BEAM DESIGN FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

DESIGN AID No. 6
BEAM FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

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Introduction

Figures 1 through 32 provide a series of shear and moment diagrams with accompanying formulas for design of beams under various static loading conditions.

Shear and moment diagrams and formulas are excerpted from the Western Woods Use Book, 4th edition, and are provided herein as a courtesy of Western Wood Products Association.

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Notations Relative to “Shear and Moment Diagrams”

- $E$ = modulus of elasticity, psi
- $I$ = moment of inertia, in.$^4$
- $L$ = span length of the bending member, ft.
- $l$ = span length of the bending member, in.
- $M$ = maximum bending moment, in.-lbs.
- $P$ = total concentrated load, lbs.
- $R$ = reaction load at bearing point, lbs.
- $V$ = shear force, lbs.
- $W$ = total uniform load, lbs.
- $w$ = load per unit length, lbs./in.
- $\Delta$ = deflection or deformation, in.
- $x$ = horizontal distance from reaction to point on beam, in.
Figure 1  Simple Beam – Uniformly Distributed Load

\[
R = V = \frac{w\ell}{2} \\
V_x = w\left(\frac{\ell}{2} - x\right) \\
M_{\text{max}} \text{ (at center)} = \frac{w\ell^2}{8} \\
M_x = \frac{wx}{2}(\ell - x) \\
\Delta_{\text{max}} \text{ (at center)} = \frac{5w\ell^4}{384EI} \\
\Delta_x = \frac{wx}{24EI}(\ell^3 - 2\ell x^2 + x^3)
\]

Figure 2  Simple Beam – Uniform Load Partially Distributed

\[
R_1 = V_1 \text{ (max when } a < c) = \frac{wb}{2\ell}(2c + b) \\
R_2 = V_2 \text{ (max when } a > c) = \frac{wb}{2\ell}(2a + b) \\
V_x \text{ (when } x > a \text{ and } < (a + b)) = R_1 - w(x - a) \\
M_{\text{max}} \text{ (at } x = a + \frac{R_1}{w}) = R_1 \left(a + \frac{R_1}{2w}\right) \\
M_x \text{ (when } x < a) = R_1 x \\
M_x \text{ (when } x > a \text{ and } < (a + b)) = R_1 x - \frac{w}{2}(x - a)^2 \\
M_x \text{ (when } x > (a + b)) = R_2(\ell - x)
\]
Figure 3  Simple Beam – Uniform Load Partially Distributed at One End

\[ R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{wa}{2\ell} (2\ell - a) \]

\[ R_2 = V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{wa^2}{2\ell} \]

\[ V_x \text{ (when } x < a) \quad \ldots \ldots \ldots = R_1 - wx \]

\[ M_{\text{max}} \left( \text{at } x = \frac{R_1}{w} \right) \quad \ldots \ldots \ldots = \frac{R_1^2}{2w} \]

\[ M_x \text{ (when } x < a) \quad \ldots \ldots \ldots = R_1 x - \frac{wx^2}{2} \]

\[ M_x \text{ (when } x > a) \quad \ldots \ldots \ldots = R_2 (\ell - x) \]

\[ \Delta_x \text{ (when } x < a) \quad \ldots \ldots \ldots = \frac{wx}{24 \text{ EI}} \left( a^2 (2\ell - a)^2 - 2ax (2\ell - a) + \ell x^3 \right) \]

\[ \Delta_x \text{ (when } x > a) \quad \ldots \ldots \ldots = \frac{wa^2 (\ell - x)}{24 \text{ EI}} \left( 4x\ell - 2x^2 - a^2 \right) \]

Figure 4  Simple Beam – Uniform Load Partially Distributed at Each End

\[ R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w_1 a (2\ell - a) + w_2 c^2}{2\ell} \]

\[ R_2 = V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w_2 c (2\ell - c) + w_1 a^2}{2\ell} \]

\[ V_x \text{ (when } x < a) \quad \ldots \ldots \ldots = R_1 - w_1 x \]

\[ V_x \left( \text{when } x > a \text{ and } < (a + b) \right) \quad \ldots \ldots \ldots = R_1 - w_1 a \]

\[ V_x \left( \text{when } x > (a + b) \right) \quad \ldots \ldots \ldots = R_2 - w_2 (\ell - x) \]

\[ M_{\text{max}} \left( \text{at } x = \frac{R_1}{w_1} \text{ when } R_1 < w_1 a \right) \quad \ldots \ldots \ldots = \frac{R_1^2}{2w_1} \]

\[ M_{\text{max}} \left( \text{at } x = \ell - \frac{R_2}{w_2} \text{ when } R_2 < w_2 c \right) \quad \ldots \ldots \ldots = \frac{R_2^2}{2w_2} \]

\[ M_x \text{ (when } x < a) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_1 x - \frac{w_1 x^2}{2} \]

\[ M_x \left( \text{when } x > a \text{ and } < (a + b) \right) \quad \ldots \ldots \ldots = R_1 x - \frac{w_1 a}{2} (2x - a) \]

\[ M_x \left( \text{when } x > (a + b) \right) \quad \ldots \ldots \ldots = R_2 (\ell - x) - \frac{w_2 (\ell - x)^2}{2} \]
Figure 5  Simple Beam – Load Increasing Uniformly to One End

\[ R_1 = V_1 = \frac{W}{3} \]
\[ R_2 = V_2 = \frac{2W}{3} \]
\[ V_x = \frac{W}{3} - \frac{Wx^2}{\ell^2} \]
\[ M_{\text{max}} \text{ at } x = \frac{\ell}{\sqrt{3}} = 0.5774 \ell \]
\[ = \frac{2W\ell}{9\sqrt{3}} = 0.1283W\ell \]
\[ M_x = \frac{Wx}{3\ell^2} (\ell^2 - x^2) \]
\[ \Delta_{\text{max}} \text{ at } x = \ell \left(1 - \frac{8}{15} \right) = 0.5193 \ell \]
\[ = 0.01304 \frac{W\ell^3}{EI} \]
\[ \Delta_x = \frac{Wx}{180EI\ell^2} (3x^4 - 10\ell^2x^2 + 7\ell^4) \]

Figure 6  Simple Beam – Load Increasing Uniformly to Center

\[ R = V = \frac{W}{2} \]
\[ V_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{W}{2\ell^2} (\ell^2 - 4x^2) \]
\[ M_{\text{max}} \text{ (at center)} = \frac{W\ell}{6} \]
\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{Wx}{2} \left(\frac{1}{2} - \frac{2x^2}{3\ell^2}\right) \]
\[ \Delta_{\text{max}} \text{ (at center)} = \frac{W\ell^3}{60EI} \]
\[ \Delta_x = \frac{Wx}{480EI\ell^2} (5\ell^2 - 4x^2)^2 \]
Figure 7  Simple Beam – Concentrated Load at Center

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots = \frac{P}{2} \]
\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{P \ell}{4} \]
\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = \frac{Px}{2} \]
\[ \Delta_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{P \ell^3}{48E I} \]
\[ \Delta_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = \frac{Px}{48E I} (3\ell^2 - 4x^2) \]

Figure 8  Simple Beam – Concentrated Load at Any Point

\[ R_1 = V_1 \text{ (max when } a < b) \quad \ldots \ldots \ldots = \frac{Pb}{\ell} \]
\[ R_2 = V_2 \text{ (max when } a > b) \quad \ldots \ldots \ldots = \frac{Pa}{\ell} \]
\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pab}{\ell} \]
\[ M_x \left( \text{when } x < b \right) \quad \ldots \ldots \ldots = \frac{Pbx}{\ell} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) \quad = \frac{Pab(a + 2b)\sqrt{3a(a + 2b)}}{27E I \ell} \]
\[ \Delta_x \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pab^2}{3E I \ell} \]
\[ \Delta_x \left( \text{when } x < a \right) \quad \ldots \ldots \ldots = \frac{Pbx}{6E I \ell} (\ell^2 - b^2 - x^2) \]
\[ \Delta_x \left( \text{when } x > a \right) \quad \ldots \ldots \ldots = \frac{Pa(\ell - x)}{6E I \ell} (2\ell x - x^2 - a^2) \]
**Figure 9**  Simple Beam – Two Equal Concentrated Loads Symmetrically Placed

\[ R = V \ldots \ldots = P \]
\[ M_{\text{max}} \text{ (between loads)} \ldots \ldots = Pa \]
\[ M_x \text{ (when } x < a) \ldots \ldots = Px \]
\[ \Delta_{\text{max}} \text{ (at center)} \ldots \ldots = \frac{Pa}{24EI}(3\ell^2 - 4a^2) \]
\[ \Delta_x \text{ (when } x < a) \ldots \ldots = \frac{Px}{6EI}(3\ell a - 3a^2 - x^2) \]
\[ \Delta_x \text{ (when } x > a \text{ and } < (\ell - a)) \ldots = \frac{Pa}{6EI}(3\ell x - 3x^2 - a^2) \]

**Figure 10**  Simple Beam – Two Equal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 \text{ (max when } a < b) \ldots \ldots = \frac{P}{\ell}(\ell - a + b) \]
\[ R_2 = V_2 \text{ (max when } a > b) \ldots \ldots = \frac{P}{\ell}(\ell - b + a) \]
\[ V_x \text{ (when } x > a \text{ and } < (\ell - b)) \ldots = \frac{P}{\ell}(b - a) \]
\[ M_1 \text{ (max when } a > b) \ldots \ldots = R_1 a \]
\[ M_2 \text{ (max when } a < b) \ldots \ldots = R_2 b \]
\[ M_x \text{ (when } x > a) \ldots \ldots = R_1 x \]
\[ M_x \text{ (when } x > a \text{ and } < (\ell - b)) \ldots = R_1 x - P(x - a) \]
Figure 11  Simple Beam – Two Unequal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 = \frac{P_1(\ell - a) + P_2b}{\ell} \]

\[ R_2 = V_2 = \frac{P_1a + P_2(\ell - b)}{\ell} \]

\[ V_x \text{ (when } x > a \text{ and } < (\ell - b) \text{)} = R_1 - P_1 \]

\[ M_1 \text{ (max when } R_1 < P_1 \text{)} = R_1a \]

\[ M_2 \text{ (max when } R_2 < P_2 \text{)} = R_2b \]

\[ M_x \text{ (when } x < a \text{)} = R_1x \]

\[ M_x \text{ (when } x > a \text{ and } < (\ell - b) \text{)} = R_1x - P_1(x - a) \]

Figure 12  Cantilever Beam – Uniformly Distributed Load

\[ R = V = \frac{w\ell}{2} \]

\[ V_x = \frac{wx}{2} \]

\[ M_{max} \text{ (at fixed end)} = \frac{w\ell^2}{2} \]

\[ M_x = \frac{wx^2}{2} \]

\[ \Delta_{max} \text{ (at free end)} = \frac{w\ell^4}{8EI} \]

\[ \Delta_x = \frac{w}{24EI} (x^4 - 4\ell^3x + 3\ell^4) \]
Figure 13  Cantilever Beam – Concentrated Load at Free End

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots = P \]
\[ M_{\text{max}} \text{ (at fixed end)} \quad \ldots \ldots \ldots = P\ell \]
\[ M_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = Px \]
\[ \Delta_{\text{max}} \text{ (at free end)} \quad \ldots \ldots \ldots = \frac{P\ell^3}{3EI} \]
\[ \Delta_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{P}{6EI}(2\ell^3 - 3\ell^2x + x^3) \]

Figure 14  Cantilever Beam – Concentrated Load at Any Point

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots = P \]
\[ M_{\text{max}} \text{ (at fixed end)} \quad \ldots \ldots \ldots = Pb \]
\[ M_x \text{ (when } x > a \text{)} \quad \ldots \ldots \ldots = P(x - a) \]
\[ \Delta_{\text{max}} \text{ (at free end)} \quad \ldots \ldots \ldots = \frac{Pb^3}{6EI}(3\ell - b) \]
\[ \Delta_x \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pb^3}{3EI} \]
\[ \Delta_x \text{ (when } x < a \text{)} \quad \ldots \ldots \ldots = \frac{Pb^2}{6EI}(3\ell - 3x - b) \]
\[ \Delta_x \text{ (when } x > a \text{)} \quad \ldots \ldots \ldots = \frac{P(\ell - x)^2}{6EI}(3b - \ell + x) \]
Figure 15  Beam Fixed at One End, Supported at Other – Uniformly Distributed Load

\[
R_1 = V_1 = \frac{3w\ell}{8}
\]

\[
R_2 = V_2 = \frac{5w\ell}{8}
\]

\[
V_x = R_1 = \frac{3w\ell}{8}
\]

\[
M_{\text{max}} = \frac{w\ell^2}{8}
\]

\[
M_1 \left( \text{at } x = \frac{3\ell}{8} \right) = \frac{9w\ell^2}{128}
\]

\[
M_2 = R_1x - \frac{wx^2}{2}
\]

\[
\Delta_{\text{max}} \left( \text{at } x = \frac{\ell}{16} (1 + \sqrt{33}) = .4215\ell \right) = \frac{w\ell^4}{185EI}
\]

\[
\Delta_x = \frac{wx}{48EI} \left( \ell^3 - 3\ell x^2 + 2x^3 \right)
\]

Figure 16  Beam Fixed at One End, Supported at Other – Concentrated Load at Center

\[
R_1 = V_1 = \frac{5P}{16}
\]

\[
R_2 = V_2 = \frac{11P}{16}
\]

\[
M_{\text{max}} \left( \text{at fixed end} \right) = \frac{3P\ell}{16}
\]

\[
M_1 \left( \text{at point of load} \right) = \frac{5P\ell}{32}
\]

\[
M_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{5Px}{16}
\]

\[
M_x \left( \text{when } x > \frac{\ell}{2} \right) = P \left( \frac{\ell}{2} - \frac{11x}{16} \right)
\]

\[
\Delta_{\text{max}} \left( \text{at } x = \ell \left( \frac{1}{5} = .4472\ell \right) \right) = \frac{P\ell^3}{48EI\sqrt{5}} = .009317 \frac{P\ell^3}{EI}
\]

\[
\Delta_x \text{ (at point of load)} = \frac{7P\ell^3}{768EI}
\]

\[
\Delta_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{Px}{96EI} (3\ell^2 - 5x^2)
\]

\[
\Delta_x \left( \text{when } x > \frac{\ell}{2} \right) = \frac{P}{96EI} (x - \ell)^2 (11x - 2\ell)
\]
Figure 17  Beam Fixed at One End, Supported at Other – Concentrated Load at Any Point

\[ R_1 = V_1 \quad \ldots \quad \frac{Pb^2}{2\ell^3} (a + 2\ell) \]
\[ R_2 = V_2 \quad \ldots \quad \frac{Pa}{2\ell^3} (3\ell^2 - a^2) \]
\[ M_1 \text{ (at point of load)} \quad \ldots \quad R_1 a \]
\[ M_2 \text{ (at fixed end)} \quad \ldots \quad \frac{Pab}{2\ell^3} (a + \ell) \]
\[ M_x \text{ (when } x < a) \quad \ldots \quad R_1 x \]
\[ M_x \text{ (when } x > a) \quad \ldots \quad R_1 x - P(x - a) \]
\[ \Delta_{\text{max}} \text{ (when } a < 0.414\ell \text{ at } x = \ell \left( \frac{\ell^2 + a^2}{3\ell^2 - a^2} \right) = \frac{Pa}{6EI} \left( \frac{\ell^2 - a^2}{3\ell^2 - a^2} \right)^3 \]
\[ \Delta_{\text{max}} \text{ (when } a > 0.414\ell \text{ at } x = \ell \left( \frac{\ell^2 + a^2}{2\ell + a} \right) = \frac{Pa}{12EI\ell^3} \left( \frac{\ell^2 + a^2}{2\ell + a} \right)^2 \]
\[ \Delta_{\text{s}} \text{ (at point of load)} \quad \ldots \quad \frac{Pa^2 b^3}{12EI\ell^5} (3\ell + a) \]
\[ \Delta_{\text{s}} \text{ (when } x < a) \quad \ldots \quad \frac{Pa}{12EI\ell^3} (3\ell^2 x - 2\ell x^2 - ax^2) \]
\[ \Delta_{\text{s}} \text{ (when } x > a) \quad \ldots \quad \frac{Pa}{12EI\ell^3} (\ell - x)(3\ell^2 x - a^2 x - 2a^2 \ell) \]

Figure 18  Beam Overhanging One Support – Uniformly Distributed Load

\[ R_1 = V_1 \quad \ldots \quad \frac{w}{2\ell} (\ell^2 - a^2) \]
\[ R_2 = V_2 + V_1 \quad \ldots \quad \frac{w}{2\ell} (\ell + a)^2 \]
\[ V_2 \quad \ldots \quad wa \]
\[ V_1 \quad \ldots \quad \frac{w}{2\ell} (\ell^2 + a^2) \]
\[ V_x \text{ (between supports)} \quad \ldots \quad R_1 - wx \]
\[ V_{s_1} \text{ (for overhang)} \quad \ldots \quad w(a - x_i) \]
\[ M_1 \left( \text{at } x = \frac{\ell}{2} \left[ 1 - \frac{a^2}{\ell^2} \right] \right) \quad \ldots \quad \frac{w}{8\ell^2} (\ell + a)^2 (\ell - a)^2 \]
\[ M_2 \text{ (at } R_2) \quad \ldots \quad \frac{wa^2}{2} \]
\[ M_x \text{ (between supports)} \quad \ldots \quad \frac{wx}{2\ell} (\ell^2 - a^2 - x\ell) \]
\[ M_{s_1} \text{ (for overhang)} \quad \ldots \quad \frac{w}{2} (a - x_i)^2 \]
\[ \Delta_{x} \text{ (between supports)} \quad \ldots \quad \frac{wx}{24EI\ell} (\ell^4 - 2\ell^2 x^2 + \ell x^3 - 2a^2 \ell^4 + 2a^2 x^2) \]
\[ \Delta_{s_1} \text{ (for overhang)} \quad \ldots \quad \frac{wx}{24EI} (4a^2 \ell - \ell^3 + 6a^2 x_i - 4a x_i^2 + x_i^3) \]

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**Figure 19** Beam Overhanging One Support – Uniformly Distributed Load on Overhang

\[ R_1 = V_1 = \frac{wa^2}{2\ell} \]
\[ R_2 = V_1 + V_2 = \frac{wa}{2\ell} (2\ell +a) \]
\[ V_2 = \frac{wa}{2\ell} \]
\[ V_{x_1} \text{ (for overhang)} = wa(a - x_1) \]
\[ M_{\text{max}} \text{ (at } R_2) = \frac{wa^2}{2} \]
\[ M_x \text{ (between supports)} = \frac{wa^2x}{2\ell} \]
\[ M_{x_1} \text{ (for overhang)} = \frac{w}{2} (a - x_1)^2 \]
\[ \Delta_{\text{max}} \text{ (between supports at } x = \frac{\ell}{\sqrt{3}}) = \frac{wa^2 \ell^4}{18\sqrt{3}EI} \approx 0.03208 \frac{wa^2 \ell^4}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{wa^3}{24EI} (4\ell + 3a) \]
\[ \Delta_x \text{ (between supports)} = \frac{wa^2}{12EI\ell} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} = \frac{wa_1}{24EI} (4a^2\ell + 6a^2x_1 - 4ax_1^2 + x_1^4) \]

**Figure 20** Beam Overhanging One Support – Concentrated Load at End of Overhang

\[ R_1 = V_1 = \frac{Pa}{\ell} \]
\[ R_2 = V_1 + V_2 = \frac{P}{\ell} (\ell + a) \]
\[ V_2 = \frac{P}{\ell} \]
\[ M_{\text{max}} \text{ (at } R_2) = \frac{Pa}{\ell} \]
\[ M_x \text{ (between supports)} = \frac{Pax}{\ell} \]
\[ M_{x_1} \text{ (for overhang)} = \frac{P}{\ell} (a - x_1) \]
\[ \Delta_{\text{max}} \text{ (between supports at } x = \frac{\ell}{\sqrt{3}}) = \frac{Pa\ell^2}{9\sqrt{3}EI} = 0.06415 \frac{Pa\ell^2}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{Pa^2}{3EI} (\ell + a) \]
\[ \Delta_x \text{ (between supports)} = \frac{Pax}{6EI\ell} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} = \frac{P}{6EI} (2a\ell + 3ax_1 - x_1^2) \]
Figure 21  Beam Overhanging One Support – Concentrated Load at Any Point Between Supports

\[ R_1 = V_1 \text{ (max when } a < b) \]
\[ R_2 = V_2 \text{ (max when } a > b) \]
\[ M_{\text{max}} \text{ (at point of load) } \]
\[ M_x \text{ (when } x < a) \]
\[ \Delta_{\text{max}} \left( \text{at } x = \frac{a(a + 2b)}{3} \text{ when } a > b \right) = \frac{Pab(a + 2b)\sqrt{3a(a + 2b)}}{27EI\ell} \]
\[ \Delta_a \text{ (at point of load) } \]
\[ \Delta_x \text{ (when } x < a) \]
\[ \Delta_x \text{ (when } x > a) \]
\[ \Delta_{x_1} \]

Figure 22  Beam Overhanging Both Supports – Unequal Overhangs – Uniformly Distributed Load

\[ R_1 \]
\[ R_2 \]
\[ V_1 \]
\[ V_2 \]
\[ V_3 \]
\[ V_4 \]
\[ V_{x_1} \]
\[ V_r \text{ (when } x < \ell) \]
\[ V_m \text{ (when } a < c) \]
\[ M_1 \]
\[ M_2 \]
\[ M_3 \]
\[ M_x \left( \text{max when } x = \frac{R_1}{w} - a \right) \]
**Figure 23** Beam Fixed at Both Ends – Uniformly Distributed Load

\[ R = V = \frac{w\ell}{2} \]

\[ V_x = \frac{w}{2} \left( \frac{\ell}{2} - x \right) \]

\[ M_{\text{max}} \text{ (at ends)} = \frac{w\ell^2}{12} \]

\[ M_i \text{ (at center)} = \frac{w\ell^2}{24} \]

\[ M_x = \frac{w}{12} (6\ell x - \ell^2 - 6x^2) \]

\[ \Delta_{\text{max}} \text{ (at center)} = \frac{w\ell^4}{384EI} \]

\[ \Delta_x = \frac{w\ell^2}{24EI} (\ell - x)^2 \]

**Figure 24** Beam Fixed at Both Ends – Concentrated Load at Center

\[ R = V = \frac{P}{2} \]

\[ M_{\text{max}} \text{ (at center and ends)} = \frac{P\ell}{8} \]

\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{P}{8} (4x - \ell) \]

\[ \Delta_{\text{max}} \text{ (at center)} = \frac{P\ell^3}{192EI} \]

\[ \Delta_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{Px^2}{48EI} (3\ell - 4x) \]
Figure 25  Beam Fixed at Both Ends – Concentrated Load at Any Point

\[ R_1 = V_1 \text{ (max when } a < b) = \frac{Pb^2}{\ell^3} (3a + b) \]
\[ R_2 = V_2 \text{ (max when } a > b) = \frac{Pa^2}{\ell^3} (a + 3b) \]
\[ M_1 \text{ (max when } a < b) = \frac{Pab^2}{\ell^3} \]
\[ M_2 \text{ (max when } a > b) = \frac{Pa^2b}{\ell^2} \]
\[ M_a \text{ (at point of load) } = \frac{2Pa^2b^2}{\ell^3} \]
\[ M_x \text{ (when } x < a) = R_x - \frac{Pab^2}{\ell^2} \]
\[ \Delta_{\text{max}} \left(\text{when } a > b \text{ at } x = \frac{2\ell}{3a + b}\right) = \frac{2Pa^2b^2}{3EI(3a + b)^2} \]
\[ \Delta_x \text{ (at point of load) } = \frac{Pa^2b^3}{3EI\ell} \]
\[ \Delta_x \text{ (when } x < a) = \frac{Pb^2x^2}{6EI\ell} (3a\ell - 3ax - bx) \]

Figure 26  Continuous Beam – Two Equal Spans – Uniform Load on One Span

\[ R_1 = V_1 = \frac{7}{16} w\ell \]
\[ R_2 = V_2 + V_1 = \frac{5}{8} w\ell \]
\[ R_3 = V_1 = -\frac{1}{16} w\ell \]
\[ V_2 = \frac{9}{16} w\ell \]
\[ M_{\text{max}} \left(\text{at } x = \frac{7}{16} \ell\right) = \frac{49}{512} w\ell^2 \]
\[ M_1 \text{ (at support } R_2) = \frac{1}{16} w\ell^2 \]
\[ M_x \text{ (when } x < \ell) = \frac{wx}{16} (7\ell - 8x) \]
Figure 27  Continuous Beam – Two Equal Spans – Concentrated Load at Center of One Span

\[ R_1 = V_1 \quad \ldots \quad \ldots \quad \ldots = \frac{13}{32} P \]

\[ R_2 = V_1 + V_3 \quad \ldots \quad \ldots \quad \ldots = \frac{11}{16} P \]

\[ R_3 = V_3 \quad \ldots \quad \ldots \quad \ldots = -\frac{3}{32} P \]

\[ V_2 \quad \ldots \quad \ldots \quad \ldots = \frac{19}{32} P \]

\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \quad \ldots \quad \ldots = \frac{13}{64} P\ell \]

\[ M_1 \text{ (at support } R_1) \quad \ldots \quad \ldots \quad \ldots = \frac{3}{32} P\ell \]

Figure 28  Continuous Beam – Two Equal Spans – Concentrated Load at Any Point

\[ R_1 = V_1 \quad \ldots \quad \ldots \quad \ldots = \frac{Pb}{4\ell^1} \left(4\ell^1 - a(\ell + a)\right) \]

\[ R_2 = V_1 + V_3 \quad \ldots \quad \ldots \quad \ldots = \frac{Pa}{2\ell^1} \left(2\ell^1 + b(\ell + a)\right) \]

\[ R_3 = V_3 \quad \ldots \quad \ldots \quad \ldots = -\frac{Pab}{4\ell^1} (\ell + a) \]

\[ V_2 \quad \ldots \quad \ldots \quad \ldots = \frac{Pa}{4\ell^1} \left(4\ell^1 + b(\ell + a)\right) \]

\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \quad \ldots \quad \ldots = \frac{Pab}{4\ell^1} \left(4\ell^1 - a(\ell + a)\right) \]

\[ M_1 \text{ (at support } R_2) \quad \ldots \quad \ldots \quad \ldots = \frac{Pab}{4\ell^1} (\ell + a) \]
Figure 29  Continuous Beam – Two Equal Spans – Uniformly Distributed Load

\[ R_1 = V_1 = R_3 = V_3 = \frac{3wl}{8} \]
\[ R_2 = \frac{10wl}{8} \]
\[ V_2 = V_{\text{max}} = \frac{5wl}{8} \]
\[ M_1 = \frac{wl^2}{8} \]
\[ M_2 \left( \text{at } \frac{3l}{8} \right) = \frac{9wl^2}{128} \]
\[ \Delta_{\text{max}} \text{ (at } 0.4215l, \text{ approx. from } R_1 \text{ and } R_3) = \frac{wl^4}{185EI} \]

Figure 30  Continuous Beam – Two Equal Spans – Two Equal Concentrated Loads Symmetrically Placed

\[ R_1 = V_1 = R_3 = V_3 = \frac{5P}{16} \]
\[ R_2 = 2V_2 = \frac{11P}{8} \]
\[ V_2 = P - R_1 = \frac{11P}{16} \]
\[ V_{\text{max}} = V_2 \]
\[ M_1 = -\frac{3Pl}{16} \]
\[ M_2 = \frac{5Pl}{32} \]
\[ M_4 \text{ (when } x < a) = R_1x \]
Figure 31  Continuous Beam – Two Unequal Spans – Uniformly Distributed Load

\[ R_1 = \frac{M_1}{\ell_1} + \frac{wl_1}{2} \]
\[ R_2 = w\ell_1 + w\ell_2 - R_1 - R_3 \]
\[ R_3 = V_4 = \frac{M_1}{\ell_2} + \frac{wl_2}{2} \]
\[ V_1 = R_1 \]
\[ V_2 = w\ell_1 - R_1 \]
\[ V_3 = w\ell_2 - R_3 \]
\[ V_4 = R_1 \]
\[ M_1 = \frac{w\ell_1^3 + w\ell_2^3}{8(\ell_1 + \ell_2)} \]
\[ M_{x_1} \left( \text{when } x_1 = \frac{R_1}{w} \right) = R_1 x_1 - \frac{wx_1^2}{2} \]
\[ M_{x_2} \left( \text{when } x_2 = \frac{R_3}{w} \right) = R_3 x_2 - \frac{wx_2^2}{2} \]

Figure 32  Continuous Beam – Two Unequal Spans – Concentrated Load on Each Span Symmetrically Placed

\[ R_1 = \frac{M_1}{\ell_1} + \frac{P_1}{2} \]
\[ R_2 = P_1 + P_2 - R_1 - R_3 \]
\[ R_3 = \frac{M_1}{\ell_2} + \frac{P_2}{2} \]
\[ V_1 = R_1 \]
\[ V_2 = P_1 - R_1 \]
\[ V_3 = P_2 - R_3 \]
\[ V_4 = R_1 \]
\[ M_1 = -\frac{3}{16} \left( \frac{P_1 \ell_1^2 + P_2 \ell_2^2}{\ell_1 + \ell_2} \right) \]
\[ M_{x_1} = R_1 a \]
\[ M_{x_2} = R_1 b \]