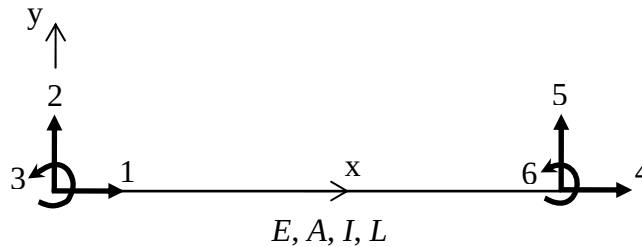


PLANAR FRAME ELEMENT



- Each node can translate and rotate in plane. The final displaced shape has 6 independent generalized displacements (i.e. translations and rotations) involved.
- Frame element resists external loads or disturbances by developing internal axial forces, shear forces, and bending moments.

The way we will find the internal generalized force (i.e. force or bending moment) vs. generalized displacement (i.e. displacement or rotation) relations is as follows:

We will displace one degree of freedom at a time while holding all other degrees of freedom fixed. The force system required to make this particular displacement pattern happen will tell us how nodal forces and moments are related to nodal displacement and rotations.

We will assume that the frame element is long and slender. This allows us to happily ignore the resistance developed due to shear strains. In other words, we will consider the element to be resisting by axial straining and by flexure.

Let's start.

Case 1) $u_i = u_i$ while other displacements and rotations = 0.

$$N_i = \frac{\Delta_i}{L} EA = \frac{EA}{L} \Delta_i \text{ and from static equilibrium } N_j = -\frac{\Delta_i}{L} EA = -\frac{EA}{L} \Delta_i$$

Note that axial deformations in the frame element do not cause shear forces or bending moments. So,

$$\begin{aligned} V_i &= 0 & M_i &= 0 \\ V_j &= 0 & M_j &= 0 \end{aligned}$$

Case 2) $u_j = u_j$ while other displacements and rotations = 0.

$$N_i = -\frac{EA}{L}\Delta_j \quad \text{and} \quad N_j = \frac{EA}{L}\Delta_j$$

$$V_i = 0 \quad M_i = 0$$

$$V_j = 0 \quad M_j = 0$$

Case 3) $v_i = v_i$ while other displacements and rotations = 0.

$$\begin{array}{lll} N_i = 0 & V_i = \frac{12EI}{L^3}v_i & M_i = \frac{6EI}{L^2}v_i \\ N_j = 0 & V_j = -\frac{12EI}{L^3}v_i & M_j = \frac{6EI}{L^2}v_i \end{array}$$

Case 4) $v_j = v_j$ while other displacements and rotations = 0.

$$\begin{array}{lll} N_i = 0 & V_i = -\frac{12EI}{L^3}v_j & M_i = -\frac{6EI}{L^2}v_j \\ N_j = 0 & V_j = \frac{12EI}{L^3}v_j & M_j = \frac{6EI}{L^2}v_j \end{array}$$

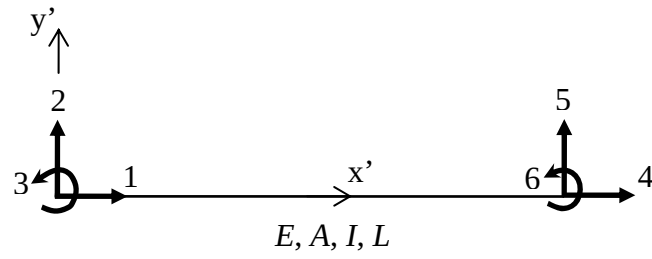
Case 5) $\theta_i = \theta_i$ while other displacements and rotation = 0.

$$\begin{array}{lll} N_i = 0 & V_i = \frac{6EI}{L^2} \theta_i & M_i = \frac{4EI}{L} \theta_i \\ N_j = 0 & V_j = -\frac{6EI}{L^2} \theta_i & M_j = \frac{2EI}{L} \theta_i \end{array}$$

Case 6) $\theta_j = \theta_j$ while other displacements and rotation = 0.

$$\begin{array}{lll} N_i = 0 & V_i = \frac{6EI}{L^2} \theta_j & M_i = \frac{2EI}{L} \theta_j \\ N_j = 0 & V_j = -\frac{6EI}{L^2} \theta_j & M_j = \frac{4EI}{L} \theta_j \end{array}$$

PLANAR FRAME ELEMENT



Member Stiffness Matrix in Local Coordinates

$$[K_{local}] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Frame Element Analytical Model

Relation between member end displacements and rotations and internal forces, moments, displacements, and rotations elsewhere in the element

Prismatic member with homogeneous, linear elastic, isotropic material properties

u : displacement of the beam element in x-direction.
 v : displacement of the beam element in y-direction.
 θ : rotation of the beam (about z-axis)

AXIAL

$$u(x) = u_i + \frac{(u_j - u_i)}{L} x$$

⇒ Axial deflected shape can vary linearly; axial strain can only be constant

FLEXURAL and SHEAR

At any given location

$$\frac{d^2 v}{dx^2} = \frac{M}{EI}$$

V & M : shear & moment
 E : modulus of elasticity
 I : moment of inertia

For a beam with no concentrated forces or moments, the differential equations of equilibrium are

$$\frac{dM}{dx} = V$$

$$\frac{dV}{dx} = q$$

q : uniformly distributed transverse load

Using the moment-curvature relationship,

$$\frac{d^3 v}{dx^3} = \frac{V}{EI}$$

$$\frac{d^4 v}{dx^4} = \frac{q}{EI}$$

For the beam element (model) under consideration, $q = 0$. Therefore,

$$\frac{d^4 v}{dx^4} = 0$$

Integrate once,

$$\Rightarrow \frac{d^3 v}{dx^3} = \frac{V}{EI} = C_1 \quad \text{where } C_1 \text{ is a constant of integration, implying that shear force is constant across the element.}$$

Continue integrating,

$$\frac{d^2 v}{dx^2} = \frac{M}{EI} = C_1 x + C_2$$

$$\frac{d^2 v}{dx^2} = \theta = \frac{1}{2} C_1 x^2 + C_2 x + C_3$$

$$v = \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4$$

where C_1, C_2, C_3 and C_4 are the four unknown constants of integration. We shall make use of the four boundary conditions to find these four unknown constants. The boundary conditions are

$$\begin{aligned} v(0) = v_i & \quad \left. \frac{dv}{dx} \right|_{@ x=0} = \theta_i \\ v(L) = v_j & \quad \left. \frac{dv}{dx} \right|_{@ x=L} = \theta_j \end{aligned}$$

Substitute and find

$$\begin{aligned} C_1 &= \frac{12}{L^3} (v_i - v_j) + \frac{6}{L^2} (\theta_i + \theta_j) & C_2 &= \frac{6}{L^2} (-v_i + v_j) - \frac{1}{L} (4\theta_i + 2\theta_j) \\ C_3 &= \theta_i & C_4 &= v_i \end{aligned}$$

Substitute to find V and M ,

$$\begin{aligned} V &= \frac{12EI}{L^3} (v_i - v_j) + \frac{6EI}{L^2} (\theta_i + \theta_j) \\ M &= \frac{6EI}{L^3} (2x - L)(v_i - v_j) + \frac{6EI}{L^2} (\theta_i + \theta_j)x - \frac{2EI}{L} (2\theta_i + \theta_j) \end{aligned}$$

Deflections in the transverse direction is given by

$$\begin{aligned}
 v(x) = & \left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1 \right] v_i & N_1 v_i \\
 & + \left(\frac{x^3}{L^2} - 2\frac{x^2}{L} + x \right) \theta_i & + N_2 \theta_i \\
 & + \left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2 \right] v_j & + N_3 v_j \\
 & + \left(\frac{x^3}{L^2} - \frac{x^2}{L} \right) \theta_j & + N_4 \theta_j
 \end{aligned}$$

The functions in parentheses in front of the displacements and rotations are called the **shape-functions**. They relate the end displacements and rotations (the only ones our “model” can see) to the deflected shape suggested (hence, only approximate) by our idealized frame model.

Recall that, longitudinal displaced shape is given by $u(x) = u_i + \frac{(u_j - u_i)}{L} x$.

To find the slope at any given point, simply take the derivative of the expression for v above with respect to x , i.e., slope @ x is $\frac{dv}{dx}$.