, because the first matrix, **A**, of the sequence **AB** has two columns and the second matrix, **B**, has two rows. However, if the sequence of the matrices is reversed, the product **BA** does not exist, because now the first matrix, **B**, has three columns and the second matrix, **A**, has two rows. The product **AB** is usually referred to either as **A** *postmultiplied* by **B** or as **B** *premultiplied* by **A**. Conversely, the product **BA** is referred to either as **B** postmultiplied by **A** or as **A** premultiplied by **B**.

When two conformable matrices are multiplied, the product matrix thus obtained will have the number of rows of the first matrix and the number of columns of the second matrix. Thus if a matrix **A** of order $m \times n$ is postmultiplied by a matrix **B** of order $n \times s$, then the product matrix **C** will be of order $m \times s$; that is,



(B.4)

As illustrated in Eq. (B.4), any element C_{ij} of the product matrix **C** can be evaluated by multiplying each element of the ith row of **A** by the corresponding element of the jth column of **B** and by algebraically summing the resulting products; that is,

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} \tag{B.5}$$

Equation (B.5) can be conveniently expressed as

$$\boxed{C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}} \tag{B.6}$$

in which n represents the number of columns of the matrix **A** and the number of rows of the matrix **B**. Note that Eq. (B.6) can be used to determine any element of the product matrix **C** = **AB**.

To illustrate the procedure of matrix multiplication, we compute the product **C** = **AB** of the matrices **A** and **B** given in Eq. (B.3) as

$$\mathbf{C} = \mathbf{AB} = \underset{2\times 2}{\begin{bmatrix} -1 & 5 \\ 7 & -3 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 2 & 3 & -6 \\ 4 & -8 & 9 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 18 & -43 & 51 \\ 2 & 45 & -69 \end{bmatrix}}$$

in which the element C_{11} of the product matrix **C** is obtained by multiplying each element of the first row of **A** by the corresponding element of the first column of **B** and summing the resulting products; that is,

$$C_{11} = -1(2) + 5(4) = 18$$

Similarly, the element C_{21} is determined by multiplying the elements of the second row of **A** by the corresponding elements of the first column of **B** and adding the resulting products; that is,

$$C_{21} = 7(2) - 3(4) = 2$$

The remaining elements of **C** are determined in a similar manner:

$$C_{12} = -1(3) + 5(-8) = -43$$
$$C_{22} = 7(3) - 3(-8) = 45$$
$$C_{13} = -1(-6) + 5(9) = 51$$
$$C_{23} = 7(-6) - 3(9) = -69$$

Note that the order of the product matrix **C** is 2 × 3, which equals the number of rows of **A** and the number of columns of **B**.

A common application of matrix multiplication is to express simultaneous equations in compact matrix form. Consider the system of simultaneous linear equations:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= P_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= P_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= P_3 \end{aligned} \tag{B.7}$$

in which x_1, x_2, and x_3 are the unknowns and A's and P's represent the coefficients and constants, respectively. By using the definition of matrix multiplication, this system of simultaneous equations can be written in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{B.8}$$

or, symbolically, as

$$\mathbf{Ax} = \mathbf{P} \tag{B.9}$$

Even when two matrices **A** and **B** are of such orders that both products **AB** and **BA** can be determined, the two products are generally not equal; that is,

$$\boxed{\mathbf{AB} \neq \mathbf{BA}} \tag{B.10}$$

It is, therefore, necessary to maintain the proper sequential order of matrices when computing matrix products. Although matrix multiplication is generally not commutative, as indicated by Eq. (B.10), it is associative and distributive, provided that the sequential order in which the matrices are to be multiplied is maintained. Thus

$$\mathbf{ABC} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{B.11}$$

and

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{B.12}$$

Multiplication of any matrix **A** by a conformable null matrix **O** yields a null matrix; that is,

$$\mathbf{OA} = \mathbf{O} \qquad \text{and} \qquad \mathbf{AO} = \mathbf{O} \tag{B.13}$$

For example,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiplication of any matrix **A** by a conformable unit matrix **I** yields the same matrix **A**, that is,

$$\mathbf{IA} = \mathbf{A} \qquad \text{and} \qquad \mathbf{AI} = \mathbf{A} \tag{B.14}$$

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ 9 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 5 & -7 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ 9 & 2 \end{bmatrix}$$

As Eqs. (B.13) and (B.14) indicate, the null and unit matrices serve the purposes in matrix algebra that are analogous to those of the numbers 0 and 1, respectively, in scalar algebra.

Inverse of a Square Matrix

The inverse of a square matrix **A** is defined as a matrix $\mathbf{A}^{-1}$ with elements of such magnitudes that the multiplication of the original matrix **A** by its inverse $\mathbf{A}^{-1}$ yields a unit matrix **I**; that is,

$$\boxed{\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}} \tag{B.15}$$

Consider, for example, the square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

The inverse of **A** is given by

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix}$$

so that the products $\mathbf{A}^{-1}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{-1}$ satisfy Eq. (B.15):

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} (-2+3) & (4-4) \\ (-1.5+1.5) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

and

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} (-2+3) & (1-1) \\ (-6+6) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

The operation of inversion is defined only for square matrices. The inverse of such a matrix is also a square matrix of the same order as the original matrix. A procedure for determining inverses of matrices is presented in the following section. The operation of matrix inversion serves the same purpose as the operation of division in scalar algebra. Consider a system of simultaneous equations expressed in the matrix form as

$$\mathbf{A}\mathbf{x} = \mathbf{P}$$

in which **A** represents the square matrix of known coefficients; **x** represents the vector of the unknowns; and **P** represents the vector of the constants. Since the operation of division is not defined in matrix algebra, we cannot solve the foregoing matrix equation for **x** by dividing **P** by **A** (i.e., $\mathbf{x} = \mathbf{P}/\mathbf{A}$). Instead, to determine the unknowns **x**, we premultiply both sides of the equation by $\mathbf{A}^{-1}$ to obtain

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{P}$$

Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{Ix} = \mathbf{x}$, we can write

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{P}$$

which indicates that a system of simultaneous equations can be solved by premultiplying the vector of the constants by the inverse of the coefficient matrix.

An important property of matrix inversion is that *the inverse of a symmetric matrix is always a symmetric matrix.*

Transpose of a Matrix

The *transpose* of a matrix is obtained by interchanging its corresponding rows and columns. The transposed matrix is usually identified by the superscript T placed on the symbol of the original matrix. Consider, for example, the 2×3 matrix

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & 4 \\ 1 & 8 & -3 \end{bmatrix}$$

The transpose of $\mathbf{A}$ is given by

$$\mathbf{A}^T = \begin{bmatrix} 6 & 1 \\ -2 & 8 \\ 4 & -3 \end{bmatrix}$$

Note that the first column of $\mathbf{A}$ becomes the first row of $\mathbf{A}^T$. Similarly, the second and third columns of $\mathbf{A}$ become, respectively, the second and third rows of $\mathbf{A}^T$. The order of $\mathbf{A}^T$ thus obtained is 3×2.

As another example, consider the 3×3 matrix

$$\mathbf{B} = \begin{bmatrix} 9 & 7 & -5 \\ 7 & -3 & 2 \\ -5 & 2 & 6 \end{bmatrix}$$

Since the elements of $\mathbf{B}$ are symmetric about the main diagonal (i.e., $B_{ij} = B_{ji}$), interchanging the rows and the columns of this matrix produces a matrix $\mathbf{B}^T$ that is identical to the matrix $\mathbf{B}$ itself; that is,

$$\mathbf{B}^T = \mathbf{B}$$

Thus, *the transpose of a symmetric matrix yields the same matrix.*

Another useful property of matrix transposition is that *the transpose of a product of matrices equals the product of the transposes in reverse order;* that is,

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \qquad \textbf{(B.16)}$$

Similarly,

$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T \qquad \textbf{(B.17)}$$

Partitioning of Matrices

Partitioning is a process by which a matrix is subdivided into a number of smaller matrices called *submatrices.* For example, a 3×4 matrix $\mathbf{A}$ is partitioned into four submatrices by drawing horizontal and vertical dashed partition lines:

$$\mathbf{A} = \left[\begin{array}{ccc:c} 3 & 5 & -1 & 2 \\ -2 & 4 & 7 & 9 \\ \hdashline 6 & 1 & 3 & 4 \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{B.18}$$

in which the submatrices are

$$\mathbf{A}_{11} = \begin{bmatrix} 3 & 5 & -1 \\ -2 & 4 & 7 \end{bmatrix} \quad \mathbf{A}_{12} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

$$\mathbf{A}_{21} = [\,6 \quad 1 \quad 3\,] \quad \mathbf{A}_{22} = [4]$$

Matrix operations such as addition, subtraction, and multiplication can be performed on partitioned matrices in the same manner as described previously by treating the submatrices as elements, provided that the matrices are partitioned in such a way that their corresponding submatrices are conformable for the particular operation. For example, suppose that we wish to postmultiply the 3×4 matrix **A** of Eq. (B.18) by a 4×2 matrix **B**, which is partitioned into two submatrices as

$$\mathbf{B} = \left[\begin{array}{cc} 1 & 8 \\ -5 & 2 \\ -3 & 6 \\ \hdashline 7 & -1 \end{array}\right] = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{bmatrix} \tag{B.19}$$

The product **AB** is expressed in terms of the submatrices as

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} \end{bmatrix} \tag{B.20}$$

Note that the matrices **A** and **B** have been partitioned in such a way that their corresponding submatrices are conformable for multiplication; that is, the orders of the submatrices are such that the products $\mathbf{A}_{11}\mathbf{B}_{11}$, $\mathbf{A}_{12}\mathbf{B}_{21}$, $\mathbf{A}_{21}\mathbf{B}_{11}$, and $\mathbf{A}_{22}\mathbf{B}_{21}$ are defined. As shown in Eqs. (B.18) and (B.19), this is achieved by partitioning the rows of the second matrix **B** of the product **AB** in the same way that the columns of the first matrix **A** are partitioned. The products of the submatrices are given by

$$\mathbf{A}_{11}\mathbf{B}_{11} = \begin{bmatrix} 3 & 5 & -1 \\ -2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ -5 & 2 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} -19 & 28 \\ -43 & 34 \end{bmatrix}$$

$$\mathbf{A}_{12}\mathbf{B}_{21} = \begin{bmatrix} 2 \\ 9 \end{bmatrix} [7 \quad -1] = \begin{bmatrix} 14 & -2 \\ 63 & -9 \end{bmatrix}$$

$$\mathbf{A}_{21}\mathbf{B}_{11} = [6 \quad 1 \quad 3] \begin{bmatrix} 1 & 8 \\ -5 & 2 \\ -3 & 6 \end{bmatrix} = [-8 \quad 68]$$

$$\mathbf{A}_{22}\mathbf{B}_{21} = [4]\,[7 \quad -1] = [28 \quad -4]$$

Substitution into Eq. (B.20) yields

$$\mathbf{AB} = \begin{bmatrix} \begin{bmatrix} -19 & 28 \\ -43 & 34 \end{bmatrix} + \begin{bmatrix} 14 & -2 \\ 63 & -9 \end{bmatrix} \\ [-8 \quad 68] + [28 \quad -4] \end{bmatrix} = \begin{bmatrix} -5 & 26 \\ 20 & 25 \\ 20 & 64 \end{bmatrix}$$

B.4 SOLUTION OF SIMULTANEOUS EQUATIONS BY THE GAUSS-JORDAN METHOD

The *Gauss-Jordan elimination method* is one of the most commonly used procedures for solving simultaneous linear algebraic equations. To illustrate the method, consider the following system of three simultaneous equations:

$$\begin{aligned} 2x_1 - 5x_2 + 4x_3 &= 44 \\ 3x_1 + x_2 - 8x_3 &= -35 \\ 4x_1 - 7x_2 - x_3 &= 28 \end{aligned} \tag{B.21a}$$

To solve for the unknowns x_1, x_2, and x_3, we begin by dividing the first equation by the coefficient of its x_1 term:

$$\begin{aligned} x_1 - 2.5x_2 + 2x_3 &= 22 \\ 3x_1 + x_2 - 8x_3 &= -35 \\ 4x_1 - 7x_2 - x_3 &= 28 \end{aligned} \tag{B.21b}$$

Next, the unknown x_1 is eliminated from the remaining equations by successively subtracting from each remaining equation the product of the coefficient of its x_1 term and the first equation. Thus, to eliminate x_1 from the second equation, we multiply the first equation by 3 and subtract it from the second equation. Similarly, we eliminate x_1 from the third equation by multiplying the first equation by 4 and subtracting it from the third equation. The system of equations thus obtained is

$$\begin{aligned} x_1 - 2.5x_2 + 2x_3 &= 22 \\ 8.5x_2 - 14x_3 &= -101 \\ 3x_2 - 9x_3 &= -60 \end{aligned} \tag{B.21c}$$

With x_1 eliminated from all but the first equation, we now divide the second equation by the coefficient of its x_2 term:

$$\begin{aligned} x_1 - 2.5x_2 + 2x_3 &= 22 \\ x_2 - 1.647x_3 &= -11.882 \\ 3x_2 - 9x_3 &= -60 \end{aligned} \tag{B.21d}$$

Next, we eliminate x_2 from the first and the third equations, successively, by multiplying the second equation by -2.5 and subtracting it from the first equation, and then by multiplying the second equation by 3 and subtracting it from the third equation. This yields

$$\begin{aligned} x_1 \qquad - 2.118x_3 &= -7.705 \\ x_2 - 1.647x_3 &= -11.882 \\ - 4.059x_3 &= -24.354 \end{aligned} \tag{B.21e}$$

By dividing the third equation by the coefficient of its x_3 term, we obtain

$$\begin{aligned} x_1 \qquad - 2.118x_3 &= -7.705 \\ x_2 - 1.647x_3 &= -11.882 \\ x_3 &= 6 \end{aligned} \tag{B.21f}$$

Finally, by multiplying the third equation by −2.118 and subtracting it from the first equation, and by multiplying the third equation by −1.647 and subtracting it from the second equation, we determine the solution of the given system of equations (Eq. (B.21a)) to be

$$\begin{aligned} x_1 \qquad\qquad &= 5 \\ x_2 \qquad &= -2 \\ x_3 &= 6 \end{aligned} \tag{B.21g}$$

That is, $x_1 = 5$, $x_2 = -2$, and $x_3 = 6$. To check that the solution is carried out correctly, we substitute the numerical values of x_1, x_2, and x_3 back into the original equations (Eq. (B.21a)):

$$2(5) - 5(-2) + 4(6) = 44 \qquad \text{Checks}$$
$$3(5) - 2 - 8(6) = -35 \qquad \text{Checks}$$
$$4(5) - 7(-2) - 6 = 28 \qquad \text{Checks}$$

As the foregoing example illustrates, the Gauss-Jordan method essentially involves successively eliminating each unknown from all but one of the equations of the system by performing the following operations: (1) dividing an equation by a scalar; and (2) multiplying an equation by a scalar and subtracting the resulting equation from another equation. These operations, which do not change the solution of the original system of equations, are applied repeatedly until a system with each equation containing only one unknown is obtained.

The solution of simultaneous equations is usually carried out in matrix form by operating on the rows of the coefficient matrix and the vector containing the constant terms of the equations. The foregoing operations are then referred to as *elementary row operations*. These operations are applied to both the coefficient matrix and the vector of the constants simultaneously, until the coefficient matrix is reduced to a unit matrix. The elements of the vector, which initially contained the constant terms of the original equations, now represent the solution of the original simultaneous equations. To illustrate this procedure, consider again the system of three simultaneous equations given in Eq. (B.21a). The system can be expressed in matrix form as

$$\mathbf{Ax} = \mathbf{P}$$

$$\begin{bmatrix} 2 & -5 & 4 \\ 3 & 1 & -8 \\ 4 & -7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ -35 \\ 28 \end{bmatrix} \tag{B.22}$$

When applying the Gauss-Jordan method, it is usually convenient to write the coefficient matrix **A** and the vector of constants **P** as submatrices of a partitioned *augmented matrix:*

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & 44 \\ 3 & 1 & -8 & -35 \\ 4 & -7 & -1 & 28 \end{array}\right] \tag{B.23a}$$

To determine the solution, we begin by dividing row 1 of the augmented matrix by $A_{11} = 2$:

$$\left[\begin{array}{ccc|c} 1 & -2.5 & 2 & 22 \\ 3 & 1 & -8 & -35 \\ 4 & -7 & -1 & 28 \end{array}\right] \tag{B.23b}$$

Next, we multiply row 1 by $A_{21} = 3$ and subtract it from row 2 and then multiply row 1 by $A_{31} = 4$ and subtract it from row 3. This yields

$$\left[\begin{array}{ccc|c} 1 & -2.5 & 2 & 22 \\ 0 & 8.5 & -14 & -101 \\ 0 & 3 & -9 & -60 \end{array}\right] \tag{B.23c}$$

Divide row 2 by $A_{22} = 8.5$, obtaining

$$\left[\begin{array}{ccc|c} 1 & -2.5 & 2 & 22 \\ 0 & 1 & -1.647 & -11.882 \\ 0 & 3 & -9 & -60 \end{array}\right] \tag{B.23d}$$

Multiply row 2 by $A_{12} = -2.5$ and subtract it from row 1; then multiply row 2 by $A_{32} = 3$ and subtract it from row 3. This yields

$$\left[\begin{array}{ccc|c} 1 & 0 & -2.118 & -7.705 \\ 0 & 1 & -1.647 & -11.882 \\ 0 & 0 & -4.059 & -24.354 \end{array}\right] \tag{B.23e}$$

Divide row 3 by $A_{33} = -4.059$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2.118 & -7.705 \\ 0 & 1 & -1.647 & -11.882 \\ 0 & 0 & 1 & 6 \end{array}\right] \tag{B.23f}$$

Multiply row 3 by $A_{13} = -2.118$ and subtract it from row 1; then multiply row 3 by $A_{23} = -1.647$ and subtract it from row 2. This yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{array}\right] \tag{B.23g}$$

Thus $x_1 = 5$, $x_2 = -2$, and $x_3 = 6$.

Matrix Inversion

The Gauss-Jordan elimination method can also be used to determine the inverses of square matrices. The procedure is similar to that described previously for solving simultaneous equations, except that in the augmented matrix, the coefficient matrix is now replaced by the matrix **A** that is to be inverted and the vector of constants **P** is replaced by a unit matrix **I** of the same order as the matrix **A**. Elementary row operations are then performed on the augmented matrix to reduce the matrix **A** to a unit matrix. The matrix **I**, which was initially the unit matrix, now represents the inverse of the original matrix **A**.

To illustrate the foregoing procedure, let us compute the inverse of the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \tag{B.24}$$

The augmented matrix is given by

$$\left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{array}\right] \tag{B.25a}$$

By multiplying row 1 by $A_{21} = 3$ and subtracting it from row 2, we obtain

$$\left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{array}\right] \tag{B.25b}$$

Next, by dividing row 2 by $A_{22} = 2$, we obtain

$$\left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -1.5 & 0.5 \end{array}\right] \tag{B.25c}$$

Finally, by multiplying row 2 by -2 and subtracting it from row 1, we obtain

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -1.5 & 0.5 \end{array}\right] \tag{B.25d}$$

Thus

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix}$$

The computations can be checked by using the relationship $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. We showed in Section B.3 that the matrix $\mathbf{A}^{-1}$, as computed here, does indeed satisfy this relationship.

PROBLEMS

Section B.3

B.1 Determine the matrix $\mathbf{C} = \mathbf{A} + 3\mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 12 & -8 & 15 \\ -8 & 7 & 10 \\ 15 & 10 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & 6 \\ 1 & 6 & 3 \end{bmatrix}$$

B.2 Determine the matrix $\mathbf{C} = 2\mathbf{A} - \mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 8 & 4 \\ 2 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 6 \\ 5 & 1 \\ 3 & -4 \end{bmatrix}$$

B.3 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = [-6 \quad 4 \quad -2] \qquad \mathbf{B} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

B.4 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & -5 \\ -5 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -3 & 4 \\ 4 & 1 \end{bmatrix}$$

B.5 Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ by using the matrices **A** and **B** given here.

$$\mathbf{A} = \begin{bmatrix} 8 & -2 & 5 \\ 1 & -4 & 3 \\ 2 & 0 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -5 \\ 7 & 0 \\ 0 & -3 \end{bmatrix}$$

Section B.4

B.6 Solve the following system of simultaneous equations by the Gauss-Jordan method.

$$\begin{aligned} 2x_1 + 5x_2 - x_3 &= 15 \\ 5x_1 - x_2 + 3x_3 &= 27 \\ -x_1 + 3x_2 + 4x_3 &= 14 \end{aligned}$$

B.7 Solve the following system of simultaneous equations by the Gauss-Jordan method.

$$\begin{aligned} -12x_1 - 3x_2 + 6x_3 &= 45 \\ 5x_1 + 2x_2 - 4x_3 &= -9 \\ 10x_1 + x_2 - 7x_3 &= -32 \end{aligned}$$

B.8 Solve the following system of simultaneous equations by the Gauss-Jordan method.

$$\begin{aligned} 5x_1 - 2x_2 + 6x_3 \qquad &= 0 \\ -2x_1 + 4x_2 + x_3 + 3x_4 &= 18 \\ 6x_1 + x_2 + 6x_3 + 8x_4 &= -29 \\ 3x_2 + 8x_3 + 7x_4 &= 11 \end{aligned}$$

B.9 Determine the inverse of the matrix shown using the Gauss-Jordan method.

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & -1 \\ -2 & 5 & 1 \\ 6 & -4 & -5 \end{bmatrix}$$

B.10 Determine the inverse of the matrix shown using the Gauss-Jordan method.

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 & -3 \\ 2 & 3 & -4 & 0 \\ 0 & -4 & 2 & -1 \\ -3 & 0 & -1 & 5 \end{bmatrix}$$