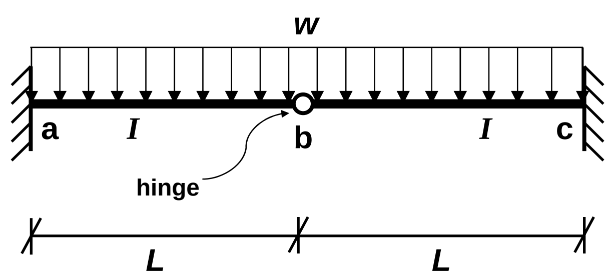
CE 474 – Structural Analysis II
Additional stiffness method problems

1) Two identical beams are connected to each other at node *b* with a hinge as shown below. The beams are fixed at their other ends (i.e. nodes *a* and *c*). Downward uniform loading of intensity *w* (load per lineal length) is applied on the beams. Both beams have modulus of elasticity *E*, moment of inertia *I*, and length *L*. Use stiffness method to analyze the system. Neglect axial deformations.

   a) Find the vertical deflection at node *b* in terms of *E*, *I*, *L*, and *w*.
   b) Draw the bending moment diagram.
   c) Draw the shear force diagram.

Note: Please indicate the degrees of freedom and sign convention you have chosen.



2) The continuous steel beam shown below has constant $EI = 100{,}000\ \text{kip}\cdot\text{ft}^2$. The beam is loaded on span A-B with a uniformly distributed load of *w* (kips/ft). The span B-C has two identical loads, *P*, applied as shown on the figure. The values of *w* and *P* are unknown.

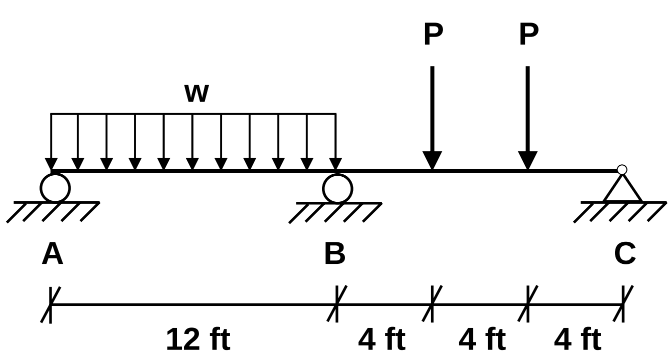
   Measurements show that the given loading results in the following beam rotations at the supports (taking counter-clockwise rotation as the positive sense rotation):

$$\theta_A = -1.68\times10^{-3}\ \text{rad}$$
$$\theta_B = +0.48\times10^{-3}\ \text{rad}$$
$$\theta_C = +0.72\times10^{-3}\ \text{rad}$$

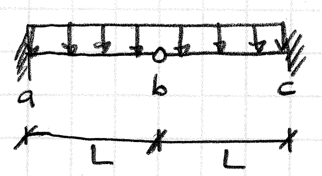
   a) Find the unique set of values of *w* and *P* that, when applied together as shown in the figure, will cause the rotations listed above.

   b) Draw the moment diagram.

   c) Draw the shear force diagram.



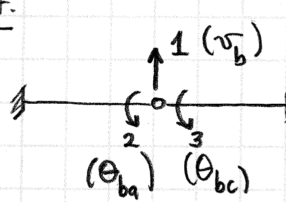
1)



Neglect axial deformations.

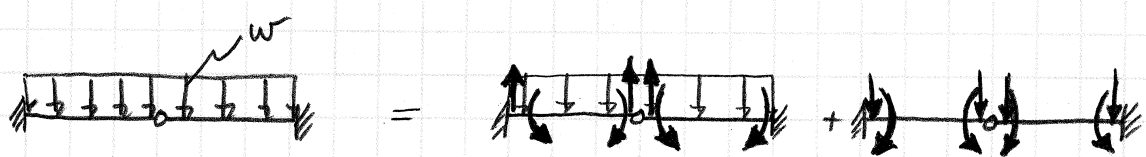
E, I const.

D.O.F.



Note that rotation on one side of the hinge @ b is dependent of the rotation on the other side.

I will solve for the displacement & rotations first by ignoring the obvious symmetry in the setup (structure & loading).



original

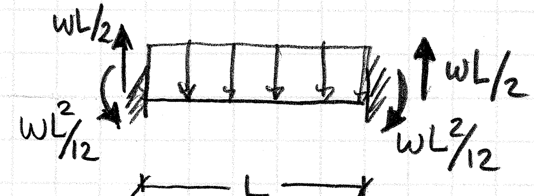
Case I

{Loading + FEM/FEF}

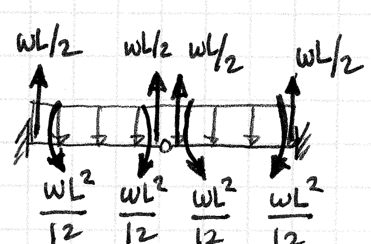
Case II

−{FEM/FEF}

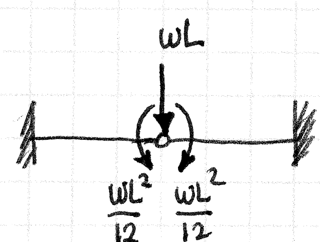
The FEM/FEF are found from



So,



+



Case I

Case II (note that only those loads along the three DOF are shown)

$$\{Disp\} = \{Disp\}_{case\ I} + \{Disp\}_{case\ II}$$

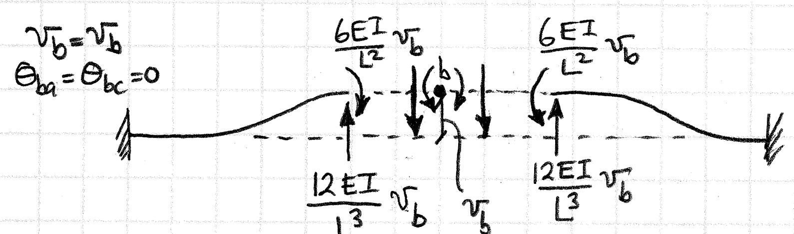
$$\begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix} = \cancel{\begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix}_{case\ I}} + \begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix}_{case\ II}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

(The FEM/FEF in Case I prevents the structure from displacing or rotating at the three DOF/coordinates we have chosen.)

So, to find the disp./rot. along DOF 1, 2, 3, all we need is to analyze Case II.

Displacement along D.O.F. 1 & rotations along D.O.F.s 2 & 3 will each cause resistance force and moments to be developed along D.O.F.s 1, and 2 and 3, respectively. They can be found by inducing a non-zero disp. or rotation along a D.O.F. while keeping others at zero disp or rot.



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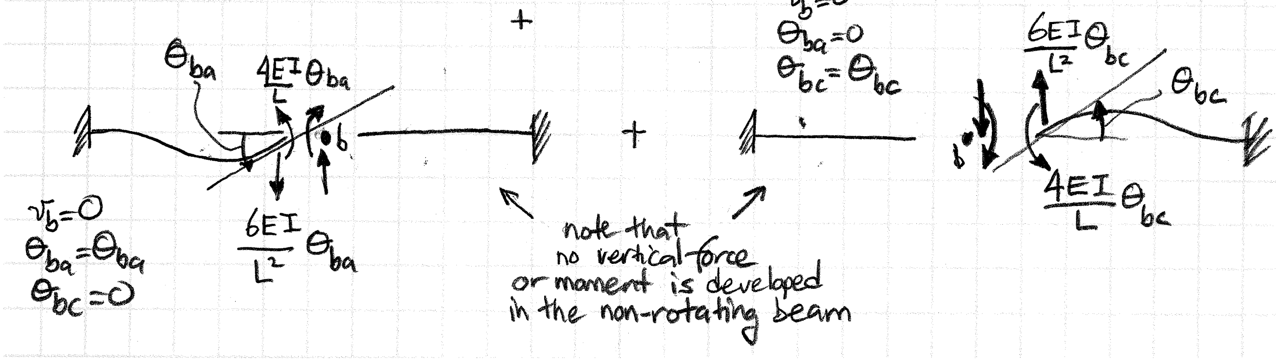


Then, the equilibrium eqns along (at) the three D.O.F. can be written as

along D.O.F. 1
$$+\frac{12EI}{L^3}v_b + \frac{12EI}{L^3}v_b - \frac{6EI}{L^2}\theta_{ba} + \frac{6EI}{L^2}\theta_{bc} = -wL \qquad (1)$$

D.O.F. 2
$$-\frac{6EI}{L^2}v_b + \frac{4EI}{L}\theta_{ba} = \frac{wL^2}{12} \qquad (2)$$

D.O.F. 3
$$+\frac{6EI}{L^2}v_b + \frac{4EI}{L}\theta_{bc} = -\frac{wL^2}{12} \qquad (3)$$

Add Eqns (1) & (2) $\Rightarrow$ $\frac{4EI}{L}\theta_{ba} + \frac{4EI}{L}\theta_{bc} = 0 \Rightarrow \theta_{ba} = -\theta_{bc}$ (4)

(this was expected thanks to symmetry in the setup)

subst. $\theta_{ba} = -\theta_{bc}$ into Eqn (1) to get

$$\frac{24EI}{L^3}v_b - \frac{12EI}{L^2}\theta_{ba} = -wL \qquad (1^*)$$

multiply Eqn (2) w/ $\frac{4}{L}$ and add to Eqn (1^*) to get

$$\frac{4EI}{L^2}\theta_{ba} = -\frac{2}{3}wL$$

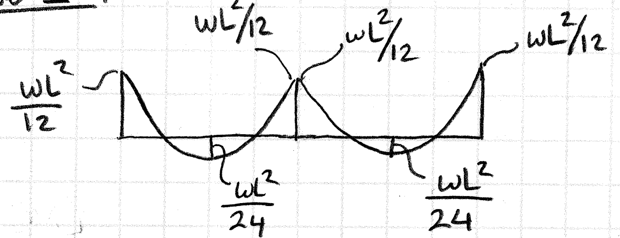
$$\Rightarrow \theta_{ba} = -\frac{wL^3}{6EI} \qquad \Rightarrow \theta_{bc} = \frac{wL^3}{6EI}$$

back subst.
$$\Rightarrow \left[v_b = -\frac{wL^4}{8EI} \right]$$

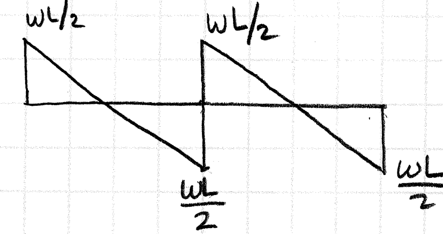
try to visualize the displaced shape — it makes sense, doesn't it?

$$\{M\} = \{M\}_{case\,I} + \{M\}_{case\,II} \qquad \{V\} = \{V\}_{case\,I} + \{V\}_{case\,II}$$

Case I:



$[M]_{case\,I}$



$[V]_{case\,I}$

Case II:

using the internal forces developed due to disp/rot along the three D.O.F.s we can find the member end bending moments & shear forces

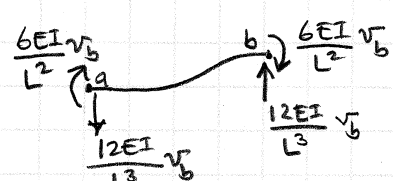
bending moments

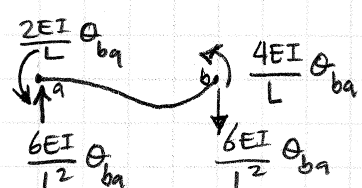
$$M_{ab} = -\frac{6EI}{L^2}v_b + \frac{2EI}{L}\theta_{ba}$$

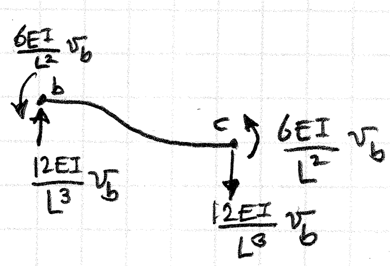
$$M_{ba} = -\frac{6EI}{L^2}v_b + \frac{4EI}{L}\theta_{ba}$$

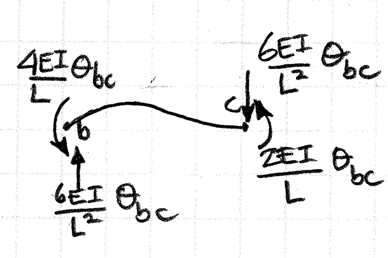
$$M_{bc} = \frac{6EI}{L^2}v_b + \frac{4EI}{L}\theta_{bc}$$

$$M_{cb} = \frac{6EI}{L^2}v_b + \frac{2EI}{L}\theta_{bc}$$









shear forces

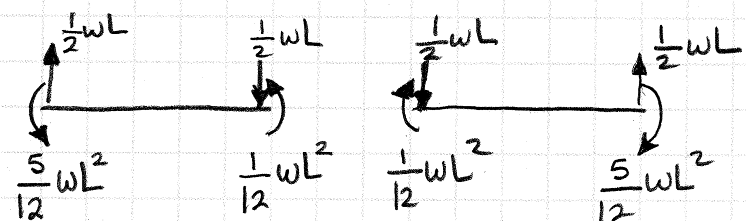
$$V_{ab} = -\frac{12EI}{L^3}v_b + \frac{6EI}{L^2}\theta_{ba}$$

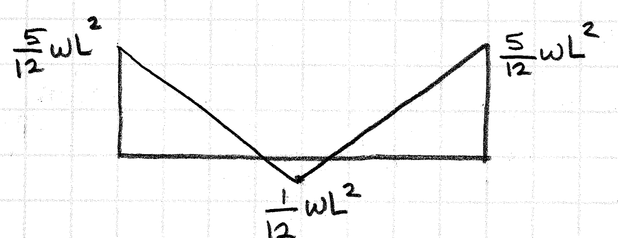
$$V_{ba} = +\frac{12EI}{L^3}v_b - \frac{6EI}{L^2}\theta_{ba}$$

$$V_{bc} = +\frac{12EI}{L^3}v_b + \frac{6EI}{L^2}\theta_{bc}$$

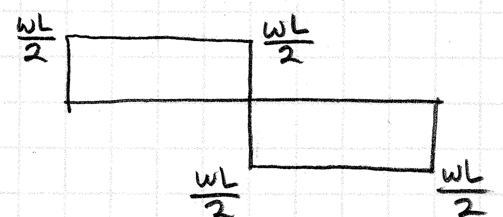
$$V_{cb} = -\frac{12EI}{L^3}v_b - \frac{6EI}{L^2}\theta_{bc}$$

substituting expressions for v_b, θ_{ba}, θ_{bc} we can find the member end forces & moments for Case II as:





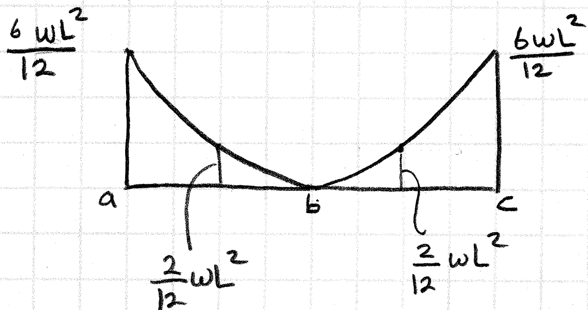
$[M]_{case\ II}$



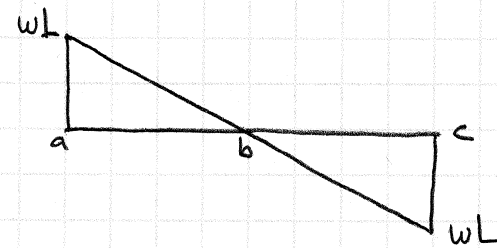
$[V]_{case\ II}$

using sign convention to draw shear force diagrams combining Case I & Case II results

$$\begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -wL^4/8EI \\ -wL^3/6EI \\ wL^3/6EI \end{Bmatrix} = \begin{Bmatrix} -wL^4/8EI \\ -wL^3/6EI \\ wL^3/6EI \end{Bmatrix}$$



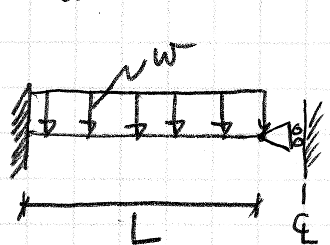
[M]



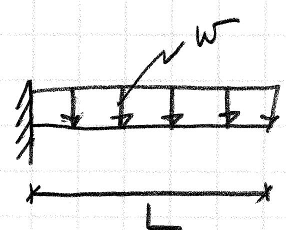
[V]

Now, if we had made use of symmetry:
looking at the setup, we see that θ_{ba} will be equal to θ_{bc} in amplitude but opposite in sense.

The problem is similar to

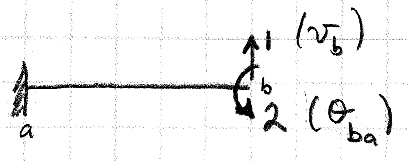


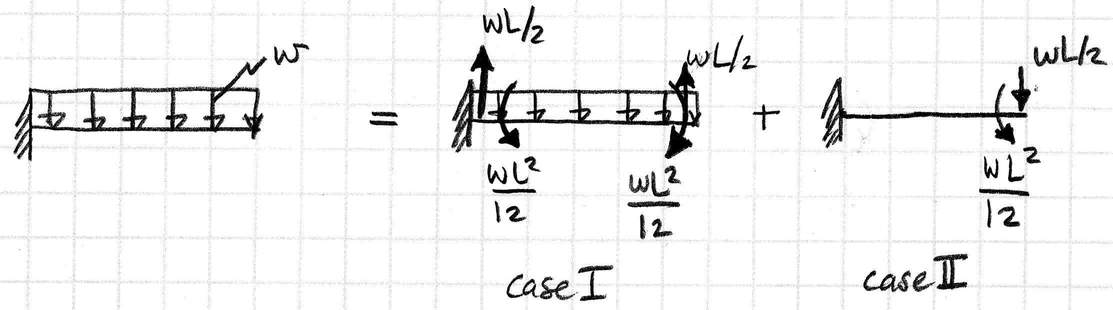
which is



and its mirror image

D.O.F.s





$$\begin{Bmatrix} v_b \\ \theta_{ba} \end{Bmatrix} = \underbrace{\cancel{\begin{Bmatrix} v_b \\ \theta_{ba} \end{Bmatrix}}_{\text{Case I}}}_{\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}} + \begin{Bmatrix} v_b \\ \theta_{ba} \end{Bmatrix}_{\text{Case II}}$$

writing equilibrium eqns (I am skipping the drawing "the deflected shapes" stage)

$$\frac{12EI}{L^3} v_b - \frac{6EI}{L^2} \theta_{ba} = -\frac{wL}{2} \qquad (5)$$

$$-\frac{6EI}{L^2} v_b + \frac{4EI}{L} \theta_{ba} = \frac{wL^2}{12} \qquad (6)$$

multiply Eqn (6) by $\frac{2}{L}$ and add to Eqn (5) to get

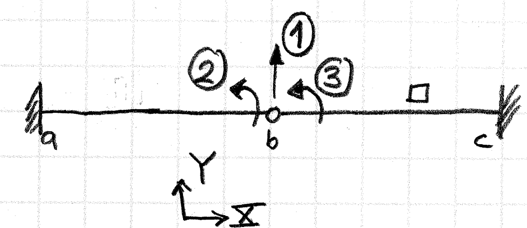
$$\frac{2EI}{L^2} \theta_{ba} = -\frac{wL}{3} \Rightarrow \theta_{ba} = -\frac{wL^3}{6EI}$$

back subst. $\Rightarrow v_b = -\frac{wL^4}{8EI}$

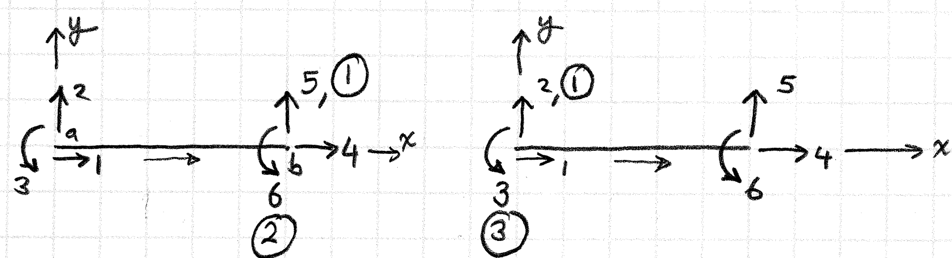
The rest follows identical steps as before — use symmetry to get the full M & V diagrams.

Now, if we were to take the automated approach and assemble the stiffness coefficients relating forces/moments along the D.O.F.s to displacements/rotations along them, we have.

D.O.F.s (global)



D.O.F.s (local)



As local coordinate systems for elements ab & bc align with the global coordinate system readily, no transformation operation is needed.

$$\Rightarrow \quad [K]^{ab}_{global} \equiv [K]^{ab}_{local} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5\;1 & 6\;2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \begin{array}{l} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5\;1 \\ 6\;2 \end{array}$$

$$
\text{and}\quad [K]^{bc}_{global} \equiv [K]^{bc}_{local} =
\begin{array}{c}
\begin{array}{cccccc} 1 & \overset{①}{2} & \overset{③}{3} & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}
\begin{array}{l} 1 \\ 2\ ① \\ 3\ ③ \\ 4 \\ 5 \\ 6 \end{array}
$$

combining stiffness coefficients corresponding to global D.O.F. we have

$$
[K]^{structure}_{global} =
\begin{array}{c}
\begin{array}{ccc} ① & ② & ③ \end{array} \\
\begin{bmatrix}
\frac{12EI}{L^3} + \frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{6EI}{L^2} \\
-\frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
\frac{6EI}{L^2} & 0 & \frac{4EI}{L}
\end{bmatrix}
\end{array}
\begin{array}{c} ① \\ ② \\ ③ \end{array}
$$

Aside: Note that the stiffness matrix is symmetric.

The equilibrium eq^{n}s are then

$$
\{\underset{\sim}{F}\} = [K]^{structure}_{global} \begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix}
$$

① : v_b

② : θ_{ba}

③ : θ_{bc}

$$
= \underbrace{\cancel{\{\underset{\sim}{F}\}_{case\,I}}}_{\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}} + \{\underset{\sim}{F}\}_{case\,II} = \begin{Bmatrix} -wL \\ wL^2/12 \\ -wL^2/12 \end{Bmatrix}
$$

$$
\underbrace{\cancel{\begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix}_{case\,I}}}_{\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}} + \begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix}_{case\,II}
$$

solving we find

$$\begin{Bmatrix} v_b \\ \theta_{ba} \\ \theta_{bc} \end{Bmatrix} = \begin{Bmatrix} -wL^4/8EI \\ -wL^3/6EI \\ wL^3/6EI \end{Bmatrix}$$

we can find member end forces/moments by using disp/rot in element local coords. Again, the local & global coord systems are oriented in the same manner

<u>Case II</u>

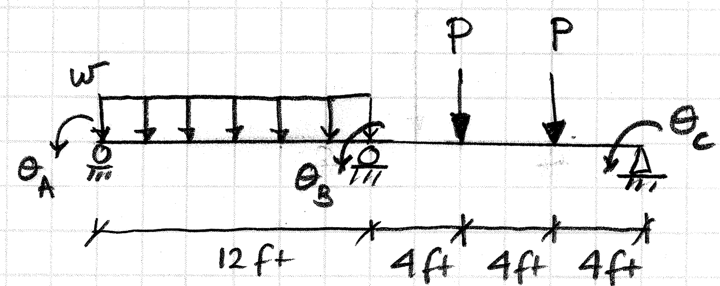
$$\begin{Bmatrix} N_{ab} \\ V_{ab} \\ M_{ab} \\ N_{ba} \\ V_{ba} \\ M_{ba} \end{Bmatrix} = [K]^{ab}_{local} \begin{array}{c} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -wL^4/8EI \\ -wL^3/6EI \end{Bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \; ① \\ 6 \; ② \end{array}$$

and

$$\begin{Bmatrix} N_{bc} \\ V_{bc} \\ M_{bc} \\ N_{cb} \\ V_{cb} \\ M_{cb} \end{Bmatrix} = [K]^{bc}_{local} \begin{Bmatrix} 0 \\ -wL^4/8EI \\ wL^3/6EI \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

To these, one adds the internal forces/moments from <u>Case I</u> to obtain the results for the <u>full case</u>.

2)

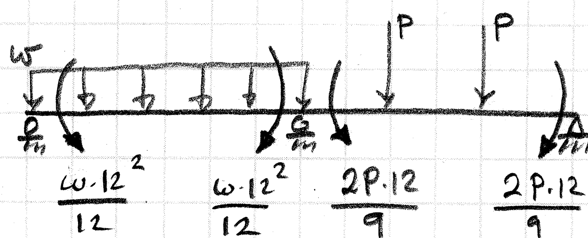


$EI = 100,000\ k\text{-}ft^2$

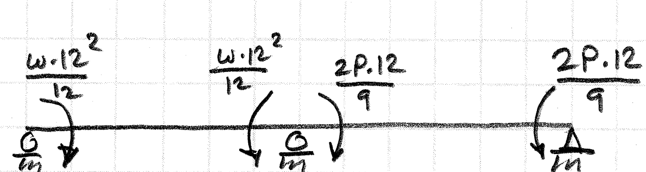
3 D.O.F. : $\theta_A, \theta_B, \theta_C$

counter-clockwise rot is considered (+)

Case I



Case II



$$\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{Bmatrix}_{orig} = \cancelto{0}{\begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{Bmatrix}_{I}} + \begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{Bmatrix}_{II} = \begin{Bmatrix} -1.68 \times 10^{-3} \\ 0.48 \times 10^{-3} \\ 0.72 \times 10^{-3} \end{Bmatrix} \text{rad.}$$

$\theta_A, \theta_B, \theta_C$ relates to the applied external loads through joint force (moment) equilibrium. Using stiffness coefficients we can write these equil. eq$\underline{n}$s as

$$M_A = -12w = \frac{4EI}{L_{AB}} \theta_A + \frac{2EI}{L_{AB}} \theta_B$$

$$M_B = 12w - \frac{8}{3}P = \frac{2EI}{L_{AB}} \theta_A + \frac{4EI}{L_{AB}} \theta_B + \frac{4EI}{L_{BC}} \theta_B + \frac{2EI}{L_{BC}} \theta_C$$

$$M_C = \frac{8}{3}P = \frac{2EI}{L_{BC}} \theta_B + \frac{4EI}{L_{AB}} \theta_C$$

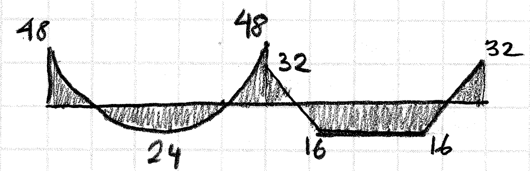
substituting $EI, \theta_A, \theta_B, \theta_C, L_{AB}, L_{BC}$ we find

$$\left.\begin{aligned} -12w &= -48 \text{ k.ft} \\ 12w - \tfrac{8}{3}P &= 16 \text{ k.ft} \\ \tfrac{8}{3}P &= 32 \text{ k.ft} \end{aligned}\right\} \quad \begin{aligned} w &= 4 \text{ k/ft} \\ P &= \frac{3 \times 32}{8} = 12 \text{ k} \end{aligned}$$

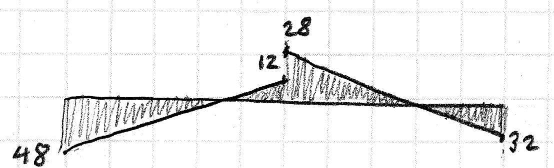
check
$12 \times 4 - \frac{8}{3} \times 12 = 16$ ✓

Moments

Case I :
[M]



Case II :
[M]



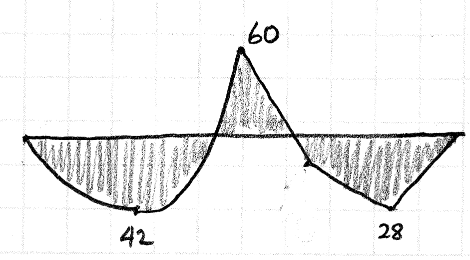
$$M_{AB} = \frac{4EI}{L_{AB}}\theta_A + \frac{2EI}{L_{AB}}\theta_B = -48 \text{ k.ft}$$

$$M_{BA} = \frac{2EI}{L_{AB}}\theta_A + \frac{4EI}{L_{AB}}\theta_B = -12 \text{ k.ft}$$

$$M_{BC} = \frac{4EI}{L_{BC}}\theta_B + \frac{2EI}{L_{BC}}\theta_C = 28 \text{ k.ft}$$

$$M_{CB} = \frac{2EI}{L_{BC}}\theta_B + \frac{4EI}{L_{BC}}\theta_C = 32 \text{ k.ft}$$

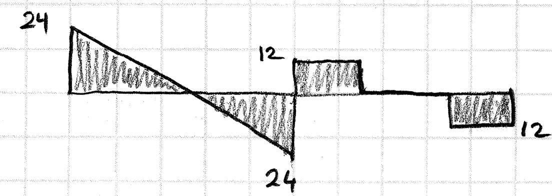
original
[M]



Shear force

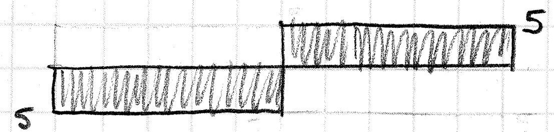
Case I

[V]



Case II

[V]



$$V_{AB} = \frac{6EI}{L_{AB}^2}\theta_A + \frac{6EI}{L_{AB}^2}\theta_B = -5$$

$$V_{BA} = -V_{AB} = 5$$

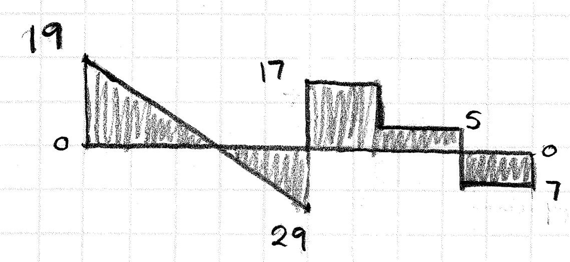
or simply from Case II moment diagram

$$V_{BC} = \frac{6EI}{L_{BC}^2}\theta_B + \frac{6EI}{L_{BC}^2}\theta_C = 5$$

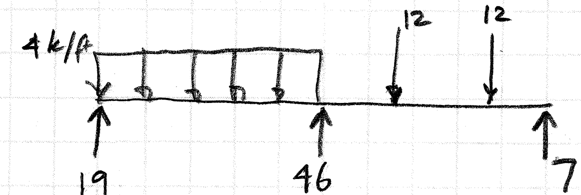
$$V_{CB} = -5$$

Original

[V]



Reactions shown on free-body diagram



$$\Sigma F_{vert} = \overbrace{-(4\times12+12+12)}^{-72} + \underbrace{19+46+7}_{72} = 0 \checkmark$$

$$\Sigma M_{@B} = -19\times12 + 4\times12\times6 - 12\times4 - 12\times8 + 7\times12 = 0 \checkmark$$