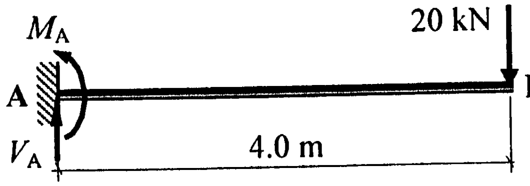
4.5.2 *Example 4.11: Deflection and Slope of a Uniform Cantilever*

A uniform cantilever beam is shown in Figure 4.44 in which a 20 kN is applied at B as indicated. Determine the magnitude and direction of the deflection and slope at B.



E and *I* are constant from A to B.

Figure 4.44

The bending moment diagrams for the applied load, a unit point load at B and a unit moment at B are shown in Figure 4.45.

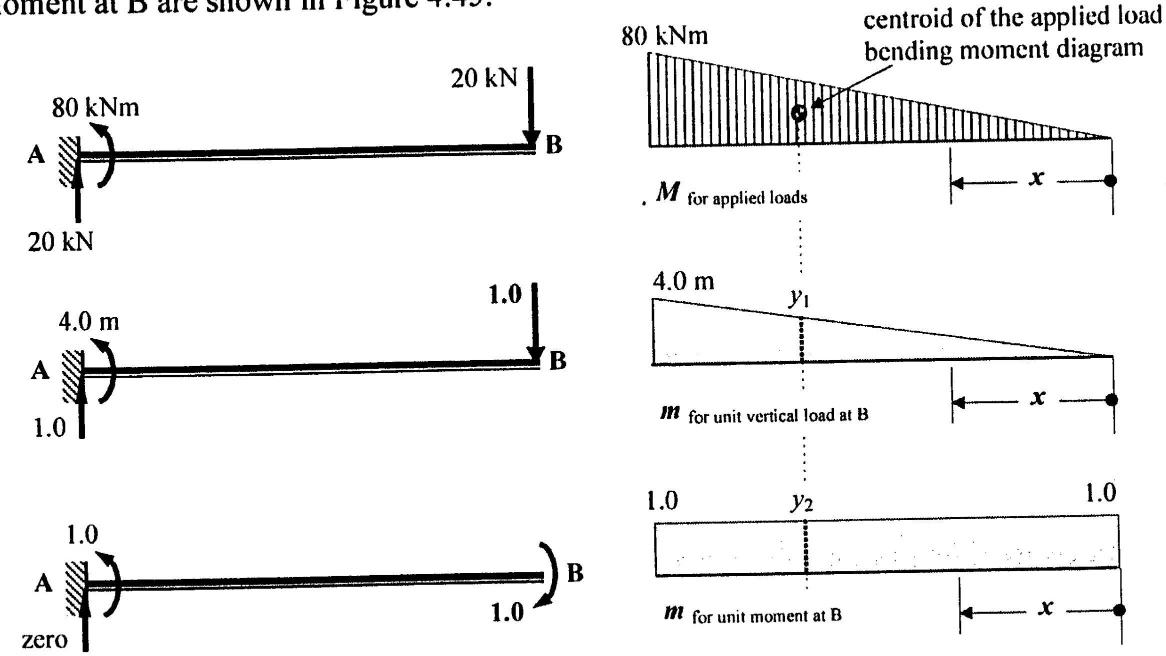


Figure 4.45

Solution:

$$\delta_B = \int_0^L \frac{Mm}{EI}\,dx$$

The bending moment at position '*x*' due to the applied vertical load $M = -20.0x$

The bending moment at position '*x*' due to the applied unit vertical load $m = -x$

$$Mm = +20x^2 \quad \therefore\ \delta_B = \int_{x=0}^{x=4} \frac{20x^2}{EI}\,dx = \left[\frac{20x^3}{3EI}\right]_0^4 = +\frac{426.67}{EI}\ \text{m} \downarrow$$

The bending moment at position '*x*' due to the applied unit moment at B $m = -1.0$

$$Mm = +20x \quad \therefore\ \theta_B = \int_{x=0}^{x=4} \frac{20x}{EI}\,dx = \left[\frac{20x^2}{2EI}\right]_0^4 = +\frac{160}{EI}\ \text{rad.} \searrow$$

The product integral $\int_0^L Mm\,dx$ can be also be calculated as:

(Area of the applied load bending moment diagram × the ordinate on the unit load bending moment diagram corresponding to the position of the centroid of the applied load bending moment diagram), e.g.

To determine the vertical deflection:

Area of the applied load bending moment diagram $A = (0.5 \times 4.0 \times 80.0) = 160\ \text{kNm}^2$

Ordinate at the position of the centroid $y_1 = 2.67\ \text{m}$

$$\int_0^L Mm\,dx = (160 \times 2.67) = 426.67 \qquad \therefore\ \delta_B = \int_0^L \frac{Mm}{EI}dx = +\frac{426.67}{EI}\ \text{m} \downarrow$$

To determine the slope:

Area of the applied load bending moment diagram $A = (0.5 \times 4.0 \times 80.0) = 160\ \text{kNm}^2$

Ordinate at the position of the centroid $y_2 = 1.0$

$$\int_0^L Mm\,dx = (160 \times 1.0) = 160 \qquad \therefore\ \delta_B = \int_0^L \frac{Mm}{EI}dx = +\frac{160}{EI}\ \text{rad.} \searrow$$

4.5.3 Example 4.12: Deflection and Slope of a Non–Uniform Cantilever

Consider the same problem as in Example 4.11 in which the cross–section of the cantilever has a variable *EI* value as indicated in Figure 4.46.

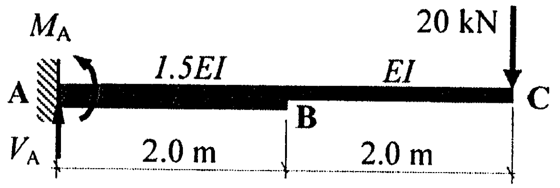


Figure 4.46

The bending moment diagrams for the applied load, a unit point load at C and a unit moment at C are shown in Figure 4.47.

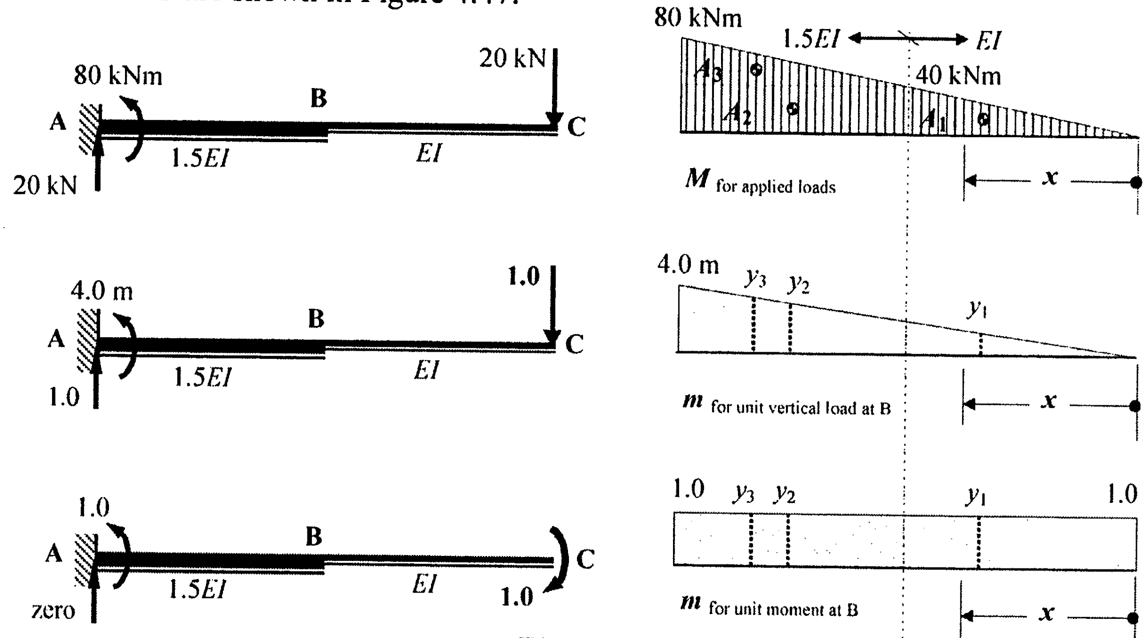


Figure 4.47

Solution:

$$\delta_C = \int_0^L \frac{Mm}{EI}\,dx$$

In this case since (Mm/EI) is not a continuous function the product integral must be evaluated between each of the discontinuities, i.e. C to B and B to A.

$$\delta_C = \int_0^L \frac{Mm}{EI}\,dx = \int_C^B \frac{Mm}{EI}\,dx + \int_B^A \frac{Mm}{1.5EI}\,dx$$

Consider the section from C to B: $\mathbf{0 \leq x \leq 2.0}$ **m**

$M = -20x \qquad m = -x \qquad \therefore\ Mm = +20x^2$

$$\int_C^B \frac{Mm}{EI}\,dx = \int_0^2 \frac{20x^2}{EI}\,dx = \left[\frac{20x^3}{3EI}\right]_0^2 = +\frac{53.33}{EI}\ \text{m}$$

Consider the section from B to A: $\mathbf{2.0 \leq x \leq 4.0}$ **m**

$M = -20x \qquad m = -x \qquad \therefore\ Mm = +20x^2$

$$\int_B^A \frac{Mm}{EI}\,dx = \int_2^4 \frac{20x^2}{1.5EI}\,dx = \left[\frac{20x^3}{4.5EI}\right]_2^4 = \left[\frac{20\times 4^3}{4.5EI} - \frac{20\times 2^3}{4.5EI}\right] = +\frac{248.89}{EI}\ \text{m}$$

$$\therefore\ \delta_C = +\frac{53.33}{EI} + \frac{248.89}{EI} = \frac{302.22}{EI}\ \text{m}\ \downarrow$$

Similarly to determine the slope:

$$\theta_C = \int_0^L \frac{Mm}{EI}\,dx = \int_C^B \frac{Mm}{EI}\,dx + \int_B^A \frac{Mm}{1.5EI}\,dx$$

Consider the section from C to B: $\mathbf{0 \leq x \leq 2.0}$ **m**

$M = -20x \qquad m = -1.0 \qquad \therefore\ Mm = 20x$

$$\int_C^B \frac{Mm}{EI}\,dx = \int_0^2 \frac{20x}{EI}\,dx = \left[\frac{20x^2}{2EI}\right]_0^2 = +\frac{40.0}{EI}\ \text{rad.}$$

Consider the section from B to A: $\mathbf{2.0 \leq x \leq 4.0}$ **m**

$M = -20x \qquad m = -1.0 \qquad \therefore\ Mm = 20x$

$$\int_B^A \frac{Mm}{EI}\,dx = \int_2^4 \frac{20x}{1.5EI}\,dx = \left[\frac{20x^2}{3.0EI}\right]_2^4 = \left[\frac{20\times 4^2}{3.0EI} - \frac{20\times 2^2}{3.0EI}\right] = +\frac{80.0}{EI}\ \text{rad.}$$

$$\therefore\ \theta_C = +\frac{40.0}{EI} + \frac{80.0}{EI} = +\frac{120.0}{EI}\ \text{rad.}\ \searrow$$

Alternatively, the applied bending moment diagram can be considered as a the sum of the areas created by the discontinuity. (In most cases this will result in a number of recognised shapes e.g. triangular, rectangular or parabolic, in which the areas and the position of the centroid can be easily calculated).
The deflection can then be determined by summing the products (area × ordinate) for each of the shapes.

$A_1 = (0.5 \times 2.0 \times 40.0)$ kNm^2, $\quad y_1 = 1.333$ m, $\quad \therefore A_1y_1 = 53.32$ kNm^3
$A_2 = (2.0 \times 40.0)$ kNm^2, $\quad y_2 = 3.0$ m, $\quad \therefore A_2y_2 = 240.0$ kNm^3
$A_3 = (0.5 \times 2.0 \times 40.0)$ kNm^2, $\quad y_3 = 3.333$ m, $\quad \therefore A_3y_3 = 133.32$ kNm^3

$$\delta_C = \int_0^L \frac{Mm}{EI}dx = (53.32/EI) + (240.0/1.5EI) + (133.32/1.5EI) = +(302.22/EI) \text{ m} \downarrow$$

The slope can then be determined by summing the products (area × ordinate) for each of the shapes.

$A_1 = (0.5 \times 2.0 \times 40.0)$ kNm^2, $\quad y_1 = 1.0$, $\quad \therefore A_1y_1 = 40.0$ kNm^3
$A_2 = (2.0 \times 40.0)$ kNm^2, $\quad y_2 = 1.0$, $\quad \therefore A_2y_2 = 80.0$ kNm^3
$A_3 = (0.5 \times 2.0 \times 40.0)$ kNm^2, $\quad y_3 = 1.0$, $\quad \therefore A_3y_3 = 40.0$ kNm^3

$$\delta_C = \int_0^L \frac{Mm}{EI}dx = (40.0/EI) + (80.0/1.5EI) + (40/1.5EI) = +(120.0/EI) \text{ rad.} \searrow$$

4.5.4 Example 4.13: Deflection and Slope of a Linearly Varying Cantilever

Consider the same problem as in Example 4.11 in which the cross–section of the cantilever has an I which varies linearly from I at the free end to $2I$ at the fixed support at A as indicated in Figure 4.48. Determine the vertical displacement and the slope at point B for the loading indicated.

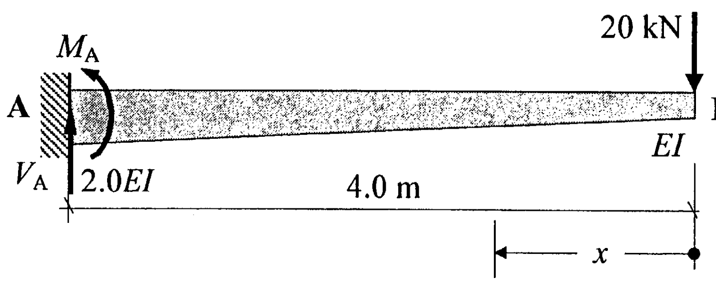


Figure 4.48

The value of I at position 'x' along the beam is given by: $I + I(x/L) = I(L + x)/L$.

In this case since the I term is dependent on x it cannot be considered outside the integral as a constant. The displacement must be determined using integration and cannot be calculated using the sum of the (area × ordinate) as in Examples 5.11 and 5.12.

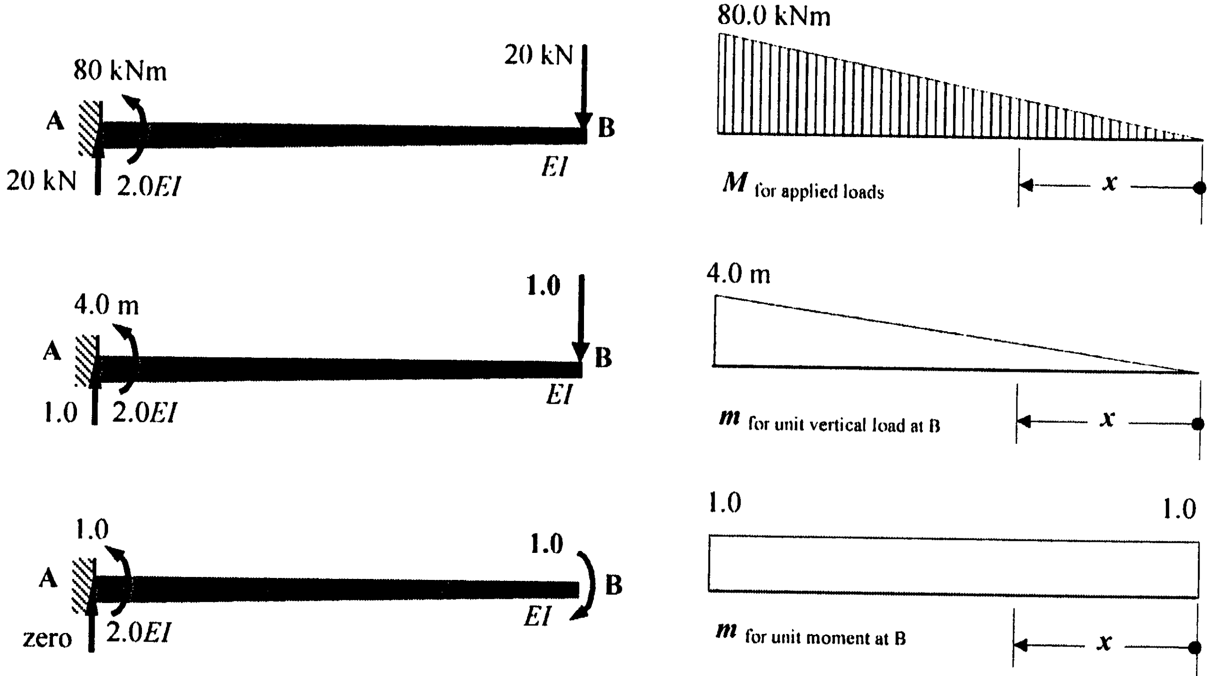


Figure 4.49

Solution:

The bending moment at position 'x' due to the applied vertical load $M = -20.0x$

The bending moment at position 'x' due to the applied unit vertical load $m = -x$

$$Mm = +20x^2 \quad \therefore\ \delta_B = \int_{x=0}^{x=4} \frac{20x^2L}{EI(L+x)}dx = \frac{20L}{EI}\int_{x=0}^{x=4}\frac{x^2}{(L+x)}dx$$

Let $v = (L+x)$ $\quad \therefore\ x = (v-L) \quad dx = dv$ and $x^2 = (v-L)^2$

when $x = 0$ $\quad v = L = 4.0$ and when $x = 4$ $\quad v = (L+4.0) = 8.0$

$$\delta_B = \frac{20L}{EI}\int_{x=0}^{x=4}\frac{x^2}{(L+x)}dx = \frac{80.0}{EI}\int_{v=4}^{v=8}\frac{(v-4.0)^2}{v}dv = \frac{80.0}{EI}\int_{v=4}^{v=8}\frac{(v^2-8.0v+16.0)}{v}dv$$

$$= \frac{80.0}{EI}\int_{v=4}^{v=8}\left(v-8.0+\frac{16.0}{v}\right)dv = \frac{80.0}{EI}\left[\frac{v^2}{2}-8v+16.0\ln v\right]_{v=4.0}^{v=8.0}$$

$$= \frac{80.0}{EI}\left\{\left[\frac{8.0^2}{2}-(8\times 8)+16.0\,ln\,8\right]-\left[\frac{4.0^2}{2}-(8\times 4)+16.0\,ln\,4\right]\right\} = +\frac{247.20}{EI}\text{ m}\downarrow$$

The bending moment at position 'x' due to the applied unit moment at B $\quad m = -1.0$

$$Mm = +20x \quad \therefore\ \theta_B = \int_{x=0}^{x=4}\frac{20xL}{EI(L+x)}dx = \frac{20L}{EI}\int_{x=0}^{x=4}\frac{x}{(L+x)}dx$$

$$\theta_B = \frac{20L}{EI}\int_{x=0}^{x=4}\frac{x}{(L+x)}dx = \frac{80.0}{EI}\int_{v=4}^{v=8}\frac{(v-4.0)}{v}dv = \frac{80.0}{EI}\int_{v=4}^{v=8}\left(1-\frac{4.0}{v}\right)dv$$

$$= \frac{80.0}{EI}\left[v-4.0\,ln\,v\right]_{v=4.0}^{v=8.0} = \frac{80.0}{EI}\{[8.0-4.0\,ln\,8]-[4.0-4.0\,ln\,4]\} = +\frac{98.19}{EI}\text{ rad.}\searrow$$

4.5.5 Example 4.14: Deflection of a Non-Uniform, Simply-Supported Beam

A non-uniform, single-span beam ABCD is simply-supported at A and D and carries loading as indicated in Figure 4.50. Determine the vertical displacement at point B.

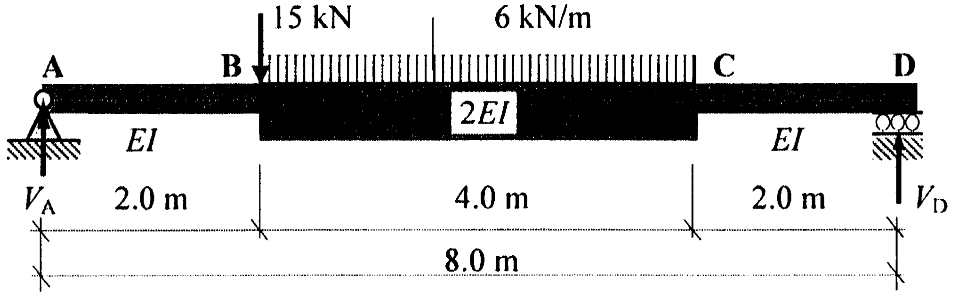


Figure 4.50

The bending moment diagrams for the applied load, a unit point load at B are shown in Figure 4.51.

The beam loading can be considered as the superposition of a number of load cases each of which produces a bending moment diagram with a standard shape. Since there are discontinuities in the bending moment diagrams the product integrals should be carried out for the three regions A to B, D to C and C to B.

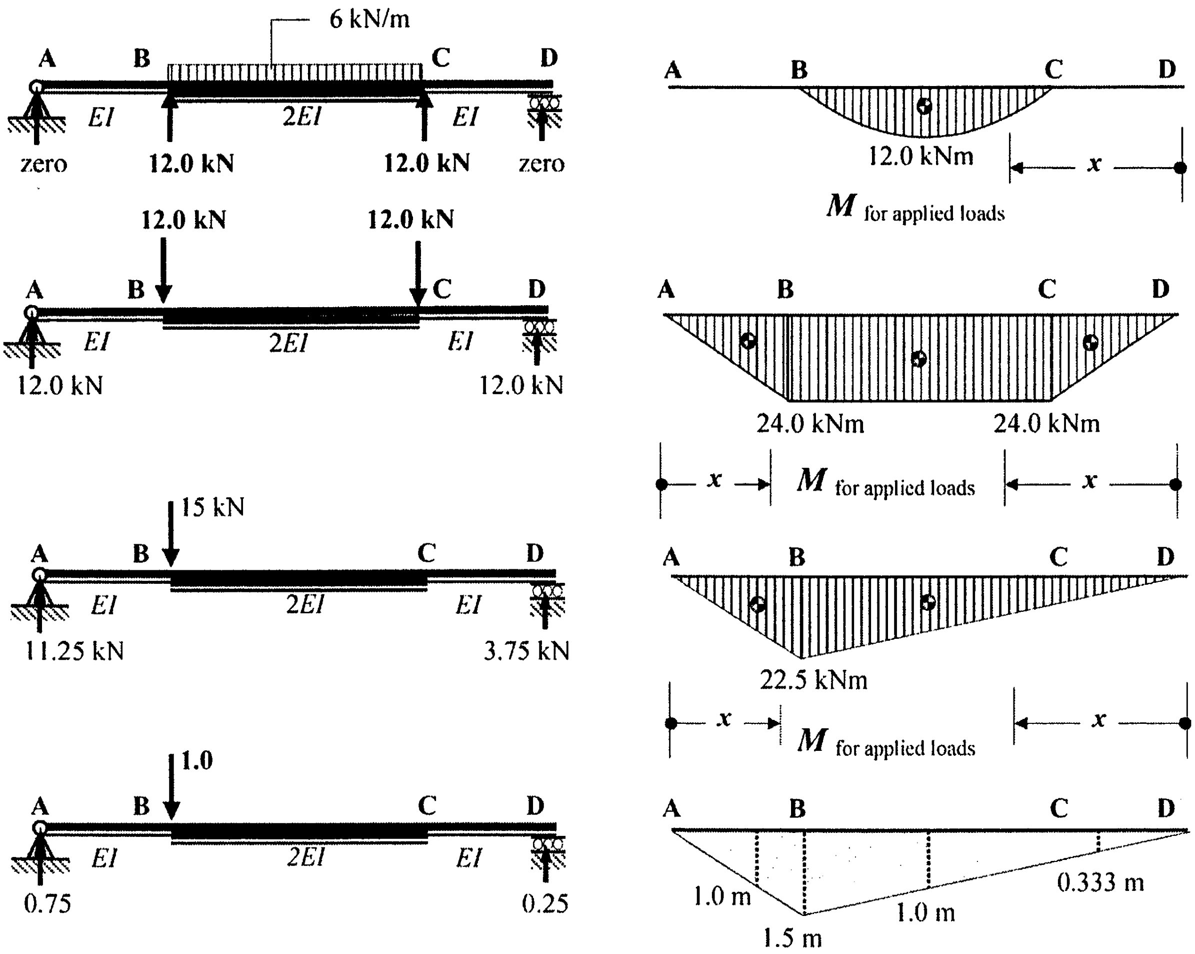


Figure 4.51 *m* for unit vertical load at B

Solution:

It is convenient in this problem to change the position of the origin from which 'x' is measured for the different regions A–B, D–C and C–B as shown in Figure 4.51.

$$\delta_B = \int_0^L \frac{Mm}{EI}dx = \int_B^A \frac{Mm}{EI}dx + \int_D^C \frac{Mm}{EI}dx + \int_C^B \frac{Mm}{2EI}dx$$

Consider the section from A to B: $\mathbf{0 \le x \le 2.0}$ **m**

$M = (12x + 11.25x) = 23.25x \qquad m = +0.75x \qquad \therefore\ Mm = +17.44x^2$

$$\int_A^B \frac{Mm}{EI}dx = \int_0^2 \frac{17.44x^2}{EI}dx = \left[\frac{17.44x^3}{3EI}\right]_0^2 = \left[\frac{17.44 \times 2^3}{3EI}\right] = +\frac{46.51}{EI}\text{ m}$$

Consider the section from D to C: $\mathbf{0 \le x \le 2.0}$ **m**

$M = (12x + 3.75x) = 15.75x \qquad m = +0.25x \qquad \therefore\ Mm = +3.94x^2$

$$\int_D^C \frac{Mm}{EI}dx = \int_0^2 \frac{3.94x^2}{EI}dx = \left[\frac{3.94x^3}{3EI}\right]_0^2 = \left[\frac{3.94 \times 2^3}{3EI}\right] = +\frac{10.51}{EI}\text{ m}$$

Consider the section from C to B: $\mathbf{2.0 \le x \le 6.0}$ **m**

$M = [12(x-2) - 6(x-2)^2/2] + [12x - 12(x-2)] + 3.75x = (27.75x - 3x^2 - 12)$

$m = +0.25x \qquad \therefore\ Mm = (6.94x^2 - 0.75x^3 - 3x)$

$$\int_C^B \frac{Mm}{EI}dx = \int_2^6 \frac{(6.94x^2 - 0.75x^3 - 3x)}{2EI}dx = \left[\frac{6.94x^3}{6EI} - \frac{0.75x^4}{8EI} - \frac{3x^2}{4EI}\right]_2^6$$

$$= \left[\frac{6.94 \times 6^3}{6EI} - \frac{0.75 \times 6^4}{8EI} - \frac{3 \times 6^2}{4EI}\right] - \left[\frac{6.94 \times 2^3}{6EI} - \frac{0.75 \times 2^4}{8EI} - \frac{3 \times 2^2}{4EI}\right]$$

$$= +\frac{96.59}{EI}\text{ m}$$

$$\therefore\ \delta_B = \left(\frac{46.51}{EI} + \frac{10.51}{EI} + \frac{96.59}{EI}\right) = \frac{153.61}{EI}\text{ m} \downarrow$$

Alternatively: considering Σ (areas × ordinates)

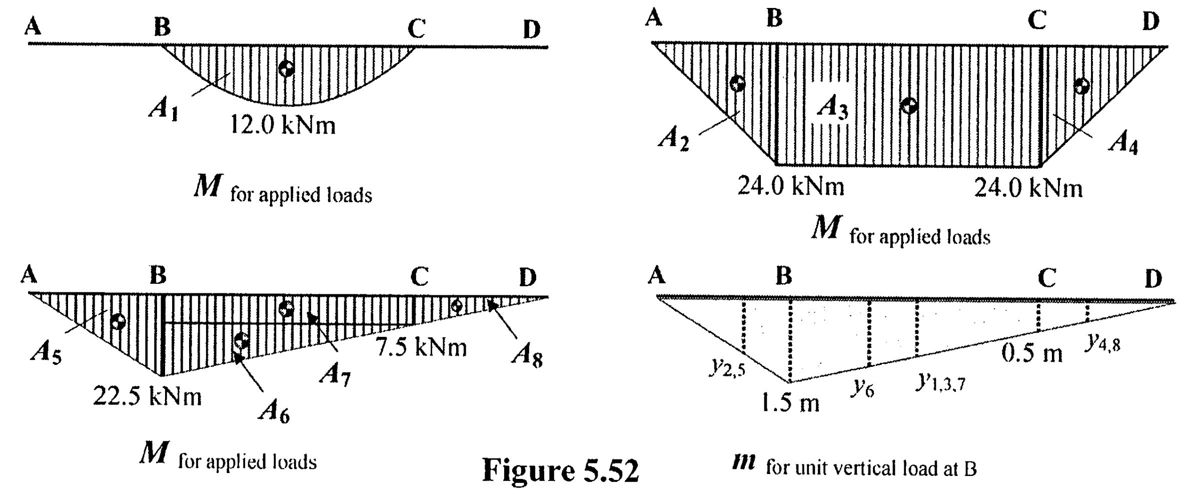


Figure 5.52

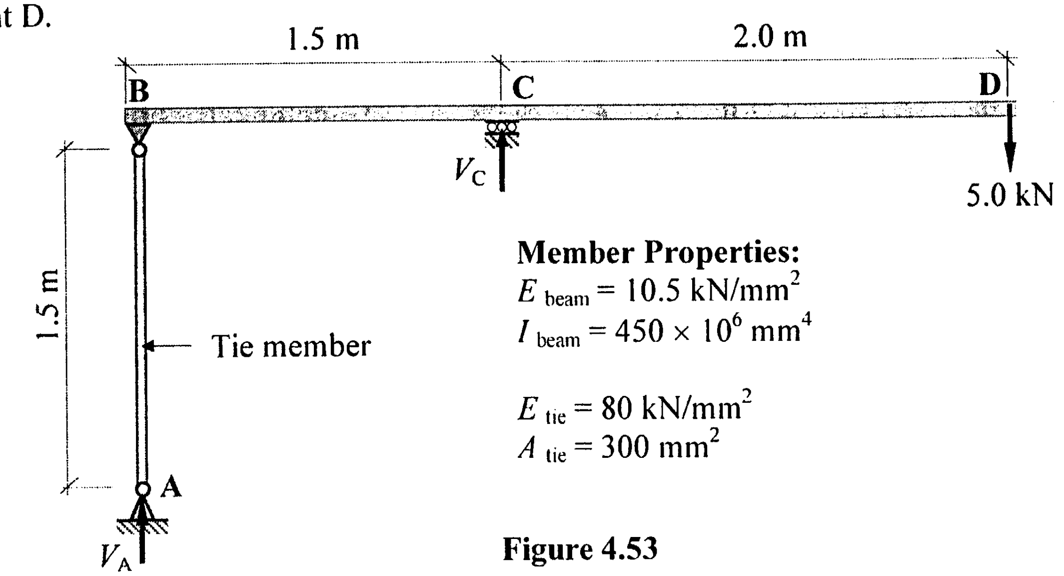
$A_1 = (0.667 \times 4.0 \times 12.0)$ kNm^2,	$y_1 = 1.0$ m,	$\therefore A_1y_1 = 32.0$ kNm^3
$A_2 = (0.5 \times 2.0 \times 24.0)$ kNm^2,	$y_2 = 1.0$ m,	$\therefore A_2y_2 = 24.0$ kNm^3
$A_3 = (4.0 \times 24.0)$ kNm^2,	$y_3 = 1.0$ m,	$\therefore A_3y_3 = 96.0$ kNm^3
$A_4 = (0.5 \times 2.0 \times 24.0)$ kNm^2,	$y_4 = 0.333$ m,	$\therefore A_4y_4 = 8.0$ kNm^3
$A_5 = (0.5 \times 2.0 \times 22.5)$ kNm^2,	$y_5 = 1.0$ m,	$\therefore A_5y_5 = 22.5$ kNm^3
$A_6 = (0.5 \times 4.0 \times 15.0)$ kNm^2,	$y_6 = 1.167$ m,	$\therefore A_6y_6 = 35.0$ kNm^3
$A_7 = (4.0 \times 7.5)$ kNm^2,	$y_7 = 1.0$ m,	$\therefore A_7y_7 = 30.0$ kNm^3
$A_8 = (0.5 \times 2.0 \times 7.5)$ kNm^2,	$y_8 = 0.333$ m,	$\therefore A_8y_8 = 2.5$ kNm^3

$$\delta_B = \int_0^L \frac{Mm}{EI}dx = (32.0/2EI) + (24.0/EI) + (96.0/2EI) + (8.0/EI) + (22.5/EI) + (35.0/2EI)$$

$$+ (30.0/2EI) + (2.5/EI) \qquad \therefore \delta_B = (153.5/EI) \text{ m} \downarrow$$

4.5.6 *Example 4.15: Deflection of a Frame and Beam Structure*

A uniform beam BCD is tied at B, supported on a roller at C and carries a vertical load at D as indicated in Figure 4.53. Using the data given determine the vertical displacement at point D.



Member Properties:
$E_{beam} = 10.5$ kN/mm^2
$I_{beam} = 450 \times 10^6$ mm^4

$E_{tie} = 80$ kN/mm^2
$A_{tie} = 300$ mm^2

Figure 4.53

Solution:

Consider the rotational equilibrium of the beam:

+ve $\circlearrowright \Sigma M_A = 0 \qquad -(V_C \times 1.5) + (5.0 \times 3.5) = 0 \qquad \therefore V_C = 11.67$ kN $\uparrow$

Consider the vertical equilibrium of the structure:

+ve $\uparrow \Sigma F_y = 0 \qquad V_A + V_C - 5.0 = 0 \qquad \therefore V_A = -6.67$ kN $\downarrow$

Since the structure comprises both an axially loaded member and a flexural member the deflection at D is given by:

$$\delta_D = \left(\frac{PL}{AE}u\right)_{\text{Member AB}} + \left(\int_0^L \frac{Mm}{EI}dx\right)_{\text{Member BCD}}$$

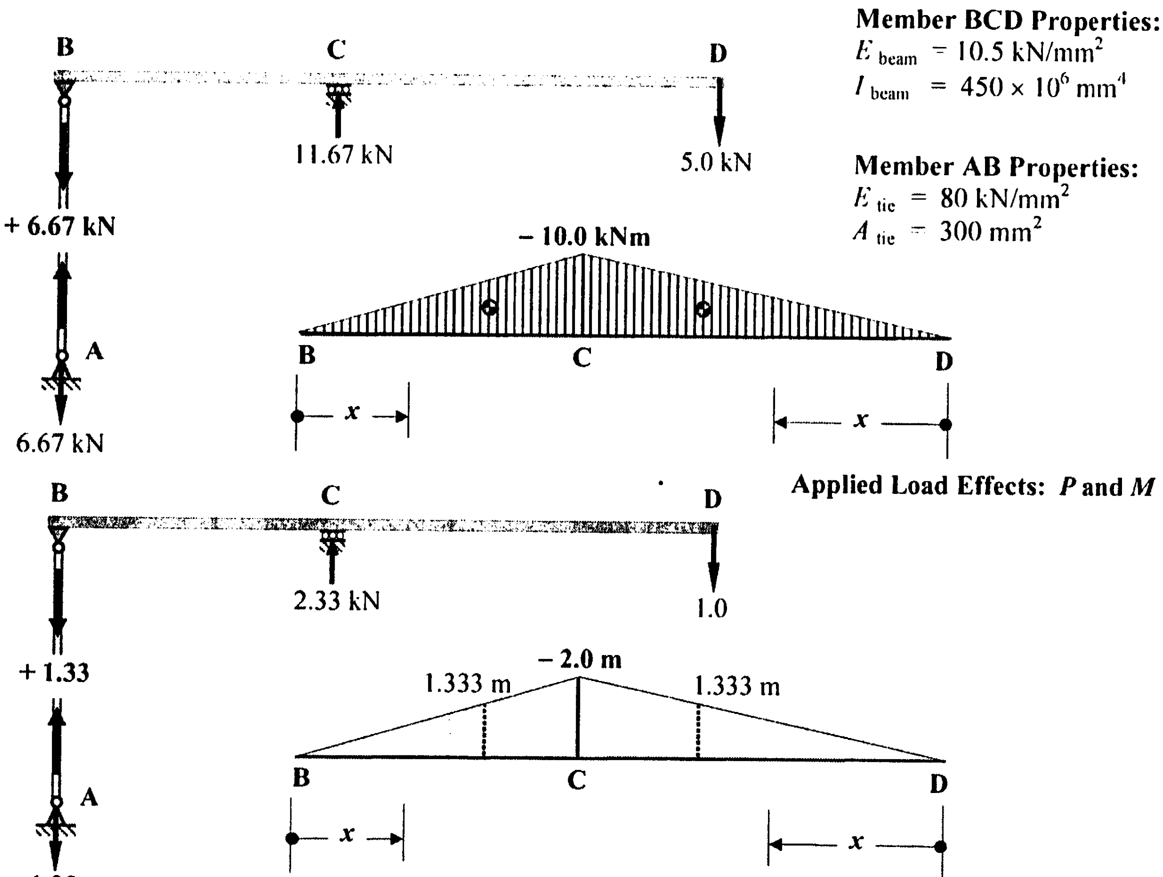


Figure 4.54

$$\left(\frac{PL}{AE}u\right)_{\text{Member AB}} = \left(\frac{6.67\times1500}{300\times80}\times1.33\right) = +\,0.55\ \text{mm}$$

$$\left(\int_0^L \frac{Mm}{EI}dx\right)_{\text{Member BCD}} = \left(\int_B^C \frac{Mm}{EI}dx\right) + \left(\int_D^C \frac{Mm}{EI}dx\right)$$

Consider the section from B to C: $\mathbf{0 \le x \le 1.5\ m}$

$M = -\,6.67x \qquad m = -\,1.33\,x \qquad \therefore\ Mm = +\,8.87\,x^2$

$$\int_B^C \frac{Mm}{EI}dx = \int_0^{1.5}\frac{8.87x^2}{EI}dx = \left[\frac{8.87x^3}{3\times EI}\right]_0^{1.5} = \left(\frac{29.94\times10^3}{3\times10.5\times450}\right) = +\,2.11\ \text{mm}$$

Consider the section from D to C: $\mathbf{0 \le x \le 2.0\ m}$

$M = -\,5.0\,x \qquad m = -\,1.0\,x \qquad \therefore\ Mm = +\,5.0\,x^2$

$$\int_D^C \frac{Mm}{EI}dx = \int_0^{2}\frac{5.0x^2}{EI}dx = \left[\frac{5.0x^3}{3EI}\right]_0^{2} = \left(\frac{40.0\times10^3}{3\times10.5\times450}\right) = +\,2.82\ \text{mm}$$

$$\delta_D = \left(\frac{PL}{AE}u\right)_{\text{Member AB}} + \left(\int_0^L \frac{Mm}{EI}dx\right)_{\text{Member BCD}} = (0.55 + 2.11 + 2.82) = \mathbf{+\,5.48\ mm}\ \downarrow$$

In the previous examples the product integrals were also determined using:

(the area of the applied bending moment diagram × ordinate on the unit load bending moment diagram).

In Table 4.1 coefficients are given to enable the rapid evaluation of product integrals for standard cases along lengths of beam where the EI value is constant.

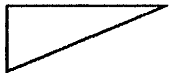
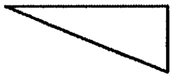
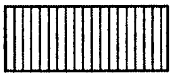
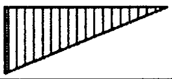
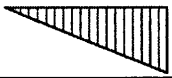
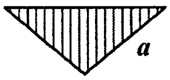
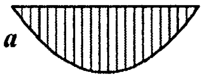
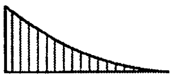
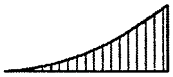
Product Integral $\int_0^L \frac{Mm}{EI}\,dx = [\text{Coefficient} \times a \times b \times L]/EI$				
M \ m	L, b 	L, b 	L, b 	L, b 
L, a 	1.0	0.5	0.5	0.5
L, a 	0.5	**0.333**	0.167	0.25
L, a 	0.5	0.167	**0.333**	0.25
L, a 	0.5	0.25	0.25	0.333
L, a 	0.667	0.333	0.333	0.417
a, L 	0.333	0.25	0.083	0.146
a, L 	0.333	0.083	0.25	0.146

Table 4.1

Consider the contribution from the beam BCD to the vertical deflection at D in Example 4.15.

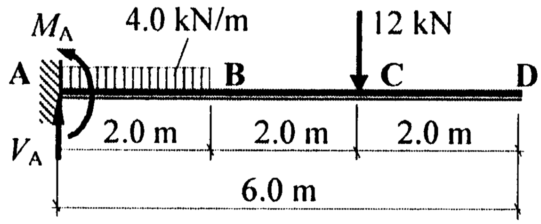
$$\text{Product Integral } \int_0^L \frac{Mm}{EI}\,dx = \Sigma\,[\text{Coefficient} \times a \times b \times L]/EI$$

From (B to C) + (D to C) = [(0.333 × 10.0 × 2.0 × 1.5) + (0.333 × 10.0 × 2.0 × 2.0)]/EI

= + 23.31/EI i.e. same as [(2.11 + 2.82] calculated above.

4.5.7 Example 4.16: Deflection of a Uniform Cantilever using Coefficients

A uniform cantilever beam is shown in Figure 4.55 in which a uniformly distributed load and a vertical load is applied as indicated. Using the coefficients in Table 4.1 determine the magnitude and direction of the deflection at D.



E and *I* are constant.

Figure 4.55

The bending moment diagrams for the applied loads and a unit point load at B are shown in Figure 4.56.

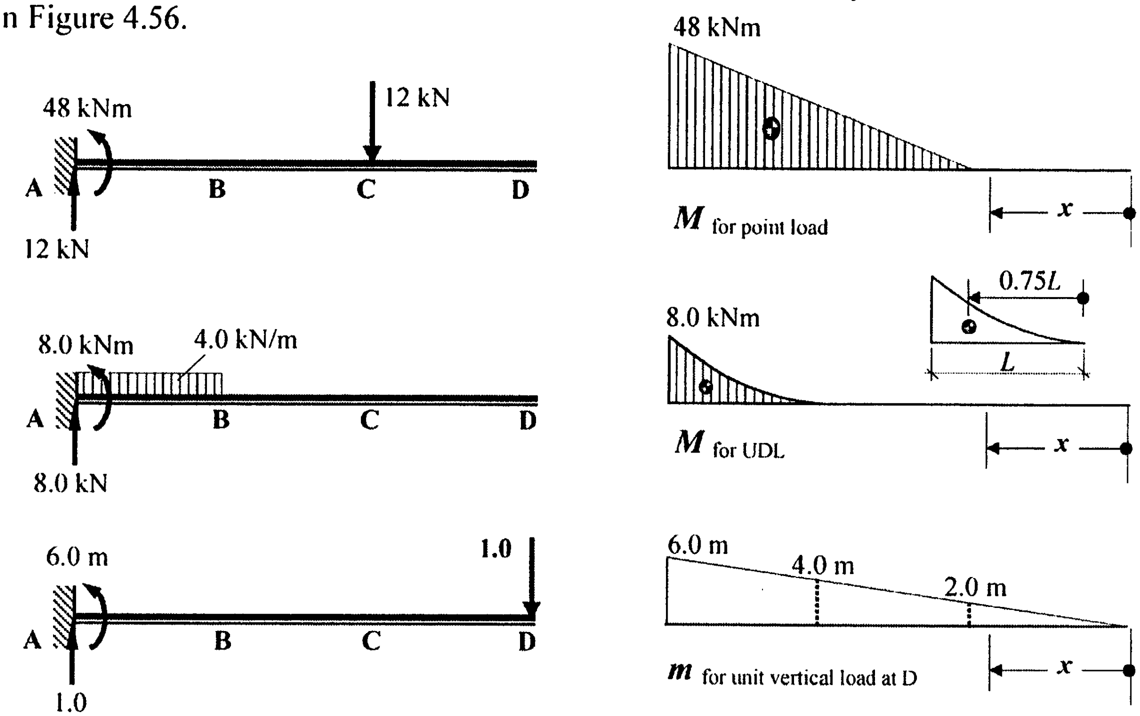
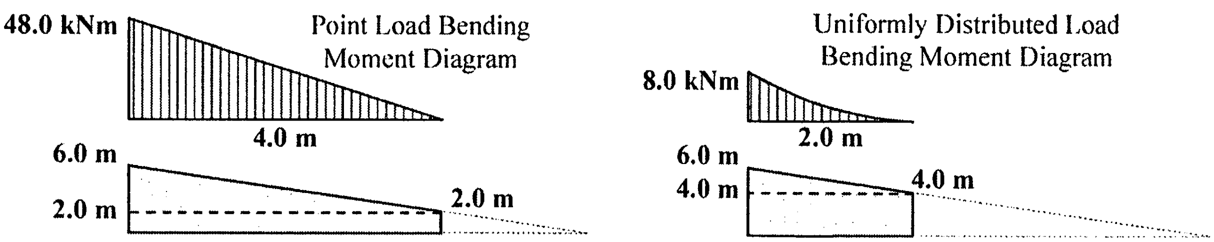


Figure 4.56

Solution:

Consider the unit load bending moment diagrams for both applied loads as the sum of rectangular and a triangular area as shown.



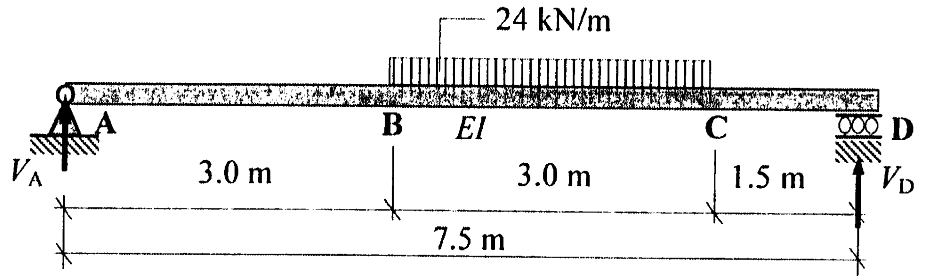
$\delta_{D,\text{ point load}} = [(\mathbf{0.5} \times 48.0 \times 2.0 \times 4.0) + (\mathbf{0.333} \times 48.0 \times 4.0 \times 4.0)]/EI = 447.74/EI$

$\delta_{D,\text{ UDL}} = [(\mathbf{0.333} \times 8.0 \times 4.0 \times 2.0) + (\mathbf{0.25} \times 8.0 \times 2.0 \times 2.0)]/EI = 29.31/EI$

$\delta_{D,\text{ Total}} = (447.74 + 29.31)/EI = \mathbf{+\ 477.05/EI}$ ↓

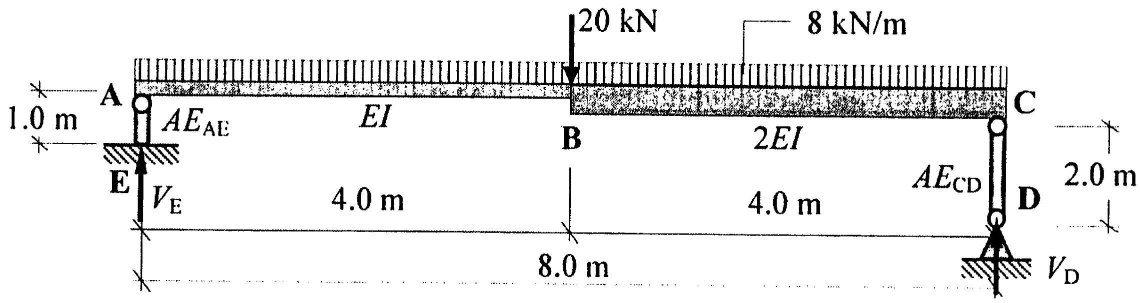
4.5.8 Problems: Unit Load Method for Deflection of Beams / Frames

A series of statically-determinate beams/frames are indicated in Problems 4.16 to 4.23. Using the applied loading given in each case determine the deflections indicated. The relative values of Young's Modulus of Elasticity (E), Second Moment of Area (I) and Cross-sectional area (A) are given in each case.



Determine the value of the vertical deflection at B given that $EI = 50.0 \times 10^3$ kNm^2

Problem 4.16

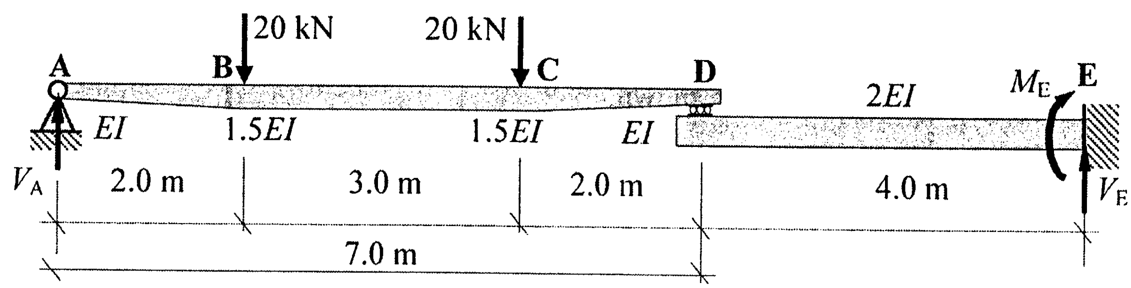


Determine the value of the vertical deflection at B given:

$E_{beam} = 9.0$ kN/mm^2 $\quad$ $I_{beam} = 14.6 \times 10^9$ mm^4

$E_{AE\ and\ CD} = 170$ kN/mm^2 $\quad$ $A_{AE} = 80$ mm^2 $\quad$ $A_{CD} = 120$ mm^2

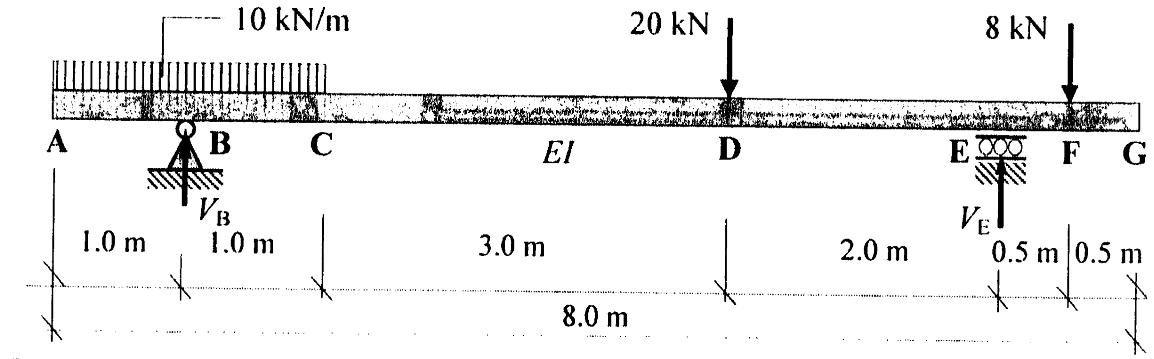
Problem 4.17



The EI value of the beam ABCD varies linearly from EI at the supports A and D to $1.5EI$ at B and C respectively and is constant between B and C.

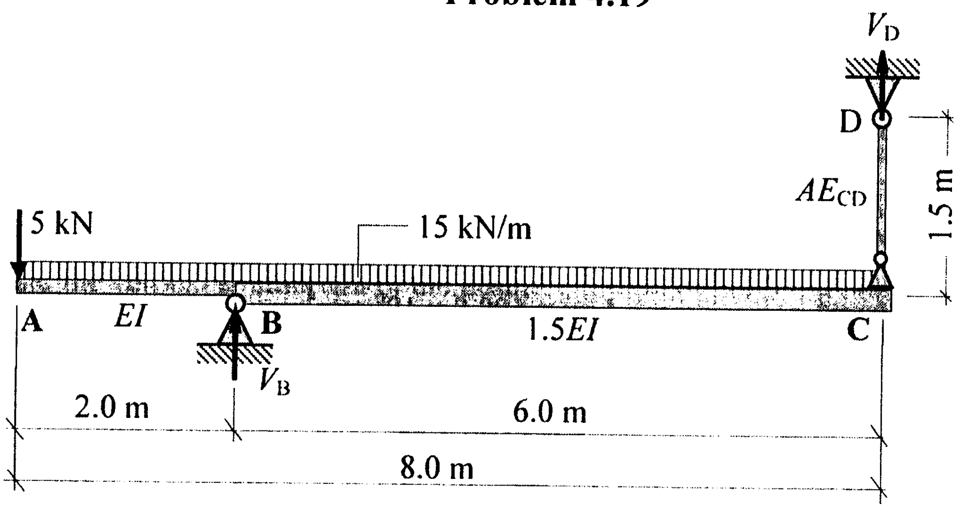
Determine the value of the vertical deflection at B given that $EI = 15.0 \times 10^3$ kNm^2

Problem 4.18



Determine the value of the vertical deflection at G given that $EI = 5.0 \times 10^3$ kNm^2

Problem 4.19

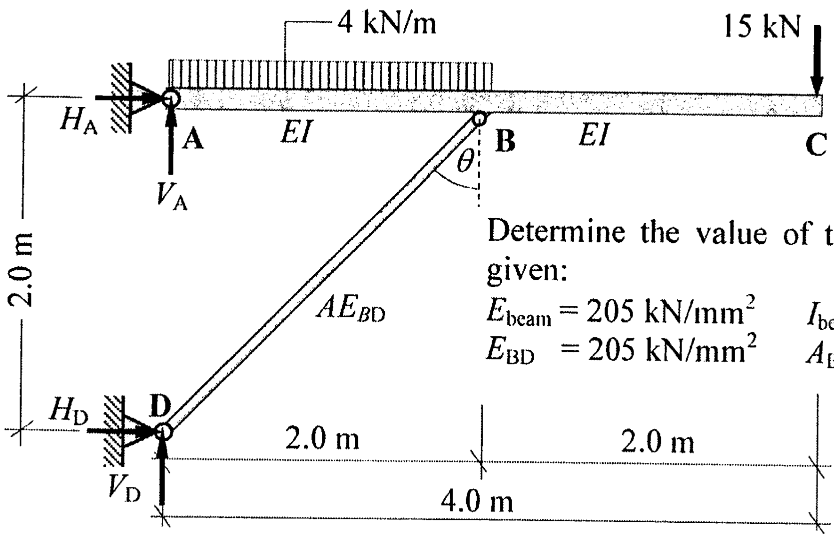


Determine the value of the vertical deflection at A given:

$E_{beam} = 205$ kN/mm^2 $\quad I_{beam} = 60.0 \times 10^6$ mm^4

$E_{CD} = 205$ kN/mm^2 $\quad A_{CD} = 50$ mm^2

Problem 4.20

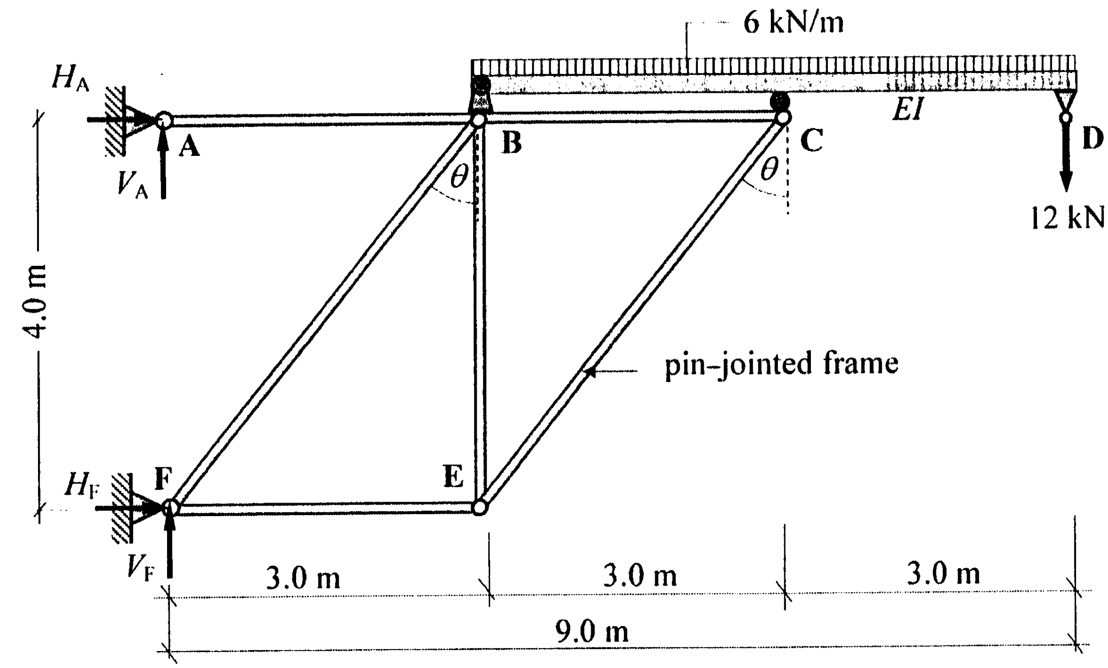


Determine the value of the vertical deflection at C given:

$E_{beam} = 205$ kN/mm^2 $\quad I_{beam} = 90.0 \times 10^6$ mm^4

$E_{BD} = 205$ kN/mm^2 $\quad A_{BD} = 1500$ mm^2

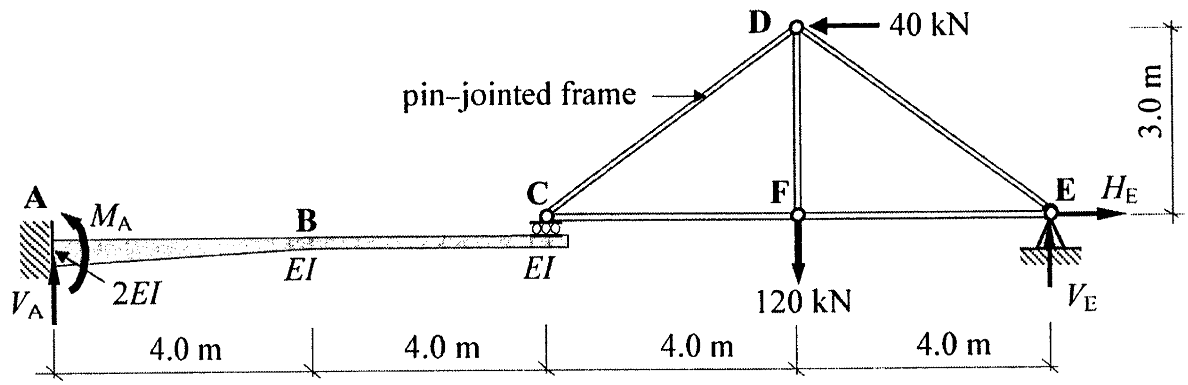
Problem 4.21



Determine the value of the vertical deflection at D given:

$E_{beam} = 205$ kN/mm^2 $\quad$ $I_{beam} = 500.0 \times 10^6$ mm^4
$E_{All\ frame\ members} = 205$ kN/mm^2 $\quad$ $A_{All\ frame\ members} = 4000$ mm^2

Problem 4.22



The *EI* value of the cantilever ABC varies linearly from 2*EI* at the fixed support to *EI* at B and is constant from B to C.

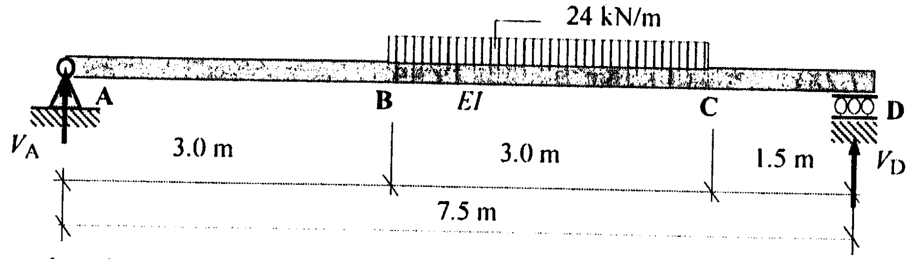
Determine the value of the vertical deflection at F and at C given:
$EI_{cantilever\ ABC} = 1080 \times 10^3$ kNm^2, $\quad$ $EA_{All\ frame\ members} = 300 \times 10^3$ kN

Problem 4.23

4.5.9 Solutions: Unit Load Method for Deflection of Beams / Frames

Solution

Topic: Statically Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.16 **Page No. 1**



Determine the value of the vertical deflection at B given that $EI = 50.0 \times 10^3$ kNm^2

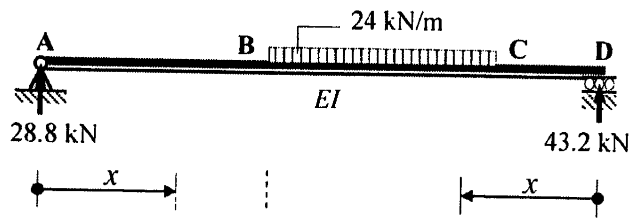
Support Reactions

Consider the rotational equilibrium of the beam:

+ve $\circlearrowright$ $\Sigma M_A = 0$ $\quad + (24.0 \times 3.0)(4.5) - (V_D \times 7.5) = 0$ $\qquad \therefore$ $\mathbf{V_D = +43.2\ kN}$ ↑

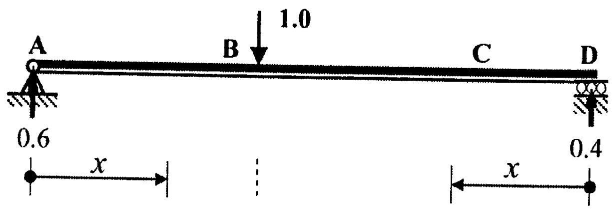
Consider the vertical equilibrium of the beam:

+ve ↑ $\Sigma F_y = 0$ $\quad + V_A - (24.0 \times 3.0) + V_D = 0$ $\qquad \therefore$ $\mathbf{V_A = +28.8\ kN}$ ↑



Applied load

Unit load



$$\delta_B = \int_0^L \frac{Mm}{EI}dx$$

(Mm/EI) is not a continuous function and the product integral must be evaluated between each of the discontinuities, i.e. A to B, D to C and C to B.

$$\delta_B = \int_0^L \frac{Mm}{EI}dx = \int_A^B \frac{Mm}{EI}dx + \int_D^C \frac{Mm}{EI}dx + \int_C^B \frac{Mm}{EI}dx$$

Consider the section from A to B: $\mathbf{0 \le x \le 3.0\ m}$

$M = +28.8x \qquad m = +0.6x \qquad \therefore\ Mm = 17.28x^2$

$$\int_A^B \frac{Mm}{EI}dx = \int_0^3 \frac{17.28x^2}{EI}dx = \left[\frac{17.28x^3}{3EI}\right]_0^3 = +\frac{155.52}{EI}\ \text{m}$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.16 **Page No. 2**

Consider the section from D to C: $\mathbf{0 \le x \le 1.5\ m}$

$M = +43.2x \qquad m = +0.4x \qquad \therefore\ Mm = 17.28x^2$

$$\int_D^C \frac{Mm}{EI}dx = \int_0^{1.5} \frac{17.28x^2}{EI}dx = \left[\frac{17.28x^3}{3EI}\right]_0^{1.5} = +\frac{19.44}{EI}\text{m}$$

Consider the section from C to B: $\mathbf{1.5 \le x \le 4.5\ m}$

$M = +43.2x - 24(x-1.5)^2/2 = 43.2x - 12(x^2 - 3x + 2.25)$

$\quad = -12x^2 + 79.2x - 27.0$

$m = +0.4x$

$Mm = -4.8x^3 + 31.68x^2 - 10.8x$

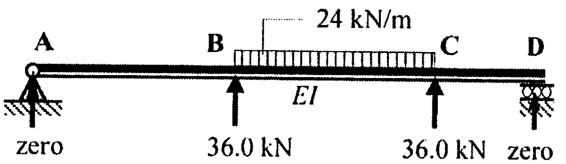
$$\int_C^B \frac{Mm}{EI}dx = \int_{1.5}^{4.5} \frac{-4.8x^3 + 31.68x^2 - 10.8x}{EI}dx = \left[-\frac{4.8x^4}{4EI} + \frac{31.68x^3}{3EI} - \frac{10.8x^2}{2EI}\right]_{1.5}^{4.5}$$

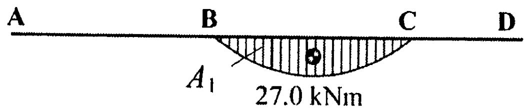
$$= \left(+\frac{360.86}{EI} - \frac{17.42}{EI}\right) = +\frac{343.44}{EI}$$

$$\therefore\ \delta_B = \left(+\frac{155.52}{EI} + \frac{19.44}{EI} + \frac{343.44}{EI}\right) = \frac{518.4}{EI} = \frac{518.4}{50.0\times10^3}\text{m} = \mathbf{10.37\ mm}\ \downarrow$$

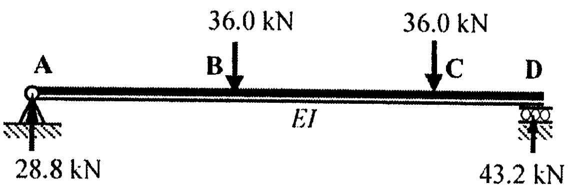
Alternatively:

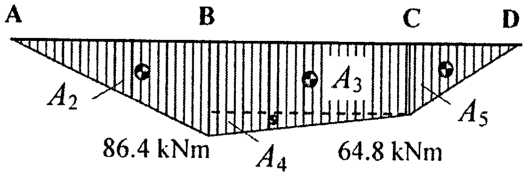
$\delta_B = \Sigma(\text{Area}_{\text{applied bending moment diagram}} \times \text{Ordinate}_{\text{unit load bending moment diagram}})$



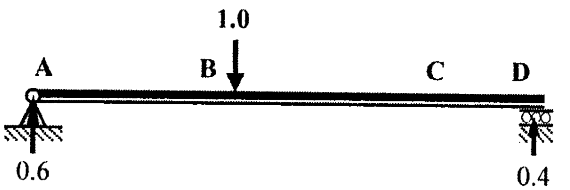


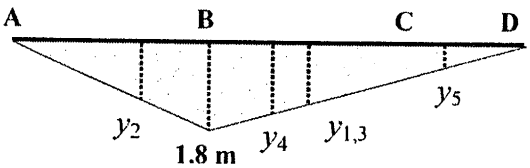
M for applied loads





M for applied loads





m for unit vertical load at B

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load

Problem Number: 4.16 **Page No. 3**

$A_1 = (0.667 \times 3.0 \times 27.0)$ kNm^2, $y_1 = 1.2$ m, $\therefore A_1y_1 = 64.83$ kNm^3

$A_2 = (0.5 \times 3.0 \times 86.4)$ kNm^2, $y_2 = 1.2$ m, $\therefore A_2y_2 = 155.52$ kNm^3

$A_3 = (3.0 \times 64.8)$ kNm^2, $y_3 = 1.2$ m, $\therefore A_3y_3 = 233.28$ kNm^3

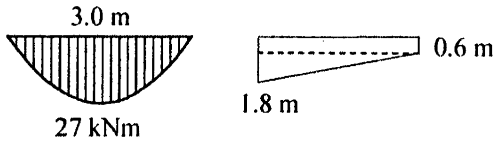
$A_4 = (0.5 \times 3.0 \times 21.6)$ kNm^2, $y_4 = 1.4$ m, $\therefore A_4y_4 = 45.36$ kNm^3

$A_5 = (0.5 \times 1.5 \times 64.8)$ kNm^2, $y_5 = 0.4$ m, $\therefore A_5y_5 = 19.44$ kNm^3

$\delta_B = (64.83 + 155.52 + 233.28 + 45.36 + 19.44)/50.0 \times 10^3 = 0.0104$ m $=$ **10.37 mm** ↓

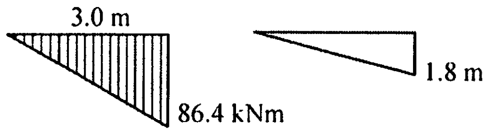
Using the coefficients given in Table 4.1:

Area A_1:



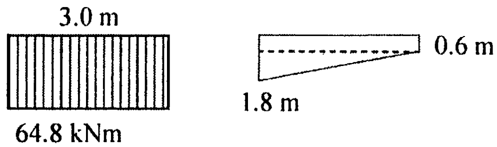
$$\int_0^L \frac{Mm}{EI}dx = [(0.667 \times 27 \times 0.6 \times 3.0) + (0.333 \times 27 \times 1.2 \times 3.0)]/EI = 64.78/EI$$

Area A_2:



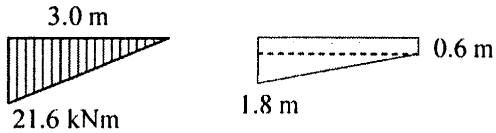
$$\int_0^L \frac{Mm}{EI}dx = (0.333 \times 86.4 \times 1.8 \times 3.0)/EI = 155.36/EI$$

Area A_3:



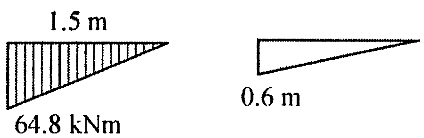
$$\int_0^L \frac{Mm}{EI}dx = [(1.0 \times 64.8 \times 0.6 \times 3.0) + (0.5 \times 64.8 \times 1.2 \times 3.0)]/EI = 233.28/EI$$

Area A_4:



$$\int_0^L \frac{Mm}{EI}dx = [(0.5 \times 21.6 \times 0.6 \times 3.0) + (0.333 \times 21.6 \times 1.2 \times 3.0)]/EI = 45.33/EI$$

Area A_5:



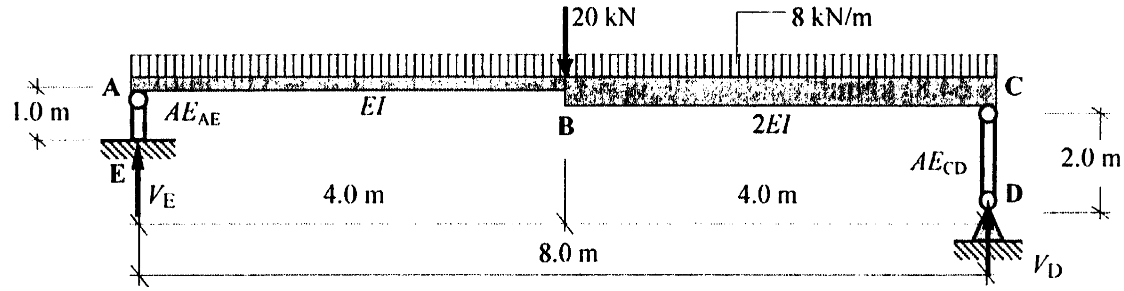
$$\int_0^L \frac{Mm}{EI}dx = (0.333 \times 64.8 \times 0.6 \times 1.5)/EI = 19.42/EI$$

$$\delta_B = \sum_A^D (Coefficient \times a \times b \times L)/EI$$

$\delta_B = (64.78 + 155.36 + 233.28 + 45.33 + 19.42)/50.0 \times 10^3 = 0.0102$ m $=$ **10.2 mm** ↓

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.17 **Page No. 1**



Determine the value of the vertical deflection at B given:

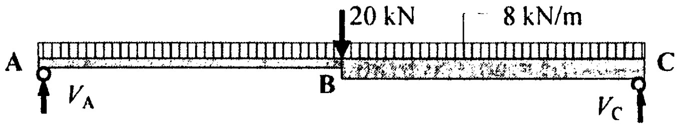
$E_{beam} = 9.0\ \text{kN/mm}^2$, $I_{beam} = 14.6 \times 10^9\ \text{mm}^4$

$E_{AE\ and\ CD} = 170\ \text{kN/mm}^2$ $A_{AE} = 80\ \text{mm}^2$, $A_{CD} = 120\ \text{mm}^2$

$EI = (9.0 \times 14.6 \times 10^9)/10^6 = 131.4 \times 10^3\ \text{kNm}^2$

$AE_{AE} = (80.0 \times 170.0) = 13.6 \times 10^3\ \text{kN}$; $AE_{CD} = (120.0 \times 170.0) = 20.4 \times 10^3\ \text{kN}$

$$\delta_B = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{AE,CD}$$



Consider the beam ABC:
Support Reactions

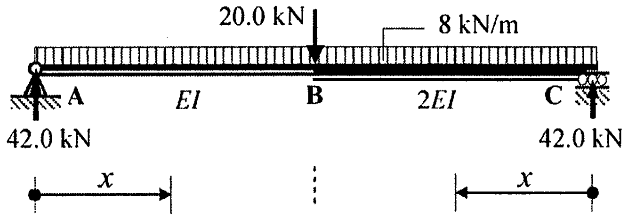
Consider the rotational equilibrium of the beam:

+ve $\circlearrowright \Sigma M_A = 0$ $+ (8.0 \times 8.0)(4.0) + (20.0 \times 4.0) - (V_C \times 8.0) = 0$

$\therefore$ $\mathbf{V_C = +\ 42.0\ kN}$ ↑

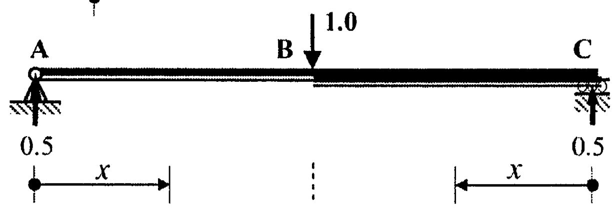
Consider the vertical equilibrium of the beam:

+ve ↑ $\Sigma F_y = 0$ $+ V_A - 20.0 - (8.0 \times 8.0) + V_C = 0$ $\therefore$ $\mathbf{V_A = +\ 42.0\ kN}$ ↑



Applied loads

Unit load



(Mm/EI) is not a continuous function the product integral must be evaluated between each of the discontinuities, i.e. A to B and C to B.

$$\delta_B = \int_0^L \frac{Mm}{EI}dx = \int_A^B \frac{Mm}{EI}dx + \int_C^B \frac{Mm}{EI}dx$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.17 **Page No. 2**

Consider the section from A to B: $\mathbf{0 \le x \le 4.0\ m}$

$M = +\,42.0x - 8.0x^2/2 = 42.0x - 4.0x^2 \qquad m = +\,0.5x$

$Mm = (42.0x - 4.0x^2)(0.5x) = 21.0x^2 - 2.0x^3$

$$\int_A^B \frac{Mm}{EI}dx = \int_0^4 \frac{21.0x^2 - 2.0x^3}{EI}dx = \left[\frac{21.0x^3}{3EI} - \frac{2.0x^4}{4EI}\right]_0^4 = +\frac{320.0}{EI}\,\text{m}$$

Consider the section from C to B: $\mathbf{0 \le x \le 4.0\ m}$

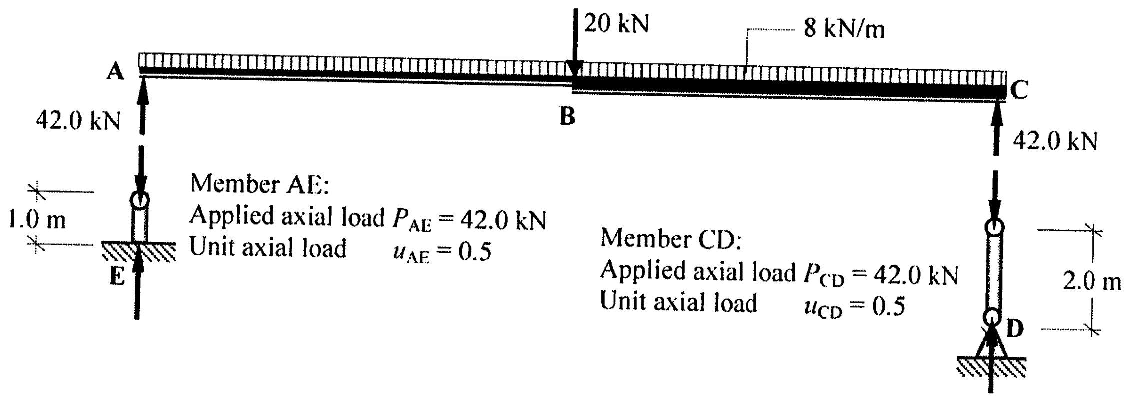
$M = +\,42.0x - 8.0x^2/2 = 42.0x - 4.0x^2 \qquad m = +\,0.5x$

$Mm = (42.0x - 4.0x^2)(0.5x) = 21.0x^2 - 2.0x^3$

$$\int_C^B \frac{Mm}{EI}dx = \int_0^4 \frac{21.0x^2 - 2.0x^3}{2EI}dx = \left[\frac{21.0x^3}{6EI} - \frac{2.0x^4}{8EI}\right]_0^4 = +\frac{160.0}{EI}\,\text{m}$$

$$\int_0^L \frac{Mm}{EI}dx = \left(+\frac{320.0}{EI} + \frac{160.0}{EI}\right) = \frac{480.0}{EI} = \frac{480.0}{131.4\times10^3}\,\text{m} = 3.65\ \text{mm}$$

Consider the columns AE and CD:



$$\sum\left(\frac{PL}{AE}u\right)_{AE,CD} = \left(\frac{42.0\times1000\times0.5}{AE_{AE}}\right) + \left(\frac{42.0\times2000\times0.5}{AE_{CD}}\right)$$

$$= \left(\frac{21.0\times10^3}{13.6\times10^3}\right) + \left(\frac{42.0\times10^3}{20.4\times10^3}\right) = +\,1.54 + 2.06 = 3.6\ \text{mm}$$

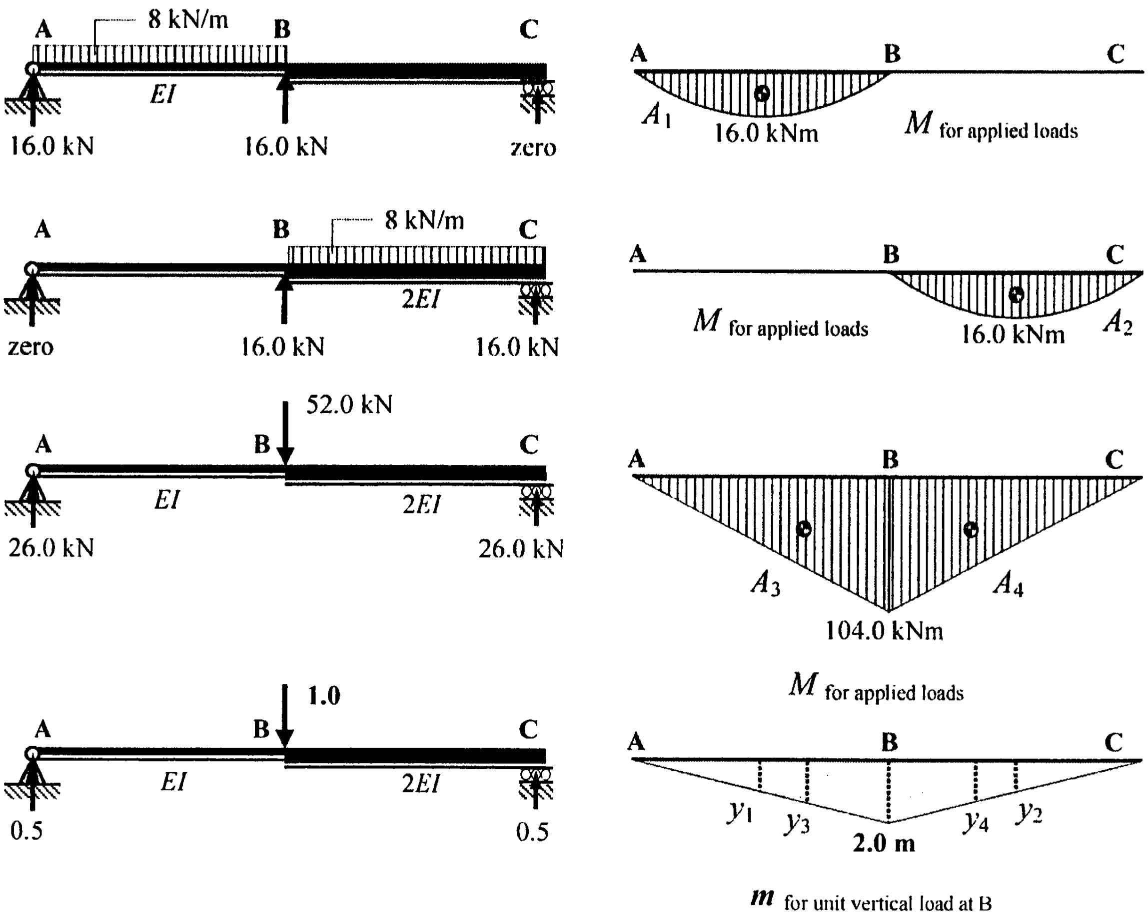
$$\delta_B = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{AE,CD} = 3.65 + 3.6 = \mathbf{7.25\ mm} \downarrow$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.17 **Page No. 3**

Alternatively for the beam ABC:

$\delta_B = \Sigma(\text{Area}_{\text{applied bending moment diagram}} \times \text{Ordinate}_{\text{unit load bending moment diagram}})$



$A_1 = (0.667 \times 4.0 \times 16.0)$ kNm^2, $y_1 = 1.0$ m, $\therefore A_1y_1 = 42.69$ kNm^3
$A_2 = (0.667 \times 4.0 \times 16.0)$ kNm^2, $y_2 = 1.0$ m, $\therefore A_2y_2 = 42.69$ kNm^3
$A_3 = (0.5 \times 4.0 \times 104.0)$ kNm^2, $y_3 = 1.33$ m, $\therefore A_3y_3 = 276.6$ kNm^3
$A_4 = (0.5 \times 4.0 \times 104.0)$ kNm^2, $y_4 = 1.33$ m, $\therefore A_4y_4 = 276.6$ kNm^3

$$\int_0^L \frac{Mm}{EI}dx = \mathbf{[(42.69 + 276.6)/\mathit{EI} + (42.69 + 276.6)/2\mathit{EI}] = 478.9/\mathit{EI}}$$

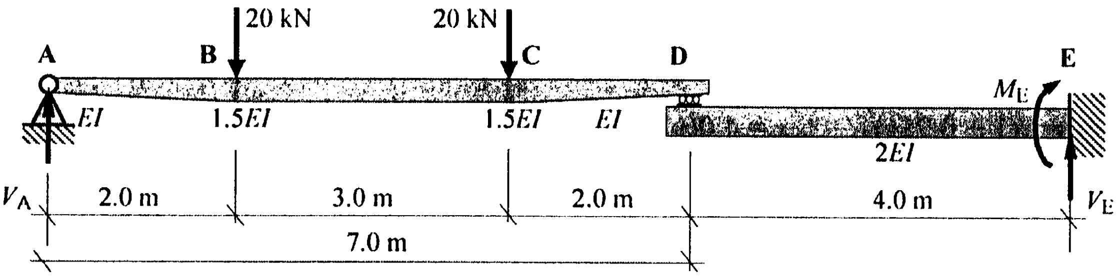
Using the coefficients given in Table 4.1:

$$\int_0^L \frac{Mm}{EI}dx = \sum_A^B(\mathit{Coefficient} \times a \times b \times L)/EI + \sum_C^B(\mathit{Coefficient} \times a \times b \times L)/2EI$$

$$= (0.333 \times 16.0 \times 2.0 \times 4.0)/EI + (0.333 \times 104.0 \times 2.0 \times 4.0)/EI$$
$$+ (0.333 \times 16.0 \times 2.0 \times 4.0)/2EI + (0.333 \times 104.0 \times 2.0 \times 4.0)/2EI$$
$$= \mathbf{479.5/\mathit{EI}}$$

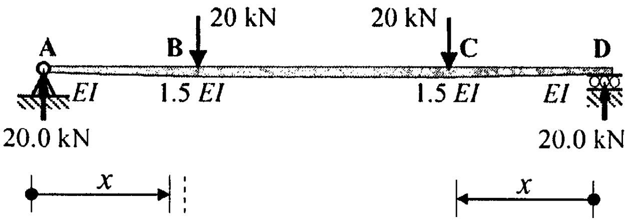
Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.18 **Page No. 1**



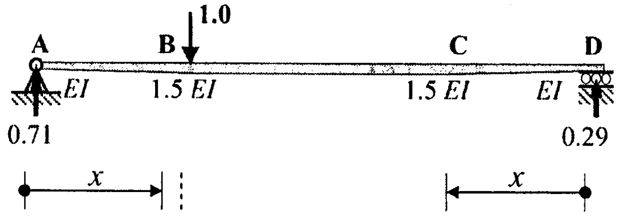
The *EI* value of the beam ABCD varies linearly from *EI* at the supports A and D to 1.5*EI* at B and C respectively and is constant between B and C.
Determine the value of the vertical deflection at B given that $EI = 15.0 \times 10^3$ kNm^2

Consider beam ABCD:
Support Reactions
Consider the rotational equilibrium of the beam:
+ve $\Sigma M_A = 0$ $+ (20.0 \times 2.0) + (20.0 \times 5.0) - (V_D \times 7.0) = 0$ $\therefore$ $\mathbf{V_D = +\ 20.0\ kN}$ ↑
Consider the vertical equilibrium of the beam:
+ve ↑ $\Sigma F_y = 0$ $+ V_A - 20.0 - 20.0 + V_D = 0$ $\therefore$ $\mathbf{V_A = +\ 20.0\ kN}$ ↑



Applied load

Unit load



$$\delta_B = \int_0^L \frac{Mm}{EI} dx$$

(*Mm*/*EI*) is not a continuous function and the product integral must be evaluated between each of the discontinuities, i.e. A to B, D to C and C to B.

$$\delta_B = \int_0^L \frac{Mm}{EI} dx = \int_A^B \frac{Mm}{EI} dx + \int_D^C \frac{Mm}{EI} dx + \int_C^B \frac{Mm}{EI} dx$$

Consider the section from A to B: $\mathbf{0 \leq x \leq 2.0\ m}$
$M = +\ 20.0x$ $\quad m = +\ 0.71x$ $\quad \therefore\ Mm = 14.2x^2$
Also
The *EI* value varies linearly between A and B and at distance '*x*' from A is given by:
$EI\,(1 + 0.25x)$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.18 **Page No. 2**

$$\int_A^B \frac{Mm}{EI}dx = \int_0^2 \frac{14.2x^2}{EI(1+0.25x)}dx$$

Let $v = (1 + 0.25x)$ $\therefore x = 4.0(v - 1)$, $dx = 4.0dv$ and $x^2 = 16.0(v - 1)^2$
when $x = 0$ $v = 1.0$ and when $x = 2$ $v = (1+ 0.5) = 1.5$

$Mm\,dx = 14.2x^2 = [14.2 \times 16.0(v - 1)^2] \times 4.0dv = 908.8(v - 1)^2\,dv$

$$= \int_0^2 \frac{14.2x^2}{EI(1+0.25x)}dx = \frac{908.8}{EI}\int_{v=1.0}^{v=1.5}\frac{(v-1)^2}{v}dv = \frac{908.8}{EI}\int_{v=1.0}^{v=1.5}\frac{(v^2-2.0v+1.0)}{v}dv$$

$$= \frac{908.8}{EI}\int_{v=1.0}^{v=1.5}\left(v-2.0+\frac{1.0}{v}\right)dv = \frac{908.8}{EI}\left[\frac{v^2}{2}-2.0v+ln\,v\right]_{v=1.0}^{v=1.5}$$

$$= \frac{908.8}{EI}\left\{\left[\frac{1.5^2}{2}-(2.0\times1.5)+ln1.5\right]-\left[\frac{1.0^2}{2}-(2.0\times1.0)+ln1.0\right]\right\}$$

$$= +\frac{27.69}{EI}\text{ m}$$

Consider the section from D to C: $\mathbf{0 \le x \le 2.0}$ **m**
$M = +20.0x$ $m = +0.29x$ $\therefore Mm = 5.8x^2$
Also
The EI value varies linearly between D and C, and at distance x from A is given by: $EI(1 + 0.25x)$

$$\int_D^C \frac{Mm}{EI}dx = \int_0^2 \frac{5.8x^2}{EI(1+0.25x)}dx$$

Let $v = (1 + 0.25x)$ $\therefore x = 4.0(v - 1)$, $dx = 4.0dv$ and $x^2 = 16.0(v - 1)^2$
when $x = 0$ $v = 1.0$ and when $x = 2$ $v = (1+ 0.5) = 1.5$

$Mm\,dx = 5.8x^2 = [5.8 \times 16.0(v - 1)^2] \times 4.0dv = 371.2(v - 1)^2\,dv$

$$= \int_0^2 \frac{14.2x^2}{EI(1+0.25x)}dx = \frac{371.2}{EI}\int_{v=1.0}^{v=1.5}\frac{(v-1)^2}{v}dv = \frac{371.2}{EI}\int_{v=1.0}^{v=1.5}\frac{(v^2-2.0v+1.0)}{v}dv$$

$$= \frac{371.2}{EI}\int_{v=1.0}^{v=1.5}\left(v-2.0+\frac{1.0}{v}\right)dv = \frac{371.2}{EI}\left[\frac{v^2}{2}-2.0v+ln\,v\right]_{v=1.0}^{v=1.5}$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.18 **Page No. 3**

$$= \frac{371.2}{EI}\left\{\left[\frac{1.5^2}{2} - (2.0\times 1.5) + ln1.5\right] - \left[\frac{1.0^2}{2} - (2.0\times 1.0) + ln1.0\right]\right\}$$

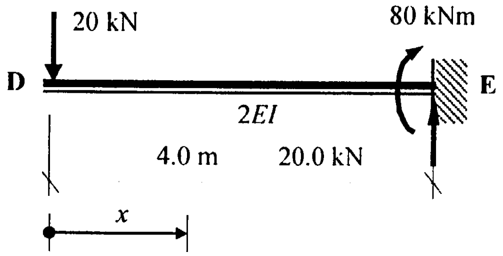
$$= +\frac{11.31}{EI}\,\text{m}$$

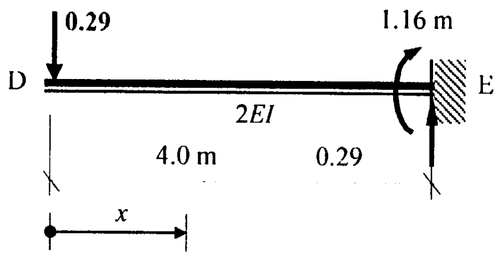
Consider the section from C to B: $\mathbf{2.0 \le x \le 5.0}$ **m**

$M = +20.0x - 20.0(x - 2.0) = 40.0 \qquad m = +0.29x \qquad \therefore\ Mm = 11.6x$

$$\int_C^B \frac{Mm}{1.5EI}dx = \int_2^5 \frac{11.6x}{1.5EI}dx = \left[\frac{11.6x^2}{3.0EI}\right]_2^5 = +\frac{81.2}{EI}\,\text{m}$$

Consider the cantilever beam DE:





$M = -20.0x \qquad m = -0.29x \qquad \therefore\ Mm = +5.8x^2$

$$\int_D^E \frac{Mm}{EI}dx = \int_0^4 \frac{5.8x^2}{2EI}dx = \left[\frac{5.8x^3}{6EI}\right]_0^4 = +\frac{61.87}{EI}\,\text{m}$$

$$\delta_B = \left(+\frac{27.69}{EI} + \frac{11.31}{EI} + \frac{81.2}{EI} + \frac{61.87}{EI}\right) = \frac{182.07}{EI} = \frac{182.07}{15.0\times 10^3}\,\text{m} = 12.14\ \text{mm}\ \downarrow$$

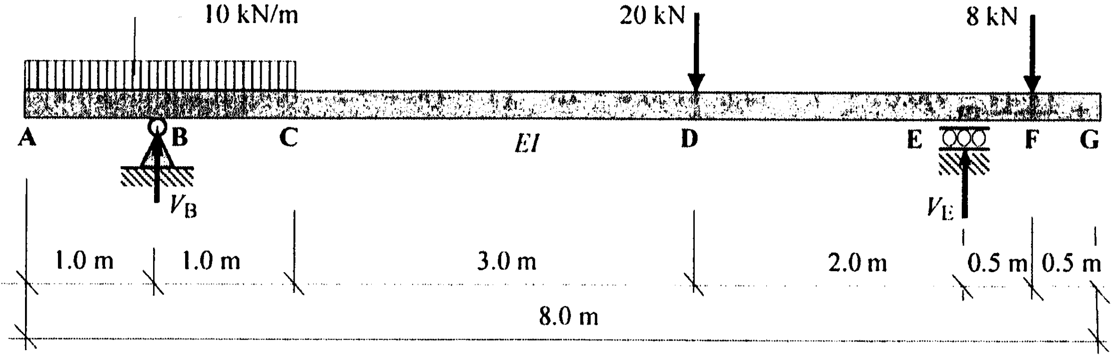
Alternatively:

Sections A to B and D to C must be carried out using the product integrals as above The terms relating to the central section C to B and the cantilever beam D to E can also be evaluated using the product (area × ordinate) or the Coefficients given in Table 4.1 since the *EI* value is constant along these lengths.

The reader should carry out these calculations to confirm the results.

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.19 **Page No. 1**



Determine the value of the vertical deflection at G given that $EI = 5.0 \times 10^3$ kNm^2

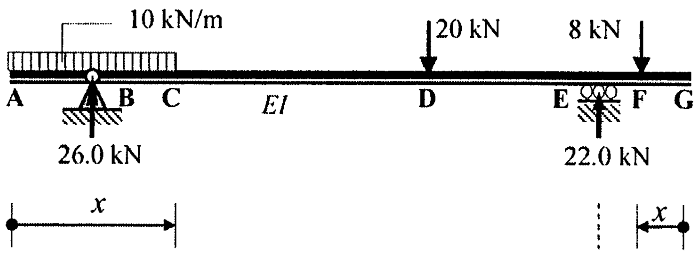
Support Reactions

Consider the rotational equilibrium of the beam:

+ve $\circlearrowright \Sigma M_B = 0$ $\quad + (20.0 \times 4.0) + (8.0 \times 6.5) - (V_E \times 6.0) = 0 \quad \therefore\ V_E = +\mathbf{22.0\ kN} \uparrow$

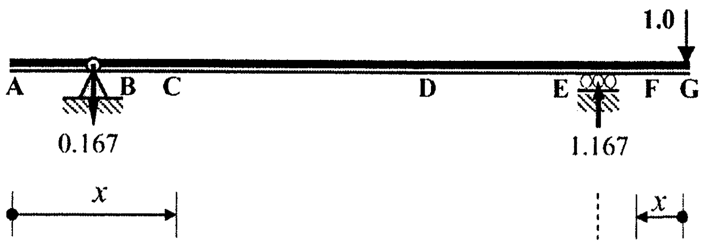
Consider the vertical equilibrium of the beam:

+ve $\uparrow \Sigma F_y = 0$ $\quad + V_B - (10.0 \times 2.0) - 20.0 - 8.0 + V_E = 0 \quad \therefore\ V_B = +\mathbf{26.0\ kN} \uparrow$



Applied load

Unit load



$$\delta_G = \int_0^L \frac{Mm}{EI} dx$$

(Mm/EI) is not a continuous function and the product integral must be evaluated between each of the discontinuities, i.e. A to B, B to C, C to D, D to E, G to F and F to E

$$\delta_G = \int_0^L \frac{Mm}{EI} dx = \int_A^B \frac{Mm}{EI} dx + \int_B^C \frac{Mm}{EI} dx + \int_C^D \frac{Mm}{EI} dx + \int_D^E \frac{Mm}{EI} dx + \int_G^F \frac{Mm}{EI} dx + \int_F^E \frac{Mm}{EI} dx$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
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Consider the section from A to B: $\mathbf{0 \le x \le 1.0\ m}$

$M = -\,10.0x^2/2 \qquad m = \text{zero} \qquad \therefore\ Mm = \text{zero}$

$$\int_A^B \frac{Mm}{EI}\,dx = \text{zero}$$

Consider the section from B to C: $\mathbf{1.0 \le x \le 2.0\ m}$

$M = -\,10.0x^2/2 + 26.0(x - 1.0) = (-\,5.0x^2 + 26.0x - 26.0)$

$m = -\,0.167(x - 1.0)$

$Mm = [(-\,5.0x^2 + 26.0x - 26.0)] \times [-\,0.167(x - 1.0)]$

$= (0.84x^3 - 5.18x^2 + 8.68x - 4.34)$

$$\int_B^C \frac{Mm}{EI}\,dx = \int_{1.0}^{2.0} \frac{0.84x^3 - 5.18x^2 + 8.68x - 4.34}{EI}\,dx$$

$$= \left[\frac{0.84x^4}{4EI} - \frac{5.18x^3}{3EI} + \frac{8.68x^2}{2EI} - \frac{4.34x}{EI}\right]_{1.0}^{2.0} = +\left[-\frac{1.77}{EI} - \left(-\frac{1.52}{EI}\right)\right] = -\frac{0.25}{EI}$$

Consider the section from C to D: $\mathbf{2.0 \le x \le 5.0\ m}$

$M = -\,(10.0 \times 2.0)(x - 1.0) + 26.0(x - 1.0) = +\,6.0(x - 1.0) \qquad m = -\,0.167(x - 1.0)$

$Mm = 6.0(x - 1.0)(-\,0.167x + 0.167) = (-\,x^2 + 2.0x - 1.0)$

$$\int_C^D \frac{Mm}{EI}\,dx = \int_{2.0}^{5.0} \frac{-x^2 + 2.0x - 1.0}{EI}\,dx = \left[-\frac{x^3}{3EI} + \frac{2.0x^2}{2EI} - \frac{x}{EI}\right]_{2.0}^{5.0}$$

$$= \left(-\frac{21.67}{EI} - \frac{0.67}{EI}\right) = -\frac{22.34}{EI}$$

Consider the section from D to E: $\mathbf{5.0 \le x \le 7.0\ m}$

$M = -\,(10.0 \times 2.0)(x - 1.0) + 26.0(x - 1.0) - 20.0(x - 5.0) = (-\,14.0x + 94.0)$

$m = -\,0.167(x - 1.0)$

$Mm = (-\,14.0x + 94.0)(-\,0.167x + 0.167) = (2.34x^2 - 18.04x + 15.7)$

$$\int_D^E \frac{Mm}{EI}\,dx = \int_{5.0}^{7.0} \frac{2.34x^2 - 18.04x + 15.7}{EI}\,dx = \left[\frac{2.34x^3}{3EI} - \frac{18.04x^2}{2EI} + \frac{15.7x}{EI}\right]_{5.0}^{7.0}$$

$$= \left[-\frac{64.54}{EI} - \left(-\frac{49.5}{EI}\right)\right] = -\frac{15.04}{EI}$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.19 **Page No. 3**

Consider the section from G to F: $\mathbf{0 \le x \le 0.5\ m}$

$M = \text{zero} \qquad m = -x \qquad \therefore\ Mm = \text{zero}$

$$\int_G^F \frac{Mm}{EI}\,dx = \text{zero}$$

Consider the section from F to E: $\mathbf{0.5 \le x \le 1.0\ m}$

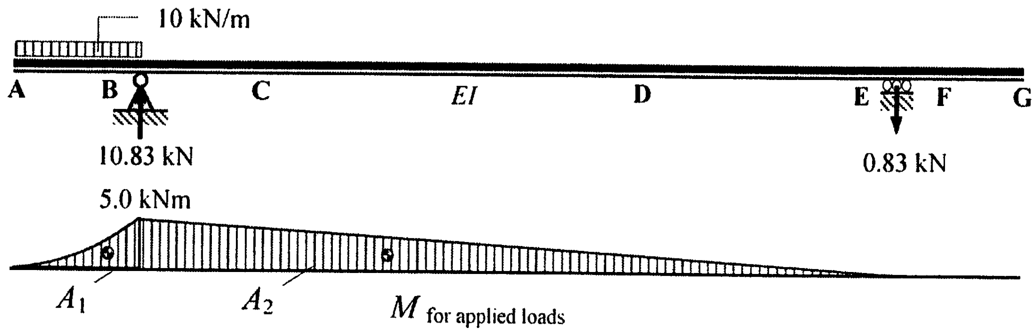
$M = -\,8.0(x - 0.5) \qquad m = -x \qquad \therefore\ Mm = (8.0x^2 - 4.0\,x)$

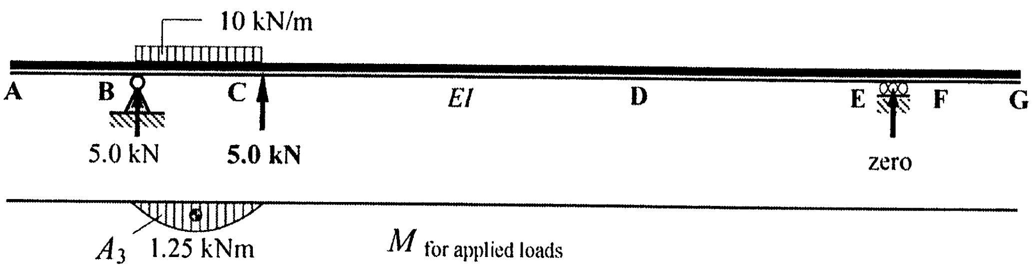
$$\int_F^E \frac{Mm}{EI}\,dx = \int_{0.5}^{1.0} \frac{8.0x^2 - 4.0x}{EI}\,dx = \left[\frac{8.0x^3}{3EI} - \frac{4.0x^2}{2EI}\right]_{0.5}^{1.0} = \left[+\frac{0.67}{EI} - \left(-\frac{0.17}{EI}\right)\right] = +\frac{0.84}{EI}$$

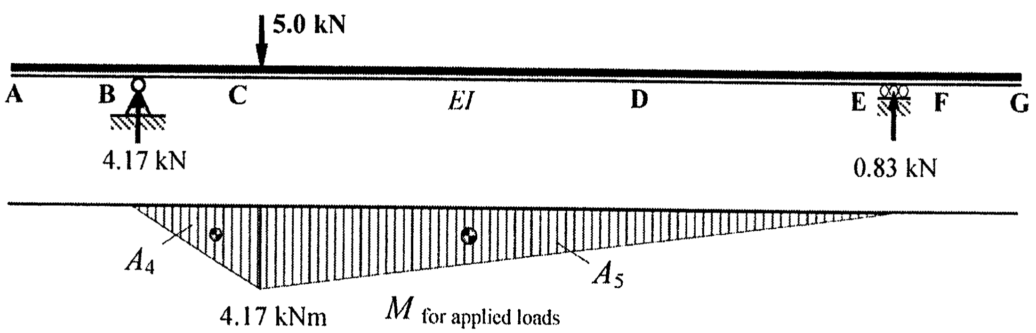
$$\delta_G = \left(-\frac{0.25}{EI} - \frac{22.34}{EI} - \frac{15.04}{EI} + \frac{0.84}{EI}\right) = -\frac{36.79}{EI} = -\frac{36.79}{5.0\times10^3}\ \text{m} = -\,\mathbf{7.36\ mm}\uparrow$$

Alternatively:

$\delta_G = \Sigma(\text{Area}_{\text{applied bending moment diagram}} \times \text{Ordinate}_{\text{unit load bending moment diagram}})$



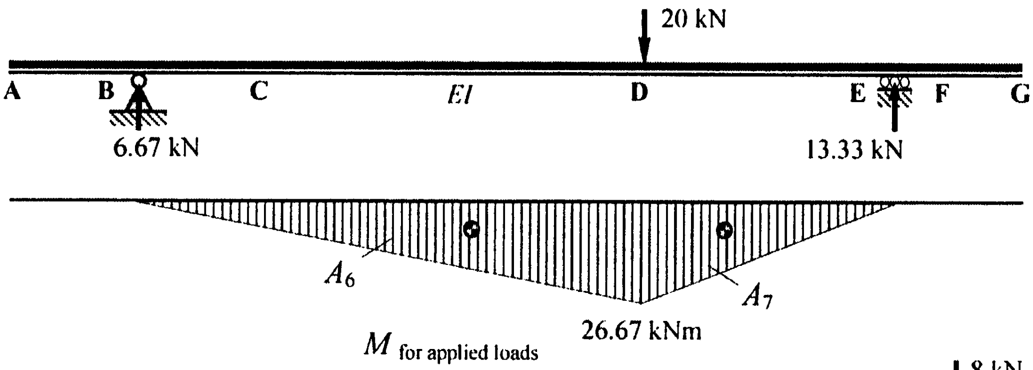


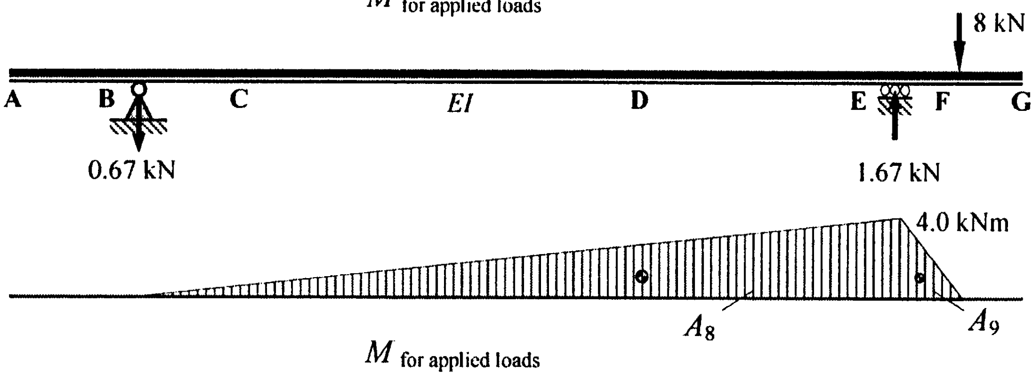


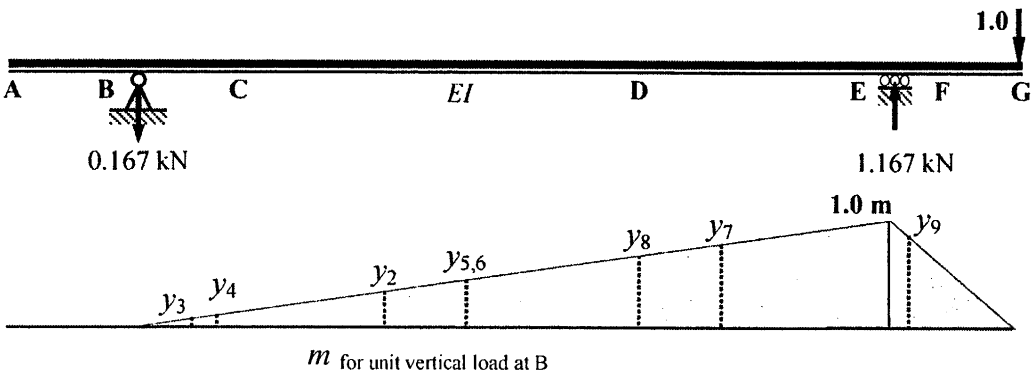
Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load

Problem Number: 4.19 **Page No. 4**







A_1 = not required since y_1 = zero

$A_2 = -(0.5 \times 6.0 \times 5.0) = -15.0$ kNm^2, $y_2 = -0.33$ m $\therefore A_2y_2 = +5.0$ kNm^3

$A_3 = (0.667 \times 1.0 \times 1.25) = 0.83$ kNm^2 $y_3 = -0.08$ m $\therefore A_3y_3 = -0.07$ kNm^3

$A_4 = (0.5 \times 1.0 \times 4.17) = 2.35$ kNm^2, $y_4 = -0.11$ m $\therefore A_4y_4 = -0.26$ kNm^3

$A_5 = (0.5 \times 5.0 \times 4.17) = 10.43$ kNm^2, $y_5 = -0.45$ m $\therefore A_5y_5 = -4.69$ kNm^3

$A_6 = (0.5 \times 4.0 \times 26.67) = 53.34$ kNm^2, $y_6 = -0.45$ m $\therefore A_6y_6 = -24.0$ kNm^3

$A_7 = (0.5 \times 2.0 \times 26.67) = 26.67$ kNm^2, $y_7 = -0.78$ m $\therefore A_7y_7 = -20.8$ kNm^3

$A_8 = -(0.5 \times 6.0 \times 4.0) = -12.0$ kNm^2, $y_8 = -0.67$ m $\therefore A_8y_8 = +8.0$ kNm^3

$A_9 = -(0.5 \times 0.5 \times 4.0) = -1.0$ kNm^2, $y_9 = -0.83$ m $\therefore A_9y_9 = +0.83$ kNm^3

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
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$$\delta_G = \int_0^L \frac{Mm}{EI}dx = \Sigma(Av)/EI$$

$= (+\,5.0 - 0.07 - 0.26 - 4.69 - 24.0 - 20.8 + 8.0 + 0.83)/EI$

$\delta_G = -\,35.99/EI = -\,(35.99/5.0 \times 10^3)$ m $= -$ **7.20 mm** ↑

Using the coefficients given in Table 4.1: $\delta_G = \sum_0^L(Coefficient \times a \times b \times L)/EI$

Area A_2: $\int_0^L \frac{Mm}{EI}dx = +\,(0.167 \times 5.0 \times 1.0 \times 6.0)/EI = +\,5.0/EI$

Area A_3: $\int_0^L \frac{Mm}{EI}dx = -\,(0.333 \times 1.25 \times 0.167 \times 1.0)/EI = -\,0.07/EI$

Area A_4: $\int_0^L \frac{Mm}{EI}dx = -\,(0.333 \times 4.17 \times 0.167 \times 1.0)/EI = -\,0.23/EI$

Area A_5: $\int_0^L \frac{Mm}{EI}dx = -\,[(0.5 \times 4.17 \times 0.167 \times 5.0) + (0.167 \times 4.17 \times 0.83 \times 5.0)]/EI$

$= -\,4.63/EI$

Area A_6: $\int_0^L \frac{Mm}{EI}dx = -\,(0.333 \times 26.67 \times 0.67 \times 4.0)/EI = -\,23.80/EI$

Area A_7: $\int_0^L \frac{Mm}{EI}dx = -\,[(0.5 \times 26.67 \times 0.67 \times 2.0) + (0.167 \times 26.67 \times 0.33 \times 2.0)]/EI$

$= -\,20.81/EI$

Area A_8: $\int_0^L \frac{Mm}{EI}dx = +\,(0.333 \times 4.0 \times 1.0 \times 6.0)/EI = +\,8.0/EI$

Area A_9: $\int_0^L \frac{Mm}{EI}dx = +\,[(0.5 \times 4.0 \times 0.5 \times 0.5) + (0.333 \times 4.0 \times 0.5 \times 0.5)]/EI$

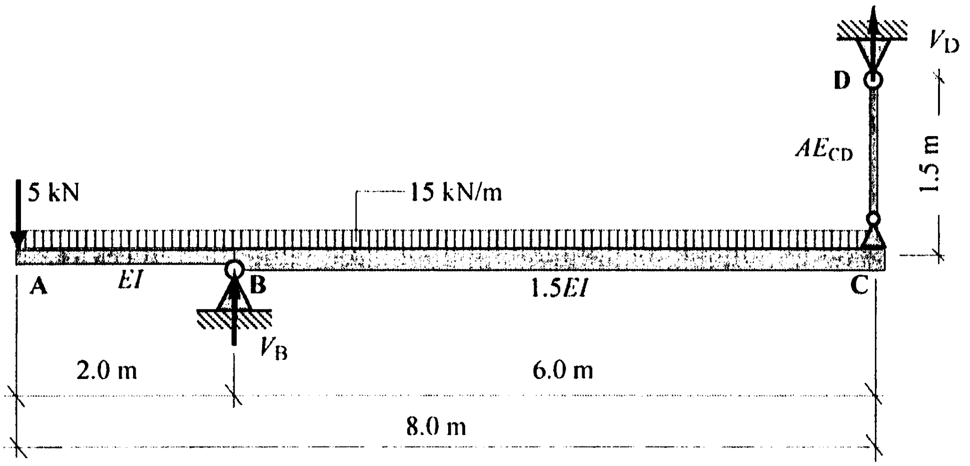
$= +\,0.83/EI$

$\delta_G = (5.0 - 0.07 - 0.23 - 4.63 - 23.80 - 20.81 + 8.0 + 0.83)/EI = -\,35.71/EI$

$= -\,(35.71/5.0 \times 10^3)$ m $= -$ **7.14 mm**

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.20 **Page No. 1**



Determine the value of the vertical deflection at A given:

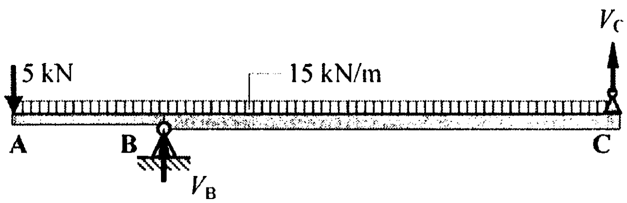
$E_{beam} = 205$ kN/mm^2, $I_{beam} = 60.0 \times 10^6$ mm^4

$E_{CD} = 205$ kN/mm^2, $A_{CD} = 50$ mm^2

$EI = (205 \times 60 \times 10^6)/10^6 = 12.3 \times 10^3$ kNm^2

$AE_{CD} = (50.0 \times 205.0) = 10.25 \times 10^3$ kN

$$\delta_A = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{CD}$$



Consider the beam ABC:

Support Reactions

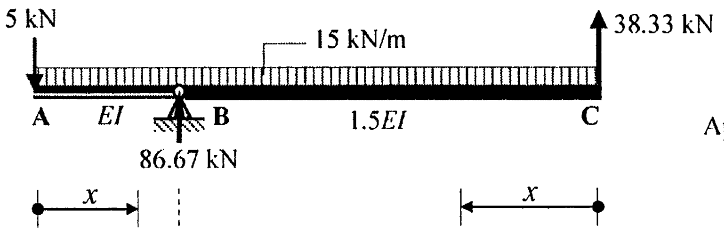
Consider the rotational equilibrium of the beam:

$+ve \circlearrowright \Sigma M_B = 0 \quad -(5.0 \times 2.0) + (15.0 \times 8.0 \times 2.0) - (V_C \times 6.0) = 0$

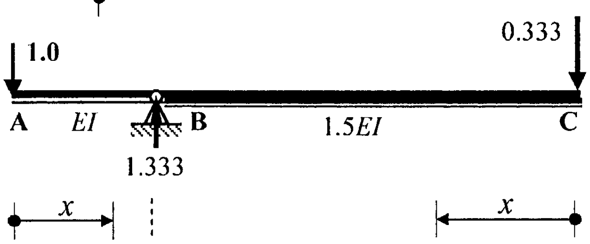
$\therefore$ $V_C = +$ **38.33 kN** ↑

Consider the vertical equilibrium of the beam:

$+ve \uparrow \ \Sigma F_y = 0 \quad + V_B - 5.0 - (15.0 \times 8.0) + V_C = 0$ $\therefore$ $V_B = +$ **86.67 kN** ↑



Applied loads



Unit load

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.20 **Page No. 2**

(Mm/EI) is not a continuous function and the product integral must be evaluated between each of the discontinuities, i.e. A to B and C to B.

$$\int_0^L \frac{Mm}{EI}dx = \int_A^B \frac{Mm}{EI}dx + \int_C^B \frac{Mm}{1.5EI}dx$$

Consider the section from A to B: $\mathbf{0 \le x \le 2.0\ m}$

$M = -\,5.0x - 15.0x^2/2 = -\,5.0x - 7.5x^2$ $\qquad m = -\,x$

$Mm = (-\,5.0x - 7.5x^2)(x) = 5.0x^2 + 7.5x^3$

$$\int_A^B \frac{Mm}{EI}dx = \int_0^2 \frac{5.0x^2 + 7.5x^3}{EI}dx = \left[\frac{5.0x^3}{3EI} + \frac{7.5x^4}{4EI}\right]_0^2 = +\frac{43.33}{EI}\ \text{m}$$

Consider the section from C to B: $\mathbf{0 \le x \le 6.0\ m}$

$M = +\,38.33x - 15.0x^2/2 = +\,38.33x - 7.5x^2$ $\qquad m = -\,0.333x$

$Mm = -\,(38.33x - 7.5x^2)(0.333x) = -\,12.77x^2 + 2.5x^3$

$$\int_C^B \frac{Mm}{EI}dx = \int_0^6 \frac{-12.77x^2 + 2.5x^3}{1.5EI}dx = \left[-\frac{12.77x^3}{4.5EI} + \frac{2.5x^4}{6EI}\right]_0^6 = -\frac{72.96}{EI}\ \text{m}$$

$$\int_0^L \frac{Mm}{EI}dx = \left(+\frac{43.33}{EI} - \frac{72.96}{EI}\right) = -\frac{29.63}{EI} = -\frac{29.63}{12.3\times10^3}\ \text{m} = -\,2.41\ \text{mm}$$

Consider member CD:

Applied axial load $\qquad P_{CD} = +\,58.33$ kN $\qquad$ (tension)

Unit axial load $\qquad u_{CD} = -\,0.333$ $\qquad$ (compression)

$$\sum\left(\frac{PL}{AE}u\right)_{CD} = -\left(\frac{38.33\times1500\times0.333}{AE_{CD}}\right) = -\left(\frac{19.146\times10^3}{10.25\times10^3}\right)\text{m} = -\,1.87\ \text{mm}$$

$$\delta_A = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{CD} = -\,2.41 - 1.87 = \mathbf{-\,4.28\ mm} \quad \uparrow$$

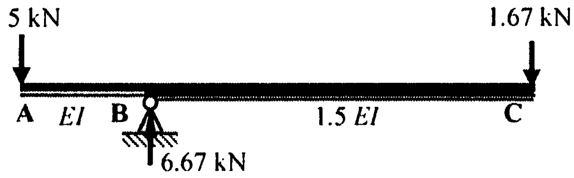
Solution

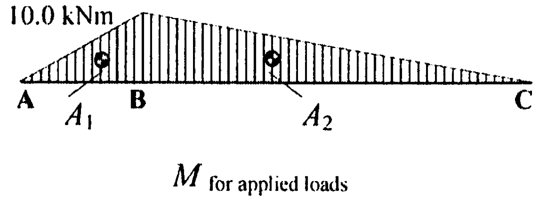
Topic: Determinate Beams/Frames – Deflection Using Unit Load

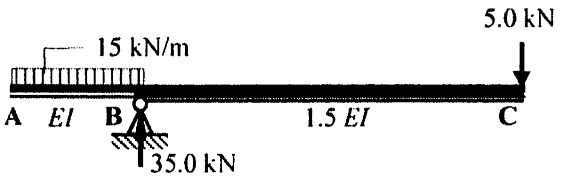
Problem Number: 4.20 **Page No. 3**

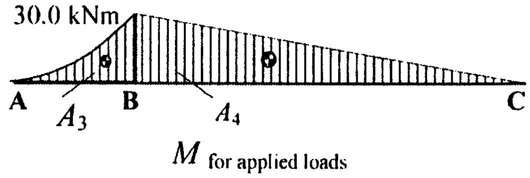
Alternatively:

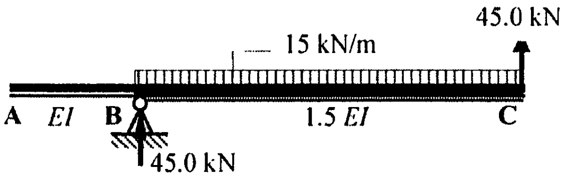
$\delta_A = \Sigma(\text{Area}_{\text{applied bending moment diagram}} \times \text{Ordinate}_{\text{unit load bending moment diagram}})$

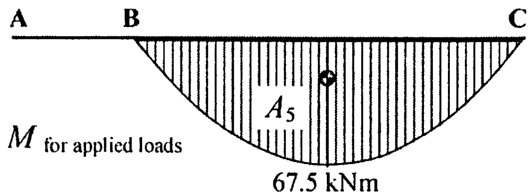


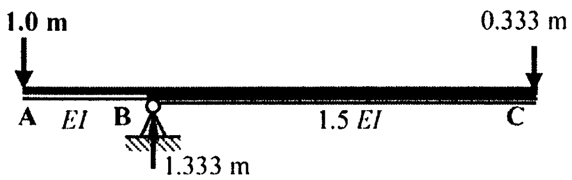


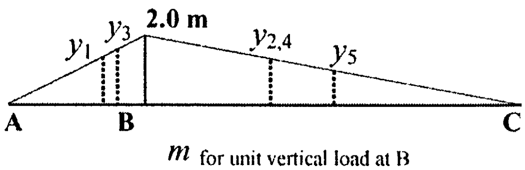












$A_1 = -(0.5 \times 2.0 \times 10.0)$ kNm^2,	$y_1 = -1.33$ m,	$\therefore A_1 y_1 = +13.33$ kNm^3
$A_2 = -(0.5 \times 6.0 \times 10.0)$ kNm^2,	$y_2 = -1.33$ m,	$\therefore A_2 y_2 = +40.0$ kNm^3
$A_3 = -(0.333 \times 2.0 \times 30.0)$ kNm^2,	$y_3 = -1.5$ m,	$\therefore A_3 y_3 = +30.0$ kNm^3
$A_4 = -(0.5 \times 6.0 \times 30.0)$ kNm^2,	$y_4 = -1.33$ m,	$\therefore A_4 y_4 = +120.0$ kNm^3
$A_5 = +(0.667 \times 6.0 \times 67.5)$ kNm^2,	$y_5 = -1.0$ m,	$\therefore A_4 y_4 = -270.0$ kNm^3

$$\int_0^L \frac{Mm}{EI}dx = (13.33 + 30.0)/EI + (40.0 + 120.0 - 270.0)/1.5EI = \mathbf{30.0/EI}$$

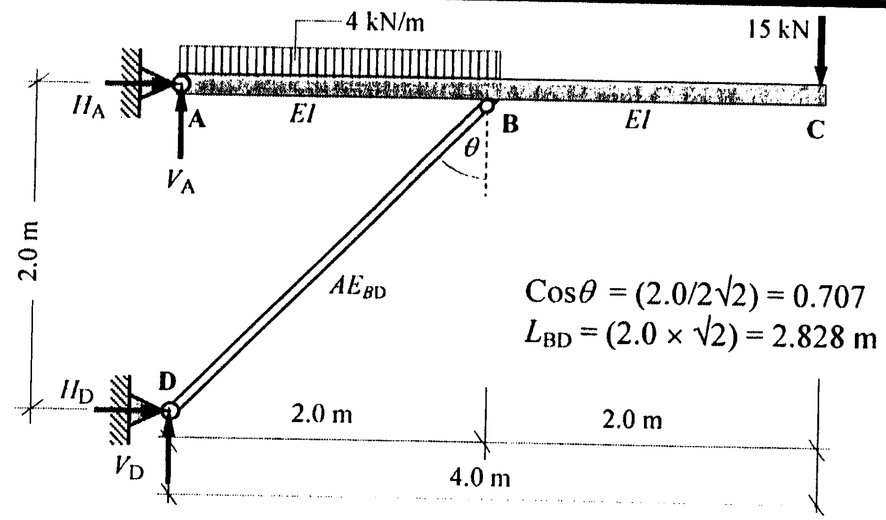
Using the coefficients given in Table 4.1:

$$\int_0^L \frac{Mm}{EI}dx = \sum_A^B (Coefficient \times a \times b \times L)/EI + \sum_C^B (Coefficient \times a \times b \times L)/2EI$$

$$= (0.333 \times 10.0 \times 2.0 \times 2.0)/EI + (0.25 \times 30.0 \times 2.0 \times 2.0)/EI$$
$$+ (0.333 \times 10.0 \times 2.0 \times 6.0)/1.5EI + (0.333 \times 30.0 \times 2.0 \times 6.0)/1.5EI$$
$$- (0.333 \times 67.55 \times 2.0 \times 6.0)/1.5EI = \mathbf{30.07/EI}$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.21 **Page No. 1**

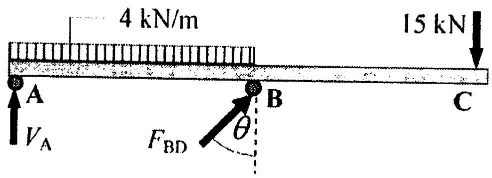


$\text{Cos}\theta = (2.0/2\sqrt{2}) = 0.707$
$L_{BD} = (2.0 \times \sqrt{2}) = 2.828$ m

Determine the value of the vertical deflection at C given:

$E_{ABC} = 205$ kN/mm^2, $I_{ABC} = 90.0 \times 10^6$ mm^4
$E_{BD} = 205$ kN/mm^2, $A_{BD} = 1500$ mm^2
$EI_{ABC} = (205 \times 90 \times 10^6)/10^6 = 18.45 \times 10^3$ kNm^2
$AE_{BD} = (1500 \times 205.0) = 307.5 \times 10^3$ kN

$$\delta_C = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{BD}$$



Consider the beam ABC:
Support Reactions

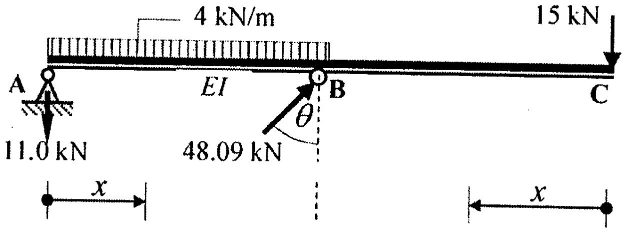
Consider the rotational equilibrium of the beam:

+ve $\circlearrowright \Sigma M_A = 0$ $+ (4.0 \times 2.0 \times 1.0) + (15.0 \times 4.0) - (F_{BD}\text{Cos}\theta \times 2.0) = 0$

$\therefore$ $F_{BD} = +$ **48.09 kN** ↗

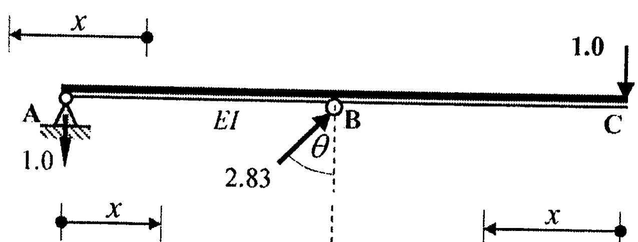
Consider the vertical equilibrium of the beam:

+ve ↑ $\Sigma F_y = 0$ $+ V_A - (4.0 \times 2.0) - 15.0 + F_{BD}\text{Cos}\theta = 0$ $\therefore$ $V_A = -$ **11.0 kN** ↓



Applied loads

Unit load



Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.21 **Page No. 2**

(Mm/EI) is not a continuous function and the product integral must be evaluated between each of the discontinuities, i.e. A to B and C to B.

$$\int_0^L \frac{Mm}{EI}dx = \int_A^B \frac{Mm}{EI}dx + \int_C^B \frac{Mm}{EI}dx$$

Consider the section from A to B: $\mathbf{0 \leq x \leq 2.0\ m}$

$M = -11.0x - 4.0x^2/2 = -11.0x - 2.0x^2$ $\qquad m = -x$

$Mm = -(-11.0x - 2.0x^2)(x) = 11.0x^2 + 2.0x^3$

$$\int_A^B \frac{Mm}{EI}dx = \int_0^2 \frac{11.0x^2 + 2.0x^3}{EI}dx = \left[\frac{11.0x^3}{3EI} + \frac{2.0x^4}{4EI}\right]_0^2 = +\frac{37.33}{EI}\text{ m}$$

Consider the section from C to B: $\mathbf{0 \leq x \leq 2.0\ m}$

$M = -15.0x$ $\qquad m = -x$ $\qquad \therefore\ Mm = +15.0x^2$

$$\int_C^B \frac{Mm}{EI}dx = \int_0^2 \frac{15.0x^2}{EI}dx = \left[\frac{15.0x^3}{3.0EI}\right]_0^2 = +\frac{40.0}{EI}\text{ m}$$

$$\int_0^L \frac{Mm}{EI}dx = \left(\frac{37.33}{EI} + \frac{40.0}{EI}\right) = +\frac{77.33}{EI} = +\frac{77.33}{18.45\times 10^3}\text{ m} = +4.19\text{ mm}$$

Consider member BD:

Applied axial load $\quad P_{BD} = -48.09$ kN $\quad$ (compression)

Unit axial load $\quad u_{BD} = -2.836$ $\quad$ (compression)

$$\sum\left(\frac{PL}{AE}u\right)_{BD} = +\left(\frac{48.09\times 2828\times 2.83}{AE_{BD}}\right) = +\left(\frac{384.88\times 10^3}{307.5\times 10^3}\right)\text{m} = +1.25\text{ mm}$$

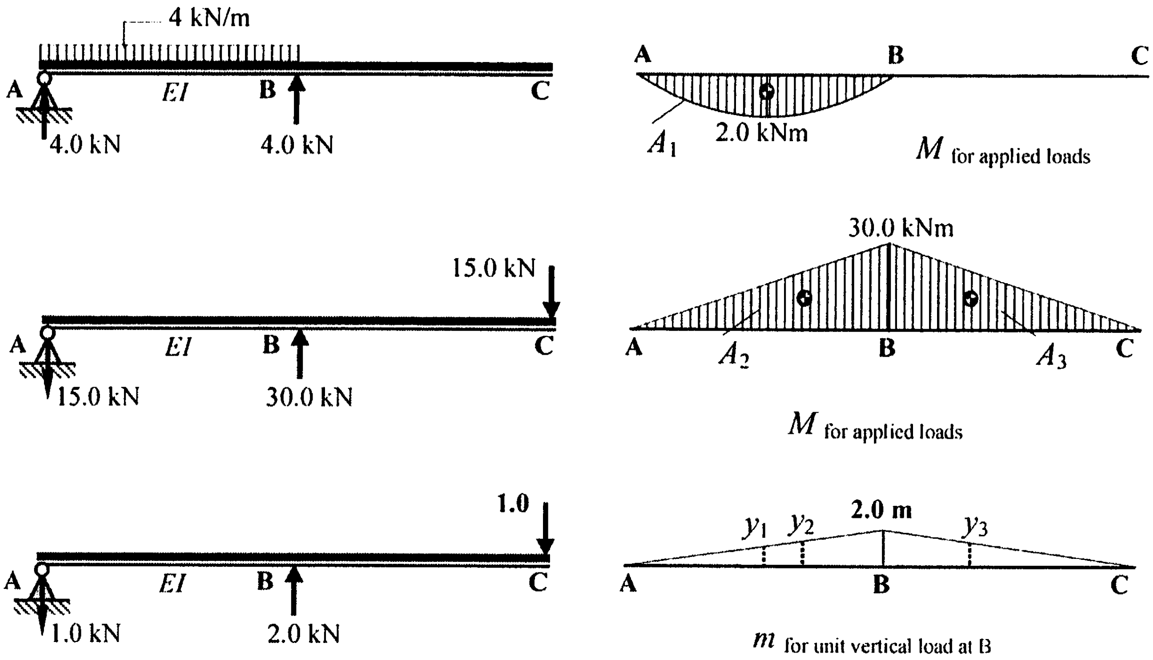
$$\delta_C = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{BD} = \mathbf{+4.19 + 1.25 = +5.44\ mm} \downarrow$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.21 **Page No. 3**

Alternatively:

$\delta_C = \Sigma(\text{Area}_{\text{applied bending moment diagram}} \times \text{Ordinate}_{\text{unit load bending moment diagram}})$



$A_1 = +\,(0.667 \times 2.0 \times 2.0)\ \text{kNm}^2,$ $\quad y_1 = -\,1.0\ \text{m},$ $\quad \therefore\ A_1y_1 = -\,2.67\ \text{kNm}^3$

$A_2 = -\,(0.5 \times 2.0 \times 30.0)\ \text{kNm}^2,$ $\quad y_2 = -\,1.33\ \text{m},$ $\quad \therefore\ A_2y_2 = +\,40.0\ \text{kNm}^3$

$A_3 = -\,0.5 \times 2.0 \times 30.0)\ \text{kNm}^2,$ $\quad y_3 = -\,1.33\ \text{m},$ $\quad \therefore\ A_3y_3 = +\,40.0\ \text{kNm}^3$

$$\int_0^L \frac{Mm}{EI}dx = (-\,2.67 + 40.0 + 40.0)/EI = \mathbf{77.33/EI}$$

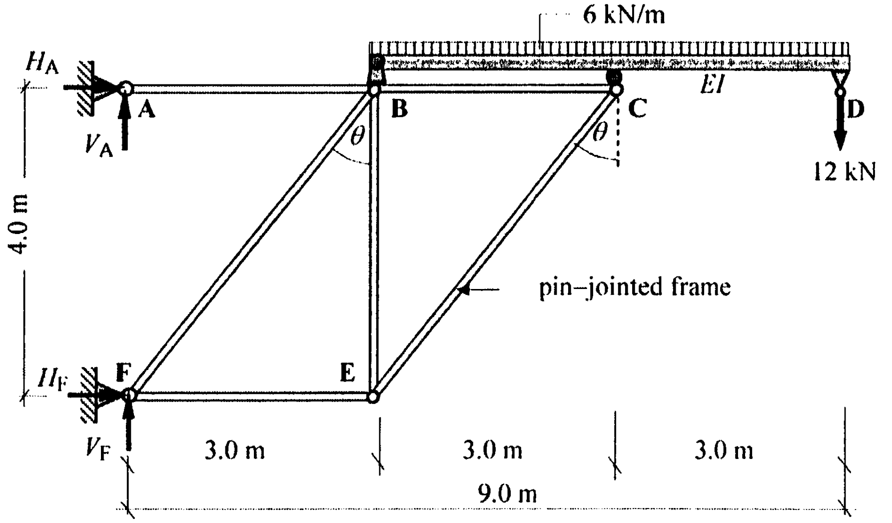
Using the coefficients given in Table 4.1:

$$\int_0^L \frac{Mm}{EI}dx = \sum_0^L \left(Coefficient \times a \times b \times L\right)/\,EI$$

$$\int_0^L \frac{Mm}{EI}dx = [-\,(0.333 \times 2.0 \times 2.0 \times 2.0) + (0.333 \times 30.0 \times 2.0 \times 4.0)]/EI = \mathbf{77.33/EI}$$

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.22 **Page No. 1**



Determine the value of the vertical deflection at D given:

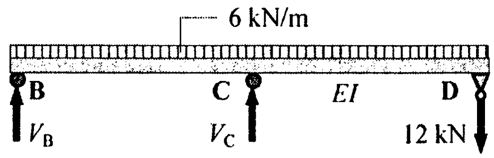
$E_{beam} = 205$ kN/mm^2, $I_{beam} = 500.0 \times 10^6$ mm^4

$E_{\text{All frame members}} = 205$ kN/mm^2, $A_{\text{All frame members}} = 4000$ mm^2

$EI_{BCD} = (205 \times 500 \times 10^6)/10^6 = 102.5 \times 10^3$ kNm^2

$AE = (4000 \times 205.0) = 820 \times 10^3$ kN

$$\delta_D = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{\text{All frame members}}$$



Consider the beam BCD:
Support Reactions

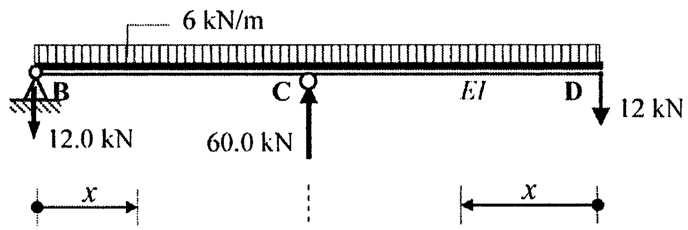
Consider the rotational equilibrium of the beam:

+ve $\sum M_B = 0$ $\quad + (6.0 \times 6.0 \times 3.0) + (12.0 \times 6.0) - (V_C \times 3.0) = 0$

$\therefore$ V_C **= + 60.0 kN** ↑

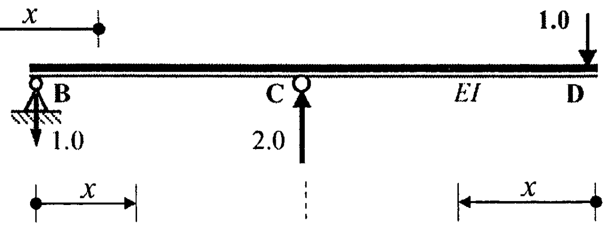
Consider the vertical equilibrium of the beam:

+ve ↑ $\sum F_y = 0$ $\quad + V_B - (6.0 \times 6.0) - 12.0 + V_C = 0$ $\quad \therefore$ V_B **= – 12.0 kN** ↓



Applied loads

Unit load



Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.22 **Page No. 2**

(Mm/EI) is not a continuous function and the product integral must be evaluated between each of the discontinuities, i.e. B to C and D to C.

$$\int_0^L \frac{Mm}{EI}dx = \int_B^C \frac{Mm}{EI}dx + \int_D^C \frac{Mm}{EI}dx$$

Consider the section from B to C: $\mathbf{0 \le x \le 3.0}$ **m**

$M = -12.0x - 6.0x^2/2 = -12.0x - 3.0x^2$ $\qquad m = -x$

$Mm = -(-12.0x - 3.0x^2)(x) = 12.0x^2 + 3.0x^3$

$$\int_B^C \frac{Mm}{EI}dx = \int_0^3 \frac{12.0x^2 + 3.0x^3}{EI}dx = \left[\frac{12.0x^3}{3EI} + \frac{3.0x^4}{4EI}\right]_0^3 = +\frac{168.75}{EI}\text{ m}$$

Consider the section from D to C: $\mathbf{0 \le x \le 3.0}$ **m**

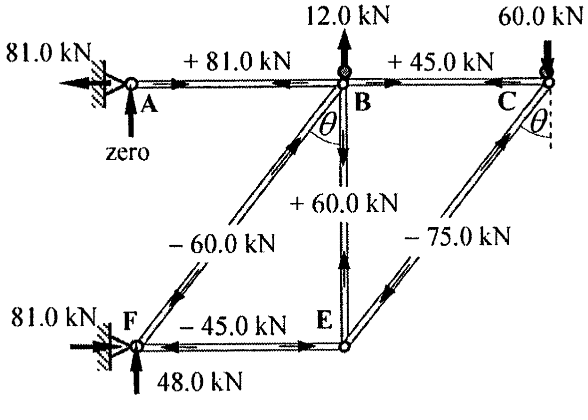
$M = -12.0x - 6.0x^2/2 = -12.0x - 3.0x^2$ $\qquad m = -x$

$$\int_D^C \frac{Mm}{EI}dx = \int_0^3 \frac{12.0x^2 + 3.0x^3}{EI}dx = \left[\frac{12.0x^3}{3EI} + \frac{3.0x^4}{4EI}\right]_0^3 = +\frac{168.75}{EI}\text{ m}$$

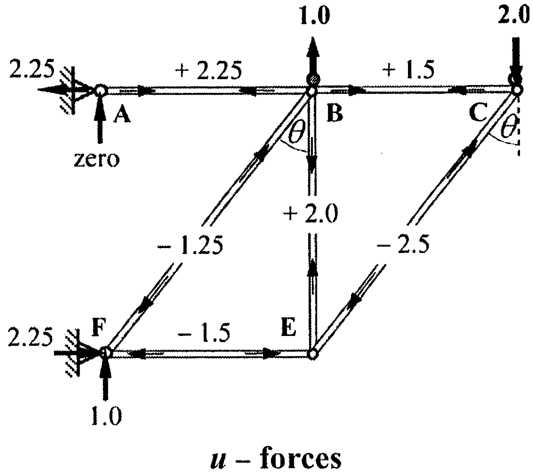
$$\int_0^L \frac{Mm}{EI}dx = \left(\frac{168.75}{EI} + \frac{168.75}{EI}\right) = +\frac{337.5}{EI} = +\frac{337.5}{102.5\times10^3}\text{ m} = +3.29\text{ mm}$$

Consider the pin-jointed frame:

The applied load axial effects (P-forces)and the unit load axial effects (u-forces) can be determined using joint resolution and/or the method of sections as indicated in Chapter 3.



***P* – forces**



***u* – forces**

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.22 **Page No. 3**

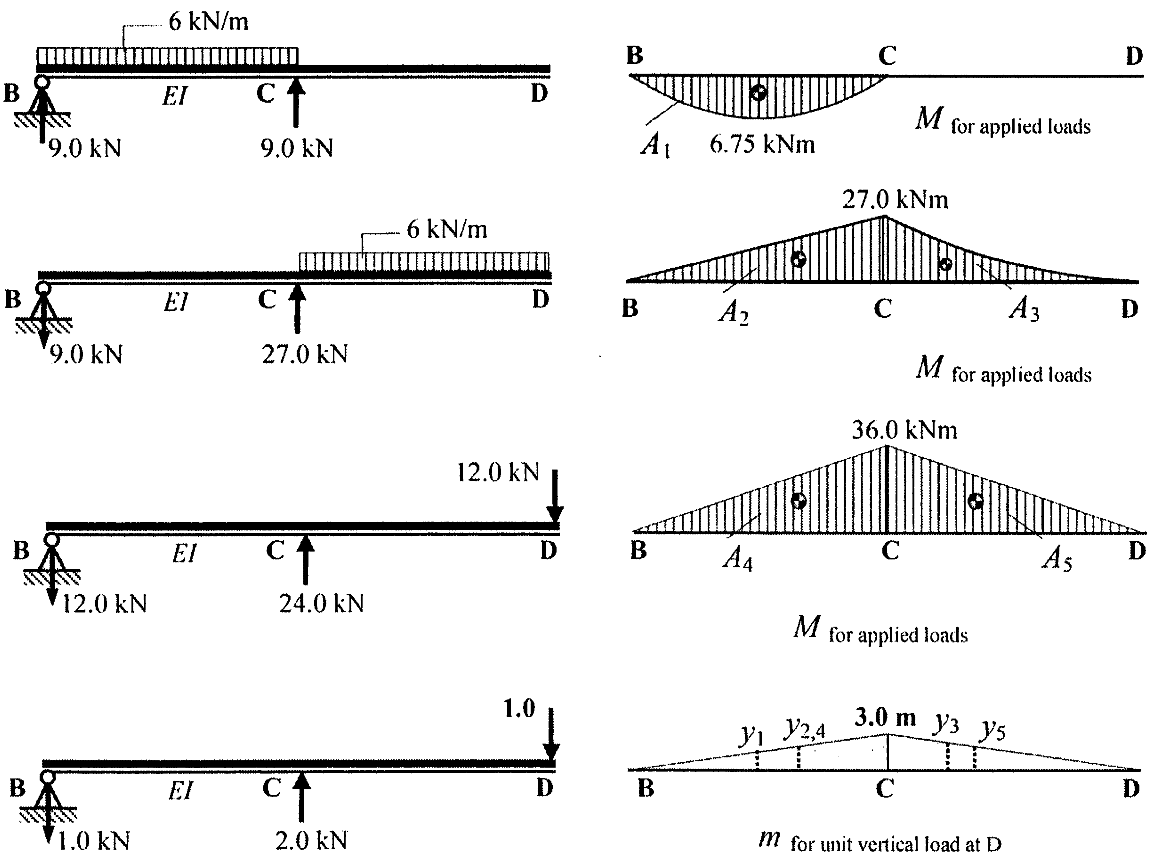
Member	Length (mm)	*AE* (kN)	*P*-force (kN)	*PL/AE* (mm)	*u*	(*PL/AE*) × *u*
AB	3000	820.0×10^3	+ 81.0	+ 0.30	+ 2.25	**+ 0.68**
BC	3000	820.0×10^3	+ 45.0	+ 0.16	+ 1.50	**+ 0.24**
BE	5000	820.0×10^3	– 60.0	– 0.37	– 1.25	**+ 0.46**
BD	4000	820.0×10^3	+ 60.0	+ 0.29	+ 2.00	**+ 0.58**
CD	5000	820.0×10^3	– 75.0	– 0.46	– 2.50	**+ 1.15**
DE	3000	820.0×10^3	– 45.0	– 0.16	– 1.50	**+ 0.24**
						Σ = + 3.35

$$\sum\left(\frac{PL}{AE}u\right)_{\text{All frame members}} = 3.35 \text{ mm}$$

$$\delta_D = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{\text{All frame members}} = \mathbf{+3.29 + 3.35 = +6.64 \text{ mm}} \downarrow$$

Alternatively:

$\delta_D = \Sigma(\text{Area}_{\text{ applied bending moment diagram}} \times \text{Ordinate}_{\text{ unit load bending moment diagram}})$



Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.22 **Page No. 4**

$A_1 = +(0.667 \times 3.0 \times 6.75)$ kNm², $y_1 = -1.5$ m, $\therefore A_1y_1 = -20.26$ kNm³
$A_2 = -(0.5 \times 3.0 \times 27.0)$ kNm², $y_2 = -2.0$ m, $\therefore A_2y_2 = +81.0$ kNm³
$A_3 = -(0.333 \times 3.0 \times 27.0)$ kNm², $y_3 = -2.25$ m, $\therefore A_3y_3 = +60.69$ kNm³
$A_4 = -(0.5 \times 3.0 \times 36.0)$ kNm², $y_4 = -2.0$ m, $\therefore A_4y_4 = +108.0$ kNm³
$A_5 = -(0.5 \times 3.0 \times 36.0)$ kNm², $y_5 = -2.0$ m, $\therefore A_5y_5 = +108.0$ kNm³

$$\int_0^L \frac{Mm}{EI}dx = (-20.26 + 81.0 + 60.69 + 108.0 + 108.0)/EI = \mathbf{337.43/EI}$$

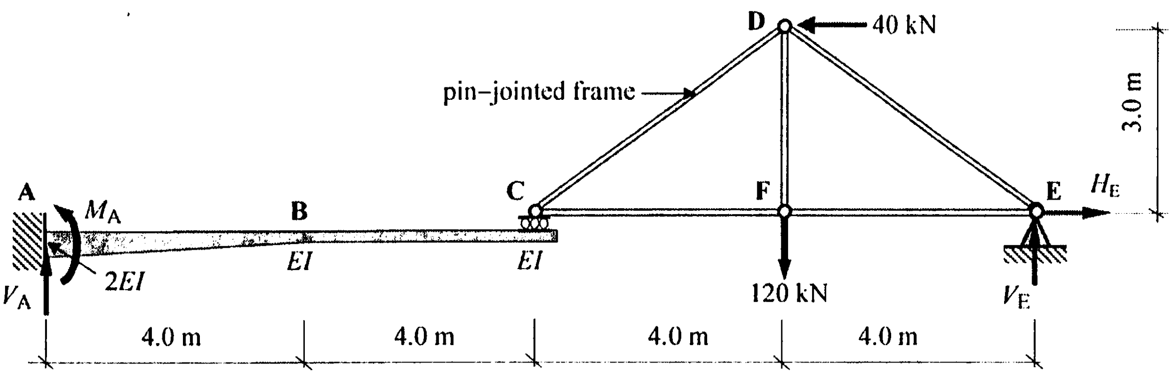
Using the coefficients given in Table 4.1:

$$\int_0^L \frac{Mm}{EI}dx = \sum_0^L (Coefficient \times a \times b \times L)/EI$$

$$\int_0^L \frac{Mm}{EI}dx = [-(0.333 \times 6.75 \times 3.0 \times 3.0) + (0.333 \times 27.0 \times 3.0 \times 3.0)$$
$$+ (0.25 \times 27.0 \times 3.0 \times 3.0) + (0.333 \times 36.0 \times 3.0 \times 6.0)]/EI = \mathbf{337.22/EI}$$

Solution

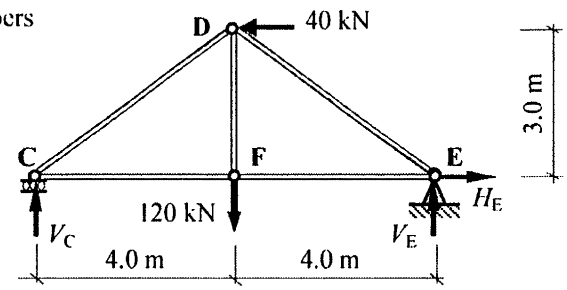
Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.23 **Page No. 1**



The EI value of the cantilever ABC varies linearly from $2EI$ at the fixed support to EI at B and is constant from B to C.
Determine the value of the vertical deflection at F and at C given:
$EI_{\text{cantilever ABC}} = 1080 \times 10^3\ \text{kNm}^2$, $EA_{\text{All frame members}} = 300 \times 10^3\ \text{kN}$

$$\delta_F = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{\text{All frame members}}$$

Consider the pin-jointed frame:



Support Reactions

Consider the rotational equilibrium of the frame:

+ve $\circlearrowright$ $\Sigma M_C = 0$ $+ (120.0 \times 4.0) - (40.0 \times 3.0) - (V_E \times 8.0) = 0$ $\therefore$ V_E **= + 45.0 kN** ↑

Consider the vertical equilibrium of the frame:

+ve ↑ $\Sigma F_y = 0$ $+ V_C - 120.0 + V_E = 0$ $\therefore$ V_C **= + 75.0 kN** ↑

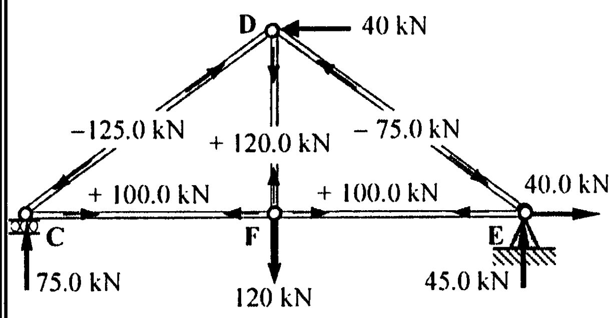
Consider the horizontal equilibrium of the frame:

+ve → $\Sigma F_x = 0$ $- 40.0 + H_E = 0$ $\therefore$ H_E **= + 40.0 kN** →

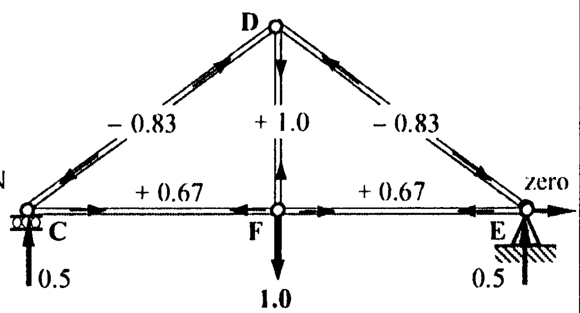
The applied load axial effects (P–forces) and the unit load axial effects (u–forces) can be determined using joint resolution and/or the method of sections as indicated in Chapter 3.

Solution

Topic: Determinate Beams/Frames – Deflection Using Unit Load
Problem Number: 4.23 **Page No. 2**



***P* – forces**

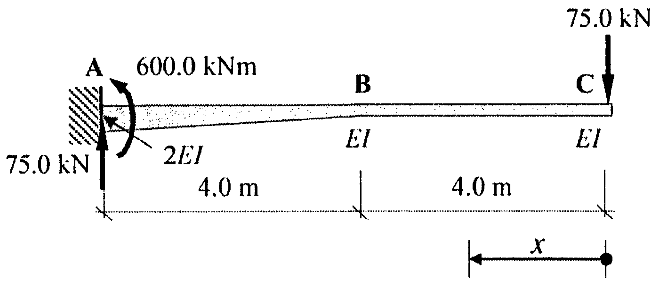


***u* – forces**

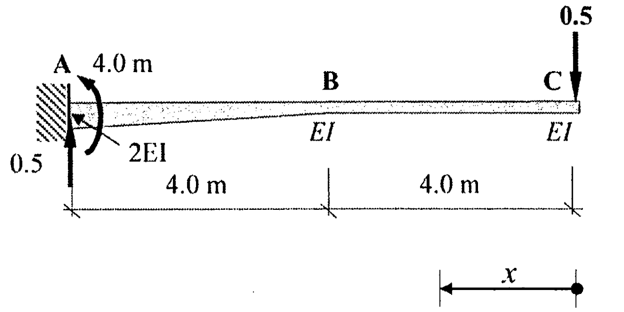
Member	Length (mm)	*AE* (kN)	*P*-force (kN)	*PL/AE* (mm)	*u*	(*PL/AE*) × *u*
CD	5000	300.0×10^3	− 125.0	− 2.08	− 0.83	+ 1.73
CF	4000	300.0×10^3	+ 100.0	+ 1.33	+ 0.67	+ 0.89
DF	3000	300.0×10^3	+ 120.0	+ 1.20	+ 1.0	+ 1.20
DE	5000	300.0×10^3	− 75.0	− 1.25	− 0.83	+ 1.04
EF	4000	300.0×10^3	+ 100.0	+ 1.33	+ 0.67	+ 0.89
						Σ = + 5.75

$$\sum\left(\frac{PL}{AE}u\right)_{\text{All frame members}} = 5.75 \text{ mm}$$

Consider the beam ABC:



M for applied loads



m for unit vertical load at F

Solution

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(Mm/EI) is not a continuous function and the product integral must be evaluated between each of the discontinuities, i.e. C to B and B to A.

The value of EI at position 'x' along the beam between B and A is given by:
$EI + EI\,[(x - 4.0)/4] = 0.25EI\,x$

$$\int_0^L \frac{Mm}{EI}dx = \int_C^B \frac{Mm}{EI}dx + \int_B^A \frac{Mm}{0.25EIx}dx = \int_C^B \frac{Mm}{EI}dx + \frac{4.0}{EI}\int_B^A \frac{Mm}{x}dx$$

Consider the section from C to B: $\mathbf{0 \leq x \leq 4.0\ m}$

$M = -75.0x \qquad m = -0.5x \qquad \therefore\ Mm = +37.5x^2$

$$\int_C^B \frac{Mm}{EI}dx = \int_0^4 \frac{37.5x^2}{EI}dx = \left[\frac{37.5x^3}{3EI}\right]_0^4 = +\frac{800}{EI}\ \text{m}$$

Consider the section from B to A: $\mathbf{4.0 \leq x \leq 8.0\ m}$

$M = -75.0x \qquad m = -0.5x \qquad \therefore\ Mm = +37.5x^2$

$$\frac{4.0}{EI}\int_B^A \frac{Mm}{x}dx = \frac{4.0}{EI}\int_{4.0}^{8.0} \frac{37.5x^2}{x}dx = \frac{150.0}{EI}\int_{4.0}^{8.0} x\,dx = \left[\frac{150.0x^2}{2EI}\right]_4^8 = +\frac{3600}{EI}\ \text{m}$$

$$\int_0^L \frac{Mm}{EI}dx = \frac{800}{EI} + \frac{3600}{EI} = \frac{4400}{EI} = \frac{4400}{1080 \times 10^3}\ \text{m} = +4.07\ \text{mm}$$

$$\delta_F = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{\text{All frame members}} = \mathbf{+4.07 + 5.75 = +9.82\ mm} \downarrow$$

Vertical deflection at C:
In this case when a unit load is applied at point C all of the u-forces for the pin-jointed frame are equal to zero.

$$\delta_C = \int_0^L \frac{Mm}{EI}dx + \sum\left(\frac{PL}{AE}u\right)_{\text{All frame members}} \text{ (zero)} \qquad \therefore\ \delta_C = \int_0^L \frac{Mm}{EI}dx$$

$M = -75.0x \qquad m = -x \qquad \therefore\ Mm = +75.0x^2$

$$\int_0^L \frac{Mm}{EI}dx = \int_0^4 \frac{75.0x^2}{EI}dx + \frac{4.0}{EI}\int_{4.0}^{8.0} \frac{75.0x^2}{x}dx = \frac{1600}{EI} + \frac{7200}{EI} = \frac{8800}{EI}$$

$$\delta_C = \frac{8800}{1080 \times 10^3}\ \text{m} = \mathbf{+8.15\ mm} \downarrow$$