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THE COLUMN ANALOGY

I. Introduction

1. *Purpose of the Monograph.*—The object of this bulletin is to present some theorems dealing with the elastic analysis of continuous frames. In ordinary cases these theorems are identical in form with the theorems, with which every structural engineer is familiar, for finding internal stresses in beams and struts.

The subject of structural mechanics is now experiencing demands for greater precision by the very accurate analyses demanded in the design of airplanes. In any case it is of the greatest importance to isolate definitely those matters which are sources of uncertainty from those which are certain and hence not proper fields of experiment.

Problems dealing with the analysis of restrained flexural members—straight beams, bents, arches—occupy a large space in structural literature. The treatment often presented involves complicated equations; in nearly all cases the method of solution is hard to remember.

If the elastic properties of the different portions of the structure are definitely known, the analysis of restrained members is essentially a problem in geometry, because the member must bend in such a way as to satisfy the conditions of restraint. The geometrical relations involved are identical in algebraic form with the general formula for determining fiber stress in a member which is bent.

Since the analysis of problems in flexure is a familiar procedure to structural engineers, it is advantageous to state the relations involved in the analysis of fixed-end beams, bents, and arches in terms of the beam formula. The advantages for structural engineers are similar to those which result from using the theorems of area-moments in finding slopes and deflections of beams; in some respects the concepts involved are identical, and the use of the beam formula in the analysis of restrained members may be thought of as an extension of the principles of area-moments. The general conception referred to in this monograph as the "column analogy" includes the principles of area-moments and also the conception of the conjugate beam.*

In this bulletin it is shown that bending moments in arches, haunched beams, and framed bents may be computed by a procedure

*See H. M. Westergaard, "Deflection of Beams by the Conjugate Beam Method," Journal of Western Society of Engineers, December, 1921.

analogous to the computation of fiber stresses in short columns subject to bending, and that slopes and deflections in these structures may be computed as shears and bending moments, respectively, on longitudinal sections through such columns.

The theorem makes available for the analysis of plane elastic structures the literature of beam analysis, dealing with the kern, the circle of inertia, the ellipse of inertia, graphical computations of moments and products of inertia, and conjugate axes of inertia.

Certain terms are defined in such a way that the method is extended to include the effect of deformations due to longitudinal stress and to shear in ribbed members, and to include trussed members.

The conceptions used in arch analysis by these methods make possible a general statement of the relations of joint displacements to joint forces, of which the familiar equation of slope-deflection* is a special case, and hence make possible the convenient extension of the method of slope-deflection, or of the theorem of three moments, to include curved members and members of varying moment of inertia.

The method here presented has application in the fields both of design and of research. In the field of design we use certain physical properties of the materials, which are necessarily assumed. In research we may either resort to the laboratory and study by empirical methods the properties of the structure as a whole, or we may study only the physical properties of the materials themselves, and depend on the geometrical relations to determine the properties of the structure. It seems obvious that the geometrical relations are not themselves a proper subject for experimental research.

The relations pointed out in this bulletin have at first been carefully restricted to geometry, and the assumptions which are necessary to apply this geometry to the design of structures are developed later in the discussion.

2. *Validity of Analyses by the Theory of Elasticity.*—The mathematical identity of the expressions for moment in an elastic ring and for fiber stress in a column section has some value in considering in a qualitative way the general validity of analyses based on the elastic theory.

It appears at times that engineers are not altogether discriminating in considering the value of elastic analyses, and seem to hold that one must either accept as precise the results of such analyses or reject entirely their conclusions.

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*"Analysis of Statically Indeterminate Structures by the Slope-Deflection Method," Univ. of Ill. Eng. Exp. Sta. Bul. 108, 1918.

Now no one but a novice accepts without discrimination the results of the beam formula. It is open to many important objections, such as lack of homogeneity of the material, effect of initial deformations, and other defects; and yet it is difficult to conceive of modern structural design existing without the beam formula, nor is anyone seriously disturbed because lack of homogeneity modifies somewhat the properties of the section, or by the fact that imperfect elasticity in the material makes invalid the superposition of stresses determined by the beam formula for different conditions of loading. Moreover the beam formula becomes a most inaccurate guide to the maximum stress in any section near the point of failure; and yet it is still true that one can scarcely conceive of modern structural design without the guidance of the beam formula.

Similarly we say that in an elastic structure the value of E may vary from section to section, that imperfect elasticity makes superposition of stresses not quite correct, and that near failure the method has only limited application. The normal process of structural design is to determine moments and shears, and from these fiber stresses. Whatever procedure is followed in the determination of the moments and shears, the beam formula is used for final determination of stress. There seem to be grounds for believing that the elastic analysis of an arch or bent with truly fixed or truly hinged ends has greater validity than does the method of analysis used later in design. The question of foundation distortion and of its effect involve engineering judgment. Elements involving judgment should be clearly isolated so that the limits of such judgment can be established.

3. *Acknowledgment.*—The bulletin was written as a part of the work of the Engineering Experiment Station of the University of Illinois, of which DEAN M. S. KETCHUM is the director, and of the Department of Civil Engineering, of which PROF. W. C. HUNTINGTON is the head. The computations were made by M. F. LINDEMAN, Research Graduate Assistant in Civil Engineering.

II. ANALYSES FOR FLEXURAL STRESS

4. *General Equation of Flexural Stress.*—Equations for stress due to flexure are usually based on the assumption that the variation of stress over the cross-section may be represented by a linear equation. This assumption is based on the assumption that the beam axis is straight and also on the assumption, based chiefly on experimental

observations, of the conservation of plane right sections and of the proportionality of stress to strain.

It will be shown later that none of these assumptions is necessary, and that the same general form of equation may be used whatever the facts as to variation of stress intensity over the section, provided the facts as to the shape assumed by deformed sections and the stress-strain relations are definitely known. For the present, however, a linear equation of stress variation over the section will be assumed. The stress will then have the general equation $f = (a + bx + cy)$, in which the coefficients a, b, c, are to be determined from the statical conditions which state that the sum of the fiber resistances must equal the applied load and that the sum of the moments of these fiber resistances about any axis in the plane of the section must equal the moment of the applied loads about that axis.

Let x, y = coördinates of any point on the cross-section along any two mutually perpendicular axes X and Y through the centroid of the section.

f = intensity of normal stress at point x, y

A = area of section

$I_x = \int x^2 dA$ = moment of inertia about axis Y (along the axis X)

$I_y = \int y^2 dA$ = moment of inertia about axis X (along the axis Y)

$I_{xy} = \int xy dA$ = product of inertia about axes X, Y

P = normal component of external forces

M_x = moment of external forces about axis Y

M_y = moment of external forces about axis X

Also write

$$M'_x = M_x - M_y \frac{I_{xy}}{I_y}$$

$$M'_y = M_y - M_x \frac{I_{xy}}{I_x}$$

$$I'_x = I_x - I_{xy} \frac{I_{xy}}{I_y}$$

$$I'_y = I_y - I_{xy} \frac{I_{xy}}{I_x}$$

All these terms are practically standard in the literature of flexure except the "skew" terms designated by primes.

Write: $f = a + bx + cy$

Then, from statics, $\Sigma V = 0$

whence: $P = \int f dA = a\int dA + b\int x dA + c\int y dA$

Since the axes are taken through the centroid, $\int x dA = 0$ and $\int y dA = 0$, by definition of centroid. Then $P = aA$, $a = \frac{P}{A}$.

The total moment about the axis of Y equals zero, whence

$$\begin{aligned} M_x &= \int f x dA = a\int x dA + b\int x^2 dA + c\int xy dA \\ &= bI_x + cI_{xy} \end{aligned} \tag{1}$$

The total moment about the axis of X equals zero, whence

$$\begin{aligned} M_y &= \int f y dA = a\int y dA + b\int xy dA + c\int y^2 dA \\ &= bI_{xy} + cI_y \end{aligned} \tag{2}$$

Multiplying (2) by $\frac{I_{xy}}{I_y}$, and subtracting from (1),

$$b = \frac{M_x - M_y \frac{I_{xy}}{I_y}}{I_x - I_{xy}\frac{I_{xy}}{I_y}} = \frac{M'_x}{I'_x}$$

Multiplying (1) by $\frac{I_{xy}}{I_x}$, and subtracting from (2),

$$c = \frac{M_y - M_x \frac{I_{xy}}{I_x}}{I_y - I_{xy}\frac{I_{xy}}{I_x}} = \frac{M'_y}{I'_y}$$

Hence

$$f = \frac{P}{A} + \frac{M'_x}{I'_x}x + \frac{M'_y}{I'_y}y \tag{3}$$

At the neutral axis $f = 0$. The equation of the neutral axis, then, is

$$\frac{P}{A} + \frac{M'_x}{I'_x}x + \frac{M'_y}{I'_y}y = 0$$

and the intercepts of the neutral axis are

$$x_1 = -\frac{\frac{P}{A}}{\frac{M'_x}{I'_x}} \qquad\qquad y_1 = -\frac{\frac{P}{A}}{\frac{M'_y}{I'_y}}$$

Equation 3 is a general equation of flexure for all cases if we assume linear variation of stress intensity. In order to apply it, it is first necessary to know the area and centroid of the section and I_x, I_y, I_{xy} for two mutually perpendicular axes through the centroid. The products of inertia should be computed first about the most convenient axes and then transferred to parallel axes through the centroid after this is located.

The foregoing brief statement of the formula contains all information necessary to its application. Like all mathematical relations, it is subject to unlimited variation of form. The general formula may be stated for principal axes, conjugate axes, neutral axis, kern axis; the S-polygon may be computed; the moments and products of inertia may be evaluated by summation, by integration, or graphically by use of the properties of string polygons; these moments and products of inertia may be combined to give the desired constants in the function for (f) by use of the circle of inertia or the ellipse of inertia, using in the latter case the polar properties of the ellipse. These are interesting lines of mathematical investigation; wherever they add appreciable convenience to the solution of the problem, any selected portions of the literature of flexure may be used.

It may sometimes be more convenient to concentrate all "prime" expressions in one term of the formula. Write

$$f = \frac{P}{A} + \frac{M'_x}{I'_x}x + \frac{M'_y}{I'_y}y = \frac{P}{A} + \frac{M_x}{I_x}x + \frac{M'_y}{I'_y}y - \left[\frac{M_x}{I_x} - \frac{M'_x}{I'_x}\right]x$$

the bracketed term reduces to

$$\frac{M_y - M_x\dfrac{I_{xy}}{I_x}}{I_y - I_{xy}\dfrac{I_{xy}}{I_x}} \cdot \frac{I_{xy}}{I_x} = \frac{M'_y}{I'_y} \cdot \frac{I_{xy}}{I_x}$$

We then write

$$f = \frac{P}{A} + \frac{M_x}{I_x}x + \frac{M'_y}{I'_y}y'$$

where $$y' = y - x\frac{I_{xy}}{I_y}$$

This expression will, however, usually be found less convenient than that given in Equation (3).

Some special cases of the general equation are more familiar.

If $I_{xy} = 0$, as in a symmetrical section,

$$f = \frac{P}{A} + \frac{M_x}{I_x}x + \frac{M_y}{I_y}y \qquad (3a)$$

If $P = 0$, and $I_{xy} = 0$,

$$f = \frac{M_x}{I_x}x + \frac{M_y}{I_y}y \qquad (3b)$$

If one of the axes is normal to the plane of bending, $I_{xy} = 0$, and $P = 0$.

$$f = \frac{M_x}{I_x}x, \text{ the usual beam formula.} \qquad (3c)$$

If there is no bending,

$$f = \frac{P}{A} \qquad (3d)$$

The kern of a section may be defined as that portion of the section within which a normal compressive force must act if all of the section is to be in compression. It is sometimes useful where it can be conveniently found.

If flexure is about one principal axis of a symmetrical section, the stress at the outer fiber is

$$f = \frac{P}{A} + \frac{M_x}{I_x}x$$

where x is the distance from the centroidal axis of Y to the outer fiber.

The distance e_k to the edge of the kern for this fiber can be found from

$$\frac{P}{A} + \frac{Pe_k}{I_x} x_1 = 0$$

$$e_k = -\frac{I_x}{Ax_1} = -\frac{\rho_x^2}{x_1}$$

where ρ_x is the radius of gyration.

Let M_k be the moment about the edge of the kern,

$$M_k = M_x + P\frac{I_x}{Ax_1}$$

Then
$$f = \frac{P}{A} + \frac{M_x}{I_x}x = \frac{x}{I_x}\left(M_x + P\frac{I_x}{Ax_1}\right) = \frac{M_k}{I_x}x \qquad (6)$$

In computing the stresses in frames it is often convenient to compute the moments about the kern points and apply formula (6) to obtain the stress directly.

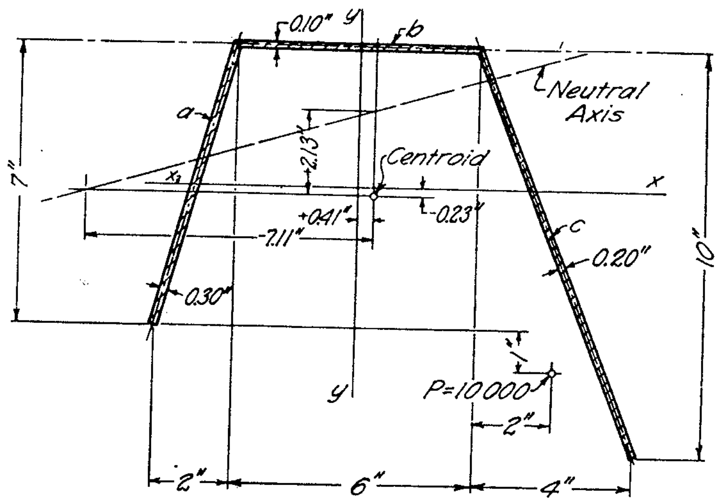
5. *Form of Computation Recommended.*—The arrangement of computations in applying the general equation for flexural stress is important where the computations are involved. It is convenient to first compute the properties of the section and the bending moments with reference to some convenient pair of axes. These properties are later corrected for axes through the centroid of the section and then corrected for lack of symmetry of the section.

A few illustrations are given to indicate the arrangement recommended. This arrangement is emphasized here because of its use later in computations required in analyses of restrained beams.

Compressive fiber stresses are taken as positive and loads which produce compression as positive. Coördinates will be taken as positive when measured upward or to the right from the axes. Positive moment produces compression on the positive side (top or right-hand side) of the section.

Problem 1.—Unsymmetrical Section Unsymmetrically Loaded

It is desired to find the neutral axis and a measure of the fiber stress in the section shown in Fig. 1. The section is unsymmetrical, and the elements have widths so small that their moments of inertia about their longitudinal axes may be neglected. The section is loaded at point P, shown in the figure with a load of 10 000 lb.



Properties of the Section											Load				
Given					Computed						Given			Computed	
Member	Length	Width	x	y	Area	Statical Moments		Products of Inertia							
					a	ax	ay	ax^2+i_x	ay^2+i_y	$axy+i_{xy}$	P	x	y	M_x	M_y
a	7.29	0.30	-4	0	2.187	-8.748	0	35.0 0.7	0 8.9	0 +2.6					
b	6.00	0.10	0	+3.5	0.600	0	+2.100	0 1.8	7.4 0	0 0					
c	10.79	0.20	+5	-1.5	2.158	+10.790	-3.237	54.0 2.9	4.9 18.0	-16.2 -7.2					
Correct to Centroid			+0.41	-0.23	4.945	+2.042	-1.137	94.4 0.8	39.2 0.3	-20.8 -0.5	+10000	+5	-4.5	+50000 +4100	-45000 -2300
Correct for Dissymmetry								93.6 10.6	38.9 4.4	-20.3				+45900 +22300	-42700 -10000
								83.0	34.5		+10000			+23600	-32700
Intercepts for N.A			-7.11	+2.13	$x_1 = -\frac{\frac{P}{A}}{\frac{M_x'}{I_x'}}$, $y_1 = -\frac{\frac{P}{A}}{\frac{M_y'}{I_y'}}$										

Fiber Stress at any Point $= \frac{M_y'}{I_y'} \times$ (Vertical Distance to Neutral Axis)

$= \frac{-32700}{34.5} \times$ (Vertical Distance to Neutral Axis)

or $= \frac{M_x'}{I_x'} \times$ (Horizontal Distance to Neutral Axis)

$= \frac{23600}{83.0} \times$ (Horizontal Distance to Neutral Axis)

FIG. 1. EXAMPLE OF AN UNSYMMETRICAL SECTION

Assume two convenient axes, in this case a horizontal axis X-X through the centroid of the portion of the section marked a and a vertical axis Y-Y through the centroid of the portion of the section marked b.

Tabulate first the known properties of the section and of the load. These are a description or key letter for each member, the length, the width, the coördinates x and y to the centroid of each member, the load, the coördinates of the load, or the moments about each of the two axes.

Now compute the statical moments of the areas of the members about each of the two axes.

Also compute the products of inertia.* These are computed as area times product of coördinates of the centroid plus the centroidal product of inertia. The latter equals $\frac{1}{12}$ area times product of projections on the two axes. Signs are entirely automatic. It should be remarked, however, that a product of inertia about the centroid is positive when $\frac{dy}{dx}$ for the member is positive, that is, when the member slopes up to the right.

Add columns to get total area, total statical moments, total products of inertia. If there are several loads or moments, add these also for total load and total moments.

Correct to the centroid. Divide the statical moments by the area to give the coördinates, $\bar{x} = +0.41$, $\bar{y} = -0.23$, of the centroid. Compute $A\bar{x}^2 = 0.8$, $A\bar{y}^2 = 0.3$, $A\overline{xy} = -0.5$. Also compute $P\bar{x} = +4100$, $P\bar{y} = -2300$.

The signs are still automatic. Subtract the corrections to give products of inertia and moments about the centroidal axes.

Correct for dissymmetry. Compute $I_{xy}\frac{I_{xy}}{I_y} = +10.6$, and $M_y\frac{I_{xy}}{I_y} = +22\ 300$, $I_{xy}\frac{I_{xy}}{I_x} = +4.4$, and $M_x\frac{I_{xy}}{I_x} = -10\ 000$.

The signs are automatic here also. Again subtract the corrections.

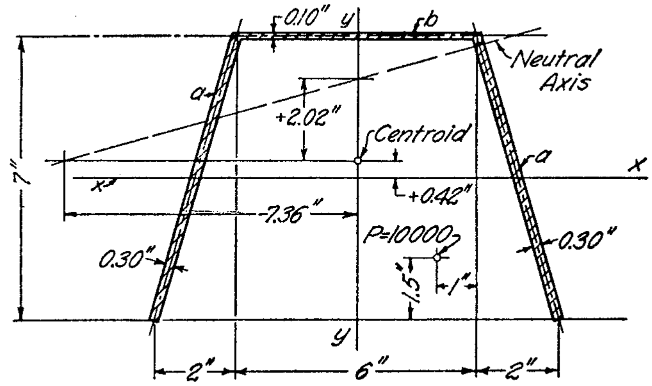
Now compute

$$x_1 = -\frac{\frac{P}{A}}{\frac{M'_x}{I'_x}} = -7.11,\ y_1 = -\frac{\frac{P}{A}}{\frac{M'_y}{I'_y}} = +2.13$$

These are the intercepts of the neutral axis on axes through the centroid parallel to the original axes.

The fiber stress at any point may be found by multiplying the vertical distance from the neutral axis by $\frac{M'_y}{I'_y} = -\frac{32\ 700}{34.5}$ or by

*A moment of inertia is merely a special case of product of inertia where the same axis is taken twice.



Properties of the Section											Load				
Given					Computed						Given			Computed	
Member	Length	Width	x	y	Area a	Statical Moments ax	Statical Moments ay	Products of Inertia ax^2+i_x	Products of Inertia ay^2+i_y	Products of Inertia $axy+i_{xy}$	P	x	y	M_x	M_y
a	7.29	0.30	-4	0	2.187	-8.748	0	35.0 0.7	0 8.9						
b	6.00	0.10	0	+3.5	0.600	0	+2.100	0 1.8	7.4 0						
a	7.29	0.30	+4	0	2.187	+8.748	0	35.0 0.7	0 8.9						
Correct to Centroid			0	+0.42	4.974	0	+2.100	73.2 0	25.2 0.9		+10000	+2	-2	+20000 0	-20000 +4200
								73.2	24.3					+20000	-24200
Intercepts for N.A.			-7.36	+2.02	$x_1 = -\frac{P/A}{M'_x/I'_x}$, $y_1 = -\frac{P/A}{M'_y/I'_y}$										

Fiber Stress at any Point $= \frac{M'_y}{I'_y} \times$ (Vertical distance to neutral axis)

$= \frac{-24200}{24.3} \times$ (Vertical distance to neutral axis)

or $= \frac{M'_x}{I'_x} \times$ (Horizontal distance to neutral axis)

$= \frac{+20000}{73.2} \times$ (Horizontal distance to neutral axis)

FIG. 2. EXAMPLE OF A SYMMETRICAL SECTION, UNSYMMETRICALLY LOADED

multiplying the horizontal distance of the point from the neutral axis by $\frac{M'_x}{I'_x} = \frac{+23\,600}{83.0}$.

It will be observed that the sign of the stress is also automatic. Since $\frac{M'_y}{I'_y}$ is negative, positive stress (compression) exists below the neutral axis. Similarly, since $\frac{M'_x}{I'_x}$ is positive, compression exists to the right of the neutral axis.

Problem 2.—Symmetrical Section Unsymmetrically Loaded

If the section is symmetrical, the same order of procedure is followed, except that there is no product of inertia about the two axes and no correction for dissymmetry.

The problem just solved has been somewhat modified in Fig. 2 for a symmetrical section. The computations follow the order of the previous problem.

Problem 3.—Symmetrical Section Symmetrically Loaded

If, further, the section is loaded symmetrically about the axis of symmetry, computations of the properties of the section about that axis may be omitted. An illustration is shown in Fig. 3.

6. *Transformed Section.*—In deriving Equation (3) for fiber stress due to flexure it has been assumed that f is a linear function of x, y. The same type of expression may also be derived if fn is a linear function of x, y where n varies for different points on the section. In this case the equations of statics are satisfied if in place of dA we write $\frac{dA}{n}$ so that $fn\frac{dA}{n} = fdA$. The differential areas dA are then to be divided by the values of n.

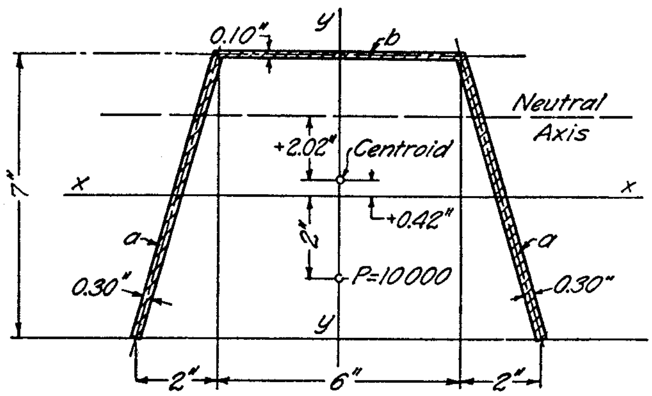
Let
$$A_t = \int \frac{dA}{n} \qquad I_{y_t} = \int \frac{dA}{n}\, y^2$$
$$I_{x_t} = \int \frac{dA}{n}\, x^2 \qquad I_{xy_t} = \int \frac{dA}{n}\, xy$$

where x and y are measured from the centroid of the section having differential areas $\frac{dA}{n}$.

The section thus defined, in which each area dA is replaced by an area $\frac{dA}{n}$, may be called the "transformed section." For it we may derive, by the algebraic process used in deriving Equation (3),

$$nf = \frac{P}{A_t} + \frac{M'_x}{I'_{x_t}}\, x + \frac{M'_y}{I'_{y_t}}\, y \tag{4}$$

the "skew" terms indicated by primes being defined in their relations



Properties of the Section											Load				
Given					Computed						Given			Computed	
					Area	Statical Moments		Products of Inertia							
Member	Length	Width	x	y	a	ax	ay	ax^2+i_x	ay^2+I_y	$axy+i_{xy}$	P	x	y	M_x	M_y
a	7.29	0.30		0	2.187		0		0 8.9						
b	6.00	0.10		+3.5	0.600		+2.100		7.4 0						
a	7.29	0.30		0	2.187		0		0 8.9						
Correct to Centroid				+0.42	4.974		+2.100		25.2 0.9		+10000	0	-2		-20000 + 4200
									24.3		+10000				-24200
Intercepts of N.A.				+2.02	$y_1 = -\frac{\frac{P}{A}}{\frac{M'_y}{I'_y}}$										

Fiber Stress at any Point $= \frac{M'_y}{I'_y} \times$ (Vertical distance to neutral axis)

$= \frac{-24200}{24.3} \times$ (Vertical distance to neutral axis)

FIG. 3. EXAMPLE OF A SYMMETRICAL SECTION, SYMMETRICALLY LOADED

to the other terms just as in Equation (3). From the values of nf we may get $f = \frac{nf}{n}$.

This method has been used in analyzing beams of reinforced concrete on the usual assumptions as to the mechanics of such beams. In this case it is assumed that plane sections remain plane and hence that $\frac{f}{E}\,ds$ is a linear function of x, y. If the two sections between which deformation occurs are parallel, as in a straight beam, ds is constant across the section. Hence $\frac{f}{E}$ is a linear function of x, y.

The value of E here may be taken as relative. The modulus of concrete in compression is taken as unity, that of concrete in tension as zero (unlimited strain without stress), and that of the steel as $\frac{E_s}{E_c}$. The section then is transformed by multiplying the area of the steel by $\frac{E_s}{E_c}$, and that of the concrete in tension by zero. Finally, the apparent stress found in the concrete in tension is multiplied by zero, and that in the steel by $\frac{E_s}{E_c}$.

This method has been traced out in detail in this elementary case because it has broader usefulness. If the stress-strain diagram is known for any material it is a simple matter to deduce stresses for given loads on a section if we accept the assumption that plane sections remain plane. In such a case we would first analyze on the assumption of Hooke's Law, then transform the section by multiplying each differential area by the relative value of E corresponding to the stress just deduced. This transformed section is then analyzed and the deduced stresses multiplied by the relative value of E. Successive revisions will furnish any desired precision.

Another important application occurs in the case of beams having sharp curvature. Here it is usual to assume that plane sections remain plane and also that Hooke's Law holds. Hence $\frac{fds}{E}$ is a linear function of x, y. The modulus E is constant, but ds varies across the section, since the two sides of the differential element are not parallel.

The value of ds varies as the distance of each fiber from the center of curvature of the beam (see Fig. 4). We may, then, transform the section by dividing the width at any point by the radius of curvature R at that point. The properties of this transformed section are then determined. It will be shown later that these may all be deduced from the known properties of the original section and the area of the transformed section.

The values Rf are then found from Equation (4). These values of Rf are then divided by the values of R to give the fiber stresses.

It should be noted here that the moment to be used in this case is the moment about the centroid of the transformed section and not about the centroid of the original section. The centroid of the original section has no special significance in this problem, and certain awkward characteristics of the so-called Winkler-Bach formula arise from neglect of this fact.

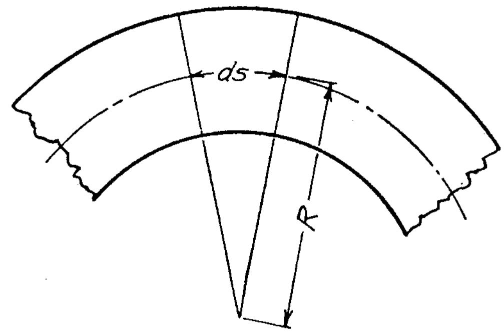


FIG. 4. SEGMENT OF A CURVED BEAM

In the usual treatment of curved beams the proportionality of stress to strain has been assumed. This, however, is not necessary. If Hooke's Law does not apply, but right sections plane before bending remain plane after bending, the value of $\frac{fds}{E}$ is a linear function of x, y and the section may be transformed by dividing the differential areas by $\frac{ds}{E}$. Hence sharply curved members of reinforced concrete, such as sometimes occur, may be analyzed by use of the transformed section. Brackets in rigid frames approximate this condition.

The transformed section could also be used either in combination with or without Hooke's Law to deal with cases in which plane sections do not remain plane in beams either straight or curved, provided we know anything or wish to assume anything as to the shape assumed by the plane after bending. In this case $\frac{fds}{E}$ is not a linear function of x, y but $\frac{fds}{rE}$ is such a function, where $r = \frac{\epsilon}{\epsilon'}$ defines the curvature of the deformed section, as shown in Fig. 5. The section

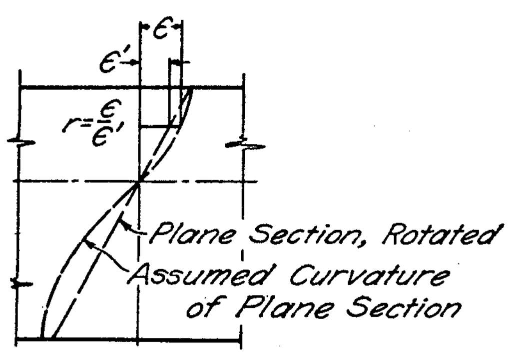


FIG. 5. RATIO, $r = \frac{\epsilon}{\epsilon'}$

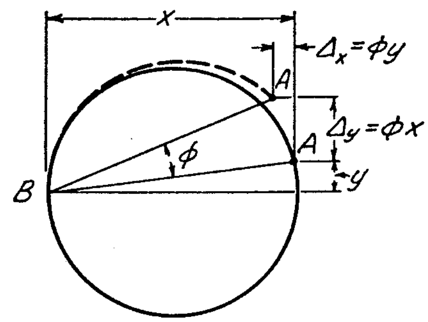


FIG. 6. CLOSED RING, CUT AND SUBJECTED TO AN ANGLE CHANGE

can then be transformed by multiplying each differential area by $\frac{Er}{ds}$.

A full understanding of the theory of the transformed section seems desirable in computing stresses in any case in which the stress distribution over the section is not linear. It has special advantage in such cases in dealing with problems involving continuity or deformations because in such cases if the transformed section is used the same methods are applicable in the analysis by the theory of elasticity as would be used where the stress distribution is linear. This will be explained in detail later.

III. THE COLUMN ANALOGY

7. *Formulas Similar to Flexure Formula.*—The flexure formula, then, is a solution not only of the problem to which it is usually applied of determining stresses where the variation of stress over the cross-section is linear and also, as shown in discussing the use of transformed sections, where this variation is not linear, but is a general type for the solution of certain algebraic problems. It will be shown that the problem of analyzing arches, bents, and beams with fixed ends in general is such a problem. The flexure formulas, at least for the special case of symmetrical flexure, are familiar tools to structural engineers, and therefore furnish a conveniently remembered routine in treating certain problems in elastic analysis.

8. *Geometry of Continuity.*—Consider any closed ring as shown in Fig. 6. Suppose this ring to be cut at A and that a certain rotation ϕ takes place at B.

At A there is now produced

(a) A relative rotation of the two sides of the cut, $= \phi$

(b) A relative vertical displacement of the two sides of the cut, $= \phi x$

(c) A relative horizontal displacement of the two sides of the cut, $= \phi y$

If the ring is continuous there is actually no relative movement of the two sides of the cut. Distinguish the rotations which occur around the ring as those which would occur if the ring were cut and those which result from the continuity at A. Call the first ϕ_s—rotations due to the forces which are statically determined—and the latter ϕ_i—rotations due to forces which are statically indeterminate. Then, if continuity exists,

$$\Sigma\phi_s = \Sigma\phi_i$$
$$\Sigma\phi_s x = \Sigma\phi_i x$$
$$\Sigma\phi_s y = \Sigma\phi_i y$$

The rotations ϕ_i are due to equal and opposite forces on the two sides of the cut section at A. Any moments in the ring due to these forces will be a linear function of x, y. Call these moments m_i indeterminate moments. If for ϕ_i we write $m_i \dfrac{\phi_i}{m_i ds} ds$,

then
$$\int m_i \frac{\phi_i}{m_i ds} ds = \int \phi_s \qquad \text{(a)}$$

$$\int m_i \frac{\phi_i}{m_i ds} xds = \int \phi_s x \qquad \text{(b)}$$

$$\int m_i \frac{\phi_i}{m_i ds} yds = \int \phi_s y \qquad \text{(c)}$$

where m_i is a linear function of x, y. (d)

The values on the right-hand side of the equations are assumed to be known constants.

These relations are satisfied by an equation of which the equation of flexure is a type.

Thus we may write $m_i = a + bx + cy$ where a, b, and c are unknown coefficients to be determined from the three relations of continuity just given. Designate the physical constants $\dfrac{\phi_i}{m_i ds}$—the rotations per unit of moment per unit of length—by the letter w (for width of the elastic section as defined later).

Then $\int m_i w ds = a\int wds + b\int wdsx + c\int wdsy = \int \phi_s$

If the axes are taken through the centroid as defined by the equations $\int wdsx = 0$, and $\int wdsy = 0$, $a = \dfrac{\int \phi_s}{\int wds}$

$$
\begin{aligned}
\text{Also } \int m_i w ds x &= a\int wdsx + b\int wdsx^2 + c\int wdsxy \\
\text{or} \qquad\qquad &= b\int wdsx^2 + c\int wdsxy \qquad = \int \phi_s x \quad (1) \\
\text{and} \quad \int m_i w ds y &= a\int wdsy + b\int wdsxy + c\int wdsy^2 \\
\text{or} \qquad\qquad &= b\int wdsxy + c\int wdsy^2 \qquad = \int \phi_s y \quad (2)
\end{aligned}
$$

Multiplying (2) by $\dfrac{\int wdsxy}{\int wdsy^2}$ and subtracting from (1)

$$
b = \frac{\int \phi_s x - \int \phi_s y \dfrac{\int wdsxy}{\int wdsy^2}}{\int wdsx^2 - \int wdsxy \dfrac{\int wdsxy}{\int wdsy^2}}
$$

Multiplying (1) by $\dfrac{\int wdsxy}{\int wdsx^2}$ and subtracting from (2)

$$
c = \frac{\int \phi_s y - \int \phi_s x \dfrac{\int wdsxy}{\int wdsy^2}}{\int wdsy^2 - \int wdsxy \dfrac{\int wdsxy}{\int wdsy^2}}
$$

Hence

$$
m_i = \frac{\int \phi_s}{\int wds} + \frac{\int \phi_s x - \int \phi_s y \dfrac{\int wdsxy}{\int wdsy^2}}{\int wdsx^2 - \int wdsxy \dfrac{\int wdsxy}{\int wdsy^2}}\, x
$$

$$
+ \frac{\int \phi_s y - \int \phi_s x \dfrac{\int wdsxy}{\int wdsx^2}}{\int wdsy^2 - \int wdsxy \dfrac{\int wdsxy}{\int wdsx^2}}\, y
$$

If we conceive a narrow strip along the axis of the arch having a variable width $w = \dfrac{\phi_i}{m_i ds}$ then it is evident that wds corresponds to a differential area. If we treat this whole strip as an area A, we may conveniently write

$$
\begin{aligned}
\int wds &= A \\
\int wdsx^2 &= I_x \\
\int wdsy^2 &= I_y \\
\int wdsxy &= I_{xy} \\
I_x - I_{xy}\frac{I_{xy}}{I_y} &= I'_x \\
I_y - I_{xy}\frac{I_{xy}}{I_x} &= I'_y
\end{aligned}
$$

Also ϕ_s corresponds to a load. If we call these known rotations along the axis elastic loads, we may conveniently write

$$\int \phi_s = P$$
$$\int \phi_s x = M_x$$
$$\int \phi_s y = M_y$$

and

$$M_x - M_y \frac{I_{xy}}{I_y} = M'_x$$

$$M_y - M_x \frac{I_{xy}}{I_x} = M'_y$$

Hence

$$m_i = \frac{P}{A} + \frac{M'_x}{I'_x} x + \frac{M'_y}{I'_y} y$$

9. *The Analogy.*—If the effects of flexure only are taken into account there exists, then, an exact algebraic parallel between the indeterminate moments in an elastic ring and the fiber stresses on plane normal sections of a short column, as follows:

1. The stiffness $\frac{\phi}{m}$ of each short length of axis corresponds to a differential area a.
2. The indeterminate moments m_i correspond to stress intensities on the column section.
3. The known angle changes correspond to loads P on the column section. These angle changes may be due to external forces acting on the structure, or to other causes, such as rotations of abutments.

Consider any single-span plane structure, with axis either straight or curved, and with any variation in cross-section, subjected to known loads. Draw any curve of moments for these loads consistent with static equilibrium.

Picture a short length of column, a section of which has the same shape as the side elevation of the beam axis and a very small width varying along the axis as the elastic width defined above ($w = \frac{\phi}{mds}$, angle change for unit moment along a unit length).

Load this column with an intensity of load over these elastic areas equal to the bending moment given by the curve of moments just computed. The change in moment produced by restraint will now equal the fiber stresses which would exist on cross-sections of this column. The total moment at any point equals the net intensity of pressure—difference between load intensity at top and reaction

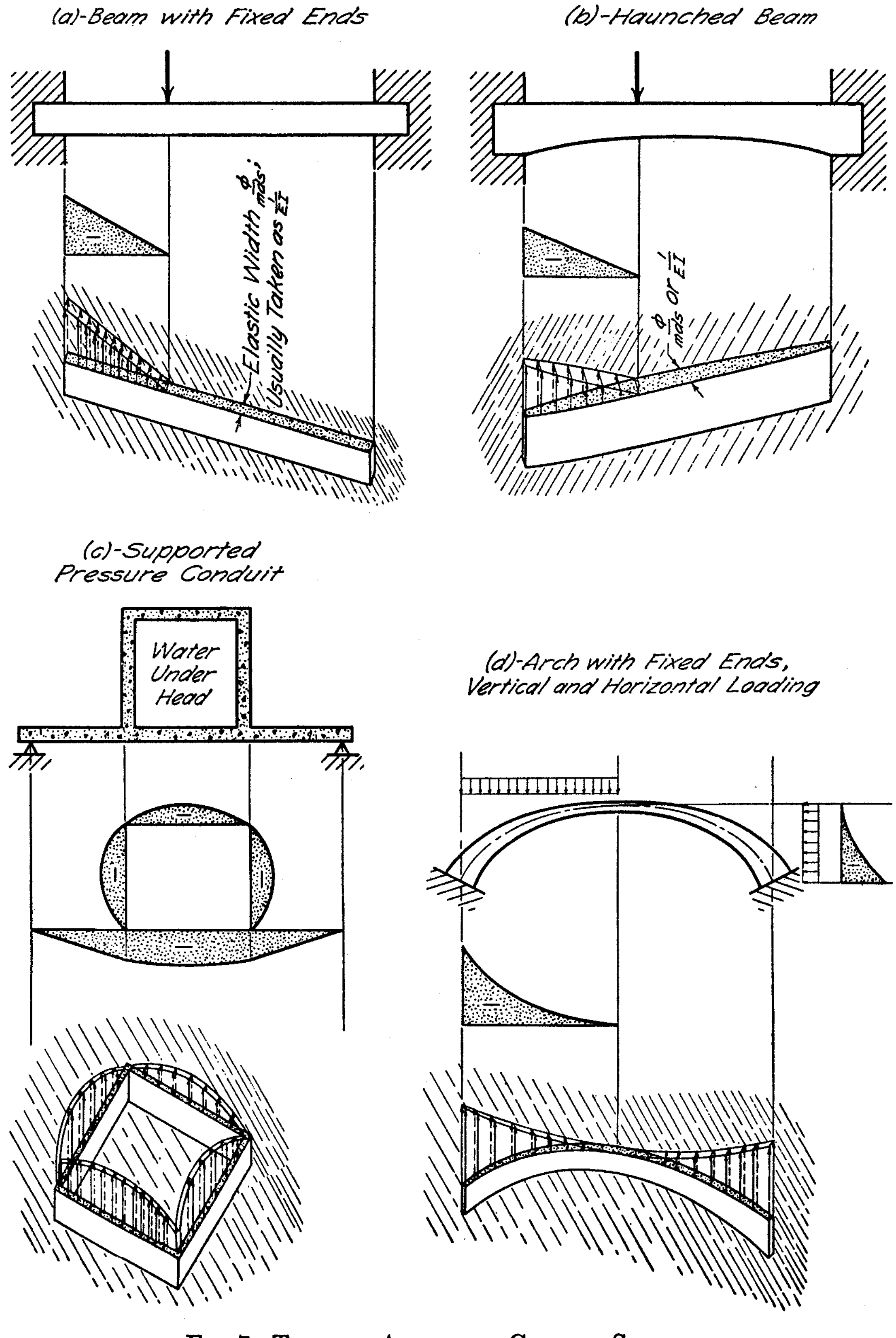


FIG. 7. TYPES OF ANALOGOUS COLUMN SECTIONS

intensity at bottom—at the corresponding point on this column section.

We have seen that a rotation in the beam corresponds to a load on the analogous column. Equal and opposite rotations about two centers produce linear movement normal to the line of centers. Hence, in the analogy, a linear displacement in the beam corresponds to a couple on the analogous column about the axis of displacement.

These rotations and displacements in the beam may be due to any cause whatever, abutment displacement, temperature change, forced distortion by jacking, or they may be imaginary displacements generating influence lines according to the principle of influence lines as stated by Müller-Breslau.

10. *The Elastic Column and Its Load.*—The analogy just stated furnishes a convenient mental picture of the relations of the moments in restrained members. In Fig. 7 are shown some sketches corresponding to these conceptions. The shaded area is the section of the column of any short length.

In Fig. 7a a straight beam of constant section and fixed at the ends is loaded with a single concentrated load. The most convenient curve of moments for such a load is obtained by treating the load as cantilevered from the nearer support. The analogous column has a width equal—or proportional—to the value $\frac{\phi}{mds}$ or $\frac{1}{EI}$ for the beam. The intensity of load on the column equals the moment curve just drawn.

In this case the average load intensity and the position of the resultant are known by inspection, and hence the total load and its moment are readily computed. Computations of the pressure intensity on the base of this short column will give the changes in bending moment resulting from the fact that the beam is not cantilevered but is fixed in position and direction at both ends.

In Fig. 7b the axis is assumed to be straight but the beam is not of constant section. The width of the column is, then, not constant, and the average load intensity and the position of the resultant load are not evident by inspection. The problem is conveniently solved by dividing the column section into a number of small lengths.

In Fig. 7a and 7b the elastic rings are closed by the earth, which has zero elastic area. In Fig. 7c the elastic ring is completed by the structure. The curve of moments due to the weight of water and to the pressure head is drawn on the assumption that the top and sides are simple beams—a stable condition. The elastic column and its load are as shown. Pressure intensities on the base of the column are changes in moment due to the fact that there are no hinges at the ends of the top and sides, but that there is really complete continuity at these points.

In Fig. 7d an arch is subjected to vertical loading on one side, and to horizontal loading on the other side. Moment curves are drawn independently for the two conditions of loading, the arch being assumed cantilevered from the left abutment for the vertical load, and from the right abutment for the horizontal load. Note that it is not necessary to assume the same statically determinate condition for the beam for all parts of the load.

11. *Signs in the Column Analogy.*—In flexural analyses it is convenient to consider compressive force and stress as positive and to measure coördinates as positive up and to the right from the centroid. Positive loads on columns, then, are downward, and positive couples are such as produce compression above and to the right of the centroid of the column section.

In applying the column analogy bending moments in the beam will be considered positive if they produce tension on the inside of the elastic ring. Shears are positive if they accompany positive rate of change of bending-moment, increase up or to the right.

Positive rotations are such as would accompany positive bending moments; positive displacements are such as would result from positive shearing forces. A clockwise abutment rotation at the left end of an arch, then, is positive; at the right end it is negative. An increase in arch span is a positive horizontal displacement. Settlement of the right abutment is a positive vertical displacement.

Note, however, that rotations and displacements which actually exist (abutment movements) are to be distinguished from those which are resisted (shrinkage or distortions due to temperature changes). In determining moments by the column analogy in the former case, the sign of the displacement may be conveniently reversed.

12. *Choice of Statically Determined Moments.*—The indeterminate forces are equal and opposite on two sides of any section, and hence satisfy the laws of statics whatever the determinate forces may be. Also the indeterminate forces are computed to have values such that continuity is preserved. Hence any set of determinate forces whatever which will support the loads may be chosen in the first place, since the conditions of both statics and continuity will be satisfied by the solution.

It is, of course, important to choose the most convenient curve of determinate moments. As a simple example, in the case of a fixed beam with a moment load, (upper part of Fig. 8) the four sets of determinate systems shown, among many, are available. In general the first or second moment curves will be more convenient.

The moment at x may be written directly by finding the indeterminate moment from the moment curve shown in (a) or in (b). Similarly the end moments may be written directly. To write M_b we use the moment curve in (a) and to write M_a we use the moment curve in (b).

In the lower part of Fig. 8 are shown three of the possible elastic loads on a rectangular bent carrying a single concentrated load. In the first case the curve of statically determinate moments is drawn as though the load were cantilevered from the left support, in the second as though the girder were simply supported on the columns, and in the third case as though the load were cantilevered from the right support. The indeterminate forces and moments will be different in the three cases, but the total moments will, of course, be the same. The second moment curve will probably be found most convenient.

13. *Components and Direction of Indeterminate Forces.*—The problem of analysis of rigid frames and arches is essentially that of finding the moments due to the indeterminate forces which are necessary to preserve continuity. These moments equal zero along the line of application of the indeterminate forces. This line of action, therefore, corresponds to the line of zero stress in the analogous column. This is the neutral axis for the given loading, and, as already found, it has intercepts on the axes through the centroid of $x_1 = -\dfrac{P/A}{\dfrac{M'_x}{I'_x}}$ and $y_1 = -\dfrac{P/A}{\dfrac{M'_y}{I'_y}}$.

The stress in a beam at unit horizontal distance from the neutral axis is $\dfrac{\delta f}{\delta x} = \dfrac{M'_x}{I'_x}$, and at unit vertical distance from the neutral axis is $\dfrac{\delta f}{\delta y} = \dfrac{M'_y}{I'_y}$.

The moment at unit vertical distance from the line of action of the indeterminate forces equals the horizontal component of the indeterminate force, and the moment at unit horizontal distance from the neutral axis equals the vertical component. Hence $h_i = \dfrac{M'_y}{I'_y}$ $v_i = \dfrac{M'_x}{I'_x}$.

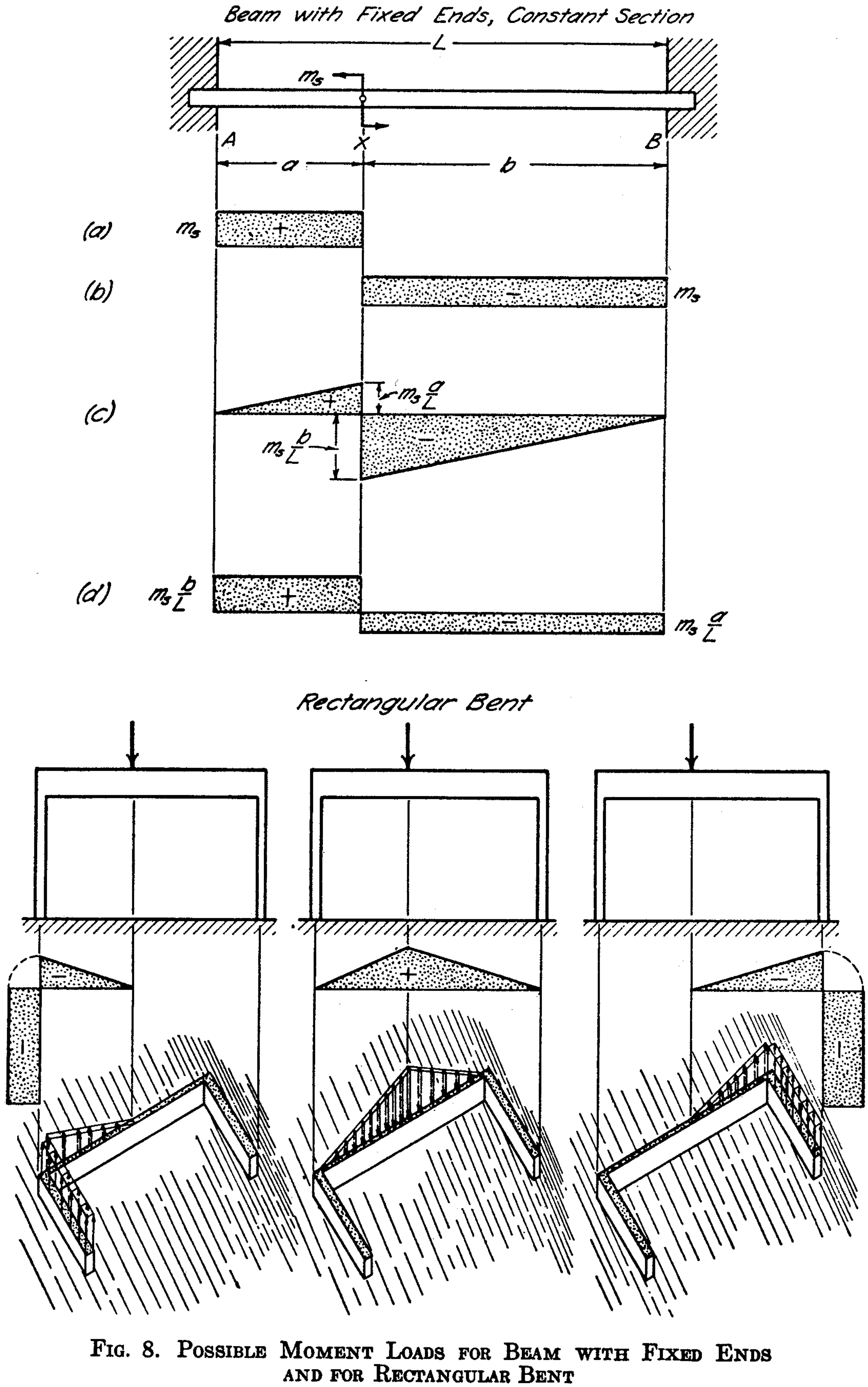


Fig. 8. Possible Moment Loads for Beam with Fixed Ends and for Rectangular Bent

A convenient basis for the computation of all indeterminate moments is to locate first the neutral axis of the analogous column and find one component along the axis.

The components of the indeterminate forces are changes in the reactions from those existing under the static conditions assumed. We may find the components of the reactions as $V = v_s - v_i$ $H = h_s - h_i$.

With the components of the reactions known it becomes a simple matter to draw the pressure line.

The pressure line for the structure may sometimes be drawn conveniently by superimposing the curve of statically determinate moments on the neutral axis to a vertical scale of $h_i = 1$ or to a horizontal scale of $v_i = 1$ according to whether the moment is laid off horizontally or vertically.

14. *Application to Simple Cases.*—The principle of the column analogy may be illustrated by application to a few simple cases. Let it be required to compute the end moments on the beam with fixed ends shown in Fig. 9a, assuming constant moment of inertia. Consider the moment curve shown, produced by the load P on a beam simply supported at its ends. Any static moment curve such as a cantilever over length (a) or (b) might equally well have been used. The centroid of a triangle using the notation of the figure may be shown to lie at a distance $\frac{L + a}{3}$ from one end.

In this case, then, the analogous column section is a narrow strip of length L and width $\frac{1}{EI}$. Both E and I, being assumed constant in this case, may be given a relative value of unity. The column section thus becomes simply L and the load $\frac{mL}{2}$. The outer fiber stresses in the column, analogous to the end moments, may be found by the usual column formula, or more conveniently in this case by taking moments about the kern points. Then, $f = \frac{M_k y}{I_o}$ where M_k is the moment about the opposite kern point, and I_o is the moment of inertia of the column section about its centroid.

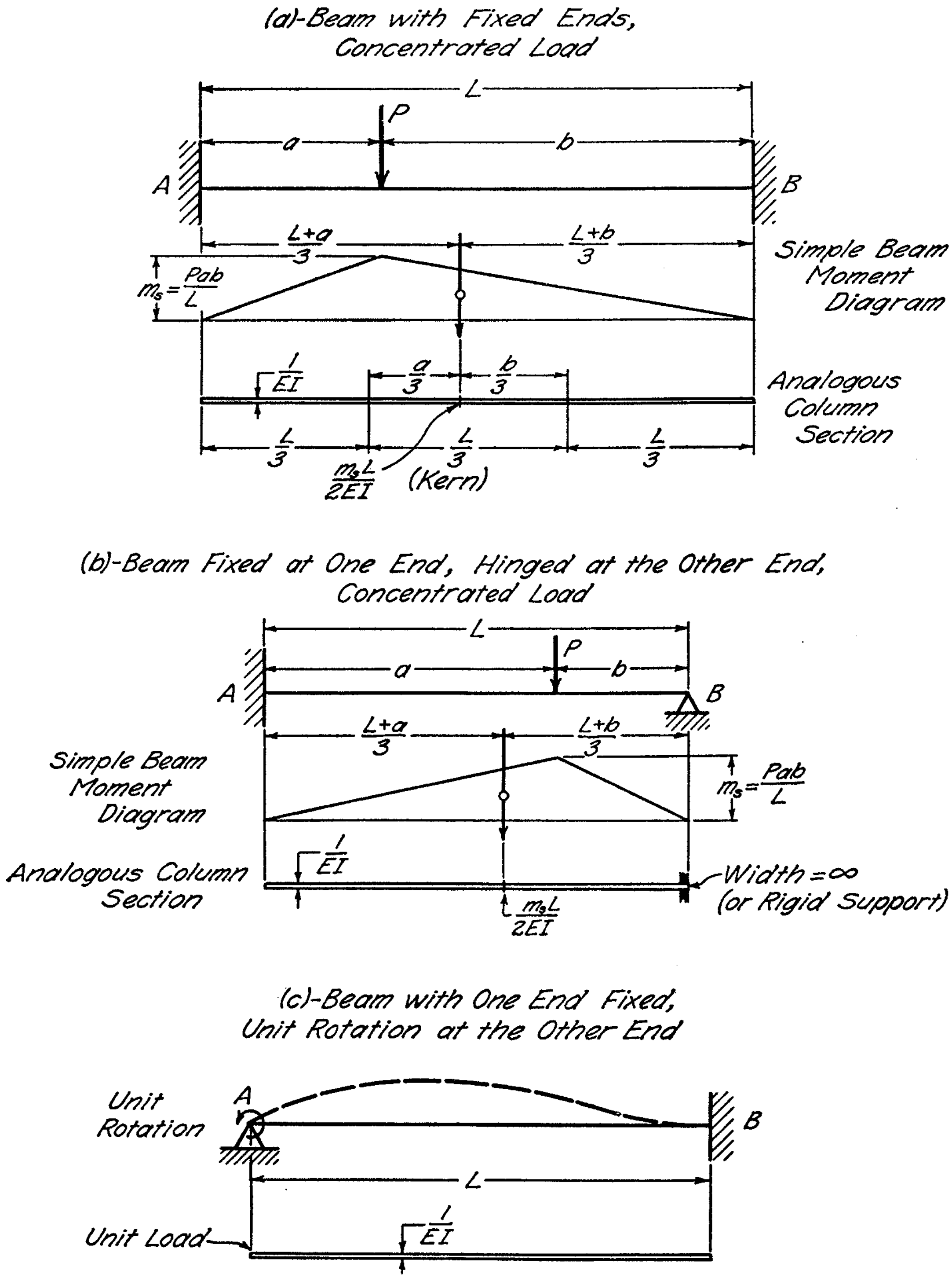


FIG. 9. APPLICATION OF THE PRINCIPLE OF COLUMN ANALOGY TO SIMPLE CASES

Further, for rectangular columns, $f = \dfrac{M_k y}{I_o} = \dfrac{M_k \dfrac{d}{2}}{\frac{1}{12}bd^3} = \dfrac{6M_k}{Ad}$

Then, at the left end, $f_a = M_a = \dfrac{6\dfrac{mL}{2}\cdot\dfrac{b}{3}}{L\,.\,L} = m_s\,\dfrac{b}{L}$.

And, at the right end, $f_b = M_b = M_s\,\dfrac{a}{L}$ where m_s is the simple beam moment, $m_s = \dfrac{Pab}{L}$.

For a beam of uniform section fixed at A and hinged at B, subject to a single concentrated load P, Fig. 9b, the hinge has an infinite elastic area (unlimited rotation due to a moment) and hence both the centroid and kern point of the infinite column section lie at the hinge.

$$\text{Whence } M_a = \frac{M_k y}{I_o} = \frac{\dfrac{mL}{2}\cdot\dfrac{L+b}{3}\cdot L}{\dfrac{L}{3}\cdot L^2} = m_s\,\frac{L+b}{2L}$$

Suppose the moment corresponding to unit rotation of the free end is required at each end of a supported beam which is fixed at one end, Fig. 9c.

The analogous column is a strip of width $\dfrac{1}{EI}$ and length L loaded with unit load at one end. Taking moments about kern points of A and B, write directly from the formula $f = \dfrac{6M_k}{Ad}$,

$$M_a = 6\,\frac{\frac{2}{3}L}{\dfrac{L}{EI}L} = 4\,\frac{EI}{L} \qquad M_b = -\,6\,\frac{L/3}{\dfrac{L}{EI}L} = -2\,\frac{EI}{L}$$

Suppose a beam with fixed ends subjected to unit relative displacement of the supports. Displacement in a beam corresponds to moment about the axis of displacement in the analogous column. For unit relative displacement of the supports, then, we write directly

$$M_a = M_b = \frac{6M_k}{Ad} = \frac{6}{\dfrac{L}{EI}L} = \frac{6EI}{L^2}$$

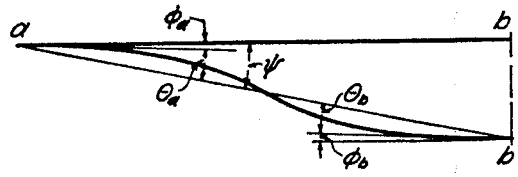


FIG. 10. ROTATION AND DISPLACEMENT OF THE ENDS OF A MEMBER

If the beam is subjected to loads and at the same time to rotations and displacements of the ends, we may write

$$M_a = M'_a + 4\phi_a \frac{EI}{L} + 2\phi_b \frac{EI}{L} - 6 \frac{\Delta EI}{L^2}$$

where M'_a is the fixed end moment at A due to the loads, ϕ_a and ϕ_b are rotations at A and B respectively, and Δ is the relative end displacement.

In place of $\frac{\Delta}{L}$ write ψ, the angle of tipping due to end displacement.

Also write $K = \frac{I}{L}$, and R_a instead of M'_a.

Then $M_a = 2EK(2\phi_a + \phi_b - 3\psi) + R_a$ which is the equation usually known in American literature as that of slope-deflection. The signs used for ϕ and ψ will depend on the convention of signs adopted.

For some purposes this may be more conveniently written as

$$M_a = 2EK(2\theta_a + \theta_b) + R_a$$

where ϕ and ψ are measured from the axis of the beam before flexure, and θ from the chord after flexure. This expression can be derived directly from the column analogy (see Fig. 10).

15. *Simple Numerical Examples.*—The column analogy is peculiarly fitted to the direct solution of numerical examples.

(a) Beam Fixed at Ends

The curve of moments will be drawn for a single concentrated load on a beam of constant section with fixed ends. Three curves of static moments will be used. The beam and its load are shown in Fig. 11. Of course for this case the formula $M = m_s \frac{a}{L}$ already derived is sufficient for finding the fixed-end moments.

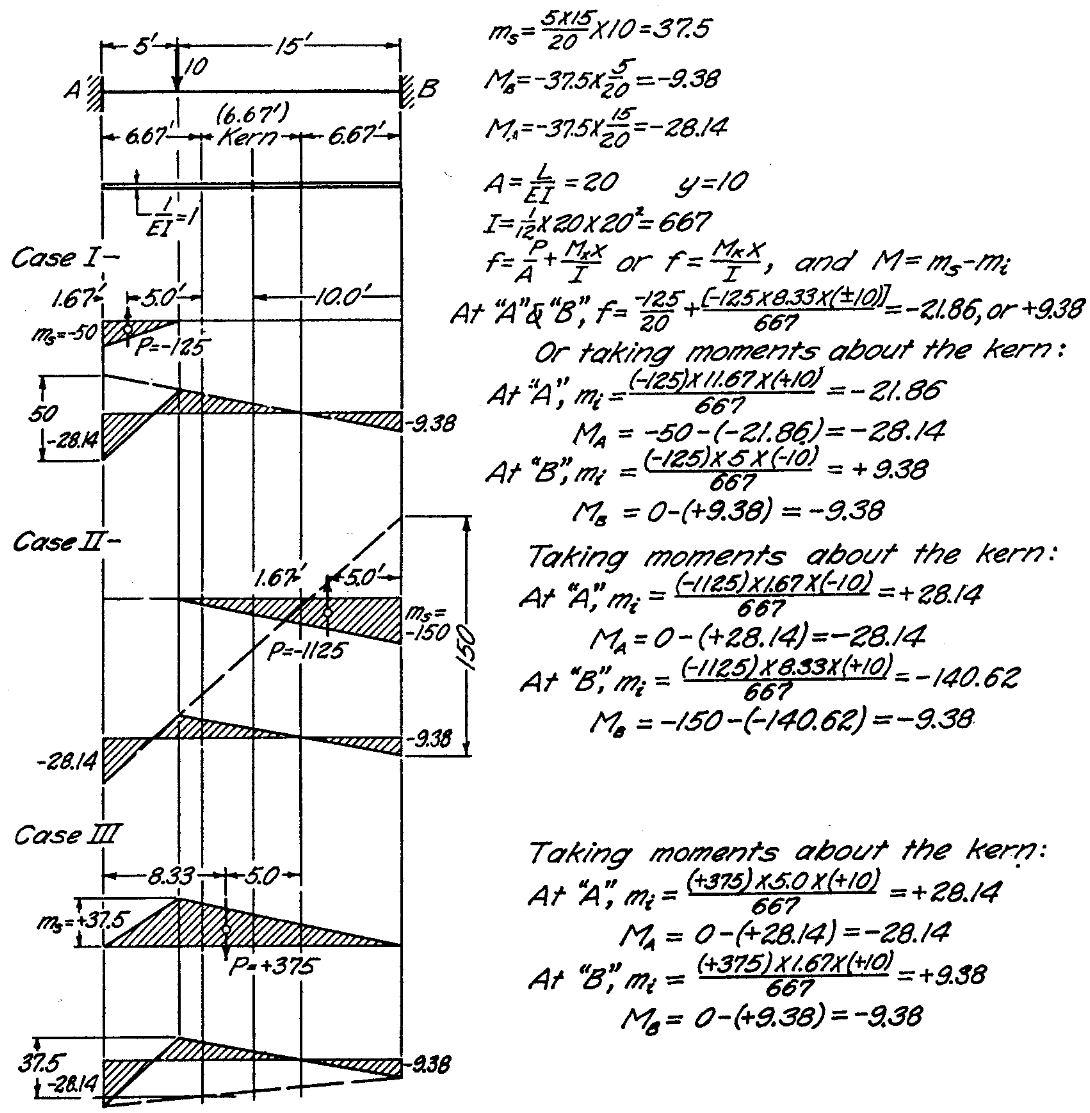


FIG. 11. BEAM WITH FIXED ENDS

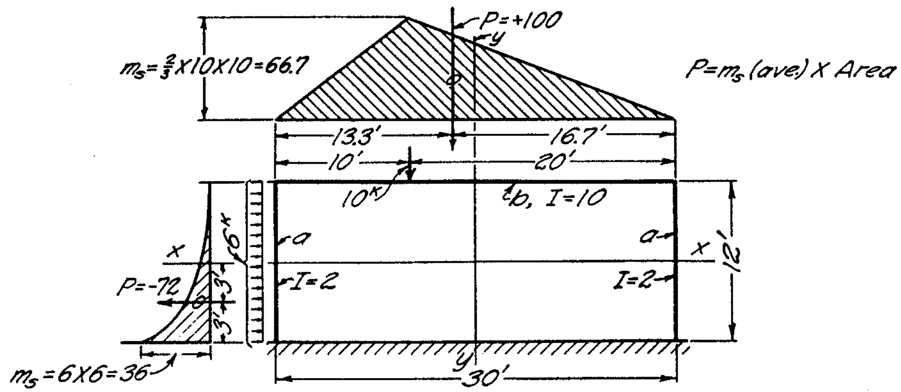
From this formula

$$m_s = \frac{5 \times 15}{20} \times 10 = 37.5 \qquad M_a = \frac{15}{20} \times 37.5 = -28.15$$

$$M_b = \frac{5}{20} \times 37.5 = -9.38$$

In case I the load is assumed cantilevered from A, in case II the load is assumed cantilevered from B, in case III the beam is treated as simply supported at A and B.

In case I, the end moment determined by statics is $-5 \times 10 = -50$, the average is $-\frac{50}{2} = -25$ acting over an area 5×1. Hence



Member	Length	I	x	y	Elastic Area a	Statical Moments ax	ay	Products of Inertia ax^2+i_x	ay^2+i_y	$axy+i_{xy}$	Vertical Given P	x	y	Vertical Computed M_x	M_y	Horizontal Given P	x	y	Horizontal Computed M_x	M_y
a	12	2	-15	0	6	-90	0	1350 0	0 72											
b	30	10	0	+6	3	0	+18	0 225	108 0											
a	12	2	+15	0	6	+90	0	1350 0	0 72											
Correct to Centroid-			0	+1.2	15	0	+18	2925 0	252 21.6		+100	-1.7	+6	-170 0	+600 +120	-72	-15	-3	+1080 0	+21.6 - 86.4
								2925	230.4					-170	+480				+1080	+302.4
								2925	230.4		+100			-170	+480	-72			+1080	+302.4
											$\frac{P}{A}$=+6.7			v_t=-0.058	h_t=+2.10	$\frac{P}{A}$=-4.8			v_t=+0.37	h_t=+1.30
											Inter-cepts	x_1 +116.0	y_1 -3.2				x_1 +13.0	y_1 +3.7		

FIG. 12. RECTANGULAR BENT

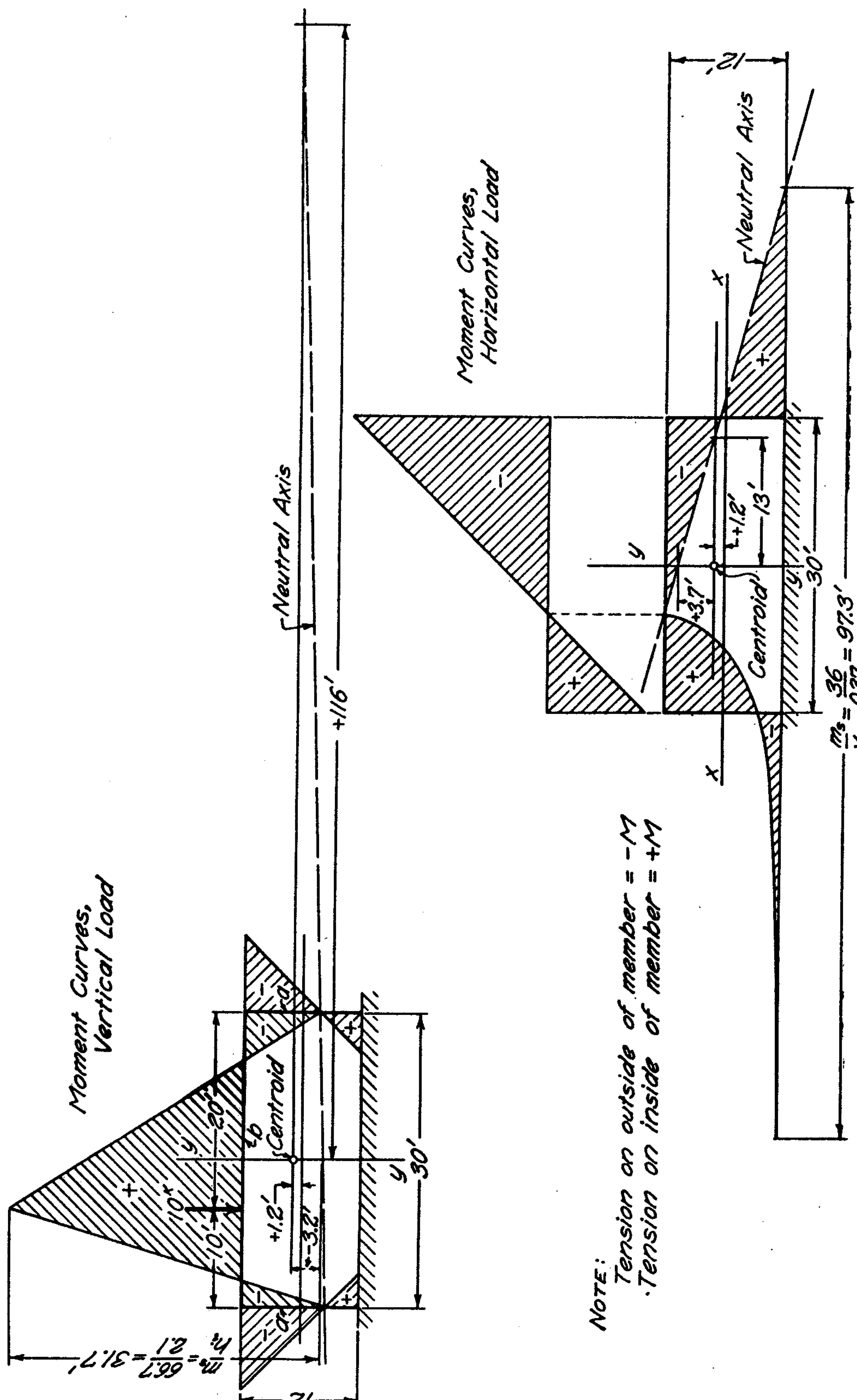


FIG. 12 (CONCLUDED). RECTANGULAR BENT

$P = -25 \times 5 = -125$. It acts at the centroid of the triangle of moments. For the column section $A = \frac{20}{1} = 20$, $I = \frac{1}{12} \times 20 \times 20^2 = 667$. Applying these values

$$f = \frac{P}{A} + \frac{M_x x}{I} = \frac{-125}{20} + \frac{(-125) \times 8.33 \times (\pm 10)}{667} =$$

$$-21.86 \text{ or } +9.38$$

Plot this moment curve, and on it as a base plot the original curve of moments.

The same procedure is shown for cases II and III.

(b) **Simple Bents**

Assume the rectangular bent shown in Fig. 12. Let the loads be a vertical load of $10k$ on the top and a horizontal load of $6k$ uniformly distributed along one side. It is desired to draw separate moment curves for the two cases of loading.

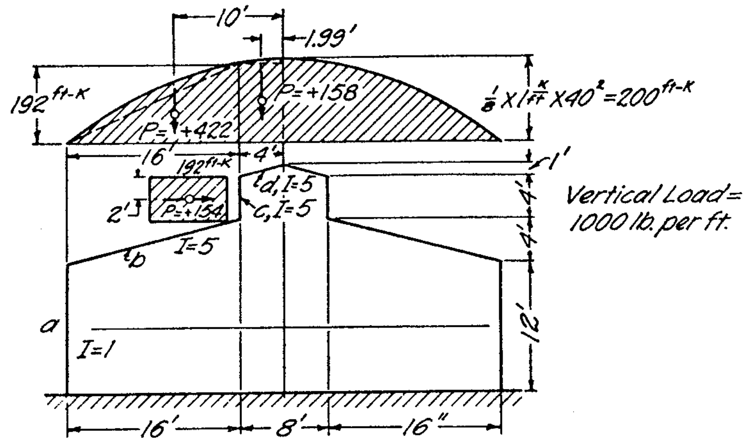
Assume as convenient axes a vertical through the center of b and a horizontal through the center of aa. Tabulate length L, moment of inertia I, horizontal coördinate of centroid x and vertical coördinate of centroid y for each of the members a, b, and a.

Also record the elastic loads and their centroids. For the load of $10k$, treating the girder as a simple beam, we have the moment curve shown, average moment $+\frac{66.7}{2}$, area loaded 3, and hence $P = +33.3 \times 3 = +100$. The centroid is as shown, and $x = -1.7$, $y = +6$. Whence $M_x = (-1.7) \times (+100) = -170.0$ and $M_y = (+6) \times (+100) = +600$.

For the horizontal load draw the curve of moments for the column as a cantilever. Average moment $-\frac{36}{3} = -12$; area loaded $= 6$. Hence $P = (-12) \times 6 = -72$. Also $x = -15$, $y = -3$. $M_x = (-15) \times (-72) = +1080$ and $M_y = (-3) \times (-72) = +216$.

Compute a, a_x, a_y, $ax^2 + i_x$, $ay^2 + i_y$ for each member. The centroidal moment of inertia (i_x and i_y) equals $\frac{1}{12} a \times$ (projection along the axis)^2. Find the totals.

Reducing to the centroid, $\bar{x} = 0$, $\bar{y} = +\frac{18}{15} = +1.2$. Find $A\bar{x}^2$, $A\bar{y}^2$, $P\bar{x}$, $P\bar{y}$, and subtract.



Properties of Section							Elastic Load			
Member	Length	I	a	y	ay	ay^2+i_y	m_s, (av.)	P	y	M_y
a	12.0	1	12.0	0	0	0 144	0			
b	16.5	5	3.3	+8	+26.4	211 4	+128	+432	+8.5	+3585
c	4.0	5	0.8	+12	+9.6	115 1	+192	+154	+12	+1850
d	4.1	5	0.8	+14.5	+11.6	168 0	+197	+158	+14.5	+2290
			16.9		+47.6	643		+734		+7725
Correct to Centroid				+2.82		134				+2070
			16.9			509		+734		+5655
							$\frac{P}{A}=$	+43.4		
							$h_i=\frac{M_y}{I_y}=$			+11.1
							$y_1=-\frac{P/A}{h_i}=$		-3.92	

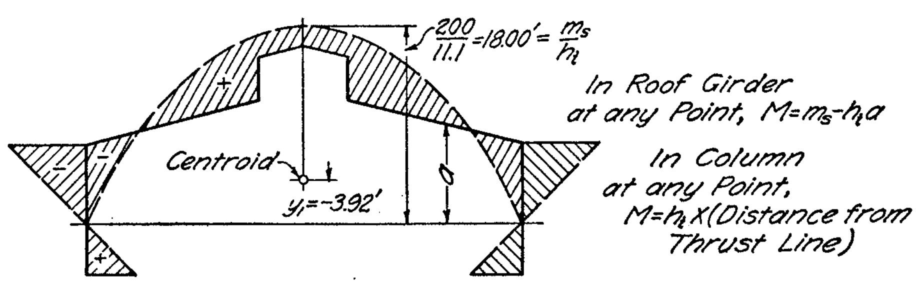
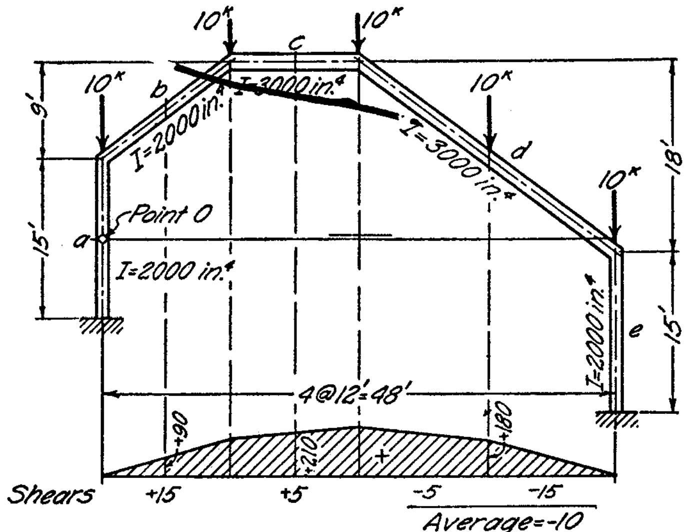


Fig. 13. Bent with Monitor

The intercepts of the neutral axis for the two conditions of loading are now found as $x_1 = -\dfrac{P/A}{M_x/I_x}$ and $y_1 = -\dfrac{P/A}{M_y/I_y}$. For the vertical load compute $h_i = \dfrac{M_y}{I_y} = +2.1$, and for the horizontal load compute $v_i = \dfrac{M_x}{I_x} = +0.37$. The neutral axes are then plotted, and on them the original curves of moments are drawn to the scale of distance.

Required:
Moment in Member a at Point O



*Coordinates of Point O - -24.0 0

Member	1 Length	2 I in 1000 in.⁴	3 a	4 x	5 ax	6 ax^2+i_x	7 y	8 ay	9 ay^2+i_y	10 $axy+i_{xy}$	11 m_s	12 v_s	13 h_s	14 P	15 M_x	16 M_y
a	15	2	7.5	-24	-180	4320 0	0	0	0 140	0 0						
b	15	2	7.5	-18	-135	2430 90	+12	+90	1080 51	-1620 + 68	+90	+15		+ 675	-12 150 + 1 350	+ 8100 + 1 020
c	12	3	4.0	- 6	-24	144 48	+16.5	+66	1090 0	-397 0	+210	+ 5		+ 840	- 5040 + 240	+13 860 0
d	30	3	10.0	+12	+120	1440 480	+ 7.5	+ 75	562 270	+ 900 - 360	+180	-10		+1800	+21 600 - 4800	+ 13 500 + 3 600
e	15	2	7.5	+24	+180	4320 0	-9.0	-67.5	608 140	-1620 0						
			36.5	-1.07	-39	13 272 42	+4.48	+163.5	3941 732	-3029 - 175				+3315	+ 1200 - 3545	+40 080 +14 850
				-22.93*		13230 2530	-4.48*		3209 614	-2854					+ 4745 -22 420	+25 230 - 1 020
						10700			2595						+27 165	+26 250

Moment at Point O:

$$m_s = \qquad\qquad = 0$$

$$m_i = \frac{+3315}{36.5} + \frac{(+27\,165)(-22.93)}{10\,700} + \frac{(+26\,250)(-4.48)}{2595} = -12.7^{\text{ft-K}}$$

$$M = \qquad\qquad = +12.7^{\text{ft-K}}$$

FIG. 14. UNSYMMETRICAL BENT

For the vertical load plot for static moments $\frac{66.7}{2.1} = 31.7$ ft.; for the horizontal loads, $\frac{36}{0.369} = 97.3$ ft.

In Fig. 13 is shown a reinforced concrete bent having a monitor. The dead load on the roof is assumed uniform at 1000 lb. per horizontal foot.

In this case both load and structure are symmetrical, and there is no need to compute moment of inertia about the axis of Y. The axes are taken on the vertical center line and through the center of members *aa*.

The moment areas and their centroids for the different members have been computed separately by breaking them up into trapezoids and parabolas.

The same procedure is followed as in the preceding problem. The neutral axis, however, is horizontal. $h_i = \frac{M_y}{I_y} = +11.1$. Intercept of neutral axis $y_1 = -\frac{P/A}{M_y/I_y} = -3.92$. The rise of the pressure line is $\frac{200}{11.1} = 18.00$ ft.

The signs of the intercepts of the neutral axis can usually be found by inspection, since the neutral axis lies on the side of the centroid opposite to the load.

In Fig. 14 is shown an unsymmetrical bent subjected to vertical loads. The tabulation of elastic properties and of elastic loads follows the procedure already explained. The elastic moments may also be conveniently computed as previously explained. They have actually been computed as the sum of the moment of the elastic load acting at the centroid of the member plus the product of shear times elastic centroidal moment of inertia. The method is explained later as an extension of the analogy, but presents few advantages. The trial axes are taken as the vertical through the center of member *c* and the horizontal through the center of member *a*.

The correction to the centroid also follows the procedure already explained.

The correction for dissymmetry is made as follows:

for I_x in column (6) write $I_{xy} \cdot \frac{I_{xy}}{I_y} = (-2854)\left(\frac{-2854}{3209}\right) = 2530,$

for M_x in column (15), write $M_y \cdot \frac{I_{xy}}{I_y} = (+25\,230)\left(\frac{-2854}{3209}\right) = -22420$

for I_y in column (9), write $I_{xy} \cdot \frac{I_{xy}}{I_x} = (-2854)\left(\frac{-2854}{13230}\right) = 614$

for M_y in column (16), write $M_x \cdot \frac{I_{xy}}{I_x} = (+4745)\left(\frac{-2854}{13230}\right) = -1020.$

Subtract these corrections to get I'_x, M'_x, I'_y, and M'_y.

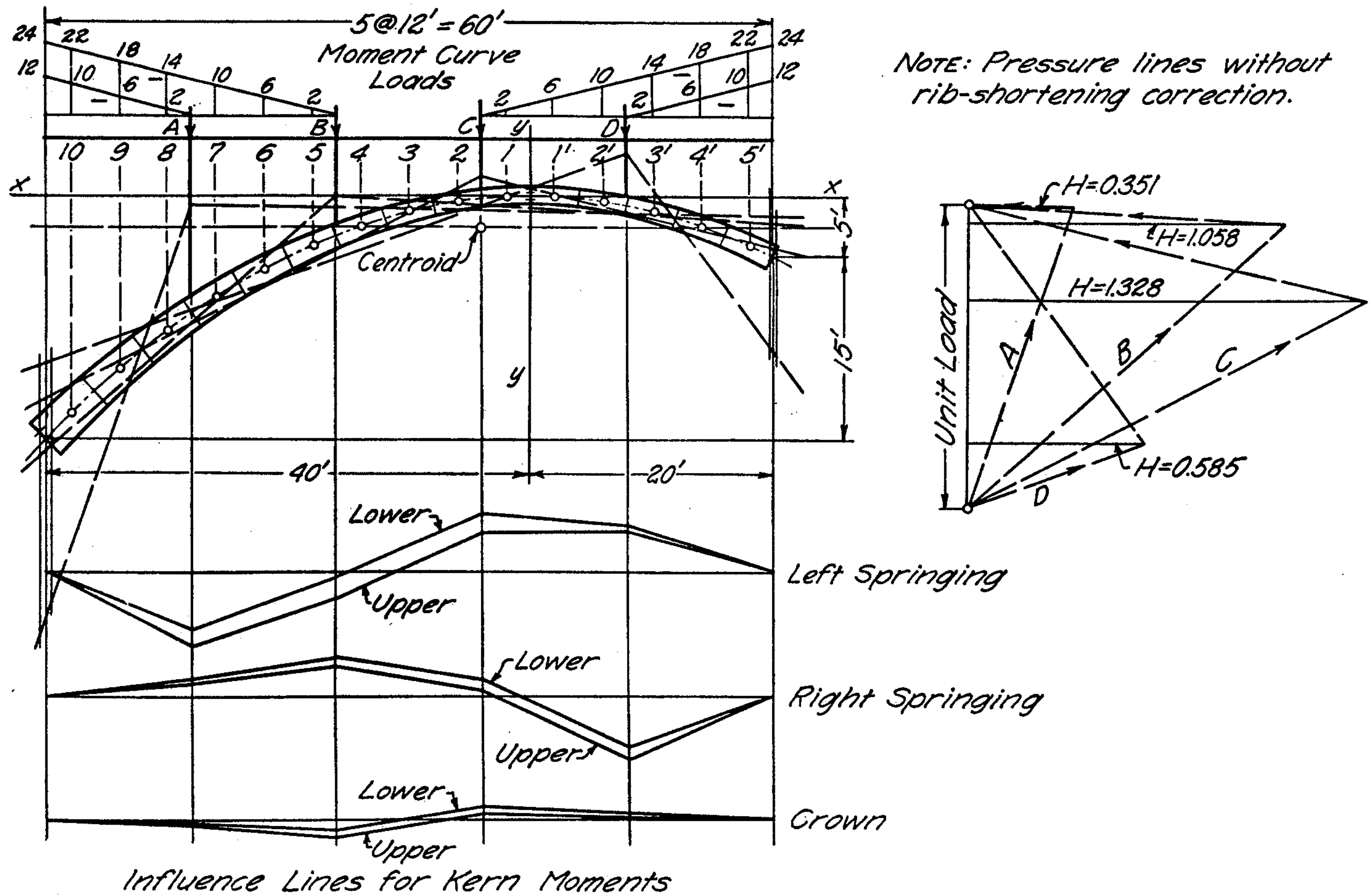


Fig. 15. Unsymmetrical Arch

Properties of the Section											Elastic Loads — Unit Load at:															
Given					Derived						A				B				C				D			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
No.	L, ft.	d, ft.	x	y	a	ax	ay	ax^2	ay^2	axy	m_s	P	M_x	M_y	m_s	P	M_x	M_y	m_s	P	M_x	M_y	m_s	P	M_x	M_y
10	5.58	3.34	-38.0	-18.05	0.150	-5.70	-2.71	217	49.0	+103.0	-10	-1.50	+57.0	+27.1	-22	-3.30	+125.5	+59.5								
9	5.23	3.04	-34.0	-14.45	0.187	-6.36	-2.70	216	39.0	+91.7	-6	-1.12	+38.0	+16.2	-18	-3.36	+114.1	+48.6								
8	5.00	2.79	-30.0	-11.25	0.231	-6.94	-2.60	208	29.3	+78.0	-2	-0.46	+13.8	+5.2	-14	-3.24	+97.2	+36.4								
7	4.78	2.54	-26.0	-8.45	0.291	-7.56	-2.46	197	20.8	+64.0					-10	-2.91	+75.6	+24.6								
6	4.55	2.33	-22.0	-6.05	0.361	-7.93	-2.18	174	13.2	+48.0					-6	-2.17	+47.7	+13.1								
5	4.45	2.16	-18.0	-4.05	0.443	-7.96	-1.79	143	7.3	+32.2					-2	-0.89	+16.0	+3.6								
4	4.26	2.00	-14.0	-2.45	0.533	-7.46	-1.31	104	3.2	+18.3																
3	4.17	1.83	-10.0	-1.25	0.684	-6.84	-0.85	68	1.1	+8.5																
2	4.05	1.67	-6.0	-0.45	0.874	-5.25	-0.39	32	0.2	+2.3																
1	4.01	1.54	-2.0	-0.05	1.102	-2.21	-0.06	4	0.0	+0.1									-2	-2.21	+4.4	+0.1				
1'	4.01	1.54	+2.0	-0.05	1.102	+2.21	-0.06	4	0.0	-0.1									-6	-6.61	-13.2	+0.3				
2'	4.05	1.67	+6.0	-0.45	0.874	+5.25	-0.39	32	0.2	-2.3									-10	-8.74	-52.4	+3.9				
3'	4.17	1.83	+10.0	-1.25	0.684	+6.84	-0.85	68	1.1	-8.5									-14	-9.57	-95.7	+12.0	-2	-1.37	-13.7	+1.7
4'	4.26	2.00	+14.0	-2.45	0.533	+7.46	-1.31	104	3.2	-18.3									-18	-9.59	-134.2	+23.5	-6	-3.20	-44.8	+7.8
5'	4.45	2.16	+18.0	-4.05	0.443	+7.96	-1.79	143	7.3	-32.2									-20	-9.75	-175.2	+39.5	-10	-4.43	-79.7	+17.9
Correct to Centroid			-4.06	-2.53	8.492	-34.49	-21.45	1714 140	174.9 54.5	+384.7 +87.0		-3.08	+108.8 +12.5	+48.5 +7.8		-15.87	+476.1 +64.4	+185.8 +40.2		-46.47	-466.3 +188.5	+79.3 +117.5		-9.00	-138.2 +36.5	+27.4 +22.8
Correct for Dissymmetry								1574 734	120.4 56.2	+297.7			+96.3 +100.4	+40.7 +18.2			+411.7 +359.2	+145.6 +77.7			-654.8 -94.2	-38.2 -123.5			-174.7 +11.4	+4.6 -33.0
					8.492			840	64.2			-3.08	-4.1	+22.5		-15.87	+52.5	+67.9		-46.47	-560.6	+85.3		-9.00	-185.1	+37.6

	A	B	C	D
$\frac{P}{A}$ =	-0.363	-1.869	-5.475	-1.059
$V_i = \frac{M_x'}{I_x}$ =	-0.0049	+0.0625	-0.667	-0.211
$h_i = \frac{M_y'}{I_y}$ =	+0.351	+1.058	+1.328	+0.585
$x_i = \frac{P/A}{V_i}$ =	-74.0	+29.9	-8.20	-5.03

Computations are based on a rib-section 12 ft. wide, since $\frac{L}{I} = \frac{L}{\frac{1}{12}bd^3}$ is taken equal to $\frac{L}{d^3}$.

Since all units are in feet, E should be taken in foot units in computing the effect of shrinkage, temperature change, or abutment movements.

Computations for temperature are for a free change in span of 100%.

Temperature, (Increase) $M_x = -15$ $M_y = +60$

Correction for Dissymmetry $+148$ -2.8

$M_x' = -163$ $M_y' = +62.8$

$V = -0.016$ $H = +0.081$

Fig. 15 (concluded). Unsymmetrical Arch

The bending moment at any point, such as joint o, may now be found as $M = m_s - m_i$, as shown.

16. *Arch Analysis.*—The column analogy affords an unusually convenient means of analyzing reinforced concrete arches, because, once understood, it furnishes a familiar order of procedure. The actual computations are, of course, the same as presented by other writers.

In the case of unsymmetrical arches the equation given in this discussion seems to offer a much more convenient order of arranging the computations than is found elsewhere. For this reason it has seemed worth while to give in some detail the essential steps in the analysis of an unsymmetrical arch.

(a) Unsymmetrical Arch

The arch analyzed is shown in Fig. 15. It has a span of axis of 60 feet divided into five panels of 12 feet each. The total rise is 20 feet and the difference in level between abutments is 15 feet.

The arch axis is first divided into fifteen segments of equal horizontal projections.

Use as convenient trial axes horizontal and vertical lines through the highest point of the arch axis.

Tabulate first the known properties of the arch. These are the length of each segment L along the arch axis, the distances x and y to its centroid, the depth of the sections at their centers.

Now compute the elastic areas a, each equal to $\frac{L}{I} = 12\frac{L}{d^3}$; this is for unit width of rib. From these compute the statical moments ax and ay about the axes of x and y, the products of inertia about these axes ax^2, ay^2, axy.

Now tabulate the m_s values at centroids of sections for unit loads at each of the panel points, A, B, C, D. The statically determinate moments will be found for these loads cantilevered from the nearer end of the arch. The statically determinate moment on any segment between the load and the nearer abutment, then, equals the distance from the load to the centroid of that segment. We then compute the elastic load $P = m_s a$, the moment of the elastic load about the axis of Y, Px, and about the axis of X, Py. Note that for any segment these three quantities may be written by multiplying m_s by columns (6), (7), (8), successively.

Sum the columns for elastic area, statical moments, products of inertia, elastic loads, elastic moments.

Correct to the centroid. Compute $\bar{x} = \frac{\Sigma ax}{A}$ and $\bar{y} = \frac{\Sigma ay}{A}$ and the corrections for the products of inertia $\bar{x}^2A$, $\bar{y}^2A$, $\overline{xy}\,A$, and for the elastic moments $x\Sigma P$, $y\Sigma P$. Subtract the corrections.

Correct for dissymmetry. Write the value of $\frac{I_{xy}}{I_y} \cdot I_{xy}$ under I_x and of $\frac{I_{xy}}{I_y} \cdot M_y$ under M_x and write the value of $\frac{I_{xy}}{I_x} \cdot I_{xy}$ under I_y and of $\frac{I_{xy}}{I_x} \cdot M_x$ under M_y. Subtract the corrections.

Draw horizontal axis through the centroid. Now compute for a load at each panel point the values of the components of the more distant reaction $v_i = \frac{M'_x}{I'_x}$ and $h_i = \frac{M'_y}{I'_y}$ and the intercept on the X axis through the centroid, $x_1 = -\frac{P}{Av_i}$.

Rib-shortening has not been included in the computations. It may be corrected for by computing the average intensity of compression for any given condition of loading and from this the change of span which would take place if the arch were free to contract. This change of span may then be treated as if it were due to change of temperature, equivalent temperature change $= \frac{f_{av}}{E\epsilon}$.

The components and location of one reaction (at the more distant abutment) now being known, it is easy to draw the pressure lines. By scaling the ordinate from any pressure line to the kern point for any particular cross-section and multiplying by the H value for that pressure line, we can compute conveniently the moment about that kern point. Thus we can draw influence lines if we wish, by plotting kern point moments at various sections for different positions of the unit load. From the kern moments the stresses can be computed directly from the formula $f = \frac{M_k c}{I}$.

For temperature changes, the horizontal change in span if the arch were free to expand multiplied by E is $E\epsilon t \times 60$ ft. and the relative vertical movement of the abutments multiplied by E is $E\epsilon t \times 15$ ft. Omit the constant multiplier $E\epsilon t$ for the time being, and use it later as a multiplier for the temperature stresses. It has been shown that

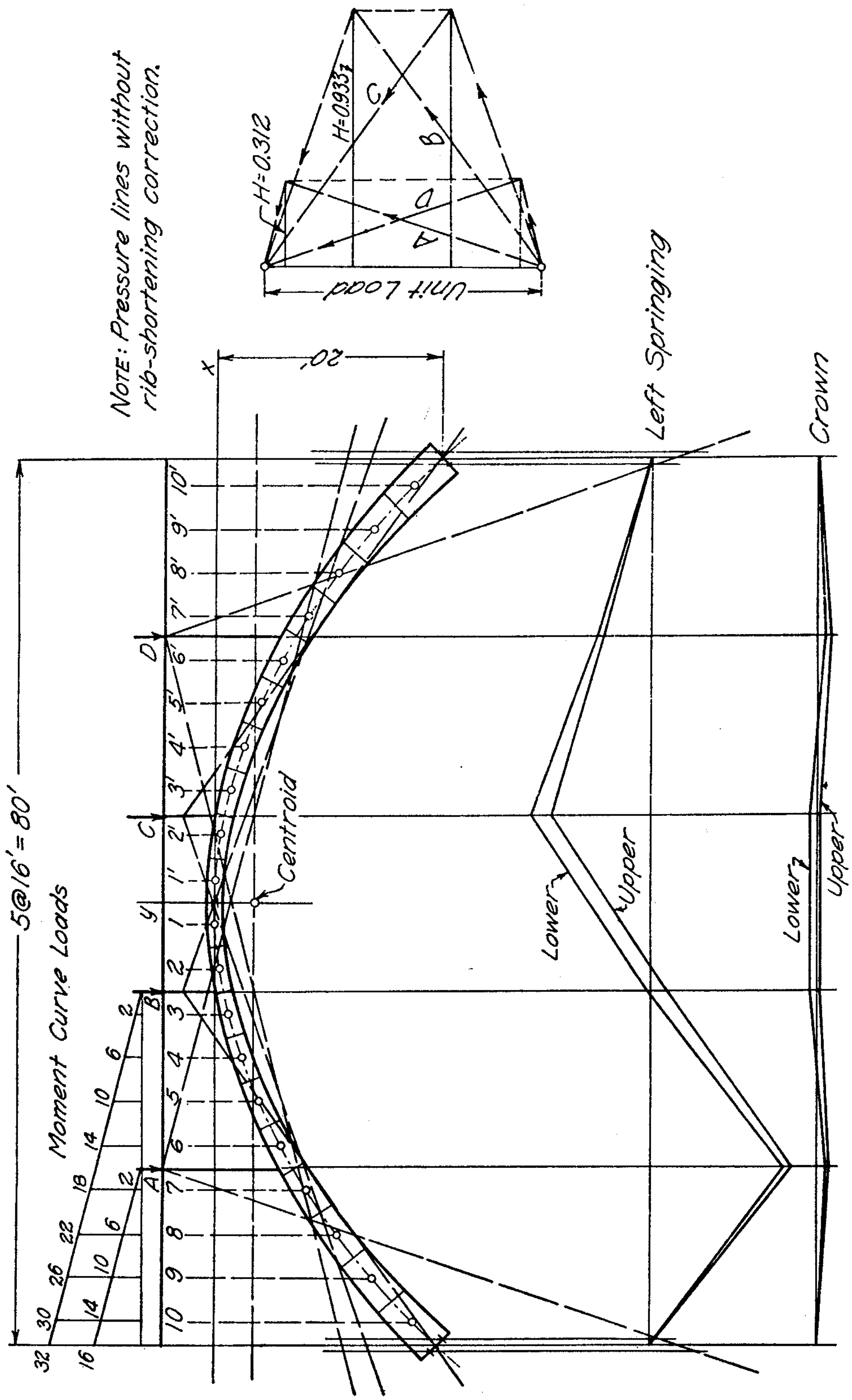


FIG. 16. SYMMETRICAL ARCH

Properties of the Section										
Given					Derived					
1	2	3	4	5	6	7	8	9	10	11
No.	L, ft.	d, ft	x	y	a	ax	ay	ax²	ay²	axy
10	5.58	3.34	-38.0	-18.05	0.150	-5.70	-2.71	217	49.0	Due to Symmetry, axy Totals Zero
9	5.23	3.04	-34.0	-14.45	0.187	-6.36	-2.70	216	39.0	
8	5.00	2.79	-30.0	-11.25	0.231	-6.94	-2.60	208	29.3	
7	4.78	2.54	-26.0	-8.45	0.291	-7.56	-2.46	197	20.8	
6	4.55	2.33	-22.0	-6.05	0.361	-7.93	-2.18	174	13.2	
5	4.45	2.16	-18.0	-4.05	0.443	-7.96	-1.79	143	7.3	
4	4.26	2.00	-14.0	-2.45	0.533	-7.46	-1.31	104	3.2	
3	4.17	1.83	-10.0	-1.25	0.684	-6.84	-0.85	68	1.1	
2	4.05	1.67	-6.0	-0.45	0.874	-5.25	-0.39	32	0.2	
1	4.01	1.54	-2.0	-0.05	1.102	-2.21	-0.06	4	0.0	
Left Half					4.856	-64.21	-17.05	1363	163.1	
Right Half					4.856	+64.21	-17.05	1363	163.1	
					9.712	0	-34.10	2726	326.2	0
			0	-3.51				0	119.5	
					9.712			2726	206.7	

	Elastic Loads — Unit Load at:															
	A				B				C				D			
1	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
No.	m_s	P	M_x	M_y	m_s	P	M_x	M_y	m_s	P	M_x	M_y	m_s	P	M_x	M_y
10	-14	-2.10	+79.7	+37.9	-30	-4.50	+171.0	+81.2	Due to Symmetry, Same as Load at B				Same as Load at A			
9	-10	-1.87	+63.5	+27.0	-26	-4.86	+165.1	+70.2								
8	-6	-1.39	+41.7	+15.6	-22	-5.09	+153.0	+57.2								
7	-2	-0.58	+15.1	+4.9	-18	-5.24	+136.4	+44.3								
6					-14	-5.05	+111.0	+30.5								
5					-10	-4.43	+79.6	+17.9								
4					-6	-3.20	+44.8	+7.8								
3					-2	-1.37	+13.7	+1.7								
2																
1																
Left Half / Right Half		-5.94	+200.0	+85.4		-33.74	+874.6	+310.8								
		-5.94	+200.0	+85.4		-33.74	+874.6	+310.8								
			0	+20.8			0	+118.0								
		-5.94	+200.0	+64.6		-33.74	+874.6	+192.8								
$\frac{P}{A}=$	-0.612				-3.47											
$v_1=\frac{M'_x}{I'_x}=$			+0.0734				+0.321									
$h_1=\frac{M'_y}{I'_y}=$			+0.312				+0.933									
$x_1=\frac{\frac{P}{A}}{v_1}=$			+8.34				+10.8									

Temperature (Increase) $M_y=+80$ $M_x=0$ $H=+0.032$

Computations are based on a rib-section 12 ft. wide, since $\frac{L}{I}=\frac{L}{\frac{1}{12}bd^3}$ is taken equal to $\frac{L}{d^3}$.

Since all units are in feet, E should be taken in foot units in computing the effect of shrinkage, temperature change, or abutment movements.

Computations for temperature are for a free change in span of 100%.

FIG. 16 (CONCLUDED). SYMMETRICAL ARCH

a linear displacement is analogous to a moment about the axis of displacement. Hence we have on the analogous column elastic moments $M_x = \pm 15$ and $M_y = \mp 60$. That the signs of these moments are opposite is determined by the fact that the change of span is analogous to a moment about a line connecting the ends of the arch axis. Such a moment on the analogous column produces compression on the top and left or on the bottom and right of the section. After the line of action of the temperature thrust has been determined, the sign of the bending moment at any point is readily determined for a rise or for a fall of temperature by observing that a thrust is required to shorten the span, and a pull to lengthen it.

These moments $M_x = \pm 15$ and $M_y = \mp 60$ are now corrected for dissymmetry. The H and V components of the temperature thrust are then computed. The thrust, of course, passes through the elastic centroid, just as the neutral axis of a beam passes through the centroid for pure bending.

(b) Symmetrical Arch

If the arch is symmetrical, the procedure just given is shortened. It is now necessary to consider only one-half of the arch ring. By inspection the product of inertia is zero and there is no correction for dissymmetry.

In Fig. 16 is shown the analysis of an arch similar to the one just shown except that the span is 5 panels of 16 ft. = 80 ft.

17. *Haunched Beams.*—Because of the occasional importance of haunched beams in continuous frames, examples of the analysis of such beams have been included. The arrangement of the computations is the same as in the analysis of arches except that there are no y coördinates.

Attention is called to the computation of end moments resulting from unit rotation of one end. These values are constants needed in certain methods of analyzing continuous frames.* In order to compute these moments a unit load is applied at one end of the section of the analogous column and the outer fiber stresses in the column are determined.

A beam symmetrically haunched is shown in Fig. 17 and is analyzed for end moments due to a uniform load and also for end moments due to a rotation at one end.

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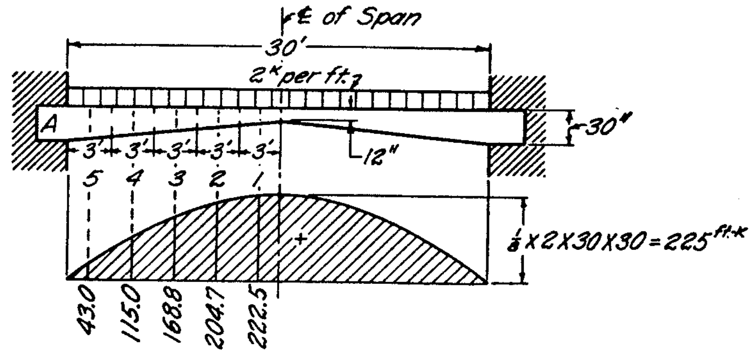
*"Continuity as a Factor in Reinforced Concrete Design," Hardy Cross, Proceedings, A. C. I., Vol. 25, 1929.

"Simplified Rigid Frame Design," Report of Committee, 301, Hardy Cross, Author-Chairman, Journal, A. C. I., Dec., 1929.

"Analysis of Continuous Frames by Distributing Fixed-End Moments," Hardy Cross, Proceedings, A. S. C. E., May, 1930.



Section	Length in ft.	d in ft.	x	a	ax	$ax^2 + i_x$	m_s	P		
1	3	1.15	±1.5	1.97	±2.96	4.43 1.47	+222.5	+438		
2	3	1.45	±4.5	0.98	±4.40	19.80 0.73	+204.7	+200		$a = 12\frac{L}{d^3}$
3	3	1.75	±7.5	0.56	±4.20	31.50 0.42	+168.8	+ 94		$a = \frac{L}{d^3}$, in this
4	3	2.04	±10.5	0.35	±3.68	38.65 0.26	+115.0	+ 40		computation. Hence b=12 ft.
5	3	2.35	±13.5	0.23	±3.10	41.90 0.17	+43.0	+ 10		E=1
				4.09	0	139.33		+782		
Multiplied by—				2		2				
				8.18		278.7				

$m_{i_A} = m_{i_B} = \frac{+782}{4.09} = +191^{ft\text{-}k}$

$M_A = M_B = m_s - m_i = 0 - (+191) \doteq -191^{ft\text{-}k}$

If support "A" rotates one unit.—

Find the moment necessary to give unit rotation by applying a unit load at "A" on the analogous column

Then $m_a = \frac{1}{8.18} + \frac{(-15)(-15)}{278.7} = +0.929$

$m_b = \frac{1}{8.18} + \frac{(-15)(+15)}{278.7} = -0.687$

m_a = moment necessary to produce unit rotation at end "A".

m_b = moment at end "B", due to restraint, when end "A" is rotated one unit

or, m_b for any given rotation at end "A", when "B" is fixed will be $= \frac{-0.687}{+0.929} \times m_a$

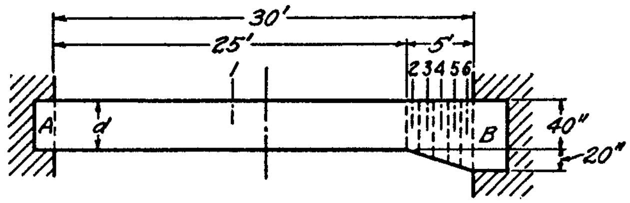
FIG. 17. BEAM SYMMETRICALLY HAUNCHED

In Fig. 18 a beam unsymmetrically haunched is shown and this is analyzed for end moments due to rotations at the ends.

18. *Slopes and Deflections of Beams.*—

(a) Relation of the Column Analogy to Theorems of Area-Moments

The sum of the rotations, or change in slope, on the beam corresponds to shear on a section through the analogous column, because the product of moment by elastic area is rotation and the sum of load



Section	Length in ft.	d in ft.	x	a	ax	$ax^2 + i_x$				
	25	3.33	−2.5	8.15	−20.4	51 425				
2	1	3.50	+10.5	0.23	+2.4	25 0				
3	1	3.83	+11.5	0.21	+2.4	28 0				$a = 12\frac{L}{d^3}$ $E = 1$
4	1	4.16	+12.5	0.17	+2.1	27 0				
5	1	4.50	+13.5	0.13	+1.8	24 0				
6	1	4.83	+14.5	0.11	+1.6	23 0				
			−1.12	9.00	−10.1	603 11				
				9.00		592				

(a)—For Unit Rotation at "A"—apply unit load on analogous column at "A";

then: $m_i = \frac{1}{9.00} + \frac{(-13.9)(-13.9)}{592} = +0.437$, moment necessary for unit rotation at "A".

and $m_i = \frac{1}{9.00} + \frac{(-13.9)(+16.1)}{592} = -0.267$, moment at far end "B", due to restraint.

(b)—For Unit Rotation at "B"—apply unit load on analogous column at "B"

then: $m_i = \frac{1}{9.00} + \frac{(+16.1)(+16.1)}{592} = +0.549$, moment necessary for unit rotation at "B"

$m_i = \frac{1}{9.00} + \frac{(+16.1)(-13.9)}{592} = -0.267$, moment at far end "A", due to restraint.

In (a), then, for any given rotation or moment at end "A":

$m_b = \frac{-0.267}{+0.437} \times m_a$, and

In (b), then, for any given rotation or moment at end "B":

$m_a = \frac{-0.267}{+0.549} \times m_b$

FIG. 18. BEAM UNSYMMETRICALLY HAUNCHED

intensity times area is shear. From the geometrical relations previously explained, it follows that the statical moment of the rotations about any section is the displacement at that section.

Slopes along the beam, then, correspond to shears on longitudinal sections through the analogous column and deflections of the beam correspond to bending moments on longitudinal sections through the analogous column parallel to the line along which the deflection is wanted.

The theorems of area-moments, then, are a part of the analogy. The theorems dealing with the slopes and deflections of beams are among the most useful and best known in the literature of structural analysis. They deal with displacements relative either to a tangent to the curved beam or to a chord of the curved beam.

They may be conveniently stated as follows:

(1) (a) Slope at any point measured with reference to a tangent to the bent beam at another point may be found as area under the $\frac{M}{EI}$ curve between the two points; (b) deflections at any point measured with reference to a tangent to the bent beam at another point may be found as the statical moment about the first point of the area under the $\frac{M}{EI}$ curve between the two points.

(2) (a) Slope at any point measured with reference to any chord of the bent beam may be found as shear at that point due to the area under the $\frac{M}{EI}$ curve as a load on the chord acting as a beam simply supported at its ends; (b) deflection at any point measured with reference to any chord of the bent beam may be found as bending moment at that point due to the area under the $\frac{M}{EI}$ curve treated as a load on the chord acting as a beam simply supported at its ends.

All of these theorems are merely theorems of geometry stated in terms convenient to the structural engineer. They neglect the effect of distortions other than rotations, and are applicable to any curved line where the angle changes are very small if we substitute "angle changes as loads" for "area under the $\frac{M}{EI}$ curve as a load" in the theorems.

In the column analogy the angle changes are treated as loads on the analogous column. From the geometrical relations already pointed out it is evident that the conception involved in the column analogy is essentially that used in finding slopes and deflections.

If we use the column analogy in the extended form explained later in which the angle changes are represented as forces on the analogous column equal to moment times elastic area, and the linear distortions are represented as couples on the analogous column equal to shear times elastic centroidal moment of inertia, we may include at once the effect of both angular and linear distortions. This is sometimes a useful theorem. It is applied in the example of Fig. 14.

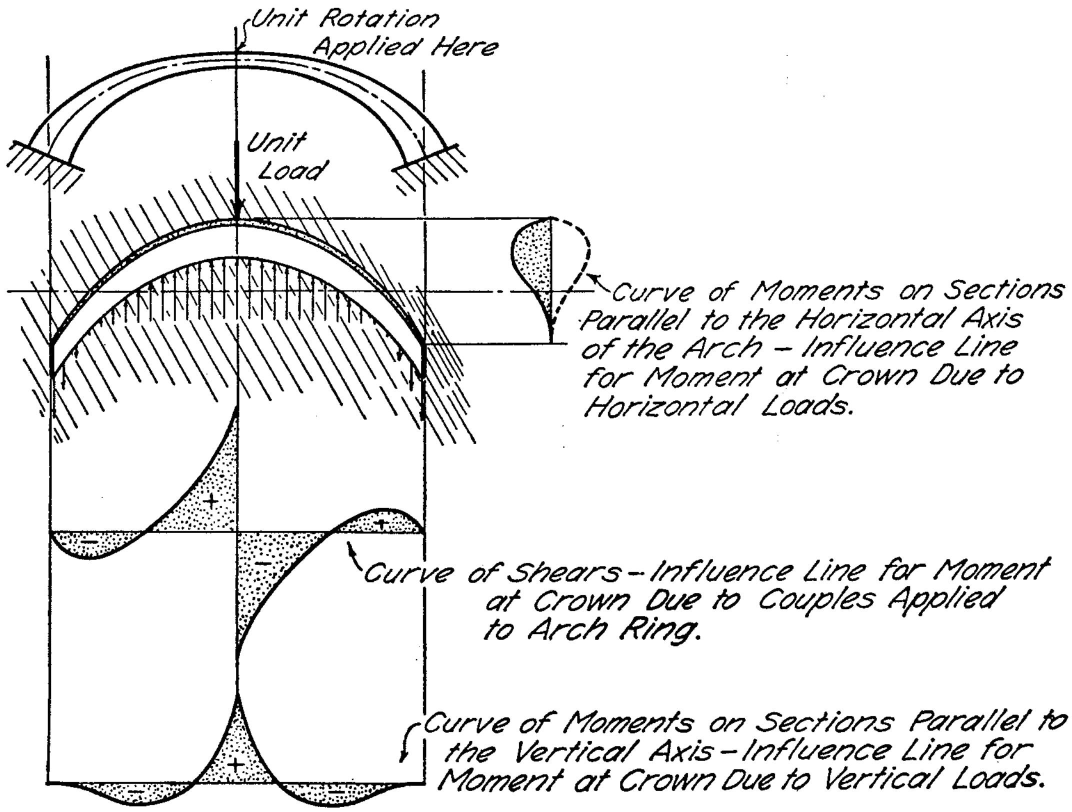


FIG. 19. INFLUENCE LINES FOR SHEAR AND MOMENT IN AN ARCH WITH FIXED ENDS

(b) Influence Lines by the Column Analogy

The column analogy may be combined with Müller-Breslau's principle to compute influence ordinates. According to this principle, the influence ordinates equal the displacements of the load line which would result from a unit distortion corresponding to the stress function under investigation. For an influence line for moment, then, we would apply a unit rotation, for shear a unit displacement, and so on.

The application for crown moment in an arch is shown in Fig. 19. A unit rotation is applied at the crown and the displacements of points on the arch axis are determined. Vertical displacements along the arch axis are influence ordinates for crown moment due to vertical loads, horizontal displacements are influence ordinates for horizontal loads, rotations of the arch axis are influence ordinates for applied couples.

All of these may be found as shears and bending moments on sections through the analogous column when the column is loaded with a unit load at the crown, as shown in the figure.

Similarly for unit crown shear, apply to the column at the crown a unit couple about the vertical axis of the arch; for crown thrust, apply to the column at the crown a unit couple about the horizontal axis of the arch.

This procedure has great value in sketching influence lines. For numerical computation it is probably as rapid and convenient to compute the influence ordinates by the elementary procedure of applying unit loads at successive points on the arch axis.

19. *Supports of the Analogous Column.*—

(a) Types of Supports

The analogous column is supported on an elastic medium. The intensity of resistance offered by this medium is the indeterminate moment, resisting distortion, or the moment resulting from the restraint. The intensity times the area is the rotation produced by this resistance.

If there is a hinge in the beam, the rotation would be infinite if there were a constant moment at the hinge; the elastic area of the hinge is infinite. But it is inconvenient to represent an infinite area; instead we may choose a small area of infinite stiffness—a rigid point support. A hinge in the beam, then, may be represented by a rigid point support of the column.

A roller nest is equivalent to a rocker—two hinges on a line normal to the roller bed. Hence a roller nest or rocker may be represented by two point supports on a line normal to the roller bed.

A free end would be produced by three hinges not in line. Hence, it may be represented in the analogous column by three point supports not in line or by a fixed end.

It is interesting, though perhaps not very important, to note that if the beam is unstable, the column is statically indeterminate; if the beam is statically determinate, so also is the column; if the beam is statically indeterminate, the column would be unstable if it were not supported by the elastic medium.

In Fig. 20a is shown the analogous column and its load in the case of a beam simply supported. The area-moment relation is familiar.

Figure 20b shows a three-hinged arch and its analogous column. Note that in statically determinate structures only one stable curve of moments is possible and that there is no pressure on the elastic base—no indeterminate moment.

Figure 20c shows an arch having a crown hinge. In analyzing this case the rigid support is given an infinite area. The centroid, then, lies at the hinge and the total elastic area is infinite.

Figure 20d shows a cantilever beam. Note that the free end of the beam is rigidly fixed in the analogous column. The moment area

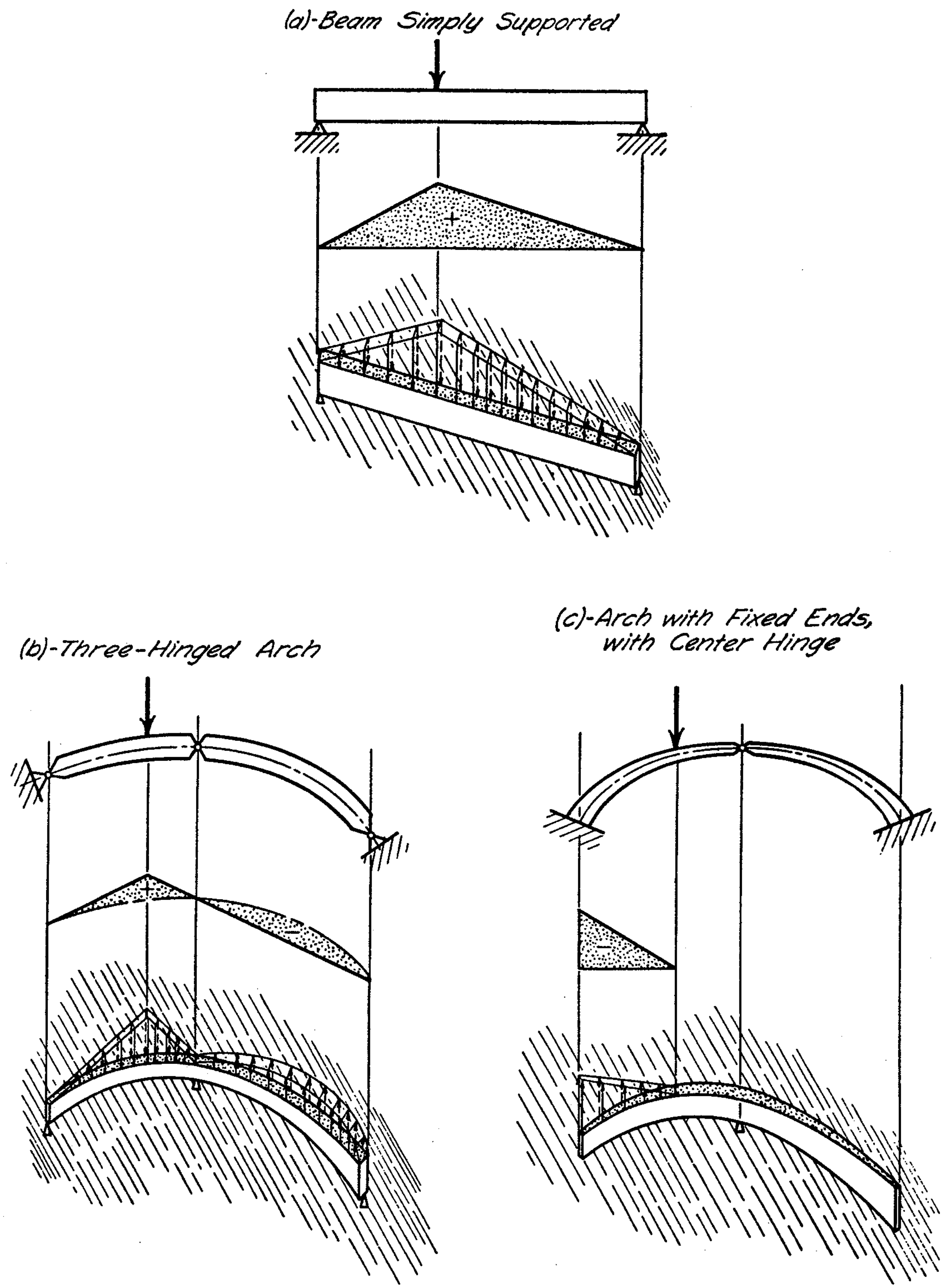


FIG. 20. TYPES OF SUPPORTS FOR ANALOGOUS COLUMNS

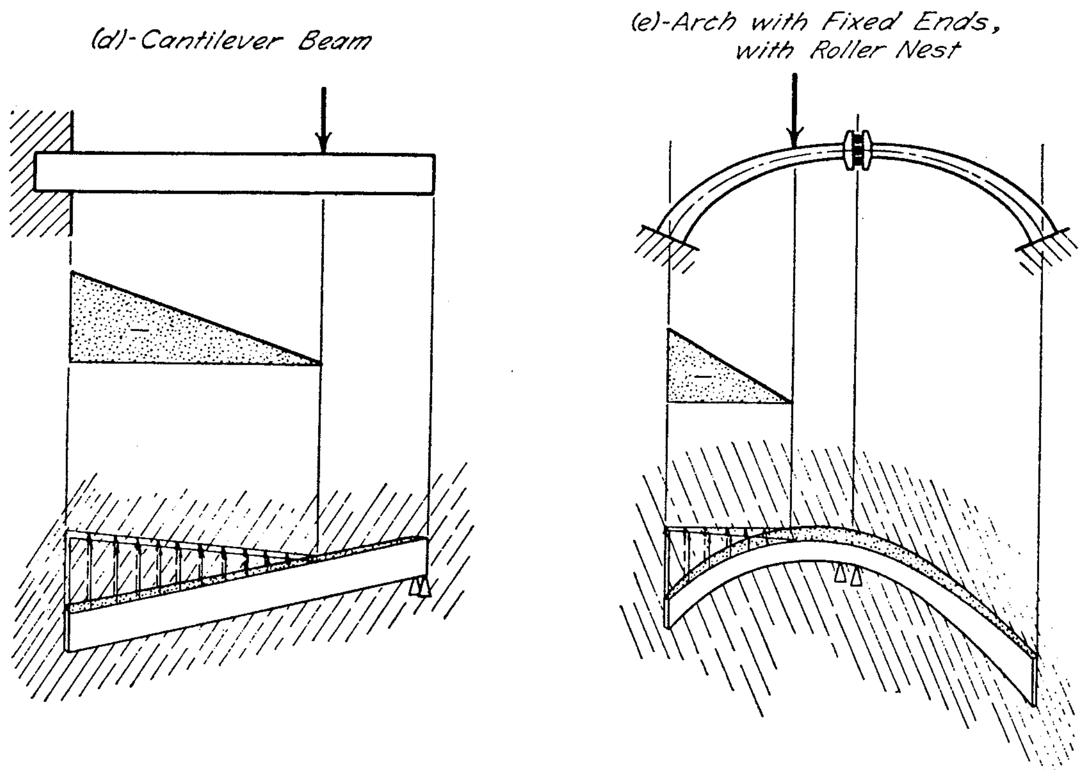


FIG. 20 (CONCLUDED). TYPES OF SUPPORTS FOR ANALOGOUS COLUMNS

relation is familiar to most students, as is also the conjugate beam relation.

In Fig. 20e is shown an arch having a roller nest equivalent to two hinges on a line normal to the roller bed. The elastic area is infinite and the elastic moment of inertia about the axis of the roller bed—the Y axis—is also infinite.

(b) Reciprocality of Hinges and Supports in Beam and Analogous Column

Just as hinges in the girder or arch may be represented by rigid supports in the analogous column, so rigid supports of the girder may be represented by hinges or combinations of hinges in the analogous column, since bending moment in the column corresponds to deflection in the beam. If the restraint is in two directions, the column at that point would contain two hinges at an angle with each other, which is equivalent to a universal joint.

These relations do not, however, seem to be very useful, since it is not convenient to analyze the stresses at the bases of such columns. The analogy, however, is illustrated in Figs. 21a and 21b. In Fig. 21b the analysis for moments in the beam is not affected by the hinge and joint in the analogous column, while analysis for slopes and deflections in the beam is facilitated by their use.

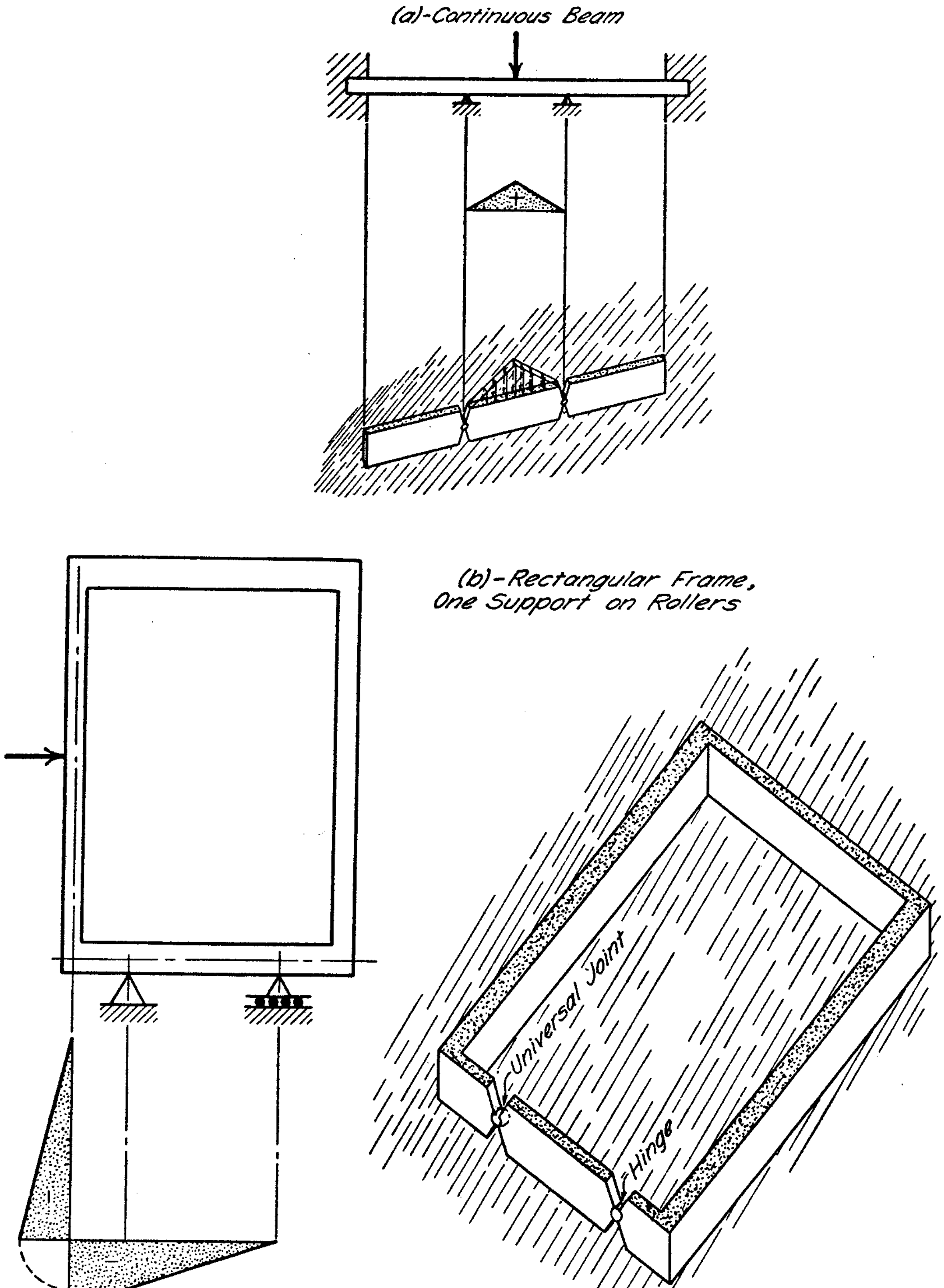


FIG. 21. TYPES OF SUPPORTS FOR ANALOGOUS COLUMNS

(c) The Conjugate Beam

The term "conjugate beam" is due to Professor H. M. Westergaard. In an article entitled "Deflection of Beams by the Conjugate

Beam Method"* he has shown a reciprocal relation between straight loaded beams and certain imaginary beams which are subject to loads equal in intensity to the curves of moments on the original beams.

The conjugate beam as presented by Professor Westergaard will be seen to be the analogous column section. The conception, however, can be extended to curved as well as to straight beams.

IV. Extension of the Analogy

20. *Introduction.*—The analogy between the computation of moments in beams, bents, and arches and the computation of fiber stresses on a section of an eccentrically loaded column as previously stated is simple and convenient. The analysis thus far explained, however, either neglects the effect of linear distortions within the structure or makes separate allowance for the effect of such distortions. In this section of the bulletin it is explained that by an extension of the analogy it is possible to include the effect of these linear distortions directly in the analysis. In doing this, however, the simplicity of the picture is marred. The extension here presented is, then, probably to be thought of as a very special tool to be used infrequently, if at all. The completeness of analysis furnished by this method of treatment justifies the inclusion of the material.

21. *Internal Distortions.*—The beam formula may be modified as follows: Moments may be separated into two parts, due to loads and due to couples and the products of inertia may be separated into two parts, due to the areas which make up the section and due to the centroidal products of inertia of those areas. The terms in the beam formula may then be redefined.

Write

$$\begin{aligned} P &= \Sigma pa \\ A &= \Sigma a \\ M_x &= \Sigma pae_x + \Sigma M_x \\ M_y &= \Sigma pae_y + \Sigma M_y \\ I_x &= \Sigma ax^2 + \Sigma i_x \\ I_y &= \Sigma ay^2 + \Sigma i_y \\ I_{xy} &= \Sigma axy + \Sigma i_{xy} \end{aligned}$$

Thus far in dealing with internal distortions only the angular rotations which take place about the centroids of the small sections into which the axis of the beam is divided have been considered.

*Loc. cit., page 7.

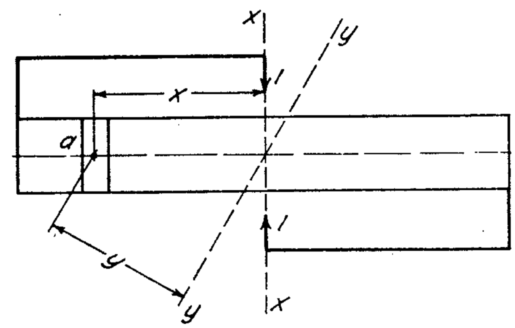


FIG. 22. TERMS USED FOR INTERNAL DISTORTIONS

Angular rotations have been treated as loads on the column section and an elastic body which will suffer angular distortion from moment has been treated as a part of the area of the column section. But a linear movement will result from two equal and opposite angular movements about different centers, and if these rotations are treated as loads, they will be parallel loads of opposite sense and will constitute a couple. Hence any linear distortion within the body corresponds to a moment load on the analogous column section.

The bending moment on any differential portion of a member due to unit forces through the elastic centroid of the member is x and the relative displacement of the ends along any axis Y due to this moment is axy (see Fig. 22). The total displacement then is Σaxy and is the product of inertia of the elastic areas about the elastic centroid. But it makes no difference whether this displacement is due to flexural or to shearing distortions. Whatever its cause, we can still treat it as if it were Σaxy and call it the centroidal product of inertia about the axes X and Y.

Just as any portion of a section may be represented for purposes of beam analysis by six quantities, namely, two coördinates of its centroid, its area, and its products of inertia about two centroidal axes, so the elastic properties of an elastic body may be completely represented by corresponding elastic quantities. And just as these six elementary properties of an area may be deduced from six other properties—namely, the statical moments about any three axes in the plane and the products of inertia about any three pairs of axes in the plane, so the elementary elastic properties of the section may be deduced if we have three displacements due to unit moment on the body and three displacements due to equal and opposite forces at the ends of the body.

This makes it possible to include automatically all types of stress distortion in beams directly in the analysis, if this seems desirable.

Moreover it is possible in members containing straight segments subject to constant shear to evaluate the bending effects by separating the effect of the total rotation and the relative displacement of the ends due to bending without finding the centroid of the moment curve.

The elastic area of a body, as the term is used here, is the total rotation produced in the body by opposite unit moments at its ends. The elastic centroid is the center about which the rotation occurs.

The total displacement of opposite ends of a member along one axis due to opposite unit terminal forces along any other axis is the elastic product of inertia of the body about these axes. If the axes are through the centroid, the displacement is an elastic centroidal product of inertia. If the axes are coincident through the centroid, the displacement is an elastic centroidal moment of inertia.

In computing the properties of a body as a whole, these centroidal products of inertia may be treated just as they are treated in the computation of the properties of beam sections. The elastic products of inertia of the analogous column, then, may be computed as the sum of the elastic areas times the products of their distances from the axes under consideration, plus the centroidal products of inertia of the elastic areas.

These centroidal products of inertia are useful also in computing the elastic moments on the analogous column, though the procedure has no simple analogue in beam analysis. It has been explained that known linear displacements may be treated as moment loads about the axis of displacement. Such displacements are produced by thrusts and shears on the portions of the elastic body. The displacement along any axis due to a force along any line equals the product of the force by the product of inertia of the body about the axis of displacement and the line of action of the force, as this product of inertia has been defined.

The elastic moments on the analogous column may be taken for each portion of the beam as equal to the moments at the elastic centroid of that portion times the elastic area plus a couple equal to the shear on that portion of the beam times the elastic centroidal moment of inertia. In this way the elastic moments have been computed in Fig. 14.

V. Physical Constants of Deformation for Structural Members

22. *Nature of Physical Constants.*—Thus far consideration of the physical properties of the members has been restricted. No theory

has been propounded to predict the values of the elastic areas. If we are dealing only with distortions produced by external loads, it does not make any difference what are the absolute values of these elastic areas, or moduli of distortion; it is the relative values only that are wanted.

It seems somewhat unfortunate that the theory used in the analysis of continuous frames has come to be known as the theory of elasticity. In its simplest form it has nothing to do with elasticity in the ordinary sense of the word. The theory of elasticity merely states the geometrical conditions essential to continuity in terms of the physical properties of the structure. If these physical properties are known for the conditions of stress which actually exist, then the theory may be applied. It is thus possible to apply the theory of continuity in a perfectly definite way to plastic materials, such as concrete, taking account of the variation of the plastic distortions with both the intensity and the duration of the stress, provided the properties of the material are accurately and definitely known.

Thus, suppose an exact analysis of a concrete arch is desired and that the ratio of total stress to total deformation is known as a function of both intensity and duration of stress. We first assume a constant value of the ratio of stress to total deformation, which we will call E, throughout the arch rib, and find all stresses. Since the value of E is now a variable over each section, we transform the sections as explained previously. Using the transformed sections, the stresses are again found and the process is repeated to any desired degree of precision.

We now know accurately the stress conditions at the time of loading. After an interval all values of E will be changed and we can repeat the process just outlined. By successive repetitions we could trace out the complete stress history of the structure.

Whether we could ever know the physical properties of the material with enough accuracy to take account of their variations is a question of fact to be considered separately. Whether the structure is sensitive enough to such variations in properties to make the variation in results secured from any such analysis appreciable is also another matter. Of what importance such information would be in the safe and economical design of the structure is still another matter.

Qualitative thinking along these lines will disperse certain illusions which seem to be current as to mysterious results from plastic flow and from time yield of concrete. The subject will not be pursued here, since the monograph is restricted to geometrical relations. The important point just now is to clearly distinguish those facts which are purely geometrical from those which are necessarily a subject for laboratory equipment.*

23. *Method of Determining Physical Constants.*—Determination of the elastic constants themselves involves a knowledge of the properties of the material. In general they are to be determined as follows: (a) apply a unit moment at each end of the body and determine the magnitude of the rotation and the center of rotation (this gives the elastic area and its centroid); (b) apply unit forces along an axis Y at the ends of the member and through the elastic centroid and determine the relative linear displacements of the ends along and normal to axis Y. Similarly apply unit forces along an axis through the elastic centroid along axis X, normal to axis Y, and determine displacement along axis X (this gives I_x, I_{xy}, I_y).

The elastic properties of a body as just defined—the elastic area, elastic centroid, and elastic products of inertia—are true physical properties for the stress condition in the body. If they are known for each portion of the body, they can be computed for the whole. If certain assumptions are made we can predict the properties of the individual parts.

In the deductions which follow a constant value of E and the conservation in bending of plane right sections is assumed. In most cases these assumptions are very nearly correct. But these values cannot be predicted with absolute precision because of chance variations in the properties of the material.

The elastic properties of a body are defined for two ends at which alone forces are supposed to be applied to the body. For another pair of termini another set of elastic properties would be deduced.

The elastic properties for a given pair of termini may be determined experimentally as follows:

Hold one of the ends rigidly and apply through a bracket attached to the other end a vertical force and a horizontal force successively at each of two points. In each case measure the vertical displacement and the horizontal displacement of each of two points on the bracket (see Fig. 23).

We now have sixteen quantities from which to deduce the six quantities desired. If Hooke's Law holds for the material, only ten of these quantities will be different and only six will be needed. Many

*For a treatment of these matters, see "Neglected Factors in the Analysis of Stresses in Concrete Arches," by Lorenz G. Straub, presented as a thesis for the degree of Doctor of Philosophy at the University of Illinois in 1927 and later published in part as "Plastic Flow in Concrete Arches," Proc., A. S. C. E., Jan., 1930.

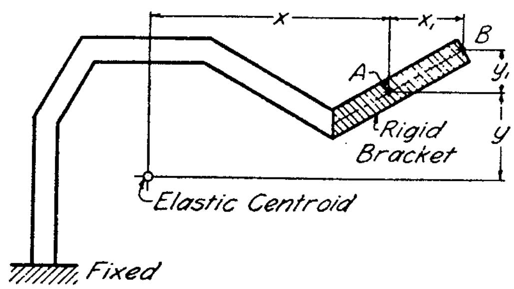


FIG. 23. EXPERIMENTAL DETERMINATION OF PHYSICAL CONSTANTS

different combinations of measurements may be used. The six quantities may all be deflections due to unit forces, as follows:

Vertical at A due to vertical force at $A = I_x + Ax^2$
Vertical at A due to vertical force at $B = I_x + Ax\,(x + x_1)$
Vertical at A due to horizontal force at $A = I_{xy} + Axy$
Vertical at B due to vertical force at $B = I_x + A(x + x_1)^2$
Horizontal at A due to horizontal force at $A = I_y + Ay^2$
Horizontal at B due to horizontal force at $B = I_y + A(y + y_1)$

From these the values of A, x, y, I_x, I_y, and I_{xy} may be deduced by simple algebra for known values of x_1, y_1.*

If Hooke's Law does not hold, different sets of measured displacements will not be consistent even though the measurements are made with absolute accuracy, because the body does not have any one set of elastic properties but has a different set for each different condition of stress.

24. *Computation of the Constants.*—Only a few of many possible illustrations of the computation of elastic constants is given here.

(a) Hinges and Roller Nests

Since forces acting on solid foundations produce no deformation, the elastic constants for the earth are taken as zero if we assume immovable abutments.

Eccentric forces acting on a frictionless hinge will produce no linear movement of the hinge but will produce unlimited rotation; hence the elastic area of a hinge is infinite, and its elastic centroidal products of inertia are zero.

Forces acting on a roller nest and inclined to its bed will produce no rotation, no displacement normal to the bed, and unlimited displacement along the bed; hence a roller nest has zero elastic area, zero

*A procedure which is simpler algebraically is as follows: Apply unit moment and measure rotation and horizontal and vertical displacement of one end. This gives the elastic area and locates the elastic centroid. The elastic centroidal products of inertia may then be measured directly as displacements along the centroidal axes due to unit loads through the centroid.

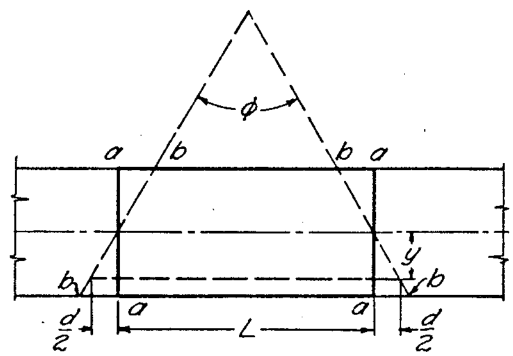


FIG. 24. FLEXURAL DISTORTION OF A BEAM

elastic centroidal moment of inertia about an axis normal to the bed, infinite elastic centroidal moment of inertia about an axis lying in the bed.

For a free end all elastic constants are infinite. The fact has no particular significance except that the theorem which states the column analogy is of perfectly general application to all plane beams, whether statically determinate or not, both simply supported and cantilevered.

(b) Flexural, Longitudinal, and Shearing Distortions in Straight Beams

The angle of flexure in a length of straight beam of constant section subjected to a given bending moment is readily computed by geometry if sections plane before bending remain plane after bending.

In Fig. 24 let *a-a* be given length L of the beam before bending and *b-b* after bending. The angle change ϕ equals the change in length of any fiber d divided by its distance from the neutral axis y.

Then $\phi = \frac{d}{y}\cdot$ From the definition of E, $d = \frac{fL}{E}$ where f is the fiber stress along any fiber. But $\frac{f}{E} = \frac{My}{I_t}$, where I_t is the moment of inertia of the transformed section obtained by dividing the areas of the original section by their values of E. Then $\phi = \frac{ML}{I_t}$ for straight beams.

Flexure of curved beams is discussed below.

Longitudinal distortion is computed as $\frac{PL}{A_t}$ for a force through the centroid of the transformed section.

The shearing distortion per unit of length of beam equals the shear divided by the continued product of area, the shearing modulus of elasticity and a factor depending on the shape of the section.

(c) Straight Homogeneous Beam of Uniform Section

Consider a straight segment of a homogeneous beam of uniform section. Let its properties be represented by length $= L$, area $= A$, moment of inertia $= I$, modulus of elasticity $= E$, radius of gyration $= \rho$.

Apply unit moment, unaccompanied by shear. Rotation is $\frac{L}{EI}$ and centroid of rotations is at the mid-point.

Apply unit transverse shear, otherwise unaccompanied by moment. Transverse displacement of end

(a) due to bending $= +\int x^2 da = \frac{1}{12}\frac{L}{EI}L^2$

(b) due to shearing distortions $= +\frac{L}{nAG}$, where n is the factor referred to above and G is the shearing modulus of elasticity. Normally, this displacement is unimportant. There is no longitudinal displacement due to transverse shear.

Apply unit longitudinal force. Longitudinal displacement $= +\frac{L}{AE} = +\rho^2\frac{L}{EI}$. No transverse displacement.

The elastic properties, then, are:

Elastic area $= \frac{L}{EI}$

Elastic centroidal moment of inertia about longitudinal axis

$$= \frac{1}{12}\frac{L}{EI}L^2 + \frac{L}{AGn}$$

Elastic centroidal moment of inertia along longitudinal axis $= \frac{L}{AE}$

Elastic centroidal product of inertia about two axes $= 0$

Elastic centroid on axis at mid-point.

These six quantities completely describe, on the ordinary assumptions of mechanics, the elastic properties of this body for terminal forces.

(d) Bars of Trusses

The strain of a bar in a truss produces a rotation at the moment center for that bar $\frac{\Delta}{r}$. If a unit moment acts at that center, $\phi = \frac{L}{AEr}$. This determines the elastic area.

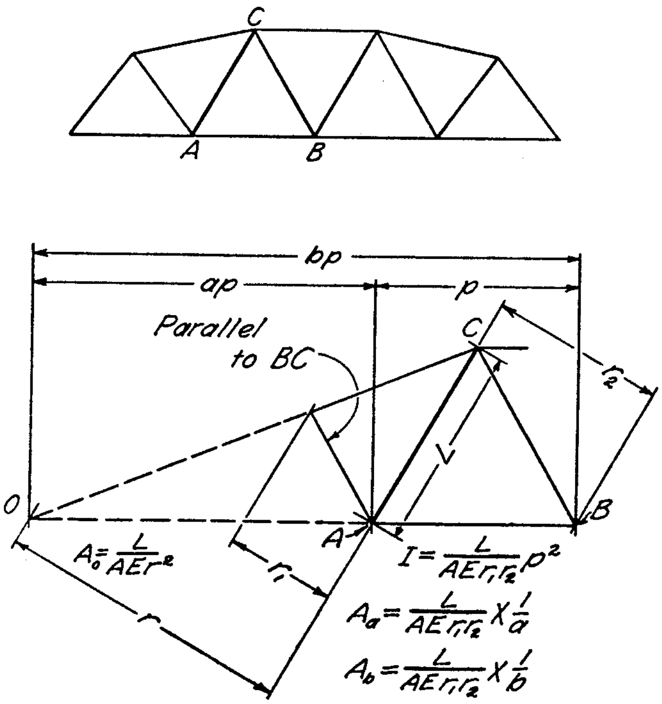


FIG. 25. DEFORMATION CONSTANTS FOR WEB MEMBERS IN A TRUSS

This expression is sometimes inconvenient for web members where chords are not parallel, because the moment center does not lie near the panel in which the section for stress is passed. If the chords are parallel, it leads to indeterminate expressions.

In these cases it is convenient to replace the distant elastic weight by two elastic weights and an elastic moment of inertia lying in the panel. Thus the true elastic area at O is $+\frac{L}{AEr^2}$. If the areas at A and B are to have that at O as a resultant (see Fig. 25)

$$A_a = +\frac{L}{AEr^2}\frac{bp}{p} = +\frac{L}{AEr_1r_2}\frac{1}{a}$$

$$A_b = -\frac{L}{AEr^2}\frac{ap}{p} = -\frac{L}{AEr_1r_2}\frac{1}{b}$$

since $$\frac{r}{bp} = \frac{r_1}{p} \text{ and } \frac{r}{ap} = \frac{r_2}{p}$$

For product of inertia about axes normal to the lower chord to be the same as that of A_o,

$$I_{o-o} = 0 = A_a(ap)^2 + A_b(bp)^2 + I$$

$$I = +\frac{L}{AEr_1r_2}p^2$$

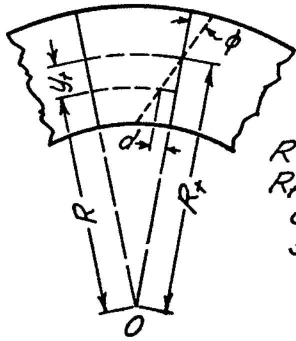


FIG. 26. SEGMENT OF A CURVED BEAM

Evidently no correction for the product of inertia is necessary if either axis is parallel to the lower chord.

The elastic areas A_a and A_b may be located on either chord and the moment of inertia taken along that chord provided ap and bp are computed along that chord. The positive elastic weight always lies on the side of the panel next to the moment center.

If the chords are parallel, the elastic areas become zero, the moment of inertia along the chord is still $I = +\frac{L}{AEr_1r_2}p^2$, but $r_1 = r_2$.

This method of treating truss bars is sometimes advantageous for computing stresses in indeterminate structures. In most cases indeterminate trusses can be analyzed conveniently by other methods than the column analogy.

(e) Beams Sharply Curved

Beams having a radius of curvature small compared with their depth are common in machine parts; in structural design they occur in thick arched dams and at the haunched junction of beams and columns. The formula of Winkler, usually presented for the solution of such beams, leads to certain complications when used to compute deformations or for the analysis of such beams when they are statically indeterminate. For this reason, a detailed explanation of the use of the transformed section seems desirable here.

In Fig. 26 is shown a differential length of a beam having a center of curvature at O.

It has been shown that if the section is transformed by dividing each differential area by its radius of curvature R we can write

$$fR = \frac{P}{A_t} + \frac{M_t}{I_t} y_t$$

where f is the fiber stress at any distance y_t from the centroid of the transformed section

R is the radius of curvature of this fiber

P is the normal load

M_t is the moment about the centroid of the transformed section

A_t is the area of the transformed section

I_t is the moment of inertia of the transformed section about its centroid.

Also let

A = the area of the original section

R_o = the radius of curvature to the centroid of the original section

R_t = the radius of curvature to the centroid of the transformed section.

By definition $A_t = \int \frac{dA}{R}$

Taking moments about O

$$R_t = \frac{\int R \frac{dA}{R}}{\int \frac{dA}{R}} = \frac{A}{A_t}$$

The moment of inertia about O =

$$\int \frac{dA}{R} R^2 = \int R dA = AR_o$$

Reducing this to the centroid of the transformed section

$$I_t = AR_o - A_t R_t^2 = AR_o - A_t \frac{A}{A_t} R_t = A(R_o - R_t)$$

If, then, the area and centroid of the original section are known, it is necessary to find only the area of the transformed section in order to compute directly all quantities needed.

The angle of rotation about the centroid of the transformed section is

$$\phi = \frac{y_t}{d} = \frac{Ey_t}{f} \times \text{(length of differential fiber under consideration)}$$

But $f = \frac{M_t y_t}{I_t R}$

$$\text{Length of any fiber} = \frac{R}{R_t}\,ds$$

Where ds is the length of the differential fiber along the axis of the centroid of the transformed section.

Hence $\phi = \dfrac{ds}{EI_tR_t}$ is the elastic area of the segment, or in general $A = \dfrac{\text{angle of arc}}{EI_t}$.

If $M_t = 0$, the section moves parallel to itself and

$$\Delta = \frac{P}{E(A_tR_t)}\,ds = \frac{P}{AE}\,ds$$

Hence $\dfrac{ds}{EA}$ is the elastic centroidal moment of inertia for rib-shortening correction.

It will be seen, then, that all expressions for beams of sharp curvature take the same form as where the beams are straight if the axis of the beam is taken as the axis defined by the centroids of the transformed sections instead of by the centroids of the original sections.

(f) Compound Members—Bifurcated Members

The elastic properties of any elastic ring may be defined with reference to the termini from the principles indicated. Thus, to determine the elastic properties of the ring $ABCDA$, Fig. 27, with reference to terminals AC, apply at C a unit moment, the ring being cut at C and held at A. Now treat as a column section cut at A and uniformly loaded along ABC and compute the shear at C and the bending moment on vertical and horizontal sections through C. This gives the elastic weight and its static moments about C, from which the coördinates of the centroid may be computed.

Next apply at C a unit vertical force, the ring being cut at C and held at A. Apply the column analogy and compute the bending moments on the analogous columns on horizontal and vertical sections at C, the section being cut at A. This gives moment of inertia for vertical axis through C and product of inertia for vertical and horizontal axes through C. These may then be reduced to the centroid.

Finally apply at C a unit horizontal force, the ring being cut at C and held at A. Compute the bending moment on a horizontal section through C on the analogous column section cut at A. This is the moment of inertia for a horizontal axis through C. This may now be reduced to the centroid.

If loads occur within the ring, the elastic load on the ring may be determined from the shear (elastic load) and moments on vertical and horizontal axes through C (static moments of the elastic load about C) for the elastic column section if cut at A as above.

The elastic load and its point of application and also the elastic properties of the ring being known, the ring may be treated as is any other member.

This procedure is relatively simple but involves a good deal of computation. The problem, however, is not a very simple one. Anyone who has occasion to use the method for numerical problems will find that it leads to comparatively simple expressions.

(g) Successive Compounding

The compounding procedure just indicated may be extended indefinitely to include any number of branches. The most common and important case is that of a series of continuous arches or bents. The general procedure by this method in such a case is as follows:

(a) Apply a unit moment at the junction point of the outside bent, find the elastic area and elastic centroid.

(b) Apply a unit vertical force at the junction and find the moment of inertia about a vertical axis through this junction and the product of inertia about horizontal and vertical axes through this point. Reduce these to the centroid.

(c) Apply a unit horizontal force at the junction and find the centroidal moment of inertia of the combination for a horizontal axis through the junction. Reduce this to the centroid.

(d) Substitute these elastic properties for those of the pier in the next bent; proceed as above and continue to include any desired number of bents.

(e) In a similar way the elastic loads are to be combined for one bent after another.

(f) The reactions having been determined for the last bent of the series (which may be the center bent, combinations having been made from both ends) these can be resolved back successively through the series.

This method has enough value to justify reference to it, though, even in the case of continuous arches, other methods are more convenient.

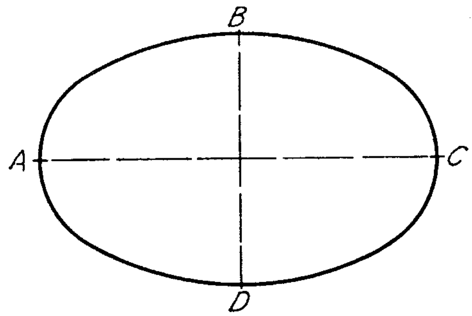


FIG. 27. CLOSED ELASTIC RING

VI. APPLICATIONS OF THEOREM

25. *Fields of Application of Theorem.*—The theorem here presented has several fields of application:

(1) In the routine analysis of symmetrical and unsymmetrical arches and bents for loads, either gravity or inclined, for temperature, for shrinkage, or for abutment displacement.

(2) In determining moments at the ends of beams fixed at ends and in determining other properties of such beams, such as the end moments corresponding to unit rotation at end, for use in connection with various methods of analysis of continuous girders or frames.*

(a) It is possible thus to determine constants for use in the general equations of displacements. These equations state that terminal forces and moments on a member are the sum of those due to known loads on the member or distortions in the member when the ends are fixed and to any displacements or rotations of the ends, known or to be determined. A special case of these equations, applicable where the members are straight and of uniform section, is known in American literature as the equation of slope-deflection.

Values of terminal forces in terms of the unknown terminal displacements may be substituted in the equations of static equilibrium of the joints. These equations may then be solved for the terminal displacements.

When the joint displacements have been found the terminal forces may be computed from the original equations of displacement.

(b) Constants may be determined for equations which state the existence of continuity at the joints. Of these the theorems of

*Loc. cit., page 48.

three moments and of four moments are special examples of limited application.

(c) Terminal forces may be determined for known loads or internal distortions on the assumption that the joints are not displaced and then the unbalanced terminal forces may be distributed among the connecting members in proportion to their resistance to end displacement. In a special case this is done by the method of moment distribution. The constants needed may be determined by the column analogy.

(d) In routine computations of slopes and deflections.

26. *Methods of Analysis of Continuous Frames.*—This bulletin does not deal primarily with methods of analysis of continuous frames, but only with the analysis and elastic properties of the individual members of which the frame may be made up. Some comments on methods of analysis of such frames is needed to explain applications of the column analogy indicated.

Methods of analysis of continuous frames may be divided into those involving internal work of the frame, such as the method of least work, and those involving the geometry of continuity. That all methods are really the same will at once be realized but their relations to each other will not be discussed here.

Of the geometrical methods of analysis we may distinguish those in which the displacements of the joints are treated as unknowns and those in which the terminal forces acting on the members at the joints are treated as unknowns. The former are represented in a special case by the method of slope-deflection; the latter are represented by the theorem of three moments and by the theorem of four moments.

The method in which the joint displacements are treated as unknowns might be called the Method of Joint Displacements. In this method the terminal shears and moments are written in terms of the joint displacements. The equations of statics—the forces balance at each joint—are then written for each joint. From these the terminal displacement is computed. The terminal displacement being known, the shears and moments may be found.

In the other method of analysis the terminal slopes and displacements are stated in terms of the terminal shears and moments. The equations of continuity which state that the displacement of a joint is common to the ends of all members meeting at the joint are then written for each joint. From these equations the terminal shears and moments are found directly. The equation of three moments is a familiar illustration of an equation of joint continuity. A less familiar illustration is the equation of four moments. This method might be called the Method of Continuity.

It is well to distinguish the method of slope-deflection from the equation of slope-deflection; the method, which seems to have been restricted to straight members, consists in treating joint rotations and displacements as unknowns in equations of static equilibrium for the joints. The equation of slope-deflection is merely one of many forms of the equation which relates the terminal forces to the loads and terminal displacements of a straight beam. The method of slope-deflection may be used without using the particular form of equation known as the equation of slope-deflection or the equation of slope-deflection may be used without using the method of slope-deflection.

If we write the end moments and forces in terms of loads and end displacements, we can derive from these expressions for the end rotations and displacements in terms of loads and end moments and forces. Displacements of the ends of all members at a joint are equal and angular rotation at the ends of all members meeting at a joint are equal. These displacements and rotations being in terms of end moments and forces on the beams and of the physical properties of the beams, the equations can be solved for the end moments and forces.

The Method of Terminal Force Distribution will not be discussed here. It is more closely related to the Method of Continuity than to the Method of Joint Displacements.

27. *General Equation of Displacements and Slope-Deflection.*—The general equations of end forces at any joint, A, of a structure may be written as follows, provided, as is usually true, the principal axes of the members are parallel and normal to each other:

$$M_a = \phi_a N_a - \phi_b r_a N_a + (\Delta_a - \Delta_b)\frac{d_a}{I_o} + M'_a$$

$$F_a = \frac{\Delta_a - \Delta_b}{I_o} + \frac{\phi_a d_a - \phi_b d_b}{I_o} + F'_a$$

M_a and F_a are total end moment and end force (thrust or shear) in the member

(Note that in general there will be two equations of force at any joint, one for horizontal and one for vertical forces.)

M'_a = moment which would exist in the member if there were no rotation or displacement of the joints.

F'_a = force which would exist in the member if there were no rotation or displacement of the joints.

ϕ_a, Δ_a are respectively rotation and displacement at the joint considered.

ϕ_b, Δ_b are corresponding quantities at the other end, B, of each member successively.

N_a is the moment at joint A corresponding to a unit rotation of this joint, the other end being fixed.

r_a is the carry-over factor at A (the ratio of the moment at B due to a unit rotation at A to the moment at A due to such a rotation.)

I_o is the elastic moment of inertia about the centroid for any member.

d is the distance of either end from the centroidal axis of a member.

From these general forms and the equations of static equilibrium we may write the equations for every joint in a complex structure such as a continuous arch series or a Vierendeel truss. These equations may then be solved simultaneously for displacements and from these the moments, shears and thrusts may be determined.

The use of such equations, requiring a carefully selected convention of signs with resulting possibility of error from this source and involving simultaneous solution, is to be thought of as a research tool and rarely as a tool of design.

If there are no joint displacements, or if it is convenient to make separate allowance for such displacements, the process is much simplified. We then have, from $\Sigma M = 0$,

$$\phi_a = \frac{\Sigma M'_a}{\Sigma N_a} + \frac{\Sigma \phi_b r_a N_a}{\Sigma N_a}$$

the signs depending on the convention used.
This, perhaps, is more conveniently written,

$$\phi_a = \Sigma \frac{M'_a}{\Sigma N_a} + \Sigma \phi_b \frac{r_a N_a}{\Sigma N_a}$$

If connecting members are treated as prismatic,

$$N_a = 4\frac{I}{L} \qquad r_b = -\tfrac{1}{2}$$

$$\phi_a = \tfrac{1}{2}\Sigma \frac{M'_a}{2\Sigma \frac{I}{L}} - \Sigma \phi_b \frac{\frac{I}{L}}{2\Sigma \frac{I}{L}}$$

If, further, the values M'_a are due to known rotations of the bars,

$$M'_a = \frac{6\Delta}{L\dfrac{L}{IE}} = 6E\,\frac{I}{L}\,\psi = 6EK\psi$$

$$\phi_a = \Sigma 3\psi\,\frac{K}{2\Sigma K} - \Sigma\phi_b\,\frac{K}{2\Sigma K}$$

which expression is convenient in finding secondary stresses in trusses. Or this may be derived from the fundamental equation above

$$\phi_a = \frac{\Sigma M'_a}{\Sigma N_a} + \frac{\Sigma\phi_a r_b N_a}{\Sigma N_a} - \frac{\Sigma(\Delta_a - \Delta_b)\,\dfrac{d}{I}}{\Sigma K_a}$$

$$M'_a = 0,\; N_a = 4\,\frac{I}{L},\; r_a = -\tfrac{1}{2},\quad \frac{\Delta_a - \Delta_b}{L} = -\psi$$

$$\frac{d}{I} = 6\,\frac{I}{L}\cdot\frac{1}{L}$$

$$\phi_a = \frac{3\Sigma K\psi}{2\Sigma K} - \frac{\Sigma K\phi_b}{2\Sigma K}$$

These discussions include any combination of bars of any shape or form provided the axes are parallel or normal to each other, and it is not very difficult to extend the method to include skewed axes. The slope-deflection equation for prismatic beams,

$$M = \frac{2EI}{L}\,(2\phi_a + \phi_b - 3\psi)$$

may be derived in a number of ways and follows from the first general equation when

$$N_a = 4K,\; r_a = -\tfrac{1}{2},\; \psi = -\frac{\Delta_a - \Delta_b}{2d},\; \frac{d^2}{I} = \frac{\dfrac{L^2}{4}}{\dfrac{L^2}{12}\cdot\dfrac{L}{I}} = 3K$$

The interest in the equations at present, however, is chiefly in the utility of the column analogy in evaluating the constants M'_a, F'_a, d_a, I_o, N_a, and V_a for the members.

VII. Conclusion

28. *Conclusion.*—The paper is restricted to geometrical relations and does not discuss applications to design. Its thesis is simply that moments, shears, slopes, and deflections of beams due to any cause may be computed in just the same way and by the same formulas as are used in computing reactions on a short column eccentrically loaded or as would be used to compute shears and bending moments on longitudinal sections through such a column.

The column analogy is a convenient tool of mechanics, a somewhat mechanical device for a structural engineer, but one which seems to give desired results with a minimum of thought as to method of procedure or sign conventions. In the study of continuous frames it becomes auxiliary to methods for the analysis of such frames.

The most obvious application is in the analysis of single span structures; perhaps its most important applications occur in the study of continuous bents, arches, and beams.