STATICALLY INDETERMINATE STRUCTURES

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FOREWORD TO REISSUE

These notes were prepared and used by Hardy Cross during the period of his service as Professor of Structural Engineering at the University of Illinois, 1921-1937. Since 1937 Professor Cross has been Chairman of the Department of Civil Engineering at Yale University.

Although the original supply of these notes was exhausted many years ago, the writer has continued to refer the students in his classes to the library copies. This has resulted in an increasing number of inquiries about the possibility of securing personal copies. Students quickly discover that the viewpoints expressed in the notes are more suggestive and the methods of presentation clearer than in more generally available material.

As a result of pressure that could not be resisted or diverted the writer requested from Professor Cross permission for reproduction of the notes as originally presented. That permission has been granted although in his letter of authority to do so Professor Cross said: "You of course realize that the book is not exactly as I should like to see it. Some parts I certainly would now state very differently."

The writer and his students sincerely hope that Professor Cross will find the time to issue the book with all parts stated as he would state them now and exactly in the form he would "like to see it." In the meantime they wish to take this opportunity to express their gratitude for the permission for this reissue of the notes in their original text.

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August 30, 1950

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INTRODUCTION

What is an Indeterminate Structure? Indeterminate structures are those in which either the reaction or the internal stresses cannot be found by the laws of statics alone. They may be divided into several classes.

(a) Standard types indeterminate as to reactions and commonly solved as indeterminate structures. Continuous bridge girders and trusses including swing bridges and continuous turntables, hingeless and two-hinged arches, most suspension bridges.

(b) More common types in which the external indetermination is often neglected or only approximately allowed for. Mill-building bents, elevated-railway bents, portals and similar structures, building frames of reinforced-concrete and to a less extent those of steel.

(c) Trusses which are statically indeterminate as regards the internal stresses. These are not now of frequent occurrence in this country and are usually solved by the use of approximate methods. Whipple trusses and other trusses with multiple cancellation systems.

(d) Slabs and ribbed slabs in which the main problem is to determine the variations of the shears and bending moments. Flat slabs, slabs supported on four sides, slabs with concentrated loads. An exact solution is quite complicated, but the laws of statics and a few general principles will go far toward a satisfactory design.

(e) Common problems of internal indetermination solved by commonly accepted approximations. Internal stresses in beams.

(f) Special problems of internal stress and especially of localized stress. Complex solutions.

What Principles are Fundamental in its Analysis? One sees at times this or that principle stated as a fundamental of indeterminate structures. In reality there can be only one fundamental principle of indeterminate structures, and that is the fact of continuity. If the fibre stresses in a beam can be computed for given moments and the modulus of elasticity be known, the angle changes resulting from these moments follow as a matter of geometry, and the relations of these angle changes to other angle changes necessary to preserve continuity follow also as a matter of geometry, and similar statements apply to effects of shear and direct stress. But the theory of fibre stress from known moments is not a fundamental of indeterminate structures, and the value of the modulus of elasticity is a fact associated with a definition and not a principle at all, and it is difficult to say what can be considered a fundamental of geometry. All of the principles usually referred to as fundamentals of indeterminate structures- $\frac{d^2y}{dx^2} = -\frac{M}{EI}$, the equation of three moments, $\int M\frac{ds}{EI} = 0$ in fixed beams, $M = 2EK(2\phi_a + \phi_b - 3\psi)$ in slope deflection-are merely deductions from the fact of continuity, based on assumptions-usually on an invariable value of E and on the validity of the beam formula.

What is Theory? It is perhaps worth while to call attention to the double use of the word "theory" in scientific discussions. In some cases it is used to mean a body or group of facts the truth of which is not questioned, in others it means a hypothesis which has strong evidence in its favor through its truth is still open to some question. Thus the theory of elasticity is a group of geometrical relations which are not open to debate, but the idea that time yield of the concrete will delay failure from temperature stresses in a concrete arch is a theory in quite a different sense. Other debatable points in indeterminate structures are not theories at all, but merely convenient assumptions; thus no one holds any theory that the modulus of elasticity is constant throughout an arch ring, the only question being whether such variations as do occur produce any important effect on the results.

Much confusion of thought has come from misuse of this term. We may further cite: as groups of facts not open to experimentation or debate, the theory of the elastic arch, the theory of continuous girders, the theory of deflection; as hypotheses strongly supported but as yet not fully proved, the theories of fatigue failure, the theory of earth pressure, the theory that the strength of concrete in a structure is the same as that shown by a cylinder in a testing machine or that rate of application of load is a negligible factor in producing failure; and finally, as misuses of the word, the "theory" that the moment of inertia of a concrete beam varies as bd^3, that the tension rods do not slip in concrete beams, that there is no distortion due to shear. The first group of "theories" is not debatable, the second depend usually on experimental verification, while in the case of the third the important question is, how significant is the error. The data often needed in the third group are elementary; when these are available, deductive processes furnish a definite answer as to the importance of the error.

Practical Application of the Theory. Reluctance to apply the theory of elasticity to structural design is based on two objections, one that it is too arduous to be justified by its results, the other that it is not sufficiently flexible to permit evaluation of the effect of such factors as variable E, brackets, gusset plates, imperfect elasticity and phenomena beyond the yield point. If the latter objection holds, the former becomes important since we have only our labor for our pains. In many cases the objection is valid, and this tool of design is useful only in proportion as it permits the evaluation of the effect of physical uncertainties. By the methods presented in this text, such uncertainties can be included in the analysis of plane continuous frames. This is a fruitful field of research. In some cases large uncertainties in the data may be shown to produce small variations in the results. In other problems, notably in the case of deformation constants in arch foundations, the uncertainties may so seriously affect the results as to make precise computations illusory.

Geometrical Relations. The study of statically indeterminate structures will have at its foundation two elementary fundamental conceptions.

1. Any point at rest is in static equilibrium.
2. Any line that is continuous preserves its elastic properties.

From these two simple basic ideas follow the laws of statics and the laws of continuity. Statics gives us the three conditions of equilibrium of non-concurrent forces: $\sum H = 0$, $\sum V = 0$ and $\sum M = 0$. From continuity follow simple geometric relations between the displacements and angle changes in the structure.

It should be emphasized that the principles of geometry and the most elementary calculus together with the laws of statics are sufficient for solving all the ordinary problems in indeterminate structures that occur in practice. It is of vital importance in the solution of such problems to be able to visualize the action of the structure under load. A qualitative study of the probable action of the structure will invariably lead to the quickest estimate of the quantities involved. A qualitative sketching of influence lines and of the approximate shape of the deformed structure will not only make it possible to avoid certain unnecessary lines of investigation as irrelevant to the problem in hand but will in many cases reveal the critical point or points to be investigated and usually the critical loading involved.

Certainly the first thing and perhaps the most important thing to be considered in the design of any structure is the determination of what needs to be figured - what is worth while. Associated with this question is that of the precision that is desirable as well as that which it may be practicable to obtain.

Limitations - Physical and Mathematical. The designer will need to keep clearly in mind two entirely independent types of limitations:

1. Limitations of a physical nature.
2. Limitations of a mathematical nature.

These two are sometimes confused with annoying consequences. It is worth while at all times to realize the fact that such things as moment, shear, stress, moment of inertia, etc., are merely definitions--convenient mathematical expressions, not physical phenomena. Loads are not transmitted nor are stresses carried by this member or that, and while such expressions may be justified by their convenience, it is important that our mental picture be accurate and that we do not lose sight of the actual geometry of the problem.

Physical limitations such as the action of rivets and the effect of gusset plates in steel design, the effect of spandrel columns, of T-slabs, time yield and variations in E in concrete, all such, need to be distinguished in our thinking from limitations of a purely mathematical nature. Each will affect the significance of the numerical result but in a different way.

General Method of Analysis. The common method of attacking problems in indeterminate structures has been based on the general idea of making the structure statically determinate by cutting redundant members at points of continuity, and then tying these static groups together by the principles of continuity solving these resulting groups of equations simultaneously for the various unknowns. The method outlined in this text is usually the reverse; preserving at all times the continuity, distributing the unbalanced moments according to the laws of that continuity, and finally adjusting as need be, to conform to the laws of statics.

Three Important Principles. The method here employed of attacking indeterminate structures involves three easily established principles which may be summarized as follows:

1. Column Analogy. If any restrained elastic ring (fixed beam, bent, arch, etc.) is treated in its outline as the cross-section of an analogous column whose differential areas are the ds/EI values of the ring, the bending moments due to continuity at any point are analogous to the corresponding fibre stresses in the column due to the angle changes as loads.

2. Distribution of Moments and other forces. If the fixed ended moments are computed for any joint the unbalanced moments may be distributed among the bars in proportion to their stiffness, and a portion carried over to the other end. This involves the distribution of a succession of unbalanced moments, but the method is quite general in its application and the series commonly converges rapidly to any degree of accuracy desired. In its general application it will be seen to include, with slight modification, the distribution of joint rotations, displacements and shears, as well as the distribution of moments.

3. Virtual Work. Displacement, linear or angular, at any point in a loaded structure is equal to the virtual internal work - this being described as the sum of the products of the existing distortions and the virtual resistances to a unit coincident force of displacement. The reactions

to this hypothetical force of displacement will fix the reference by which the displacement may be measured. The direct application of these principles coupled with the laws of statics affords an immediate solution of most of the ordinary problems in indeterminate structures. Variable moment of inertia offers no peculiar difficulty. Signs may be readily taken care of by the usual conventions. Indeed in many cases they may be determined by inspection at the end of the solution. The most common language of the engineer - moments, shears, stresses, constitutes the greater part of the work. The real action of the structure is readily visualized and limitations, mathematical and physical, become apparent. Close approximate results may be obtained by the same processes which if carried further will give any degree of precision.

CHAPTER I

THE COLUMN ANALOGY

General Theory. The column analogy is essentially a theorem for finding the indeterminate moments, - i.e., the moments due to continuity - in a restrained beam, straight or curved. It might be defined as a conception wherein these indeterminate moments become analogous to fibre stresses in a column, one advantage of the analogy being that the language and habit of thought of the engineer is preserved and the necessity for formulae reduced. Primarily it is useful in finding fixed ended moments, particularly as these are treated in this text as a convenient starting point in the analysis of the unknown forces at any joint. The conception of the "column" in the theorem forms the basis for a convenient mental picture in the process, despite the fact that there is no physical similarity.

The angle change produced in any short length of the axis of a beam by a given bending moment in this length can usually be definitely, or at least proportionately, predicted. That is, it can be predicted definitely that a given section, say one inch long, subject to a given bending moment, will have twice or three times the angle change of another section subject to a different bending moment. It is the relative angle change and not its absolute value (a small quantity in radians) that we are usually interested in. But the angle change is a function of the moment and of the physical properties of the section. Hence it is the relative and not the absolute values of such constants as modulus of elasticity and moment of inertia that are needed except where absolute displacements are required, which is not the usual problem.

If sections plane before bending remain plane after bending, then the angle changes (Fig. 1) can be visualized as a geometrical consequence of the shortening of the fibres on one side of the neutral axis and the lengthening on the other. Then since the change in the length of any fibre over a section of the beam of length ds will be $\frac{f}{E}$ ds, from the geometry of small angles the value of $d\phi$ will be $d\phi = \frac{f}{Ey}$ ds where y is the distance of the fibre from the neutral axis. Of if $f/y = M/I$ we may also write $d\phi = \frac{M}{EI}$ ds.

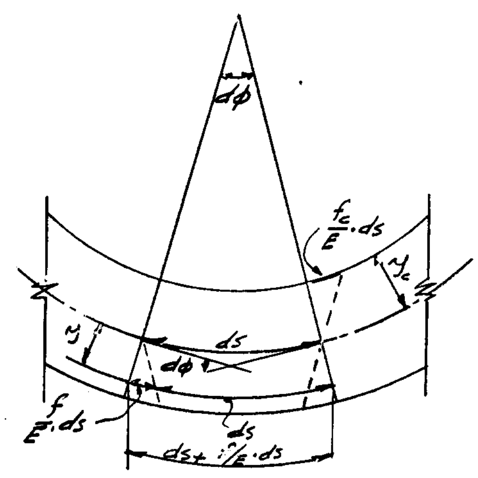


Fig. 1

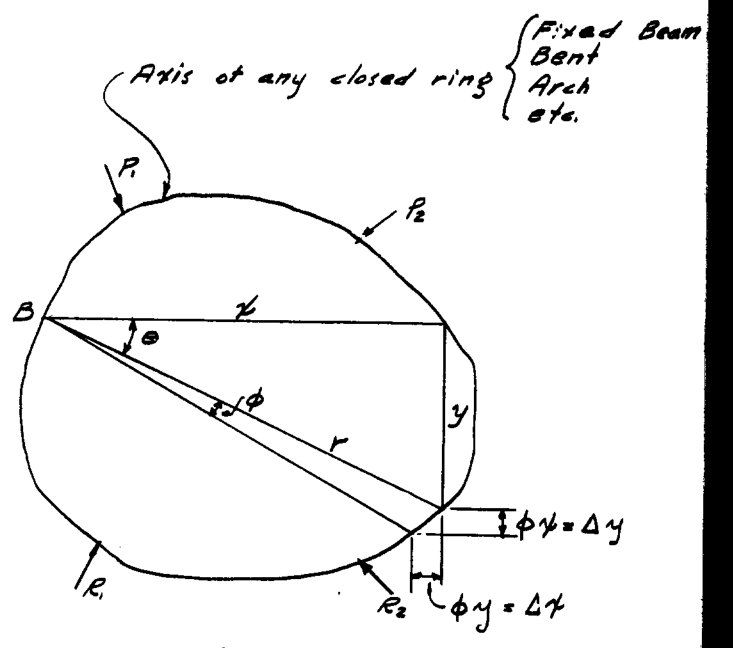


Fig. 2

The theorem which is conveniently described by the term column analogy may be presented in its simplest and most useful form as applying to any single closed ring.

Relations of the Angle Changes. Consider the axis of any closed elastic ring as shown (Fig. 2) acted on by some system of external forces. Imagine the ring cut at any point A. Then any angle change occurring in a short length of the axis at any point B will produce at A -

(a) A relative rotation of the cut ends ϕ
(b) A relative vertical movement of the cut ends ϕx
(c) A relative horizontal movement of the cut ends ϕy

Since no relative movement of these ends actually occurs in the structure it follows that, if distortions due to shear be neglected, -

(a) $\sum\phi = 0$, (b) $\sum\phi x = 0$, (c) $\sum\phi y = 0$

If the angle changes within the ring be considered as loads on the axis normal to the plane of the paper, these conditions correspond to the three conditions of static equilibrium for such forces, -

(a) $\sum V = 0$, (b) $\sum M_x = 0$, (c) $\sum M_y = 0$

Further suppose these angle changes to be made up of two parts, ϕ_s and ϕ_i, ϕ_s (static) being assumed to be known and ϕ_i (indeterminate) to be produced by the continuity of the ring. If the angle changes ϕ_s are produced by loads, then ϕ_s may be determined as $\phi_s = m_s\, ds/EI = m_s\, w$, where m_s is the bending moment on any elastic weight ($ds/EI = w$) for any set of bending moments statically consistent with the given loading.

The indeterminate moments m_i will now be a result of the internal stresses - a moment, a shear, and a thrust - at the cut section and if plotted at the proper point on the axis and normal to the paper they will obviously have planar distribution. Evidently then if the known angle changes ϕ_s be considered as loads normal to the paper on a column section wholse plan has the shape of the axis of the ring and the differential areas of which $ds/EI = w$, then, -

(a) the indeterminate angle changes $\phi_i = m_i w$ will correspond to total stresses in this column section, and

(b) the indeterminate moments m_i will correspond to fibre stresses in this column section. Positive values of m_i will then represent compression on the section, and since they are opposed to the load the total moment at any point will be the algebraic difference between m_i and the static moment m_s, i.e., $M = m_s - m_i$.

Signs and Other Relations. The column analogy holds strictly with regard to signs for moments due to loads, or due to angular distortions. The angle changes in the unrestrained ring are the loads, those which produce tension on the inside of the ring being positive. Positive load produces positive stress (indeterminate moment) at the centroid and the sign elsewhere is determined by the location of the point of stress just as in any other column. The indeterminate moment is to be subtracted (algebraically) from the static moment at any point. For lineal distortions such as shrinkage or settlement of abutments of arches, the signs of the moments are best determined by inspection.

Usually the known angle changes ϕ_s will be computed from known bending moments. Any angle change due to any cause, however, may be treated as a load on the section. Moreover, the value $\sum\phi x$ is a vertical movement of the imaginary cut ends and corresponds to a moment about the vertical axis. And in general any lineal distortion within the ring corresponds to a bending moment about an axis parallel to the line of distortion.

One side of the ring may be and usually is the earth which is infinitely stiff and hence has an elastic weight of zero. Since the elastic weights which constitute the differential areas of the analogous column are the angle changes produced in any length of the beam by a unit couple, the elastic weight of a hinge is infinite.

The column analogy then is primarily a device for computing the bending moments in any single span of a beam either straight, broken or curved. The solution of unsymmetrical cases follows directly when reference is made to the principal axes of the column section. It is particularly useful in the solution of arches and beams of variable moment of inertia within the span where it has direct application in finding moments, shears and thrusts, and in the construction of influence lines. A few simple cases will illustrate its application.

Fixed Beam Any Type of Loading. Let it be required to compute the end moments on the fixed ended beam shown, Fig. 3, assuming constant moment of inertia. Consider the moment curve shown, produced by the load P on a beam simply supported at its ends. Any static moment curve such as a cantilever over length a or b might equally well have been used. The centroid of any triangle or any pair of triangles having a common base, using the notation of the figure may be shown to lie at a distance $\frac{L+a}{3}$ from one end.

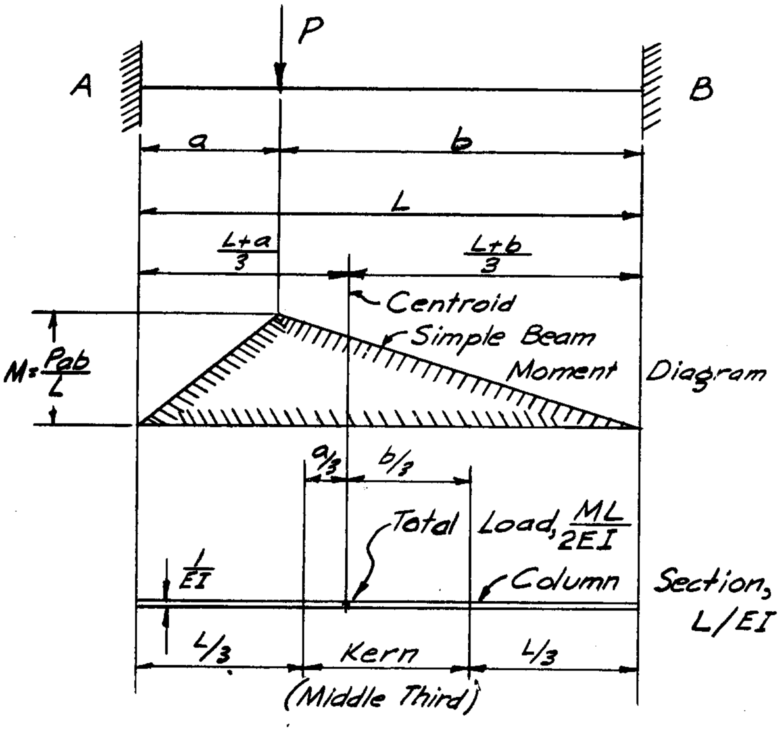


FIG. 3

In this case then the analogous column section is a narrow strip of length L and width L/EI. Both E and I being assumed constant in this case they will have no effect on the moments and may be given a relative value of unity. The column section thus becomes simply L and the load ML/2. The outer fibre stresses in the column, analogous to the end moments, may be found by the usual column formula or what is more convenient in this case by taking moments about the kern points. That is, $-f = P/A \pm My/I = M_k y/I_o$ where M_k is the moment about the opposite kern point and I_o is the moment of inertia of the column about its centroid.

Further, for rectangular columns, $-f = \frac{M_k \cdot y}{I_o} = \frac{M_k \cdot d/2}{1/12\, bd^3} = \frac{6 M_k}{Ad}$

Then at the left end, $f_a = M_a = \dfrac{6\,\frac{ML}{2}\cdot\frac{b}{3}}{L\cdot L} = M\,\dfrac{b}{L}$

And at the right end, $f_b = M_b$ - - - - - - - - - - $= M\,\dfrac{a}{L}$ where M is the simple beam moment, $M = Pab/L$.

With the aid of very simple integration this formula is sufficient for the determination of fixed-ended moments on prismatic beams for all conditions of loading. For a combined or multiple loading the separate moments may be obtained for each loading and added together. Applying this formula to the case of uniform load from one support to the center line, Fig. 4,

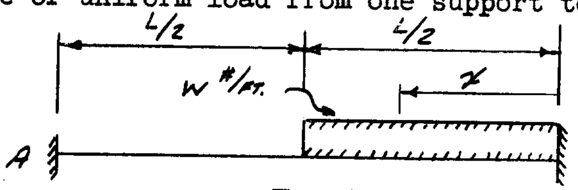


FIG.4

$$M_a = \int_0^{L/2} w\,dx\,\frac{x(L-x)}{L}\cdot\frac{x}{L} = \frac{w}{L^2}\int_0^{L/2}(Lx^2 - x^3)\,dx = \frac{5}{192}wL^2$$

$$M_b = \int_0^{L/2} w\,dx\,\frac{x(L-x)}{L}\cdot\frac{L-x}{L} = \frac{w}{L^2}\int_0^{L/2}(L^2x - 2Lx^2 + x^3)\,dx = \frac{11}{192}wL^2$$

For full uniform load the end moments may be found as above or more easily by the direct application of the column analogy. In this case the moment diagram is a parabola and the total load $2/3 \cdot 1/8\,WL^2 \cdot L$

Whence $f = M = P/A = \dfrac{2/3\cdot 1/8\,WL^3}{L} = \dfrac{WL^2}{12}$ ✓

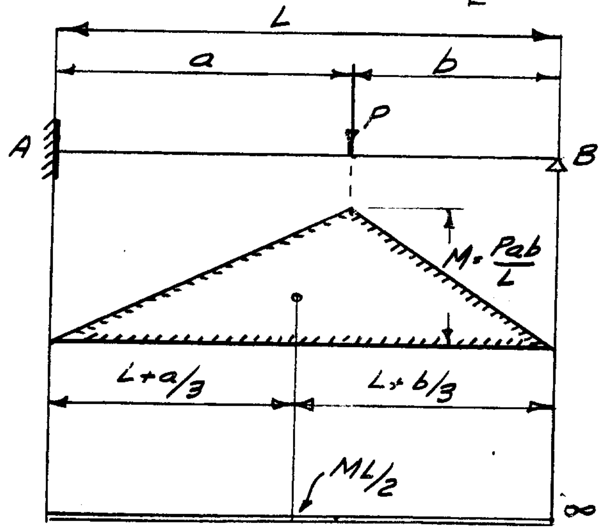


FIG. 5

For a beam of uniform section fixed at A and hinged at B, subject to a single concentrated load P, Fig. 5.- The hinge has an elastic weight of ∞ and hence both the centroid and kern point of the infinite column section lie at the hinge.

Whence $M_a = \dfrac{M_K\cdot y}{I_o} = \dfrac{\frac{ML}{2}\cdot\frac{L+b}{3}\cdot L}{\frac{1}{3}L\cdot L^2} = M\,\dfrac{L+b}{2L}$

This relation might also be used as a general formula, although it will usually be more convenient to obtain this moment from the value for the fixed ended condition.

Fixed Beam Unit Rotation at One End. Consider a fixed beam of uniform section subject to, - i.e. loaded with - a unit rotation at one end A (Fig. 6). This rotation is equivalent to an angle and therefore a load unity on the column.

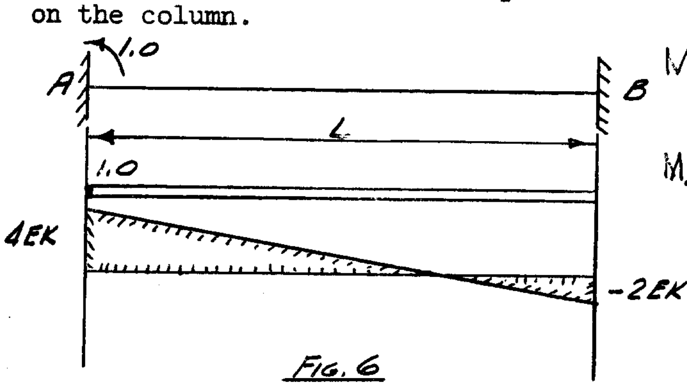


FIG. 6

$$M = \frac{P}{A} \pm \frac{Per}{I_o} = \frac{P}{A}\left(1 \pm \frac{6e}{d}\right) = \frac{1}{L/EI}\left(1 \pm \frac{6\cdot L/2}{L}\right) = \frac{EI}{L}(1 \pm 3)$$

$$M_a = \frac{4EI}{L} = 4EK \qquad M_b = -\frac{2EI}{L} = -2EK = -\tfrac{1}{2}M_a$$

Where $K = I/L$

That is, if B is made a hinged

joint, its moment being released, 1/2 of it will be thrown over to A with opposite sign. That is, - for hinge at B,- M_a = 4EK - 1/2 2EK = 3EK

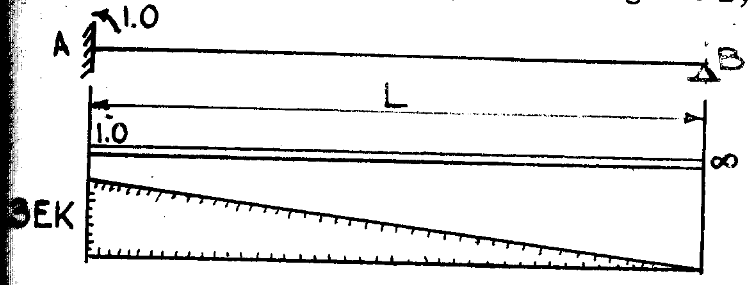


FIG. 7

This may be obtained directly from the column analogy Fig. 7,-

$$M_a = \frac{M_K \cdot y}{I_o} = \frac{1 \cdot L \cdot L}{\frac{1}{3} \cdot \frac{L}{EI} \cdot L^2} = \frac{3EI}{L} = 3EK$$

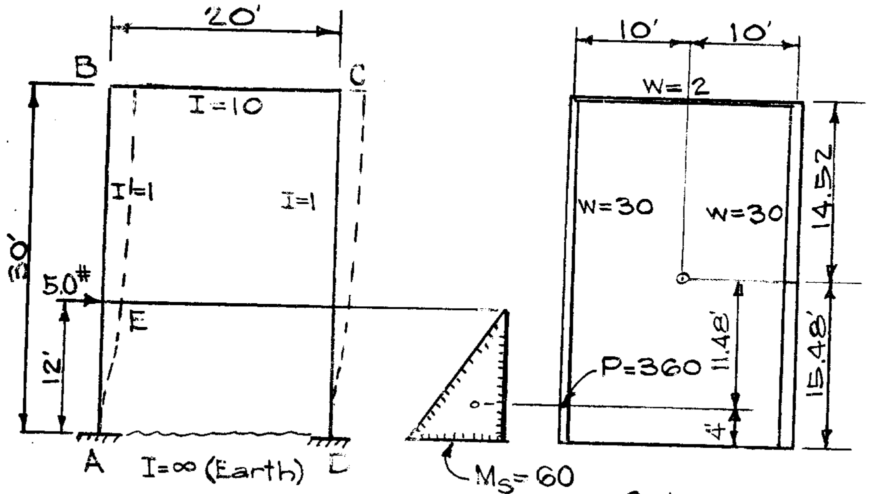
This shows that a beam hinged at one end is 3/4 as stiff, i.e. offers 3/4 the resistance to rotation, of a beam fixed at both ends.

The column analogy can be extended similarly to beams of variable moment of inertia within the span. It is particularly useful in such cases in finding the end moments and the effect on these moments of unit rotations at the joints. It is this latter factor that enables us to distribute properly the unbalanced moments in continuous structures. These and other applications will be taken up as required in the body of the text.

Application to Numerical Examples. The application of the theorem to a few simple numerical examples will further illustrate its usefulness. Literal examples are purposely avoided except in very general cases as these tend to formulas. Formulas are always less desirable than direct solutions unless the solutions are quite involved. It will be found that the column analogy is a ready tool affording a simple direct solution to otherwise extended cases, minimizing the need of formulas.

In the examples that follow, as elsewhere throughout this text, an ordinary slide rule has been used in the calculations with no studied attempt at refinement in reading. No greater accuracy is desirable because the refinement in itself is meaningless as an answer to the physical problem, and therefore rather misleading than illuminating. The examples themselves for the most part are illustrated by the figures accompanying in each case. Kip units of 1000 pounds are generally assumed, although the designation # may be indicated for convenience. As previously explained, relative values of I will be sufficient, and where not specified E may be taken as unity. Except where it has seemed especially desirable, no attempt has been made to draw to scale. Units are easily discovered, and are therefore generally ignored in the partial steps of the solution.

Problem - Rectangular Bent   Fixed Bases   Horizontal Load



Column

Centroid $\frac{60}{62}$ x 15 = 14.52

Moment of Inertia,-

$2/3 \times 14.52^3 = 2045$

$2/3 \times 15.48^3 = 2470$

$2 \times 14.52^2 = 420$

$I_h = 4935$

60 x 100 = 6000

1/12 x 2 x 400 = 67

$I_v = 6067$

The statically indeterminate moment M_i at any point is then analogous to $f = P/A \pm My/I \pm M'y'/I'$.

At A,- $m_i = \frac{360}{62} + \frac{360 \times 11.48 \times 15.48}{4935} + \frac{360 \times 10 \times 10}{6067} = 5.8 + 13.0 + 5.9 = +24.7$

$$M_A = m_s - m_i = -60 + 24.7 = -35.3^{\#'}$$

$$M_B = 5.8 - \frac{360 \times 11.48 \times 14.52}{4935} + 5.9 = 5.8 - 12.2 + 5.9 = -0.5$$

$$M_C = 5.8 - 12.2 - 5.9 = -12.3$$

$$M_D = 5.8 + 13.0 - 5.9 = +12.9$$

$$M_E = 5.8 + \frac{360 \times 11.48 \times 3.48}{4935} + 5.9 = +14.6$$

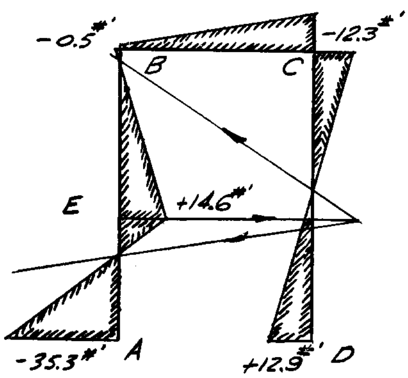


Fig. 9

Moment diagrams may be conveniently constructed on the tension side of the members. With the critical moments known the points of contraflexure are established and the shears and reactions may be calculated by statics. The sign of the moments and the shape of the distorted structure will be seen to be consistent if inward bending of any member is taken as positive.

Application to Trusses. If the effect of the web members is neglected, the treatment of trussed members by the methods here presented presents no special difficulty. The elastic weights of the chord members lie at the centers of rotation for those members (centers of moment) and are equal to the angle changes at those points due to unit moments existing there. The angle changes are $\frac{\Delta L}{r} = \frac{\frac{S}{A} \cdot \frac{L}{E}}{r} = \frac{\frac{1}{r} \cdot \frac{L}{AE}}{r} = \frac{L}{EAr^2}$. If E be omitted, $w = \frac{L}{Ar^2}$

The treatment of the web members in this way presents the difficulty that their centers of moments either lie well outside the truss or, if the chords are parallel, they lie at infinity and have zero values and expressions involving them become indeterminate.

In the case of parallel chords it is probably best to include the effect of web members separately. This presents the added advantage that, since the effect of the web is not very large, it may be omitted in preliminary computations.

In using the column analogy either to analyze trussed bents and arches or to determine fixed-ended moments and stiffness relations to be used in continuous frame computations where the chords are parallel allowance is made for the web members as follows,-

To the moment of the elastic load about the vertical axis (or the axis normal to the chords) add the total vertical displacement occurring in the web system due to the moment curve used. This is $\sum \frac{SuL}{AE}$ Where S is the stress in any web bar due to the static moment curve used, u is the stress in any web bar due to a unit shear in any panel. L, A and E have their usual significance.

To the moment of inertia of elastic weights about the vertical axis (or the axis normal to the chords) add $\sum \frac{u^2 L}{AE}$ for the web where u is as defined above.

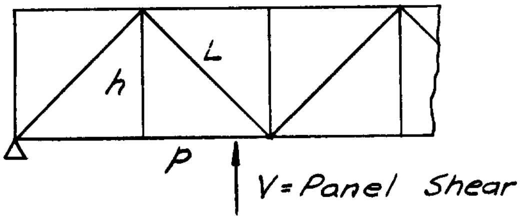


Fig. 10

If, as is usual, the parallel chords have the same direction (horizontal) in all panels (Fig. 10)

For diagonals, $S = V\frac{L}{h}$; $u = \frac{L}{h}$

$$\sum \frac{SuL}{A} = \sum VL \frac{L}{AE} L$$

$$\sum \frac{u^2 L}{A} = \sum \frac{L}{Ah^2} L^2$$

For verticals $S = V$; $u = 1$

$$\sum \frac{SL}{A} = \sum VL \frac{L}{AE} L \qquad \sum \frac{u^2 L}{A} = \sum \frac{L}{Ah^2} L^2$$

Here L and h are the same.

It is doubtful whether frames involving trusses will often justify an exact analysis including the web, though the trussed girder (roof truss) is the standard type of horizontal member in the mill-building frame, in many bents of elevated railways, and in other cases. The reference here is not to large continuous truss or arched bridges.

In the simple cases mentioned, where exact analysis is desired, it is best to make a few analyses of typical cases in order to determine the effect of the web and thereafter make allowance accordingly.

The corrections in numerator and denominator, in the column analogy, are like the original expressions in that

Elastic weight = $\frac{L}{Ah^2}$
Distance from axis to elastic weight = L
Load intensity on elastic weight = VL
Elastic load = Vw = $V \frac{L}{Ah^2}$

The term in the denominator is always positive, while that in the numerator should be determined by inspection as in the case of rib-shortening in arches. It will be noted that the web-correction is very similar to that for rib-shortening.

This suggests that corrections in the moment terms of the column analogy gives it greater generality. To the numerator add (algebraically) deformation due to shear and direct stress due to the assumed moment distribution. To the denominator add the deformation along the axis considered due to unit thrust and to unit shear.

Useful Geometrical Relations - The Kern. The kern of a section is defined as that area within which a load must lie in order to produce exclusively compressive stresses over the section. It will in general be a polygon each side of which corresponds to one corner of the section. The most familiar case is, of course, the middle third of a rectangular section.

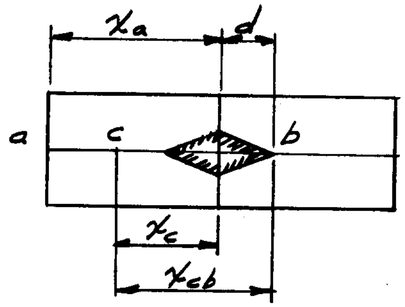


FIG. 11

Actually the kern is as shown, Fig. 11, each side of the kern corresponding to one corner of the section.

There is usually no great simplification in using the kern, but it is sometimes convenient in connection with rectangular columns eccentrically loaded. Since point b is located from the fact that a load applied there produces zero stress at a, $\frac{P}{A} - \frac{P \cdot d \cdot x_a}{I} = 0$; $\frac{d \cdot x_a}{\rho^2} = 1$; $d = \frac{\rho^2}{x_a}$

If now a load is applied at any other point c,-

$$f_a = \frac{P}{A} + \frac{P \cdot x_a \cdot x_c}{I} = \frac{P \cdot x_a}{I}\left(\frac{\rho^2}{x_a} + x_c\right) = \frac{P \cdot x_a \cdot x_{cb}}{I}$$

Hence it follows from the definition of the kern that external fibre stresses may be found by use of the beam formula if we substitute moment about the kern for moment about the centroidal axis.

This is in some ways a convenient conception in studying fibre stress in arches, and is introduced here because of that.

It is important to realize that the idea of the kern adds no new nor necessary conception. It is a mental device of a geometrical nature, interesting if convenient. Like all other such geometrical concepts it is subject to unlimited elaboration and complication including the ellipse of inertia and all the properties which projective geometry attributes to the ellipse.

Graphical Construction - Eccentric Bending. This property of the kern gives rise to the following construction for maximum fibre stresses in eccentric bending.

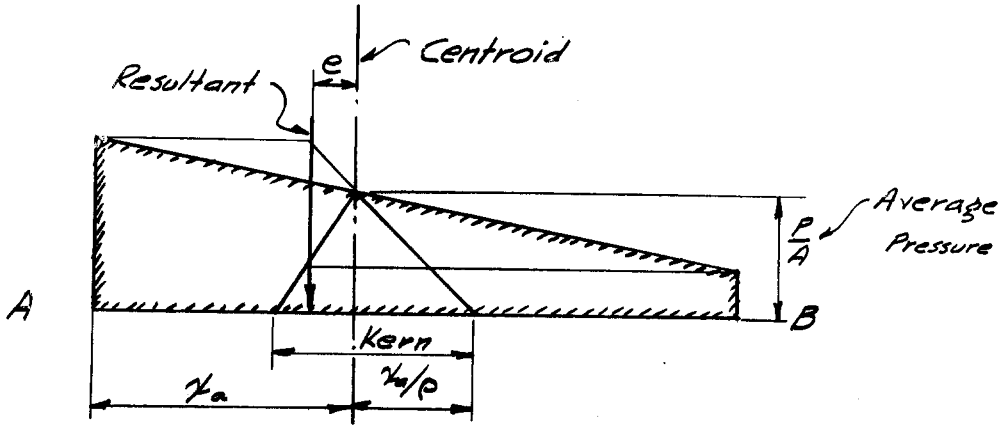


FIG. 12

Lay off the average pressure on the centroidal axis, Fig. 12. Lines drawn from the kern points through this ordinate will intersect the resultant normal at ordinates giving the fibre stress at the point corresponding to that kern point.

Proof,- $$f_a = \frac{R}{A} + \frac{R \cdot e \cdot x_a}{I} = \frac{R}{A}\left(1 + \frac{e \cdot x_a}{\rho^2}\right) = \frac{R}{A}\,\frac{e + \frac{\rho^2}{x_a}}{\frac{\rho^2}{x_a}}$$

Except for occasional applications, the utility of the kern in structural engineering may be summarized by saying that the kern is a device for computing fibre stresses by a method different from the ordinary method, whereas it is found by the method which its use is intended to avoid. Many tools of analytical mechanics are of this nature.

The Circle of Inertia. In analyzing unsymmetrical structures by the column analogy it is necessary to locate the principal axes and to find the principal moments of inertia. This is readily done by means of the familiar construction of the circle of inertia, Fig. 13. The moments of inertia I_x and I_y are laid off from any reference point and the product of inertia in a perpendicular direction from the end of the shorter of these values. The construction of the circle, as indicated, then gives the direction of the principal axes and the magnitude of the principal moments of inertia.

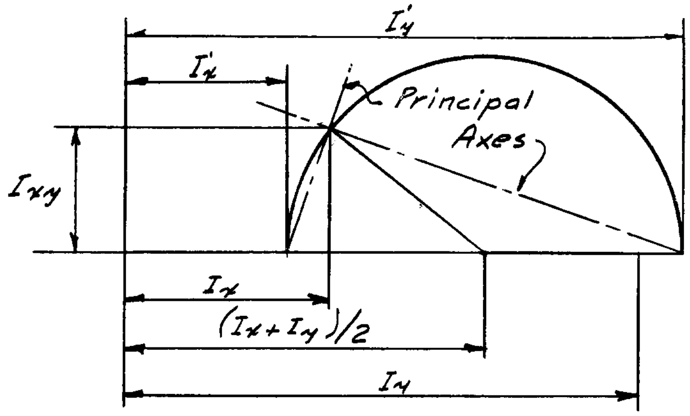


FIG. 13

Product of Inertia. In this connection it will help to remember that the moment of inertia is only a special case of the product of inertia. The product of inertia of a line about any two axes through its centroid is $\pm \frac{1}{12} Aab$ (Fig. 14) where a and b are the projections of the line parallel to the two axes. The product product of inertia then about parallel axes A' and B' is $A(\pm \frac{ab}{12} + a'b')$

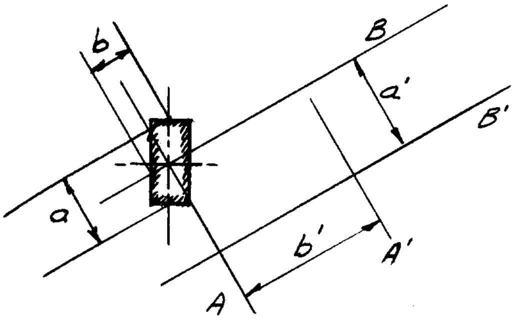


FIG. 14

The signs may be taken care of by the usual convention of geometry, plus reading to the right and up or any other consistent arrangement.

Another useful relation is that the product of inertia of two areas (Fig. 14a) about their centroid is $\frac{W_a \cdot W_b}{W_a + W_b} xy$ where W_a and W_b are the two areas. To this should, of course, be added the centroidal products of inertia.

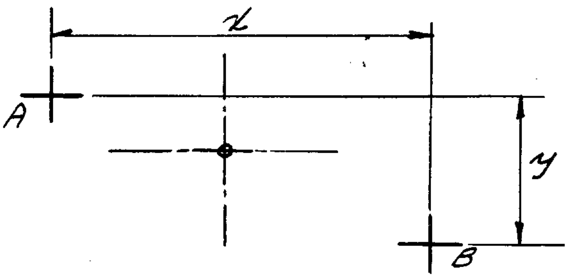


FIG. 14a

Product of Inertia in Deflection Computation. The conception of the elastic properties of a beam as concentrated at its centroid gives rise to certain relations of the deformations which are sometimes useful. Assume a cantilever beam CD fixed at D (Fig. 14b) Assume connected to the end C two inelastic brackets CA and CB. Let it be required to find the displacement of A along the axis AA' due to a force acting at B along the axis BB'.

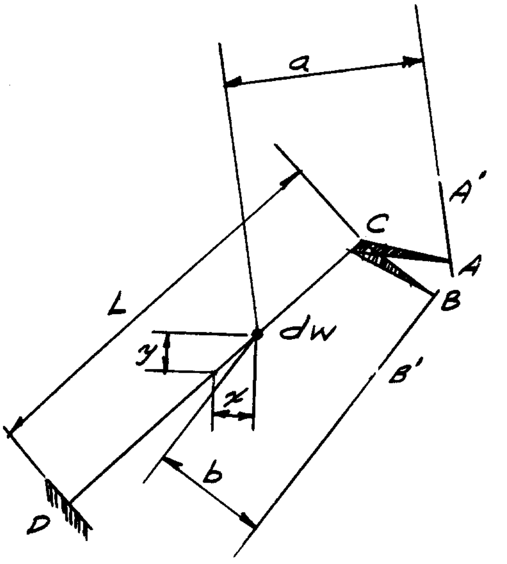


Fig. 14b

$$W = \int \frac{ds}{EI}$$

$$I_y = \int \frac{ds}{EI} x^2$$

$$I_b = \int \frac{ds}{EI} y^2$$

This will be $\int dw\ a \cdot b$. This is the product of inertia of all elastic weights about axes AA' and BB'.

This statement is general. If AA' and BB' are coincident, the product of inertia becomes a moment of inertia. If the force at A is a moment we get the difference of two products of inertia about axes AA' - BB' and A"A" - BB' where A"A" is an axis parallel to and very near AA'. The coordinates from the axis AA' will now drop out and the deflection alóng BB' is the statical moment of elastic weights about BB'. If the rotation at B for a moment at A is wanted, coordinates for both axes drop out and the result is the total elastic weight.

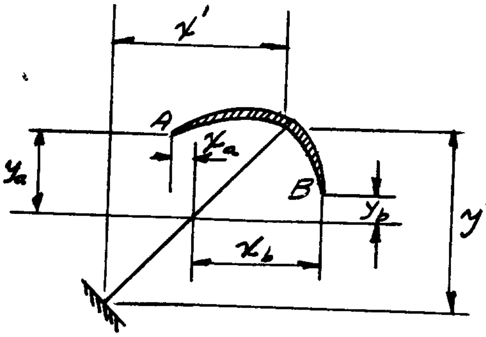


Fig. 14c

As a specific case, consider the cantilever shown in Fig. 14c. Due to a unit horizontal force at B, the movement of A in a vertical direction is

$$\overset{a}{\Delta} V = \frac{L}{I}\left(\frac{1}{12} x' y' - x_a y_b\right)$$

$$H_b = 1$$

Similarly the movement of B in a vertical direction due to a unit vertical force at B is

$$\overset{b}{\Delta} V = \frac{L}{I}\left(\frac{1}{12} x'^2 - x_b^2\right)$$

$$V_b = 1$$

Using similar notation we may write for the rotation of A due to a vertical unit force at B and a unit moment at B, respectively.

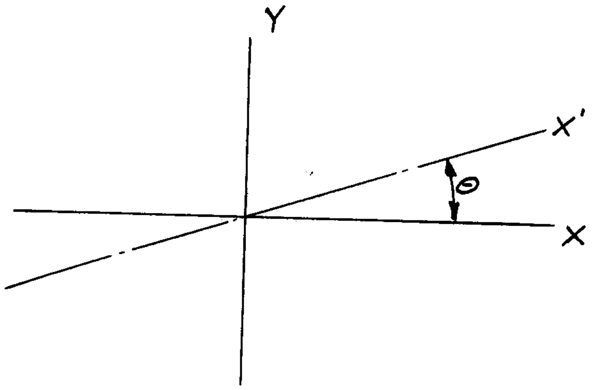
$$\Delta^a \phi = \frac{L}{I} x_b$$

$$V_b = 1$$

$$\Delta^a \phi = \frac{L}{I}$$

$$M_b = 1$$

<u>Theory of Conjugate Axes</u>. Conjugate axes of inertia are axes about which the product of inertia is zero. Any number of pairs of conjugate axes may be drawn; one pair will be mutually perpendicular and these are principal axes. If X and Y are any two axes and the products of inertia for these axes are I_x, I_y, I_{xy}, then the axis conjugate to Y makes with X an angle.



$$\theta = \tan^{-1} \frac{I_{xy}}{I_y}$$

$$\text{and } I_{x'} = \left(I_x - \frac{I_{xy}^2}{I_y}\right) \cos \theta$$

FIG. 14 d

Similarly the axis conjugate to X makes with Y an angle,-

$$\tan^{-1} \frac{I_{xy}}{I_x} \quad \text{and } I_{y'} = \left(I_y - \frac{I_{xy}^2}{I_x}\right) \cos \left(\tan^{-1} \frac{I_{xy}}{I_x}\right)$$

These relations are easily proved or they may be found in standard texts. They are sometimes useful in simplifying expressions in such cases as continuous arches on elastic piers.

CHAPTER II

DISTRIBUTION OF MOMENTS

A. PRISMATIC SECTIONS

General Method. The general method of moment distribution described in this chapter is applicable to all problems in the analysis of indeterminate frames - continuous girders, bents, arches, viaducts - with members straight or curved, solid or trussed, with constant or varying section, due to vertical or transverse loading, to settlement of supports, or to internal distortion such as that resulting from temperature changes or such as result in secondary stresses in bridge trusses, or are used where influence lines are to be determined. It is also general in the sense that it applies equally to moments and all other joint forces and movements, but is here described with particular reference to moments, after which the distribution of other forces will be considered.

Each member is first considered to be held fixed at its ends and all terminal forces determined for this condition. The method then depends on a single general theorem, which is obvious, that if any joint of a frame be allowed to move until equilibrium is set up at that joint, the other joints of the frame being held rigid during the movement, then the unbalanced forces or moments at that joint will be distributed among the members there connected in proportion to their resistance to such movement. This resistance is measured by the force or moment necessary in each member to produce a unit movement of that end of the member. A definite proportion of this change in force or moment at the end of the member considered appears, as a result of the movement, as an unbalanced force or moment at the other end. If then, each unbalanced joint successively and alone be allowed to move and the distributed forces or moments be "carried over" in the proper ratio, and if the process be continued until all imaginary restraints are removed, each joint will finally be balanced.

The method, then, is one of successive distribution of unbalanced forces and therefore yields results that form a series of successive approximations to the final one. It is in no sense an approximate method, for the final result may be found with any degree of precision desired.

No matter how complicated the structure, no general formulas are needed, nor is it necessary to apply graphical constructions or solve simultaneous equations. Continuity of the structure is always preserved and temporary restraints are imposed to preserve equilibrium. The rather complex mental device of breaking the structure, allowing it to deform, and then pulling it together again is avoided, and attention is concentrated on the successive removal of the restraints. Attention is therefore centered on the laws of statics which are familiar tools, rather than on the more complex equations of elastic displacement.

From this it follows that each step in the solution has a definite physical meaning, making it possible to readily visualize the action of the structure. This is a matter of great importance and merits emphasis in the solution of all problems in indeterminate structures. The more complex the structure the more desirable it is to be able to picture its action under load - particularly to the extent of being able to intelligently follow and interpret the computations. It is too easy to become lost in a maze of figures and make a mistake in the answer not of 5 per cent or 10 per cent but perhaps of many times that amount, due to the meaninglessness of the various intermediate steps.

Further, in the method outlined the usual convention of signs is preserved in the girders. In general the signs are determined either automatically or by inspection in the solution without resort to any convention of clockwise rotation.

Summarized, the method preserves continuity at the joints allowing the statics to be temporarily unbalanced. By successively balancing the joints the equations of statics are satisfied and the approximate results are obtained in a converging series. The method is rapid and accurate and a perfect check is indicated if desired. In some cases considerable simplification is possible by the use of special relations, but these special relations are not necessary. The method lends itself readily to approximate solutions which are always of great value, particularly in complicated structures.

Continuous Prismatic Beams. The simplest method of preserving continuity in the frame is as follows:-

1. Consider all joints locked against rotation. The moments at the ends of the loaded spans will then be fixed-ended moments. These may be found conveniently by the formula $M' = m\frac{a}{L}$ (See Fig. 15), applicable to any type of loading, as derived in the chapter on the column analogy.

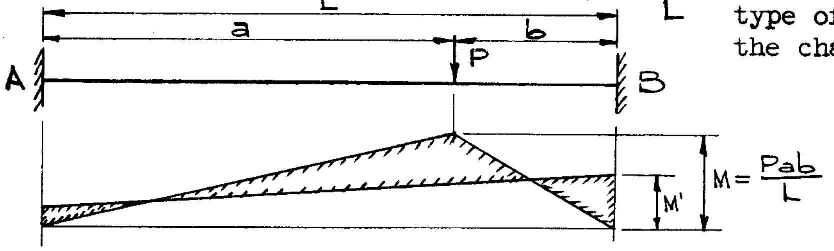


FIG. 15

2. Now unlock any one joint, that is, allow it to freely rotate. Since, in general, the fixed-ended moments at the joint are not balanced, the joint will rotate until equilibrium is established. All members meeting at the joint will have the same rotation. This rotation is directly proportional to the moment at the joint and inversely proportional to K (or I/L) for that member. The difference of the fixed-ended moments will then be distributed among the members in proportion to the K values, and these moments must be added to those previously existing. Moreover, the rotation will produce at the far end of each member a moment equal to one-half that at the joint and of opposite sign (since all other joints are still locked against rotation). See also page 9.

3. Each joint in succession can thus be unlocked, all other joints being temporarily locked. Since this procedure will unbalance joints which are already balanced, it must be repeated until all joints are balanced.

4. Evidently any joint which is once balanced will be unbalanced only by the moments coming in from other joints and these only need further distribution.

5. The process may be performed simultaneously for all joints, may be continued to any desired degree of accuracy, and is universally applicable. If the joints move horizontally or vertically a correction is made for this as indicated on page

6. A final check on the results may be secured from the condition that the unbalanced moment at a joint is finally distributed among the connecting members in proportion to their K/C values where K is the I/L for the member and $C = 1 + 1/2\,\frac{M_b}{M_a}$. M_a is the change from the fixed-ended moment at the end

of the member under consideration and M_b is the corresponding change at the other end of the member. When the other joints remain locked as under the procedure in 2, the C values are equal and hence the distribution is proportional to K. This may be readily demonstrated as follows (See Fig. 16): If a beam is loaded only with moments at its two ends, the bending moment diagram is a straight line. The slope at one end A, then, is the reaction (shear) due to two triangular loadings.

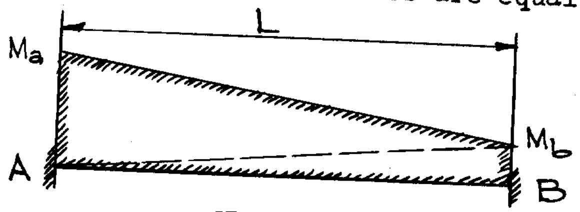
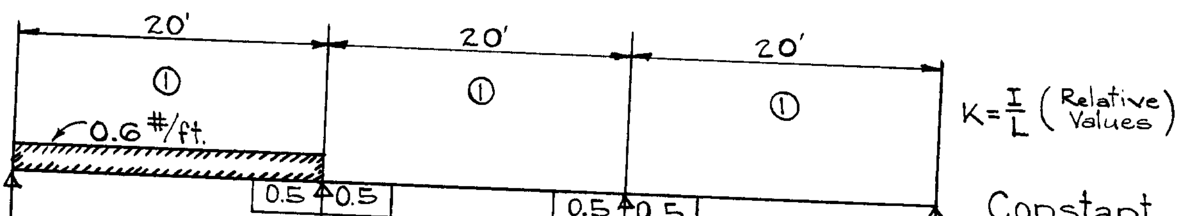


FIG. 16

$$\phi_a = \left(\frac{2}{3} M_a \frac{L}{2} \frac{L}{2} \frac{1}{3} M_b \frac{L}{2}\right) EI = M_a \frac{L}{3EI}\left(1+\frac{1}{2}\frac{M_b}{M_a}\right) = \frac{M_a}{3EK} C = \frac{1}{3E}\frac{M_a}{K/C}$$

Illustrative Example. Three Spans. Ends Free. Fig. 17. The fixed-ended moments are first computed considering each span separately as a single fixed-ended span, and the joints considered locked with these moments. Continuity is thus preserved but the moments at the joints are unbalanced. Unlock the joints in any order and distribute the unbalanced moments. Beginning at the left - 20.0 released throws + 20.0 into this end of the beam as there is no other member at the joint. At the second joint the unbalanced moment (the difference between the fixed-ended moments on each side of the joint) is likewise - 20.0. When this is released, the K values of the two spans being equal it will divide + 10.0 on the left and - 10.0 on the right, the signs being determined by the required static balance.



Constant I

$M' = \frac{1}{12}wL^2$						
-20.0	-20.0	0	0	0	0	Fixed Ended Moment
+20.0	+10.0	-10.0	0	0	0	Balancing Moment
-5.0	-10.0	0	+5.0	0	0	Distributed Moment
+5.0	+5.0	-5.0	-2.5	+2.5	0	Balancing Moment
-2.5	-2.5	+1.3	+2.5	0	-1.2	Etc.
+2.5	+1.9	-1.9	-1.2	+1.3	+1.2	
-0.9	-1.2	+0.6	+0.9	-0.6	-0.6	
+0.9	+0.9	-0.9	-0.7	+0.8	+0.6	
-0.4	-0.4	+0.3	+0.4	-0.3	-0.4	
+0.4	+0.3	-0.4	-0.4	+0.3	+0.4	
-0.1	-0.2	+0.2	+0.2	-0.2	-0.1	
+0.1	+0.2	-0.2	-0.2	+0.2	+0.1	
0	-16.0	-16.0	+4.0	+4.0	0	Actual Moment #'

FIG. 18

At joint 3 and at the right end the moments being zero, no unlocking is required.

The joints having been unlocked it will be noticed that the joints are now balanced and the net result at each joint would give us the actual moments except for the portion carried over to the other end of the beam in each case of unlocking. That is, our results so far are 0, -10, 0 and 0. In the first span when +20.0 was released at the left end, one-half of opposite sign or -10.0 was carried over to the other end. When the right end received moment of +10.0, one-half of opposite sign of -5.0 was carried over to the left end. Similarly considering each joint in turn we get the third line of distributed moments which become on again locking the joints the new unbalanced moments. These need to be released, balancing the joints, and distributed as before. Our answers after the second unlocking if we cared to look at them, summing up the columns thus far, are 0, -15, +2.5 and 0. The third approximation is 0, -15.6, +3.8 and 0 and the fourth 0, -15.9, +4.0 and 0. The series converges rapidly and may be continued if desired until there is nothing further to distribute. The column of figures is added readily by considering the increment after balancing each joint. Obviously this amounts to adding on only one side of the joint. If desired, a check is readily obtained at the end but it is scarcely necessary.

While this method is perfectly general and perhaps as simple as need be desired, certain modifications are apparent. For instance, it is needless to keep locking and unlocking the end joint. When once unlocked it should be left free to rotate. At the next joint, then, any unbalanced moment is to be distributed between two members one of which is free at its far end and the other fixed at its far end. For the same end moment the free ended member will rotate more, and its K value, which measures its resistance to rotation, should be reduced accordingly. This reduction has been shown in Chapter I to be 3/4 of its former value.

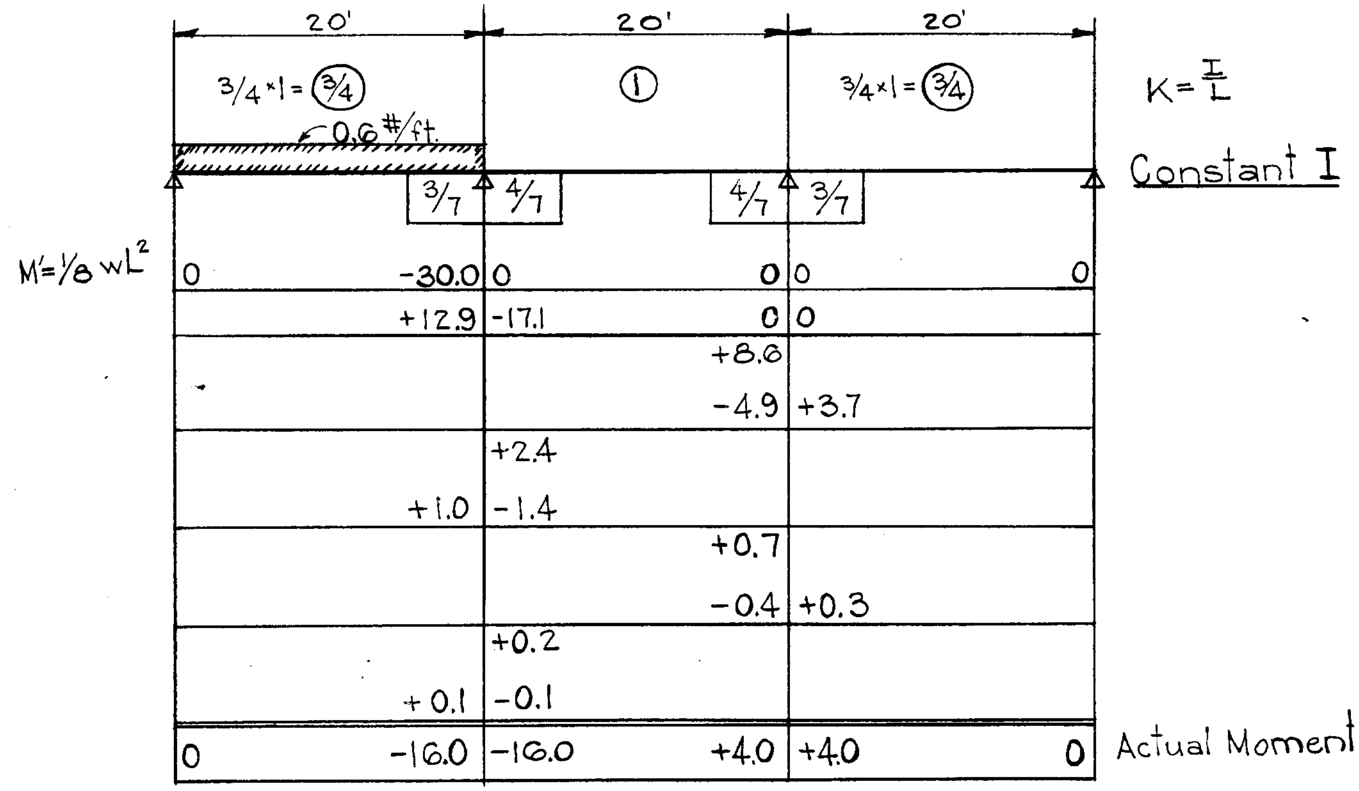


Fig. 18

Using this reduced K value in the end spans, Fig. 18, the rotations at the intermediate supports will be divided between the two adjacent spans in the proportion of 3/7 and 4/7. In the case of the center span (or any intermediate span), one-half of the released or balancing moment will be carried over to the other end in each case. In the end spans nothing will be carried over as the outer ends always remain free. Obviously, the outside rows of figures contribute nothing to the solution and could be omitted. The numerical work is thus considerably simplified. Tabulating the figures with line work has no advantage unless it be for explanation. The tabulation shown in the figure therefore, covers needless space.

Where irregular loadings occur on the end span, one end being free, the general formula $m' = m\frac{a}{L}$ for fixed-ended moment will still be found useful. On obtaining these values the moment at the continuous end will be increased by one-half the moment found at the free end by first assuming it fixed. This is illustrated in Fig. 19.

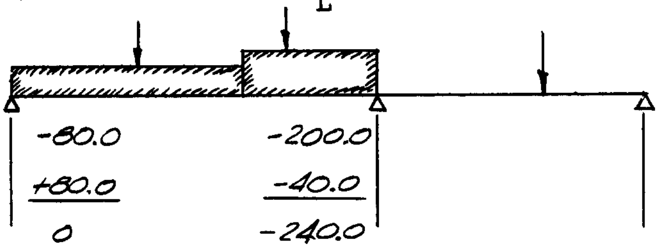


FIG. 19

Problem - Four Spans. Cantilever End. Fig. 20 illustrates a more general problem. The left end is fixed. It may be locked and unlocked as before or as will be found simpler, it may remain locked. The difference between the final moment at the second joint -6.8 and the fixed-ended moment in the first span -16.3 is +9.5. One-half of this value of opposite sign or -4.7 will then need to be added to the moment at the other end.

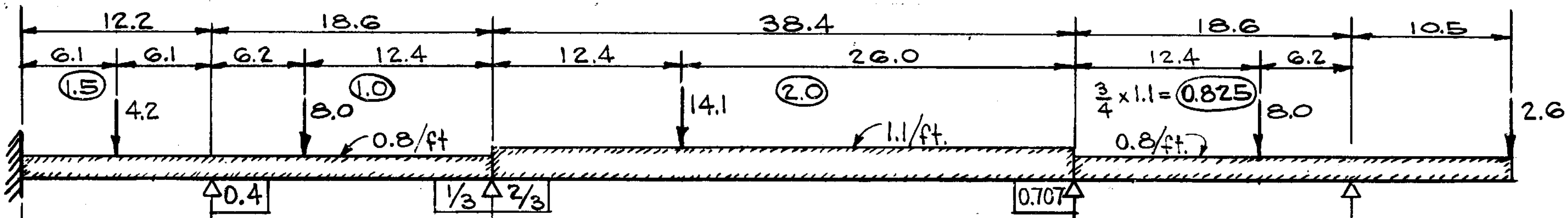
The right end is an overhanging beam. The moment at the support will then be that computed from the cantilever. The unbalanced moment at this joint -26.3 will carry over +13.1 to the other end. The moment being known at the end support it is best treated as a free end, the K value in the adjacent span being reduced accordingly.

The moment diagram if desired is easily constructed, as shown in the figure, by combining the simple beam moment curve with that due to continuity.

It is interesting to note that the moments at the supports may be roughly approximated by interpolating between the fixed-ended moments. The difference between the fixed-ended moments is the unbalanced moment which is divided between the adjacent spans in proportion to their stiffness. This distribution may be obtained graphically by successive approximation, as described in this text, or by any other method. The important thing to note is that continuity must be preserved. The exact division between the two spans may be fairly accurately estimated. Indeed, such a simple approximation as the mean of the two fixed-ended moments is a very useful one and perhaps all that we need in some cases for purposes of design.

The problem shown is somewhat abnormal in loading and in ratio of span lengths, giving rise to high reductions in the adjacent spans and at the second support. Such a reduction is precarious to say the least, as any under-load or over-load in the long span will affect these moments abnormally.

Estimating End Moments. Evidently continuity may be preserved in the girder when the fixed-ended moments are released on other assumptions than that all joints except the one in question are restrained against rotation,



M′ $-\frac{1}{12}WL$	$-9.9 = \frac{0.8\times\overline{12.2}^2}{12}$	$= -9.9$	$-23.1 = \frac{0.8\times\overline{18.6}^2}{12}$	$= -23.1$	$-135.2 = \frac{1.1\times\overline{38.4}^2}{12}$	$= -135.2$			$-44.1 = 0.8\times\frac{\overline{10.5}^2}{2}$
M′ $-M\frac{a}{L}$	$-6.4 = \frac{4.2\times\overline{12.2}^2}{8}$	$= -6.4$	$-22.0 = \frac{8.0\times6.2\times\overline{12.4}^2}{\overline{18.6}^2}$	$= -11.0$	$-80.2 = \frac{14.1\times12.4\times26}{\overline{38.4}^2}\times26$	$do\times12.4 = -37.0$			$-27.3 = 2.6\times10.5$
	-16.3	-16.3	-45.1	-34.1	-215.4	-172.2	-34.1 +13.1	-45.1 -26.3	-71.4
	-16.3	-16.3	-45.1	-34.1	-215.4	-172.2	-21.0		
			+11.5	-60.4	+120.9	+107.0			
			+30.2	-5.8	-53.5	-60.4			
			-12.1	-15.9	+31.8	+42.7			
			+8.0	+6.0	-21.3	-15.9			
			-3.2	-9.1	+18.2	+11.2			
			+4.5	+1.6	-5.6	-9.1			
			-1.8	-2.4	+4.8	+6.4			
			+1.2	+0.9	-3.2	-2.4			
			-0.5	-1.4	+2.7	+1.7			
			+0.7	+0.2	-0.8	-1.3			
			-0.3	-0.3	+0.7	+0.9			
			+0.1	+0.1	-0.4	-0.3			
	-4.7		0	-0.2	+0.3	+0.2			
	-21.0		-6.8	-120.8		-91.5		-71.4	Actual Moments

+9.5 (Net Rotation)

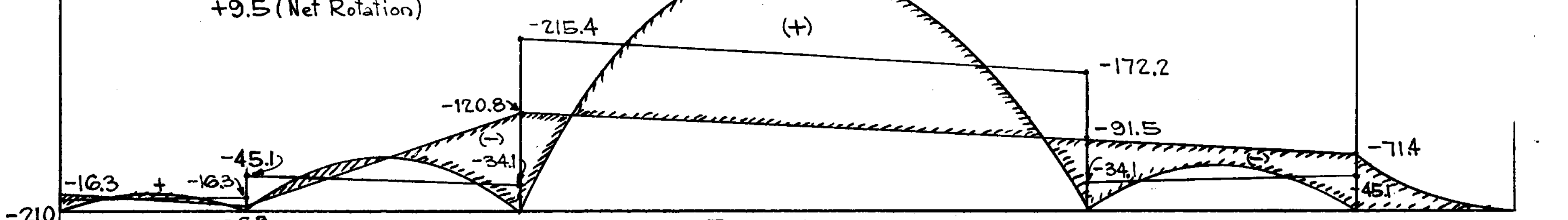


FIG. 20

but the usefulness of such an assumption would depend on its accuracy. If, however, we can estimate the ratio of the moments producing rotation at the end of the members connected at any joint, then, as has been shown, the difference of the fixed-ended moments will be distributed at that joint in proportion to the K/C values for the members, where:

$$C = 1 + 1/2 \frac{\text{moment producing rotation at far end}}{\text{moment producing rotation at near end}}.$$

It needs to be especially emphasized here that the moments to be used in computing C are those due to rotation of the joints; they are, therefore, not the total moments at the ends but the difference between the total moments and the fixed-ended moments. The moments at the far ends of the members may be produced by loads in other spans, in which case they must be either estimated directly or by successive approximation, or they may be produced by partial fixation of the far end by unloaded spans beyond.

In the latter case the value of C may be determined as follows:

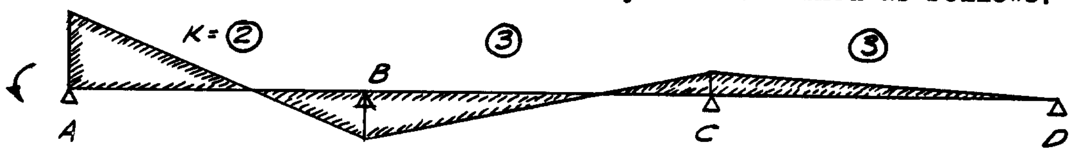


Fig. 21

Assume that it is required to determine the ratio of the moment in $\overline{AB}$ at B (Fig. 21) produced by a moment at A to the moment at A, all spans being otherwise loaded, that is, to determine the ratio M_{ba}/M_{ab}.

The K values have been assumed at random. Now a unit moment at B, if $\overline{AB}$ is cut free and omitted, will produce at C a moment - 1/2 if C is locked against rotation. If C is unlocked, the moment in $\overline{CD}$ at C will be,

$$M_{cd} = \frac{5}{5+3} \cdot \frac{1}{2} = \frac{5}{16}$$

and, of course, M_{cb} will be the same. Similarly a unit moment applied at A will produce a moment at B of -1/2 if B is locked. When B is unlocked and allowed to rotate, this moment of 1/2 will be distributed between $\overline{BA}$ and $\overline{BC}$ in proportion to their K/C values.

For $\overline{BA}$ $\quad C = 1 + (1/2).0 = 1 \quad K/C = 2$

For $\overline{BC}$ $\quad C = 1 - 1/2\ (5/16) = 27/32 \quad K/C = 3.56$

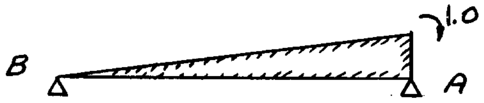
The value 5/16 was previously determined as the ratio M_{cb}/M_{bc}.

Hence, $M_{bc} = -1/2\ (\frac{3.56}{2+3.56}) = -0.320 = M_{ba}$

Whence, for $\overline{AB}$, $C = 1 - 1/2\ (\frac{0.320}{1}) = 0.840$ and $K/C = 2.38$

The following values of 1/C are useful and easy to remember, (Fig. 22).

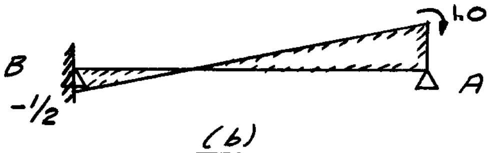
(a) For a beam simply supported at the far end:



$C_a = 1 + (1/2)\,0 = 1 \qquad 1/C = 1$

(a)

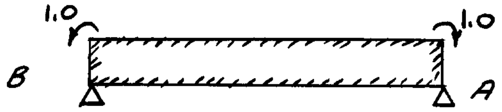
(b) For a beam fixed at the far end:



$C_a = 1 - 1/2(1/2) = 3/4 \qquad 1/C = 4/3$

(b)

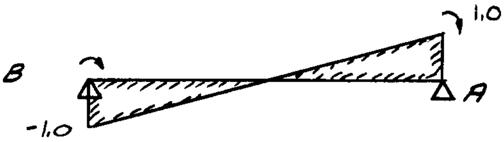
(c) For equal moments at the two ends:



$C_a = 1 + 1/2(1) = 3/2 \qquad 1/C = 2/3$

(c)

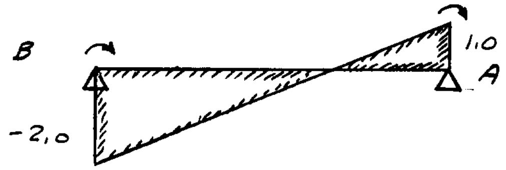
(d) For equal and opposite moments at the two ends:



$C_a = 1 - 1/2\,(1) = 1/2 \qquad 1/C = 2$

(d)

(e) Where moment at far end is double and of opposite sign:

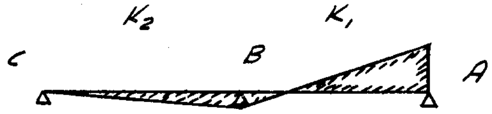


$C_a = 1 - 1/2\,(2) = 0 \qquad 1/C = \infty$

This indicates that there is no rotation at A and the joint may be treated as fixed.

(e)

(f) For a beam continuous with other beams:



$C_a = 1 - \left[1/2 \cdot 1/2 \frac{K_2}{K_1+K_2}\right]$ and varies from

3/4 to 1

1/C varies from 1 to 4/3

Where the K values are equal,

$C_a = 1 - \left[1/2 \cdot 1/2 \cdot 1/2\right] = 7/8 \qquad 1/C = 8/7$

(f)

Fig. 22

Examples - Using the Factor 1/C -- Three Symmetrical Spans. Assume the fixed-ended moments to be -200 and -300 and the K values to be 2 and 6 as shown in Fig. 23. The value of 1/C for BB' is 2/3 by symmetry and the value of 1/C for AB and A'B' is 1 by the physical conditions. The unbalanced moment at B will then be distributed between AB and BB' in proportion to their K/C values or as 2 is to 4.

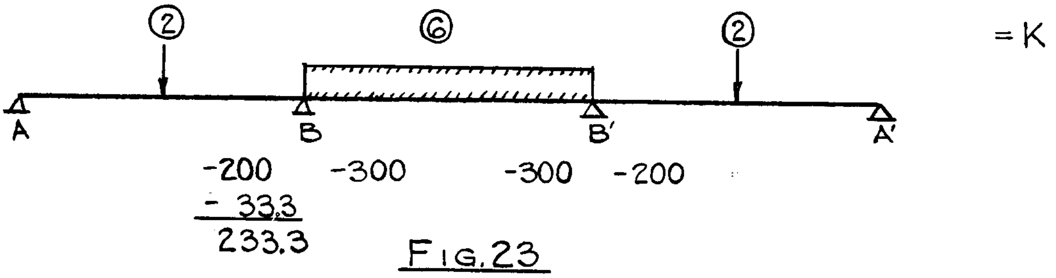


FIG. 23

2/6 · 100 = 33.3 to AB

4/6 · 100 = 66.7 to BB'

Moment at B = -233.3

If A and A' are fixed ends, 1/C for BA and B'A' is 4/3 and K/C = 2.67. The fixed-ended moments in these spans for symmetrical loading are 2/3 (-200) = -133.3 and the moment at B is:

$$-133.3 - (300 - 133.3)\,\frac{2.67}{6.67} = -200$$

Five Symmetrical Spans. Assume the K values to be as shown and the fixed-ended moment in the center span to be -300. (Fig. 24).

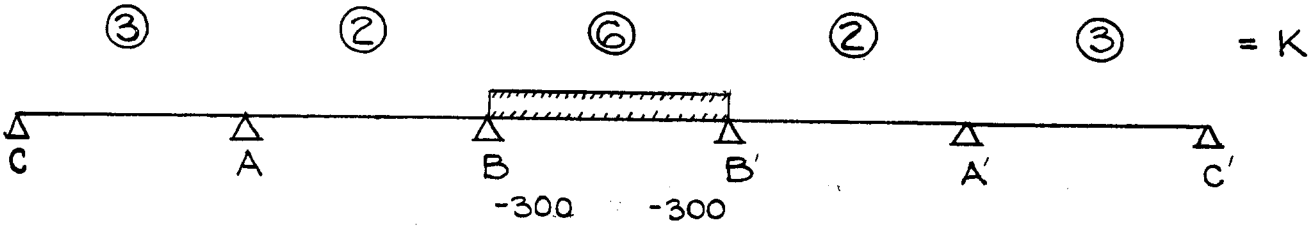


FIG. 24

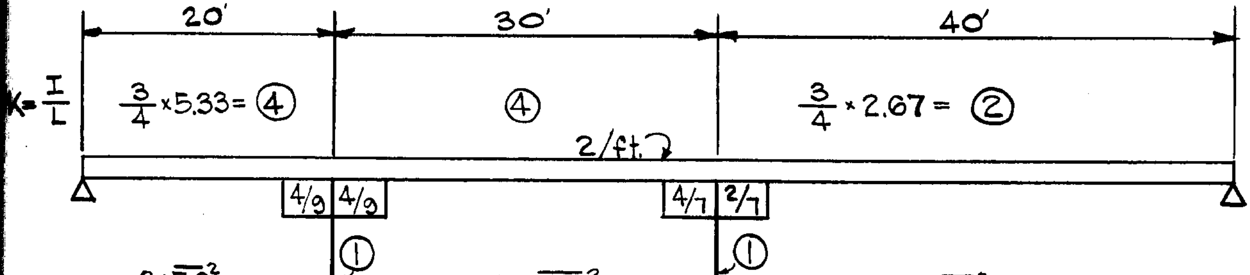
Here 1/C for BB' = 2/3 by symmetry and K/C = 4

$$1/C \text{ for } AB = \frac{1}{1 - 1/2 \cdot 3/5 \cdot 1/2} = 20/17 \text{ and } K/C = 2.35$$

$$\text{Moment at } B = \frac{2.35}{6.35}(-300) = -111$$

If spans CA and C'A' are loaded to give the same fixed-ended moments as BB', the 1/C for AB will be very nearly 2/3. And $M_B = 2/8\ (-300) = -75$.

Girder Frames and Viaducts Held Against Longitudinal Sway. Where the girder is rigidly attached to columns which are fixed or free at their far ends, it is not necessary to determine the moments in the columns by successive approximation. It is important to note that there are no moments coming



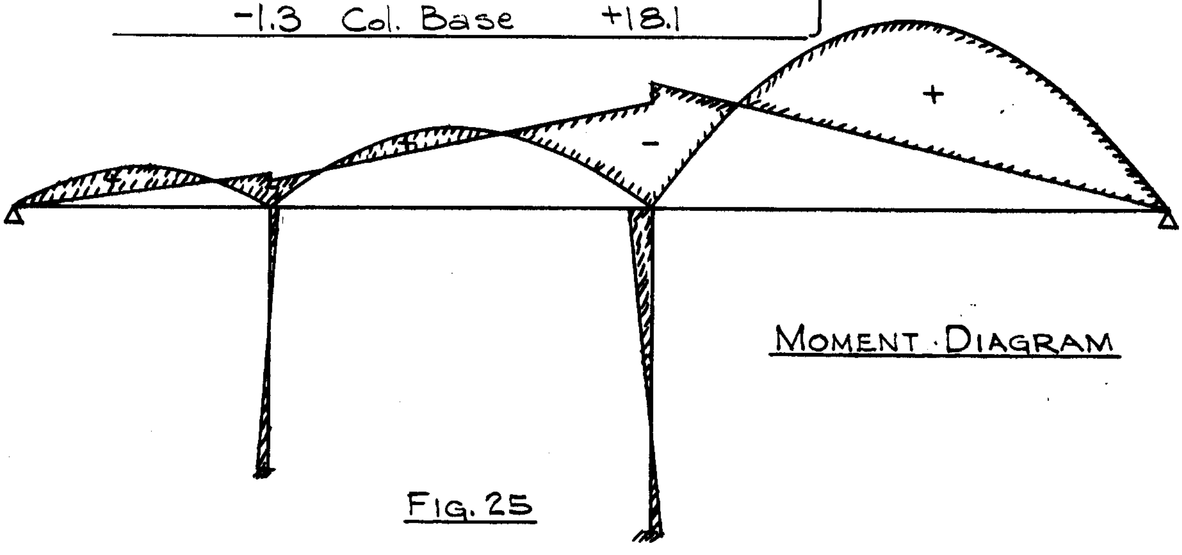
M' = 0	$\frac{2\times\overline{50}^2}{8}$ = -100.0	-150.0 = $\frac{2\times\overline{30}^2}{12}$ =	-150.0	-400.0 = $\frac{2\times\overline{40}^2}{8}$	0
	-22.2	+22.2	-143.0	+71.5	
		+71.5	-11.1		
	+31.8	-31.8	+6.3	-3.1	
		-3.1	+15.9		
	-1.4	+1.4	-9.1	+4.5	
		+4.5	-0.7		
	+2.0	-2.0	+0.4	-0.2	
		-0.2	+1.0		
	0	+0.1	-0.6	+0.3	
		+0.3	0		
	0	-0.1	0	0	

-89.8	-87.2	Girders	-290.9	-327.0
	+2.6	Column Top	-36.1	
	-1.3	Column Bottom	+18.0	

} Actual Moments

ALTERNATE SOLUTION – Omitting the two outside columns of figures

		-87.2	-290.9		Center Girder
	+12.8			+109.1	Δm - Col. and Side Girder
(1/5) =	+2.6	(1/3) =	-36.3		Δm in Columns
(4/5) =	+10.2	(2/3) =		+72.8	Δm in Side Girders
	-89.8	-87.2 Girders	-290.9	-327.2	Actual Moments
	+2.6	Col. Top	-36.3		Actual Moments
	-1.3	Col. Base	+18.1		Actual Moments



MOMENT DIAGRAM

FIG. 25

to the joints from the far ends of the columns and hence that it is only the moments coming through the girders from the unlocking of adjacent joints which need consideration. The moments in the columns can be found as the difference between the final values of the moments in the girders, and if there are two columns this difference can be distributed between them in proportion to their K's. In such cases where the column moment is very small, error results from using the method of differences. More exact values may be obtained by distributing the moments as in girders, but it must be realized that exact computations for small columns are illusory. Members are not lines and supports are not knife edges; neutral axes are not limited to established intersections and moments of inertia do not make sudden changes. Moments in columns are not necessary so serious as some computations may seem to show and very likely the values established by differences may be all we ordinarily need to know about them.

In Fig. 25 a simple girder frame is analyzed. The girder moments have been carried down on each side of the column and the moments in the column top obtained by subtraction. The moments at the bases will be one-half this value of opposite sign. It is, of course, unnecessary to carry down the moments for the outside girders. With the moments in the center girder known the others follow directly as shown in the alternate solution. Conditions of statics give the change in moment in column and flanking girder together (the sign is automatically determined by the condition that the sum of all moments is the same on both sides of the support and of the same sign). This total difference is then distributed between column and side span in proportion to their K/C values.

Since there is no standard convention of signs for bending moments in columns, it has been arbitrarily assumed that the top of a column is considered with the girder on the left, and the bottom of the column with the girder on the right. Positive moment at the top of a column then means that the column here has tension on the right hand side. Inspection of the actual moments here will show that the moments balance and so the signs are automatically determined.

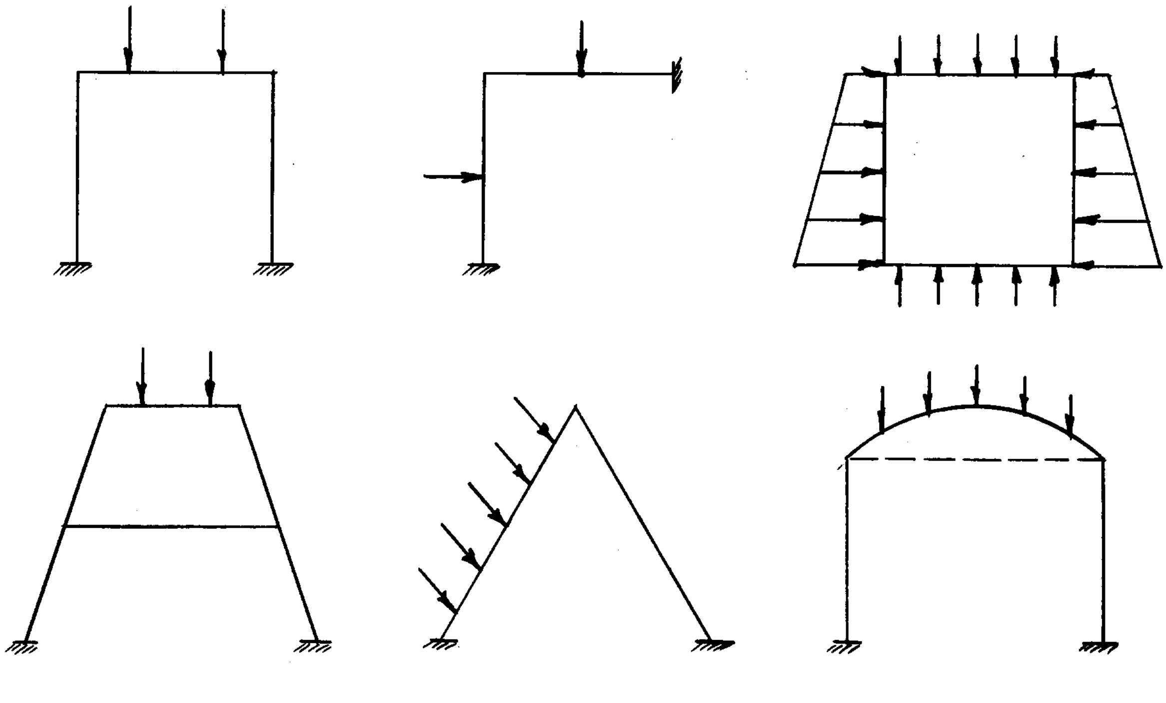


FIG. 26

<u>Simple Frames, Symmetrical or Braced Transversely</u>. It is evident that single square or trapezoidal frames, portals, L-frames, box culverts, and similar structures (see Fig. 26) act as simple continuous beams if there is no transverse deflection. If they are symmetrical as to form and loading, they will not deflect sideways and if they are restrained against sidewise movement, they cannot so deflect.

In these cases, the value of 1/C for the unloaded spans is easily determined by inspection. For the loaded span 1/C = 2/3 in symmetrical cases, and in unsymmetrical cases, this value may be taken as 2/3 and the dissymmetry corrected for, or the fixed-ended moments may be distributed by successive approximation. For the ordinary bent symmetrical as to form and loading, for the girder 1/C = 2/3, for the column 1/C = 1 if pinned at the bottom and 1/C = 4/3 if fixed at the bottom. Thus in Fig. 27, if the columns are fixed-ended, the moment in the column at the top

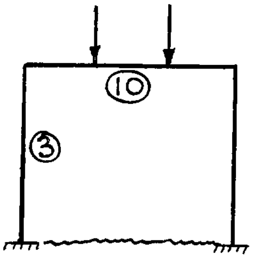


FIG.27

$$= \frac{3 \times 4/3}{3 \times 4/3 + 10 \times 2/3} = 6/16 = 37\text{-}1/2 \text{ per cent of}$$

fixed moment in girder.

If the columns are pin-ended, the moment at the top of the column is

$$\frac{3}{3 + 2/3 \times 10} = 31 \text{ per cent of the fixed moment in the girder.}$$

For the culvert shown in Fig. 28 the value 1/C = 2/3 for all members. The moment in the sides is 40 per cent of the fixed moment in top or sides. If the load is assumed to be concentrated on the top and uniform over the bottom, the value of C for the sides may be approximated closely. Successive approximation will give it exactly, if precision seems worth while.

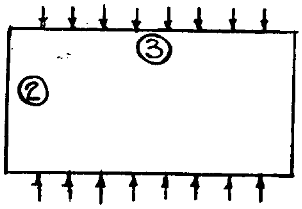


FIG.28

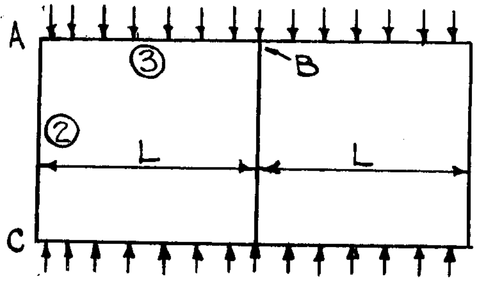


FIG.29

In the double-box culvert of Fig. 29 1/C for AB = 4/3 and for AC = 2/3.

$M_a = 2/8 \cdot 1/12\ wL^2 = 25$ per cent $(1/12\ wL^2)$

$M_b = 1/12\ wL^2 + 1/2$ relieving moment in girder at A - 137 per cent $(1/12\ wL^2)$

In the two-story bent in Fig. 30 the value of 1/C for AB may be assumed as about 120 per cent by inspection. More accurately 1/C = 2/3 for BB. 1/C = 4/3 for BC. The ratio of moments at A and B, then, is

$$M_b = -1/2\ M_a \frac{2/3\ (1) + 4/3\ (3)}{2 + 2/3\ (1) + 4/3\ (3)} = 35 \text{ per cent } M_a$$

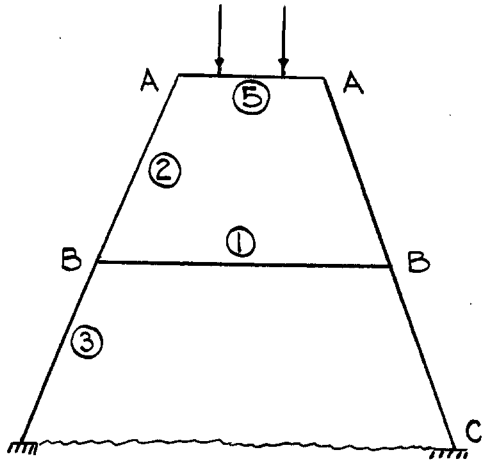


Fig. 30

$C = 1 - 1/2\ (.35) = .825 \qquad 1/C = 121$ per cent

K/C for AB = 2.42

K/C for AA = 2/3 (5) = 3.33

$M_a = \frac{2.42}{5.75}$ (fixed moment in girder) = 42 per cent (fixed moment).

Inspection in the first place would have shown this moment to be a little less than 4/9 of the fixed moment.

B. VARIABLE MOMENT OF INERTIA

Girders of Varying Section. The general method of moment distribution indicated above for prismatic beams - considering all joints fixed at first and then distributing successively the unbalanced moment among connecting members in proportion to their K values - is also applicable to beams in which the section varies within the span, provided the fixed-ended moments, stiffness, and "carry-over" factors are correctly determined.

In this case the area, location of the centroid and moment of inertia of the 1/I curve (the analogous column) are first determined. From this determine by the column analogy the end moments, the resistance to end rotation and the carry-over factor. The distribution of the moments then follows in the usual manner.

Following the procedure for prismatic beams, stiffness may be expressed in the general form $S = K/C$.

Where S represents the stiffness (moment at one end accompanying unit rotation at that end).

K is a physical constant depending on the dimensions of the beam and equal to the moment at one end accompanying a unit rotation of that end when the other end is simply supported.

C is a function of the carry-over factor and of the ratio of the end moments accompanying rotation (changes in moment from the fixed-ended condition) for one end, A. Now K may be determined directly from the column analogy by considering a hinge at B (Fig. 31). The centroid of the column then passes through B and the total elastic weight is infinite. Then for unit rotation at A, $M_a =$

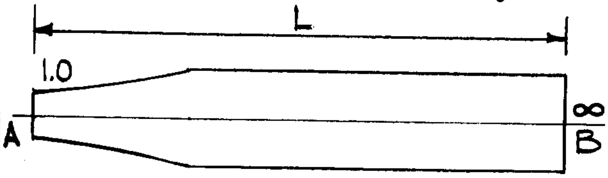


Fig. 31

$$K_a = \frac{1.L.L}{I_b} = \frac{L^2}{I_b}\ .$$

In this case $\phi_a = 1$ and ϕ_b has some other value. Now apply at B a moment M_b but hold A against rotation so that ϕ_a still equals unity. The moment at A will then be $M'_a - r_b\ M_b$ (where r_b is the carry-over factor at B or the ratio of moment at A to moment at B when A is fixed).

Then $M_a = S_a = M'_a - r_b M_b$

$$M_a + r_b M_b = M_a \left(1 + r_b \frac{M_b}{M_a}\right) = M'_a = K_a$$

$$S_a = M_a = \frac{K_a}{1 + r_b \frac{M_b}{M_a}} = \frac{K_a}{C_a}$$

Where $C_a = 1 + r_b \frac{M_b}{M_a}$

This is the general expression of which the K/C expression for prismatic beams is a special case. For prismatic beams the K value is 3EI/L (See Chapter I) the value 3 being common to all members. Hence it may be dropped with E and the K value taken as I/L.

The table shown in Fig. 33 gives values of 1/C for common cases. The values in the third column will be seen to be identical with those previously given for the 1/C factor for prismatic beams.

These relations are perfectly general and sufficient for the solution of any type of beam or condition of loading. For the common cases of variable section in concrete beams it will be found helpful to obtain these properties from tables such as those prepared by Strassner.* A conversion of these prepared by Walter Ruppel (Trans. A. S. C. E. 1926) gives values which may be converted by the following relations to give the elastic properties of the section. See Fig. 32.

The following diagram shows clearly the relations of the quantities given in Ruppel's tables to the elastic properties of the section, and to the fixed-ended moment for any condition of loading. The carry-over factor and the expression for stiffness are also given.

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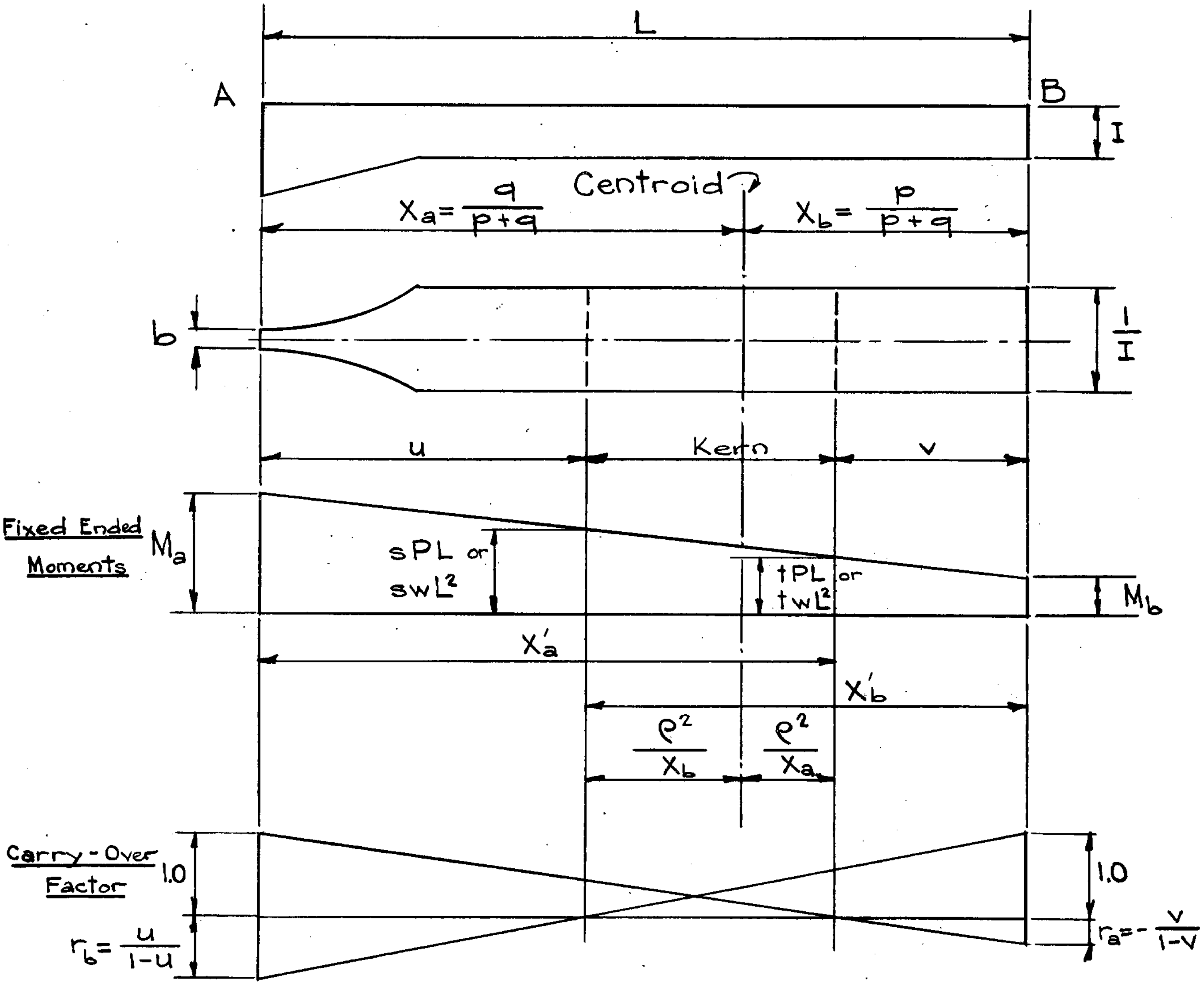
* "Neuere Methoden" by A. Strassner.



Stiffness Factor

$$K_a = \frac{1}{I_b} = \frac{1}{A x_b^2 + I_o} = \frac{1}{A\left(x_b + \frac{\rho^2}{x_b}\right)x_b} = \frac{1}{A x_b x_b'} = \frac{1}{\frac{L}{I}(p+q)\frac{p}{p+q}(1-u)} = \frac{1}{p(1-u)}\frac{I}{L}$$

$$K_b = \frac{1}{q(1-u)}\frac{I}{L}$$

FIG. 32

Tables of Properties--Moment Distribution.

Elastic Property	General Case	Prismatic Case
Stiffness	$S_a = K_a/C_a$	$S = K/C$ or $S = K$ (Relative Value)
Carry-over Factor	$r_a = M_b/M_a$ due to $\phi_a - 1$	-1/2
Stiffness Factor $K_a = M_a$ due to $\phi_a = 1$ When other end is hinged	$K_a = L^2/I_b$	$K = 3EI/L$ or $K = I/L$ (Relative Value)
End-Rotation Constant	$C_a = 1 + r_b \dfrac{M_b}{M_a}$	$C_a = 1 + 1/2 \dfrac{M_b}{M_a}$

Values of 1/C

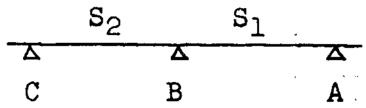
Description	Unsymmetrically Haunched	Symmetrically Haunched	Prismatic
Beam simply supported at far end	1	1	1
Beam fixed at far end	$\dfrac{1}{1 - r_a r_b}$	$\dfrac{1}{1 - r^2}$	4/3
Equal moments at two ends	$\dfrac{1}{C_a} = \dfrac{1}{1 + r_b}$	$\dfrac{1}{1 + r}$	2/3
Equal and opposite moments at two ends	$\dfrac{1}{C_a} = \dfrac{1}{1 - r_b}$	$\dfrac{1}{1 - r}$	2
Beam continuous with other beams 	$\dfrac{1}{C_a} = \dfrac{1}{1 - \frac{S_2}{S_1+S_2} r_a r_b}$	$\dfrac{1}{1 - \frac{S_2}{S_1+S_2} r^2}$	$\dfrac{1}{1 - \frac{K_2}{K_1+K_2} \cdot \frac{1}{4}}$
1/C varies from - - - - - - - -	1 to $\dfrac{1}{1 - r_a r_b}$	1 to $\dfrac{1}{1 - r^2}$	1 to 4/3
For equal S values and simply supported at C	$\dfrac{1}{1 - 1/2\, r_a r_b}$	$\dfrac{1}{1 - 1/2\, r^2}$	8/7

Fig. 33.

C. APPROXIMATE RELATIONS

Approximate Distribution--Prismatic Sections. Evidently if the C values due to continuity (not to loads) be known for the members meeting at a joint, the unbalanced moment at that joint can be distributed among the members in proportion to the values K/C and then distributed to the next joints and so throughout the frame.

For prismatic beams the value of C may vary from 1 to 3/4 (1/C varies from 1 to 4/3).

To compute more accurately, proceed as follows (Fig. 34):

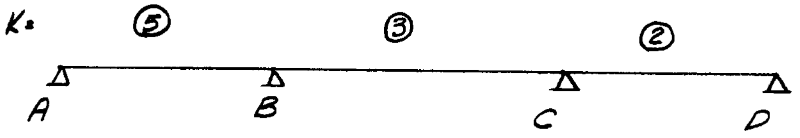


Fig. 34

A moment at B (AB being removed) will produce at C a moment $-1/2\ M_b$ if C is held fast, and if C is allowed to rotate, this will be distributed so that $M_c = +1/2 \cdot 2/5\ M_b$.

$$\text{Hence,}\quad \frac{M_c}{M_b} = -\frac{1}{2}\cdot\frac{2}{5}\ ,\quad \frac{1}{C} = \frac{1}{1-\frac{1}{2}\cdot\frac{1}{5}}\ ,\quad \frac{K}{C}\ (\text{for BC}) = \frac{3}{9/10} = 3.33$$

Apply a unit moment at A

$$M_b = -1/2\ M_a \text{ if B is locked}$$

$$M_b = -1/2\ M_a \frac{3.33}{8.33} \text{ if B rotates}$$

$$\frac{M_a}{M_b} = -.02 \qquad \frac{1}{C} = \frac{1}{1-0.1} \qquad \frac{K}{C}\ (\text{for AB}) = \frac{5}{9/10} = 5.55$$

If further it is observed that when the adjoining span is fixed-ended and its K is 1/3 that of the span considered, 1/C for the latter is 108 per cent and if the K ratio is 3, 1/C is 125 per cent, the estimates for K/C may be made even more closely. In general it will be observed that an error in K/C can result in a relative error only about one-fourth as great in distributing the moments, as may be seen by considering that the two connecting beams are respectively increased and decreased one-half of the error, the increase (decrease) being again approximately halved in the distribution.

In the case of prismatic beams we may go even further and neglect entirely the 1/C values except that K is increased 1/8 for fixed-ended beams and decreased 1/8 for beams free at the ends. This is equivalent to assuming 1/C = 1 for beams free-ended, $1/C = \frac{1}{7/8} = 114$ per cent for all interior beams and $1/C = \frac{9/8}{7/8} = 128$ per cent for beams fixed-ended. Since these values cannot be more than 10 per cent in error, we may expect the error in the moment to be not greater than two or three per cent of the unbalanced moment.

The distributed moments are readily carried on to other joints as $-1/2 \frac{K''}{K'+K''}$ (K" for the far beam, K' for the beam next to the joint) successively.

Of course, some cumulative error will occur, but the total error will be consistently within 5 per cent of the largest of the unbalanced moments.

If the moment of inertia varies within the span, such approximations as these are less satisfactory. If the carry-over factor is 0.75, the value 1/C may vary from 1 to $\frac{1}{1-(0.75)^2} = 2.28$ and intermediate values are more difficult to estimate.

We may, however, estimate approximately if we remember,

If the member is free-ended $1/C = 1$

If the member is fixed-ended $1/C = \frac{1}{1-r^2}$

If the next member is simply supported and its K value n times that of the given member $1/C = \dfrac{1}{1 - \frac{n}{n+1} r^2}$

This is equivalent to considering the stiffness of four spans in estimating the distribution at any joint instead of only two, as was suggested for prismatic beams.

It is proposed to use such estimates only for the full load moments, and hence the error in the unbalanced moments due to full load, which in general are not very large, are less serious.

It is important to consider that a small relative error in the negative moments at the supports may result in a large relative error in the positive moment at the center.

Approximate Live-Load Maxima. In the common case of building construction where the live loads are uniformly distributed, the exact determination of live load maxima involves the combination of n independent load conditions, where n is the number of spans. That is, by combining the effects of loads on individual spans we get all critical maximum positive and negative moments for live load and the moments for dead load. If the number of spans is not large, this is not tedious, either by successive moment distribution, or direct moment distribution depending on the K/C factor, or by graphical or semi-graphical methods. In most cases even then it requires more time than it is possible to give in ordinary design.

It is desirable then to determine approximate live load maxima by quick approximations, and in very many cases this is all that is practicable. The conventional rules for maxima, of course, are a recognition of this fact, but accuracy has been sacrificed to an extreme degree.

Such considerations make the following relations important.

Maximum moments due to live load may be estimated from the computed moments due to full load. There are, in this connection, many interesting relations of which the following seems the most useful.

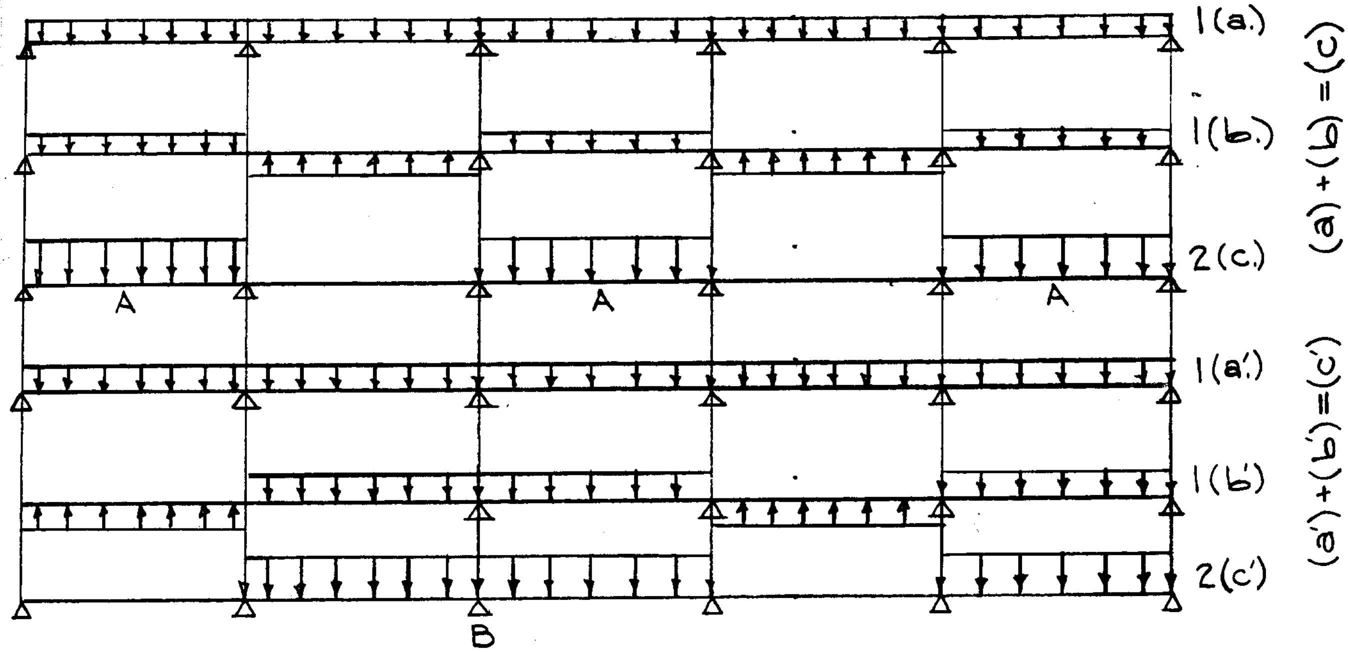


FIG.35

If for uniform load we combine the effect of full loading with that of upward and downward loading of selected spans, we get as a result, twice the maximum moment for live load. This is shown diagrammatically in Fig. 35, (a, b, c) for maximum positive moment at Points A and in (a', b', c') for maximum negative moment at the support B.

Now for uniform load, and similar beams, the loading (b) is the same as alternate upward and downward loads on a series of simple beams. The loading (b') is very nearly the same as loading on beams simply supported except at B and fixed at B.

If the beams are of unequal spans this will not necessarily be true, because the slopes at the ends of the adjoining beams when simply supported and uniformly loaded will not be equal. The end slope of such a beam due to uniform load varies as L^3/bd^3. This matter is chiefly of interest in reinforced concrete design and there in general b is approximately constant for a series of beams and d is proportional to L. Hence, in general this will be a close approximation in reinforced concrete.

If the spans are unequal and the assumption of constant breadth and L/d ratio does not hold, the load in the adjoining span which would equalize the slopes of the simple beams may be estimated and the remaining fixed-ended moment distributed. This is illustrated below.

We can now say that the maximum live load moment at center of span can be found as 1/2 (full load moment ± simple beam moment), and the maximum live load moment at support as 1/2 (full load moment ± moment at support when adjoining spans are freely supported at their ends). This is strictly true if b and L/d are constant; otherwise a correction should be considered.

<u>Approximate Effect of Haunches</u>. Of course, all approximate methods become more difficult to apply as complications are added to them and approximate solutions of haunched beams are difficult to make.

The effect of such haunching as ordinarily occurs in concrete beams may be estimated by the following rules:

Let the area of the side elevation of a symmetrically haunched beam be (1+A') times that of a beam of uniform section having the same center depth.

A. Where the haunching is at one end only:

a. Multiply the fixed-ended moment for symmetrical loading by (1+2A') at the haunched end and by (1-A') at the end not haunched.

b. The carry-over factor at the haunched end is unchanged but is increased to (1/2 + A') at the end not haunched.

c. Multiply the stiffness (moment at one end corresponding to a unit rotation of that end when the other end is free to rotate) by (1+3A') at the haunched end but leave it unchanged at the end not haunched.

B. Where the haunching is at both ends, the effects of haunching each end separately may be added -- in other words,

a. Multiply the fixed-ended moment for symmetrical loading by (1+A').

b. Increase the carry-over factor to (1/2 + A').

c. Multiply the stiffness by (1+3A').

While these rules were arrived at partly by analysis, it is evident that their proof is largely empirical. The accompanying table, Fig. 36, is evidence of their accuracy. They should not be extended much beyond the limits of this table -- that is, for haunches to the quarter point and of a depth equal to that of the beam. For haunches slightly longer and deeper than this--say to the third point and one and one-half times the beam depth the error due to taking the values for quarter haunches of unit depth will not be very great. In formulating the rules, accuracy has to some extent been sacrificed to simplicity, but the discrepancies are less serious than might appear. Errors in the fixed-ended moments appear in full as errors in the final moments at the supports, but a little consideration will show that errors in the carry-over factor or in the stiffness affect the final result relatively only about one fourth as much.

These rules are, then, probably dependable within five per cent for haunches not longer than to the quarter span nor deeper than the center depth of beam. There is, in general, little occasion for greater haunching than this and the rules probably cover three-fourths of the cases which commonly arise. It is worth noting that the rules may also be applied to continuous steel girders which are heavily strengthened at the supports by the addition of cover plates, provided the depth of equivalent haunch be determined as proportional to the cube root of the moment of inertia of the section.

For special cases and for heavy unsymmetrical loads, such tables as those prepared by Strassner or Ruppel may be used to advantage in determining the end moments and the elastic properties of the section.

Comparison of Exact and Approximate Constants for Haunches

a	d	A' (%)	Haunched One End Only: Fixed-Ended Moments: Uniform Load: Haunched End	Uniform Load: Unhaunched End	Centre Load: Haunched End	Centre Load: Unhaunched End	Third-Points: Haunched End	Third-Points: Unhaunched End	Carry-Over Factor: Haunched End	Carry-Over Factor: Unhaunched End	Stiffness Factor: Haunched End	Stiffness Factor: Unhaunched End	Haunched Both Ends: Fixed-Ended Moments: Uniform Load	Fixed-Ended Moments: Centre Load	Fixed-Ended Moments: Third-Points	Carry-Over Factor	Stiffness Factor	
0.15	0.19	2.8	105	97					50	54	110	100	105			54	110	Straight Exact
									50	53	108	100	103			53	108	Approx.
		1.9	104	98					50	53	107	100	104			53	107	Parabolic Exact
									50	52	106	100	102			52	106	Approx.
	0.49	7.4	115	93					49	60	122		110			58	122	Straight Exact
									50	57	122		107			57	122	Approx.
		4.9	110	95					50	58	113		108			56	113	Parabolic Exact
									50	55	115		105			55	115	Approx.
	0.71	10.7	121	89					49	61	128		113			62	128	Straight Exact
									50	61	131		111			61	131	Approx.
		7.1	114	93					50	58	119		110			58	119	Parabolic Exact
									50	57	121		107			57	121	Approx.
	1.02	15.3	131	85					48	65	134	100	115			63	130	Straight Exact
									50	65	146	100	115			65	146	Approx.
		10.2	120	90					50	61	123	100	112			60	123	Parabolic Exact
									50	60	130	100	110			60	130	Approx.
0.20	0.19	3.8	113	95					49	56	113	100	106			55	113	Straight Exact
			108	96					50	54	111	100	104			54	111	Approx.
		2.5	109	96					50	54	109	100	104			53	109	Parabolic Exact
			105	97					50	53	108	100	103			53	108	Approx.
	0.49	9.8	125	88					48	62	130		113			60	130	Straight Exact
			120	90					50	60	129		110			60	129	Approx.
		8.5	120	91					49	59	120		110			57	120	Parabolic Exact
			117	91					50	59	125		109			59	125	Approx.
	0.71	14.2	133	84					48	66	138		116			63	139	Straight Exact
			128	86					50	64	142		114			64	142	Approx.
		9.5	126	88					49	61	126		112			60	126	Parabolic Exact
			119	90					50	60	128		110			60	128	Approx.
	1.02	20.4	144	81					48	70	147	100	119			66	147	Straight Exact
			141	80					50	70	161	100	120			70	161	Approx.
		13.6	132	86					49	64	132	100	115			62	132	Parabolic Exact
			127	86					50	64	140	100	114			64	140	Approx.
0.25	0.19	4.8	113	94					49	57	117	100	107			55	115	Straight Exact
			110	95					50	55	114	100	105			55	114	Approx.
		3.2	109	95					49	55	111	100	105			54	111	Parabolic Exact
			106	97					50	53	110	100	103			53	110	Approx.
	0.49	11.3	131	87					48	65	138		115			62	137	Straight Exact
			123	89					50	61	133		111			61	133	Approx.
		7.5	122	88					49	61	125		111			59	125	Parabolic Exact
			115	92					50	58	122		108			58	122	Approx.
	0.71	17.8	141	82	149	79	146	80	47	70	150		118	122	121	65	149	Straight Exact
			136	82	136	82	136	82	50	68	153		118	118	118	68	153	Approx.
		11.9	131	86	135	85	133	85	48	64	134		114	117	115	62	133	Parabolic Exact
			124	88	124	88	124	88	50	62	135		112	112	112	62	135	Approx.
	1.02	25.5	151	78	159	75	155	77	46	75	165	101	122	126	123	69	163	Straight Exact
			151	74	151	74	151	74	50	76	176	100	126	126	126	76	176	Approx.
		17.1	139	83	145	74	144	76	48	68	142	100	118	119	118	64	142	Parabolic Exact
			134	83	134	83	134	83	50	67	151	100	117	117	117	67	151	Approx.

Fig. 36

Approximate Column Moments. There has been some tendency to emphasize the importance of bending moments in building columns. A conventional solution by a line diagram may be made as follows:

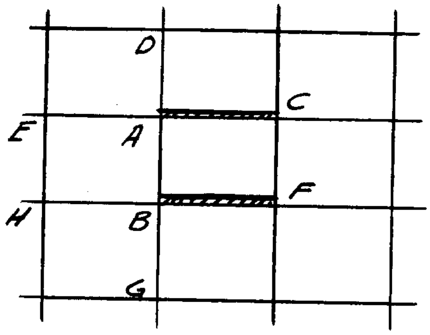


Fig. 37

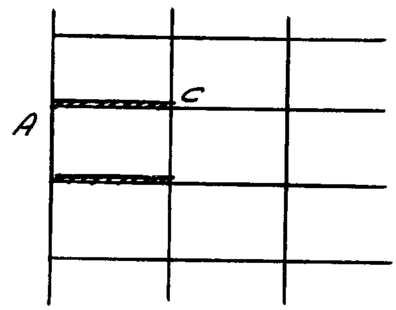


Fig. 38

Maximum moment at A, Fig. 37, occurs when, (a) the unbalanced moment at A is maximum; (b) the stiffness of the column at A is maximum and that of other members connecting at A is minimum. This indicates the loading shown. For the column 1/C = 2, for the girder AC 1/C = 2/3, for the column AD and the girder AE, 1/C is whatever the end restraints determine--say 120 per cent.

If the column is a wall column the same relations apply approximately except that 1/C for AC, Fig. 38, is slightly greater than 2/3 (say 3/4). A corner column is like any other wall column except that it is subject to bending in two directions. A roof column differs chiefly in omitting the connecting column at A. The whole subject will be found worth further study by comparing approximate and exact values.

Interior columns are usually subject to very little moment from dead load while wall columns are subject to moment from both dead and live load.

The relations are not so simple as here indicated, however, because, (a) allowance should be made for the stiffening effect of the girder on the column, the moment of inertia of the column being very great (infinite) after it joins the girder; (b) if the girders are haunched the lower column takes much more moment than the upper column; (c) the moment found above is not very significant and must be corrected to give the moment at the top or bottom of the girder; (d) time yield will relieve somewhat any moments due to dead load, but time yield may be just as significant an element in failure as is overstress; (e) column moments are less serious than direct loads because they partake of those characteristics of secondary stress which are discussed elsewhere while the fact that they are columns and are designed as such makes them relatively very important; (f) it is possible to haunch the girders to such an extent as to make the girders act largely as cantilevers from the columns and, in the lower stories, this can be done without adding any material to the columns, because of the increased stresses usually permitted in columns subject to bending. Whether such designs should be encouraged depends on the soundness of current ideas of the effect of combined stresses in columns.

An extreme case of such designing would place practically a hinge at the center of each girder or slab span and depend on the column to prevent tipping over.

Column moments may become important in some cases - in viaducts they often control the design - but their importance in buildings has sometimes been exaggerated.

Approximations for I and K. The values (relative) of I and K must be determined before any analysis is possible. It has several times been suggested above that great accuracy in this regard is not necessary. The following approximations are suggested:

In concrete beams $I \propto bd^3$.

In flat slabs $b = L/2$.

In a series of concrete beams, Relative $I \propto d^3$.

In normal concrete design L/d is approximately constant and $K \propto d^3/L \propto L^2$. (This is useful for preliminary design).

For steel members $I \propto Ah^2 \propto wh^2$.
(Roughly, h = width, A = area, w = weight per foot).

In a series of girders of uniform depth $K \propto L$.
(Based on the assumption $M_{max} \propto L^2$

$$I \propto M_{max}\ I/L \propto L^2/L \propto L).$$

For a plate girder I = (Gross area flanges + 1/3 gross area web) $(h^2/2)$.

Outline for Preliminary Design. The following outline is suggested for preliminary design of a series of reinforced concrete girders:

1. Assume $K \propto L^2$.
2. Approximately determine moments for full unit uniform load. Call these $M_{F(ull)}$.
3. Find moments for unit uniform loads on simple beams. Call these $M_{S(imple)}$.
4. Find moments at each support for unit uniform load on adjoining spans these being considered freely supported at their far ends. Call these $M_{C(olumns)}$. Estimate any corrections.
5. Distribute any moments due to cantilevers or special concentrations. Call these M_P.
6. Dead load $M_F D$ (D is dead load per foot).

Live load maxima $\dfrac{M_F \pm M_S}{2} L$ at centers of spans.

$\dfrac{M_F \pm M_C}{2} L$ at centers of supports.

(L is live load per foot).

Special moments M_P

Correct the moments at the supports by subtracting the end shear times the half column width to get moments at column face.

Design, haunching if preferred.

7. Revise for effect of haunching by increasing the computed maxima at the supports in the ratio of the area of the haunched to the unhaunched beam.
8. Revise and make final computation of moments.

D. LIMITATIONS OF THE THEORY OF CONTINUITY

Underlying Assumptions. In considering the usefulness of any method of analysis of continuous girders, it is essential to consider the limitations imposed by the assumptions.

The assumptions underlying the above analyses are:

First: The differential rotation of the two ends of the axis in a short length of the beam is directly proportional to the bending moment in this length because the elastic properties of the materials are not changed by the intensity or duration of the loading. In steel structures this is correct up to the elastic limit; the ratio of stress intensity to deformation is constant. In structures of reinforced concrete the ratio of stress intensity to deformation varies both with the intensity of the stress (Hooke's Law does not hold) and with the duration of the stress (time yield occurs). In timber structures the same is true to an even greater extent.

Second: The relative rotation of two ends of a short length of the axis is proportional to a physical constant for that section, which is L/I for homogeneous beams and which is also so treated in beams of reinforced concrete, though, of course, here the term moment of inertia has a meaning only by analogy. In steel this assumption is approximately correct, in timber it is roughly correct, while in reinforced concrete it is far from correct.

Third: Distortions due to shear and direct thrust are negligible. This is true in all cases except such short, deep girders as only a novice would attempt to analyze exactly by the above theory.

Fourth: The neutral axes of all members at any joint meet in a point and the sections which the members have outside the joint persist to this point.

Limitations on the Value of I in the Analysis of Concrete Structures. The value of I, representing the elastic properties of the section, may perhaps best be studied by a consideration of the imaginary transformed section. This section may be affected by,

(a) The value of n.
(b) Plasticity of the concrete in compression.
(c) Tension in the concrete in the zone where there are no cracks on the tension side.
(d) Tension in the concrete between the cracks in the zone where there are cracks.

Factors Affecting Continuity. Four principles need emphasis at this point:

(a) The theories in use for the safe computation of strength in beams has almost no bearing on the problem of the effect on continuity on bending moments.

(b) It is only the relative values -- not the absolute values of the differential angle changes which is to be studied.
(c) Studies of relative deflections of beams are of value only in a presumptive way -- it is the end slopes and not the deflections which are to be investigated. Since, however, in a simple beam, the end slopes correspond to end shears and the deflection to the maximum moments due to the angle changes, studies of deflection evidently have some bearing on slope computations.
(d) The conditions for which the slopes are to be studied are for the stress conditions actually existing in the beams at or near failure due to bending moment. What constitutes failure need not here be discussed. It is here considered to be at a computed fibre stress in the steel of about 30,000 lb. per sq. in.

Use of the Transformed Section in Reinforced Concrete. At and near the point of contraflexure, the concrete on the tension side is uncracked and there seems little doubt that the full transformed section can be used.

In the area of high positive moment, the full compression flange is in action and some tension exists in the concrete - the amount varying from zero at a crack to the full tensile strength half-way between cracks. It is realized that consideration of such tension together with tension in the steel invalidates the assumption of planar section conservation. At a point midway between cracks plane sections are perhaps preserved, the warping being in opposite directions as the two cracks are approached and being therefore to some extent compensating.

Measurements of deformation in the steel at the supports of concrete building girders show less elongation than would be expected on any assumption of continuity. The most probable explanation is that the floor slab carries a good deal of tension even up to high loads, and hence that there the section in action approaches the full transformed section.

In order to form some idea of the effect of these factors on the moment of inertia of the transformed section, the following figures are presented:

Girder - total depth 30 in., stem 12 in. wide, flange 60 in. x 6 in., reinforcement 2.56 sq. in.

Moment of inertia of transformed section -

Full concrete and steel in bottom	63,500 $in.^4$
Full concrete without steel	41,000
Steel in action in bottom but concrete cracked to 10 in. from bottom	32,500
Steel in top, half of flange in action	41,700
Steel in top, no flange	32,500
No steel, no flange.	

Effect of I Variations on the Moments. Without pursuing this matter further, it is possible to proceed to an investigation of the effect which variations of such magnitude as are here indicated produce.

If in a beam we assume a variation in moment of inertia from unity at the center line to one-half as much at the support, the section being constant from center to quarter-point, this is perhaps the extreme probable case. The effect of this variation will be chiefly on the fixed-ended moments and scarcely at all on the distribution of the unbalanced moments at the joints since the stiffness of all beams is probably affected in the same way.

With the variation assumed above of from one at center to one half at support, the fixed ended moment on the beam will be 92 per cent of that with constant I, for uniform load and 89 per cent for load concentrated at center, while if the I at support is 1/3 that at the center these ratios are 87 per cent and 81 per cent respectively.

In general, then, the effect of the variable I of the girder is to reduce the fixed-ended moment, the effect probably being about 10 per cent. On the other hand, this fixed-ended moment is increased by the resistance offered by the column to any bending of the girder between the center of the column and the column face. If the I at the center line of the column be assumed as ten times that at the center of girder and the width of the column one-tenth the span of the girder center to center of columns, the effect is to increase the computed end moment about 5 per cent.

In view of these facts it seems probable that the moments which would exist in the girders if the joints are not permitted to rotate are correct within about five per cent if the girder be treated as having uniform moment of inertia. Any alleged precision in computations greater than 5 per cent of the moments at the supports is probably illusory in structures of reinforced concrete.

<u>Effect of Variations in Quality of Concrete</u>. It is now necessary to consider to what extent the variation of the quality of the concrete from beam to beam will affect the distribution of the unbalanced moments at a joint. It is true that beams apparently identical may show a large variation in deflection for the same loads, but it is to be noted that if the fixed moments are as computed it is only the distribution of their difference which will be affected by such variations in E. Further it should be noted that if two adjoining girders at a joint, apparently identical, should actually differ in their E values by, say, 50 per cent, then the difference of the fixed-ended moments would be distributed (the far ends remaining fixed) 60 per cent to one and 40 per cent to the other instead of 50 per cent to each as would be indicated by their identity.

It seems then, that such uncertainties as to elastic action as one is led to expect from existing data do not seriously affect the results.

<u>Effect of Width of Flange and Stem</u>. While such investigations are not at all conclusive, they seem to lend support to the usual practice of using bd^3 as a measure of the I value for concrete beams. The value of b is very uncertain but it is probably about the same for all beams in a series. Where relative values for beams and columns are involved, the value of b for the flange seems more reasonable. It is certainly greater than that of the stem and is just as certainly dependent on the relative proportions of flange and web and other factors. A value one-half that used in the design of the flange leans to the side of safety in column design, while in studying the beam it makes little difference. Obviously, no general rule will cover all the variable elements.

It should at least be noted from the above that great precision in this work is absurd in concrete design. The moments at supports cannot be computed more exactly than $\pm$ 10 per cent, but it seems probable that they can be computed with about this accuracy.

<u>Live Loads for Maximum Moments</u>. In order that moments in continuous frames may represent anything but an academic abstraction, it is necessary to consider not only the maximum live load moments but also the probability of their occurrence. It is absurd to design with the usual working stresses for loadings the occurence of which is highly improbable. It is certainly very difficult, in structural design, to set up any scientific procedure for determining the probability of occurrence of a given condition of loading or to establish a method of increasing working stresses where the probability of occurrence is small. Little investigation seems to have been done along this line, but the principle involved has been recognized,

(a) in the consideration of stresses for maximum congested loading on long span bridges,
(b) in reducing the live loads on column carrying many stories in building,
(c) occasionally in reducing the loads on girders carrying a large area of floor space,
(d) in allowing an increase in working stresses for combinations of wind and centrifugal forces with live loads,
(e) sometimes vaguely by neglecting the effect of some forces.

It is generally felt that the worst conceivable combination of loads shall produce stresses not exceeding about 80 per cent to 85 per cent of those which are thought to represent the limit of safe usefulness, (elastic limit of steel or ultimate strength of concrete).

<u>Rules for Combined Loading in Building Construction</u>. In bridge work this matter becomes very complex. In ordinary building construction, however, it seems possible to lay down certain definite rules which are justified by common sense and which simplify the analytical procedure.

The follwing rules are suggested in this regard:

First - Vertical live loads only on continuous unbroken areas on any floor level need be considered in design.

Second - For combinations of horizontal and vertical forces, where the two are independent in their action, an increase of one-third of the working stresses may be permitted.

This clause will restrict the loading to the single span under consideration for maximum positive moment at the center and to the two adjoining spans for maximum moment at the support.

<u>Factor of Safety in Reversals of Stress</u>. Another question of importance concerns the factor of safety in those cases of loading where the dead and live load moments are of opposite values. This occurs in computing the negative moment at the center of a span, especially of short spans adjacent to long spans. In such cases an increase in the value of the live load produces a disproportionately large increase in the value of the negative moment. The case is similar to that occurring in the design of counters in bridge trusses. Some provision should be made for this, perhaps by reducing the live load for discontinuity as suggested above except in those cases where the live load stress is opposite in character to that due to dead load.

E. OTHER SOLUTIONS OF CONTINUOUS BEAMS

Principal Methods. Numerous other treatments of continuous girders may be given, all of which have their own advantages and disadvantages. As in all such cases, a little study will indicate the mathematical identity of these methods. They may be grouped as:

(a) Methods considering the reactions as redundants.
(b) Methods considering the moments at the supports as redundants.
(c) Methods considering the rotations at the supports as redundants.
(d) Treatment of a loaded span as a fixed beam with rotating abutments.
(e) Graphical analyses using the fixed points.

The Reactions as Redundants.

This is illustrative of the classic treatment of indeterminate structures. It may be illustrated by the case of a three-span beam, Fig. 39:

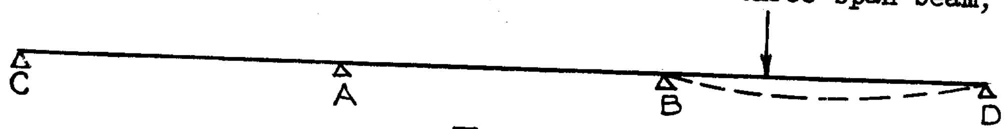


FIG. 39

If reactions A and B are taken as the redundants -

Total deflection at A = (deflection at A due to loads on simple span CD) + (deflection at A due to R_B = 1) R_b + (deflection at A due to [illegible]) + (deflection at A due to R_A = 1) R_a = 0 or, in shorthand,

$$\Delta^a_T = 0 = \Delta^a_{\Sigma P} + R_b\,\Delta^a_{R_b=1} + R_a\,\Delta^a_{R_a=1}$$

Write as many such equations as there are redundant supports and solve for the redundant reactions. All other values then follow by statics.

The method is important as an illustration of principles and is sometimes convenient in drawing influence lines.

The Moments at Supports as Redundants.

In Fig. 39 write (Slope at B of span BD due to simple beam action of the loads in span BD) + (Slope at B of span BD due to negative moment at B acting on the simple span BD) = (Slope at B of span BA due to negative moment at B acting on simple span BA) + (Slope at B of span BA due to positive moment at A acting on simple span BA), or, in shorthand,

$$\phi^{bd}_{\Sigma P} + M_b\,\phi^{bd}_{M_b=1} = M_b\,\phi^{ba}_{M_b=1} + M_a\,\phi^{ba}_{M_a=1}$$

Again write as many such equations as there are redundant supports and solve for the moments at the supports.

The above statement - in plain English, if continuity exists, the slope is the same on two sides of any support - is all that is stated by the equation of three moments. This was first stated for prismatic beams by Claperon in 1857 and has since been often modified to include such complications as settling supports and variable moment of inertia.

Though the methods used in this chapter have preferably been derived by a different method of reasoning, it will be seen that what we have been doing is to solve a series of equations of three moments by successive convergence.

Rotations at the Supports as Redundants.

This is the method which, when applied to prismatic beams, is known in American literature as the method of slope-deflection. As many equations are written as there are redundant supports, these are solved for the rotations, and from the rotations the moments are determined. The equations are the equations $\Sigma M = 0$ at each joint. The method is discussed further in the chapter on Displacements.

So nearly are the methods of solution identical in their methods of reasoning that it is difficult to say whether the method used in this chapter does not properly belong with slope-deflection rather than with the equation of three moments.

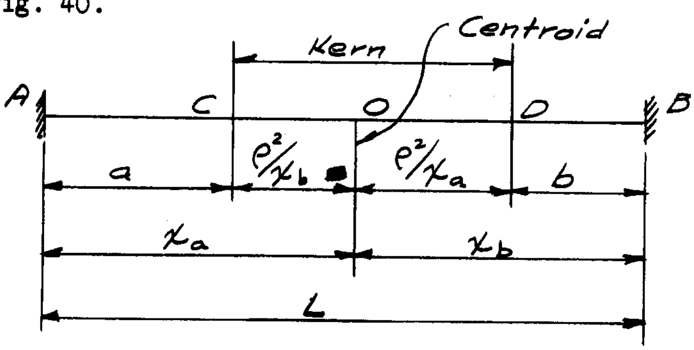
Loaded Span as a Fixed Beam with Rotating Abutments.

Where only one span is loaded, the effect of flanking spans is evidently to permit rotation at the ends of the loaded span. Hence, the flanking spans may be replaced by their elastic weights (rotation at end due to unit moment at end) and the loaded span may then be analyzed as a fixed-ended beam by the column analogy.

The analysis follows to some extent that for the computation of fixed points. For fixed points we determine the K/C values for flanking spans, which gives the moment at the end of the flanking girder accompanying unit rotation of that end. The reciprocal is the rotation due to unit moment. Hence, $w = C/K$.

The fixed points of the span under consideration are the kern points for this span considered as a column with the elastic weights of the flanking spans at the ends. This is true because a moment at one end of a fixed-ended beam will produce no moment at the kern point corresponding to that end and this fact defines the fixed point. If, then, we have completed the computation for fixed points, this gives the total area and kern points for each span considered as a column.

From the kern points the centroid is readily located by the relation, Fig. 40.



$$X_a = \frac{a}{a+b} L$$

$$X_b = \frac{b}{a+b} L$$

Fig. 40

This follows because $CO = \rho^2/X_b$ $\quad OD = \rho^2/X_a$ $\quad CO/OD = X_a/X_b$

Hence the centroid divides the span and the kern into proportional segments.

$$\left[L - (a + b)\right] X_a/L + a = X_a \qquad X_a = \frac{a}{a + b} L$$

This gives all elastic properties of the column and the usual construction for end moments then applies for any given loading, the moment load being computed for a simple beam (moment at ends zero).

Graphical Analyses Using the Fixed Points and Kern Points. The graphical analyses are derived or derivable from certain relations first indicated by Claxton Fidler. Fidler's graphical constructions have been extended and modified by others, - chiefly by Professor Ostenfeld of Copenhagen.

Fidler's construction is based on the fixed points as we have defined them, their location being a property of the structure independent of the loading and on what he calls "characteristic points" which are defined below.

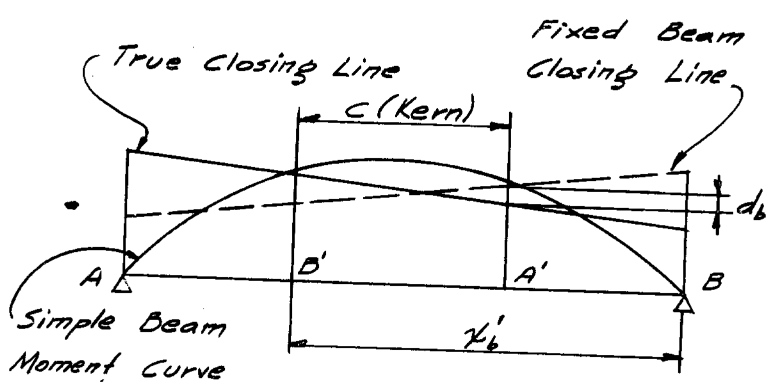


Fig. 41

If any loaded span is considered as fixed-ended and the closing line drawn, Fig. 41, it will cut verticals through the kern points of that span considered as fixed-ended (not as continuous) in the characteristic points. If now the ends of the span rotate, the closing line will cut these verticals above or below these characteristic points and the distance along a vertical from the characteristic point to closing line is a measure of the rotation of the corresponding end of the girder. This, $d_b \propto \phi_b$ in the figure.

This property of the characteristic points follows directly from the column analogy, for rotation at A produces no change in moment at kern point B', while rotation at B produces at A' a change of moment $d_b = \phi_b \dfrac{X_b'C}{I_o}$.

Also, the characteristic points are readily determined from the column analogy when the moment load and its centroid are known by drawing the closing line for the fixed-ended beam.

Fidler's main thesis was that the closing line must go as far above the left characteristic point in the span on the right of a given support as it did above the right characteristic point of the span on the left, provided the spans were alike. His idea was that this could be sketched in with all necessary accuracy. This is excellent if the complications of haunching, columns, or varying spans and depths are not present.

Fidler showed how to compute the effect of these factors and Ostenfeld has put the entire construction on a graphical basis.

Fig. 41a shows the more useful constructions. The reader is referred to Trans. Am. Soc. C. E.* 1926 for further details and for proofs as well as for references to the original sources.

The fixed points having been found, we find for a load system in any one span the end moments. The moment curve then passes in turn through the fixed points of the other spans. When moment curves for loads in each span have been determined, they can be combined to give dead load moments and maximum live load moments.

The constructions may be extended to give the moments when any number of spans are loaded, but where maximum moments are wanted with live load, the construction for loads in single spans is to be preferred. There has been much variation in the details of the construction and in the form of proof given. Space forbids further elaboration.

Moment Distribution. After this review it will be realized that the method of moment distribution has been emphasized for several reasons. It permits one to see readily what will be the effect of variations in design or in physical action of continuous girders. For determination of moments due to dead load it is as rapid as any method and more rapid than most of them and simple considerations as indicated above permit by this method a quick estimate of live load maxima from the dead load moments.

* "Moments in Restrained and Continuous Beams." Paper and especially discussion. Trans. Am. Soc. C.E., 1926.

Orig. page 45 is page 100a in 1950 reissue.

CHAPTER III

GEOMETRY OF DEFORMATIONS - VIRTUAL WORK

General Deduction - Displacement. The fundamental relations which determine in structures the distortions produced by internal strains may be stated in the mathematical language of the calculus and of geometry and in the engineering terminology of moment and shear, product of inertia, and moment of inertia. These theorems are, however, correlated by use of the principle of virtual work. While essentially a principle of geometry, the principle of virtual work has proved the most powerful tool ever applied to the analysis of statically indeterminate structures. Assume any rigid body acted on by a force and reactions and let it be required to determine the deflection Δ in the direction ab of any point a. Consider first the stretch--call it δ -- of any differential fibre. Now if we consider a unit hypothetical resistance to motion at a to produce a stress u in the fibre, then the external work will be $1 \times \Delta$ and the internal work will be $u \times \delta$, then,

$$\Delta = u\delta$$

Each fibre produces its independent deflection and hence,

$$\text{Total } \Delta = \Sigma u\delta$$

If the internal hypothetical stress resists the distortion, the internal work will be positive and the external movement opposite in direction to the hypothetical load. The same result follows from the more convenient rule which treats lengthening and tension as positive, and indicates a positive displacement in the direction of the hypothetical force when the algebraic summation, $\Sigma u\delta$, is positive.

Reference has been made to the stretch of a differential fibre. We might, without affecting the argument, have referred equally well to the shearing distortion of a differential cube, to the rotational distortion of a differential cylinder, to internal angular displacement, or to any internal distortion.

Also, it is possible to deal with external rotation at a instead of translation. The hypothetical unit force is then a unit moment. Then

$$\text{Rotation at } \underline{a} = \Sigma u\delta$$

in which u is the stress in each small particle due to a unit moment at a and is the distortion actually existing in the particle.

Since the equation of virtual work as thus stated involves no work done by the reactions from the hypothetical unit force, it must have been assumed that they do no work, and hence that they do not move. Or, it may be stated that these reactions do not have any motion which appears as a part of the described displacement, and hence the reactions to the hypothetical unit force exist at those points which fix the line with reference to which the desired displacement is measured.

Commonly this line is the line joining the supports of the structure, but it might equally well be the line joining any two points on the deflected structure or it might be any tangent to the deflected structure. In the first case the unit load or moment acts on a structure simply supported at the two given points; in the latter case, the structure is fixed, or cantilevered, at the point of tangency of the fixed tangent.

The General Principle of Virtual Work. This principle may be summarized as follows: Displacement, linear or angular, at any point in a structure is equal to the sum of the product of the internal distortions by imaginary internal resistances to such distortion produced by a unit hypothetical force of displacement, - i.e., a unit force at the point and in the direction of the displacement. The reactions to this hypothetical force of displacement will fix the reference by which the displacement may be measured.

The internal distortions may be due to any condition of stress and they may be either elastic or plastic, or they may be due to temperature or inaccurate workmanship. The external movement may be either of rotation or of translation. Rotation may be measured with reference to any line in the deflected structure. Translation may be measured in any direction with reference to any given line in the deflected structure and any given point on that line.

Virtual Work Applied to Trusses. The broad statement of the relations of external movements to internal distortions thus presented leads directly to the usual theorems dealing with slopes and deflections. In trusses it takes the form,

$$\Delta = \Sigma \frac{SuL}{AE}$$

except that u needs to be defined with reference to the fixed or reaction points. Thus, it is possible to find the deflection of any panel point A with reference to any other two panel points B and C by considering the hypothetical unit load applied at A, the truss being supported at B and C. Either B or C may be taken as fixed, the other point being for hypothetical loads, on rollers. Similarly, the relative deflection of opposite corners of any quadrilateral of a truss may be found by taking $\Sigma \delta u$ where u is the stress in any bar due to a unit load at one corner in the direction of the other when the truss is supported at the other corner. This is a familiar problem in internal indetermination. The trusses in Fig. 42 illustrate in a simple way some of the applications of this theorem.

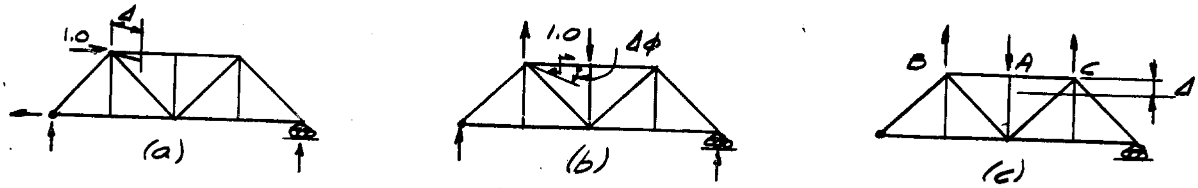


Fig. 42

Virtual Work Applied to Beams. In the case of beams the displacements due to moment only are:

$$\Delta = \int M' d\phi$$

in which, Δ is the external relative movement of any point, $d\phi$ is the rotation (produced by the bending moments) in any differential length, and M' is the imaginary moment over this differential length produced by a unit resistance to the external movement. See Fig. 43.

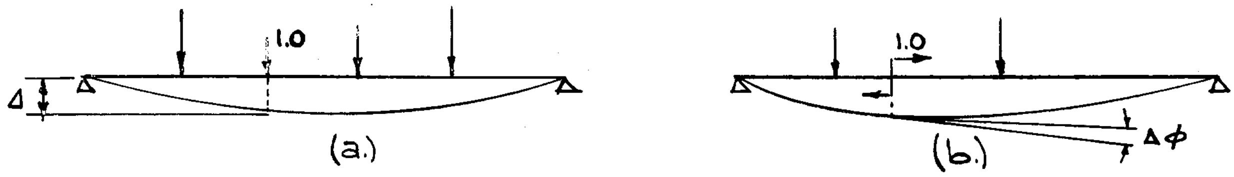


FIG. 43

If the deflection at any point from a chord of a structure, due to moments only, is to be found, the values, M', will be identical with the ordinates to an influence line for bending moment at this point on the chord. From this it follows that the deflections away from its original position produced by slightly curving a line will be the bending moments on the chord, treated as a beam, simply supported at its ends, due to the angle changes as loads. This is applicable in any case to a beam straight or curved, a floor line, or a truss chord, if the angle changes can be computed. From it the moment-area theorems of Mohr and Greene may be shown to follow directly, or they may be taken as direct corollaries of the principle of virtual work, as will be shown.

Moreover, the curve of displacements due to transverse distortions (shear slip, for example), may be computed as a moment diagram due to the displacement considered as a moment load on the beam. If Δ is the transverse distortion at A, Fig. 44, u is the shear at A due to a unit load at the point where the deflection is desired, B. But the shear at A due to a load at B is the same as the shear at B due to a load at A. The deflection diagram then is the curve of shears due to the displacement as a load at A, which is the same as the curve of moments due to the displacement as a moment load at A.

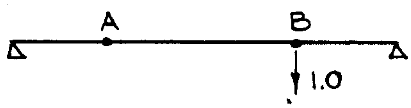


FIG. 44

This theorem would give a convenient method of including shearing distortions in deflection computation, if it were worth doing.

The angle change in a differential length of the axis of a beam, if plane sections remain plane, is the strain of an outer fibre, divided by the distance of this fibre from the neutral axis. The strain of the outer fibre is $\frac{f}{E}$ dl, in which f is the intensity of stress in the outer fibre. Then

$$\Delta\phi = \frac{f}{Ey}\, dl$$

in which y is the distance from outer fibre to neutral axis.

In all the theorems dealing with moment-areas, the term, f/y, may be substituted for M/I. Sometimes this is convenient. It also makes possible a clearer understanding of the deflection of reinforced concrete beams and, in general, gives a clearer view of the assumptions that are involved in the analysis of indeterminate structures.

If the beam formula applies, then,

$$f = \frac{My}{I}\ ; \quad d\phi = \frac{f}{Ey}\, dl = \frac{M}{EI}\, dl$$

Hence,

$$\Delta = \int M'M \frac{dl}{EI}$$

If applied to a part of a beam containing a frictionless hinge, this expression is evidently indeterminate, since at the hinge both M and I are zero. Moment-area theorems are, therefore, indeterminate for those parts of a beam in which a hinge occurs unless the change of angle at the hinge is computed independently.

<u>The Reciprocal Theorem</u>. If the expressions, $\sum \frac{SuL}{AE}$ for trusses and, $\int M'M \frac{dl}{EI}$ for beams, be used to find absolute displacements due to loads on the structure, the interchangeability of the terms, S and M, which are due to the loads, with u and M', respectively, which are due to the hypothetical external unit resistances to displacement, indicates at once the general theorem of reciprocal displacements. If "displacement" is interpreted in a general sense as either linear or angular, and "load" in a general sense as either force or moment, then in any structure the displacement at A due to a load at B has the same value as the displacement at B due to the same load at A, provided that both at A and B the force and the displacement are of the same nature, linear displacement corresponding to force along its line of action, and rotation corresponding to moment. This is illustrated in Fig. 44a.

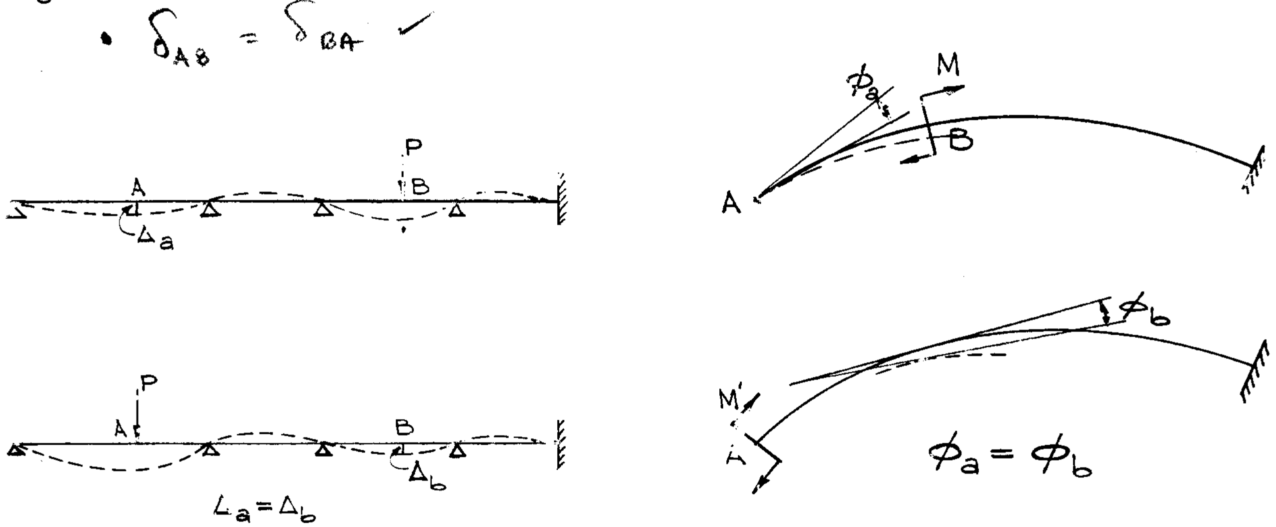


FIG. 44a.

This theorem is useful principally in interpreting as influence ordinates those displacements of the load line of a structure which would be produced by an imaginary unit internal distortion corresponding to the function for which the influence line is desired.

<u>Area Moments</u>. <u>Greene's Theorems</u>. If the relative rotation at one point on a beam referred to the tangent at another point is desired, consider the beam cantilevered from one point and loaded with a unit moment at the other point. Then, M' = constant = 1, and

$$\frac{MM'}{EI} dl = \frac{1}{EI} \times \text{(area under moment curve between the points of reference).}$$

If, however, the deflection of the second point with reference to a tangent from the first is wanted, consider the beam to be loaded and cantilevered from the first point, with a unit load at the second. Then M' equals the distance of each section from the second point and $(\frac{M}{EI}\,dl)\,M'$ is evidently $\frac{1}{EI}$ times the statical moment of the area of the moment curve about the second point.

Mohr's Theorems. In the case of a beam on fixed supports the change of slope at any point, A, relative to the line joining these supports may be found by applying a unit moment at the point, the beam being simply supported at the fixed points. The moment curve for M' will be found to be identical with an influence line for shear at A. Hence,

$$\frac{MM'}{EI}\,dl = \int \frac{M\,dl}{EI} \times \text{(shear at A due to a unit load at each section)}$$

$$= \int \text{(shears at A due to each } \frac{M\,dl}{EI})$$

$=$ shear at A on a simple beam due to the $\frac{M}{EI}$ curve considered as a load.

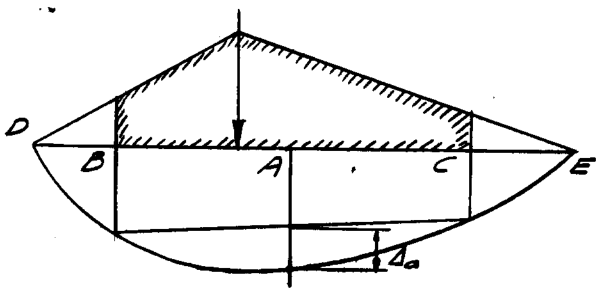


Fig. 45

For the deflection at A, on a simply supported beam, DE, apply a unit load at A. The curve for M' is then identical with an influence line for bending moment at A due to the M/EI curve considered as a load.

Evidently, any line on the beam other than DE may be considered as fixed. Thus, consider a beam bent as shown in Fig. 45. The deflection of point A with reference to the line BC - Δ_a in the diagram - is the bending moment at A produced by the M/EI curve between B and C on a simply supported beam, BC. This is sometimes a convenient theorem.

Also, the slope of the tangent at A with reference to the line, BC, is the shear at A on the simply supported beam, BC due to the M/EI - curve between B and C acting as a load.

Angle Changes in Trusses. Another interesting application of the principle of virtual work is in finding the change, $\Delta\alpha$ of any angle, α of a triangle (Fig. 46) due to the stresses in the three sides.

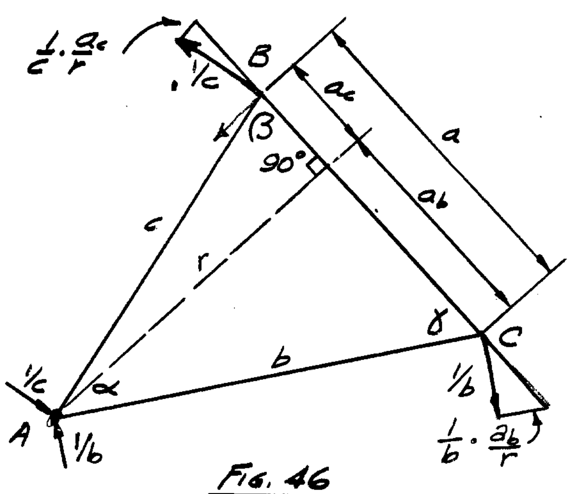


Fig. 46

Apply at A and C the elements of a unit couple, causing reactions such as to hold AB fixed in direction. Tension and increase of angle are taken as positive. Then,

$$\Delta\alpha = \Sigma\,\delta u = \Sigma\,\frac{fL}{E}\cdot u$$

$$E\,\Delta\alpha = \Sigma\, f L u$$

Resolving the forces at joints B and C,

$$u_a = \frac{1}{r}\,;\quad u_b = -\frac{1}{b}\cdot\frac{a_b}{r}\,;\quad u_c = \frac{1}{c}\cdot\frac{a_c}{r}$$

$$E\,\Delta\alpha = f_a\cdot a\cdot\frac{1}{r} - f_b\cdot b\cdot\frac{1}{b}\cdot\frac{a_b}{r} - f_c\cdot c\cdot\frac{1}{c}\cdot\frac{a_c}{r}$$

$$= f_a\cdot\frac{a}{r} - f_b\cdot\frac{a_b}{r} - f_c\cdot\frac{a_c}{r}$$

$$= \left[f_a - \left(f_b\frac{a_b}{a} + f_c\frac{a_c}{a}\right)\right]\frac{a}{r}$$

This is probably the most convenient formula for the angle changes. It may be modified to:

$$E\Delta\alpha = f_a\frac{a_b}{r} + f_a\frac{a_c}{r} - f_b\frac{a_b}{r} - f_c\frac{a_c}{r}$$

$$= (f_a - f_b)\frac{a_b}{r} + (f_a - f_c)\frac{a_c}{r}$$

$$= (f_a - f_b)\cot\gamma + (f_a - f_c)\cot\beta$$

This is the familiar formula used in secondary stress computations. Evidently, the method may be readily extended to the quadrilaterals which occur in sub-divided trusses and K-trusses. (See Chapter IX)

Slope-Deflection. The application of this method to the special case of a beam or part of a beam loaded only with moments at its two ends gives directly either the end slopes in terms of the end moments or the latter in terms of the former. In the first case given the loads on the beam, the reactions (end shears) are desired; in the second, the magnitude of the loads is to be determined for given reactions. By taking moments about B (Fig. 47), there follows:

$$\theta_a = \frac{2}{3}M_a\frac{L}{EI} - \frac{1}{3}M_b\frac{L}{2EI} = \frac{L}{6EI}(2M_a - M_b)$$

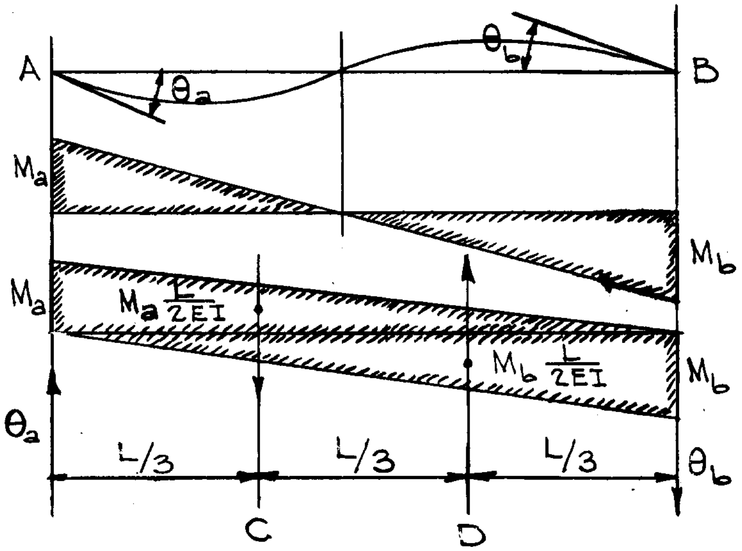


FIG. 47

By taking moments about D,

$$\frac{M_a L}{2EI} = 2\theta_a + \theta_b$$

or,

$$M_a = \frac{2EI}{L}(2\theta_a + \theta_b)$$

Both forms of the equation have been found convenient in evaluating secondary stresses. The moments due to the displacements of the joints may be found by the column analogy in which the displacement ψL, Fig. 48 is equivalent to a

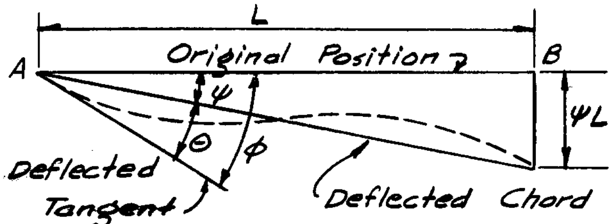


FIG. 48

bending moment about the parallel axis or, -

$$M = f = \frac{6M}{Ad} = \frac{6\psi L}{L/_{EI} \cdot L} = 6E\psi K$$

Combining this with the moments due to the rotations of the joints and noting that $\theta = \phi - \psi$ we have, -

$$M_a = 2EK(2\phi_a + \phi_b - 3\psi)$$

In these equations

θ is the deflection of the tangent from the chord in the deflected structure - the primary angle in the analysis of continuous structures.

ϕ is an angle of reference - the deflection of the tangent from the original position, or any assumed line of reference.

ψ is likewise an angle of reference - the deflection of the chord from an assumed original position.

Angle Weights. The principle of virtual work furnishes directly a method of computing angle weights for determining the deflected load line for trusses. Assume that it is desired to draw the deflected load line for a unit load as shown in Fig. 49(a), that is, an influence line for horizontal reaction. The angle change at a may be computed by applying a unit moment resisting this angle change. This is effected by applying loads as shown in Fig. 49(b), acting at the points on the load line or floor as indicated by circles, and then computing $\sum \delta u$, in which, δ is the change in length of any bar due to the horizontal reaction, and u is the stress for the loading shown. These angle changes may then be treated as loads at the panel points and, when corrected for the deflection of the ends of the load line, the moment curve thus produced will have the shape of the influence line.

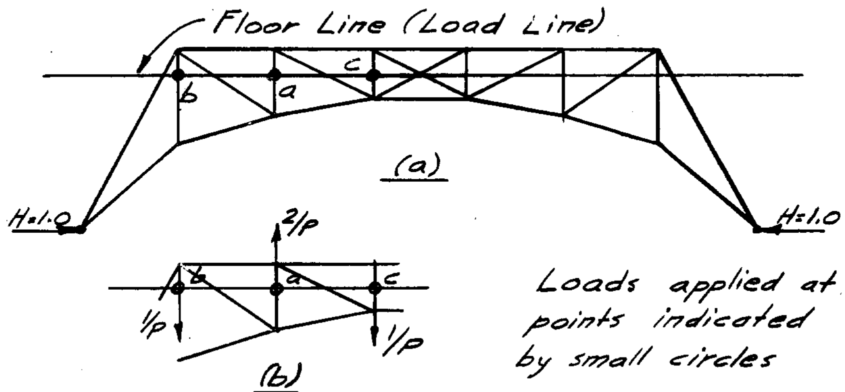


FIG. 49

This method presents advantages in directness in some cases as where the floor-beams frame into the verticals between upper and lower panel points.

Practical Considerations.

These illustrations indicate the broad usefulness of the general principle of virtual work. More definite applications of the principle will be made as various problems are discussed in succeeding chapters. The purely geometrical nature of all the correlated theorems needs emphasis. None of these theorems bears on the accuracy of the physical assumptions regarding the action of the structures.

To apply them to engineering structures, it is simply necessary either to show that for given values of the moments, shears, and thrusts, the strains can be predicted, or, if exact prediction is not possible, to determine by direct computation what error results from the inexactness. In analyzing indeterminate structures it is also to be noted that usually the relative and not the absolute values of these strains are in question.

Because of the definiteness of the moments, shears, and thrusts, strain measurements on statically determinate structures seem for this purpose more valuable and dependable than similar data derived from measurements on indeterminate frames. When it is established that, for any type of construction, these strains - or their relative values - can be definitely predicted, the whole theory of indeterminate stress analysis follows from the relations of the distortions as a matter of geometry. In order to be of real value in designing indeterminate structures, however, the load-strain relations must be those for conditions approaching failure and not merely those that exist at working stresses. An understanding of these facts will make clearer the limitations of the theory of elasticity as applied in much of the literature dealing with indeterminate structures of reinforced concrete, will make possible the application of more correct theory to such structures, and will give greater confidence in the results obtained by its use.

Internal Work - Least Work. It may be well here to distinguish virtual work, internal work and least work. The internal work done in a structure is evidently the continued product of the internal forces by the internal distortions. In the case of a beam the work of the moments is the sum of the products of the differential rotations by the bending moments. Each section of length ds has a relative rotation of its two ends $\frac{mds}{EI}$ and the work done in it is $\frac{m}{2}\frac{m}{EI}$ ds, if the load, and hence the moment, be gradually applied. The total internal work then, is $\frac{m}{2}\frac{m}{EI}$ ds, which is also, in the case of a prismatical beam, the statical moment about the base of the moment curve times 1/EI, provided we consider all statical moments as positive.

Now suppose that any structure has acting on it forces and reactions which satisfy the laws of statics. As a consequence of the law of conservation of energy the redundant reactions will so adjust themselves that the internal work stored in the structure must be a minimum. This is the principle of least work. Quantitatively this means that the first derivative of the internal work with reference to any internal stress or external force considered as redundant is zero.

The equations derived by least work are identical with those given by a direct consideration of the distortions, but, except in the case of trusses, they are likely to be unwieldy. The theorem has, however, great analytical value in some cases. One illustration of its use is the analysis of arches as given by Professor Spofford. Perhaps the best illustration is the proof of the so-called line of pressure method of arch analysis.

As distinguished from the internal work done in a structure, which we rarely have any occasion to compute, and from the principle of least work, which cannot be used to compute the internal work, though it deals with it, the principle of virtual work has nothing to do with the true internal work, but is simply a mental device for deducing certain purely geometrical relations. The internal distortions may be either elastic or plastic, and may or may not be accompanied by internal work.

Summary of Principles.

An effort has been made to make this chapter brief. The important points are

First - Virtual work is a convenient tool in developing the theory of placements.

Second - In its direct form, as used chiefly in trusses, we apply a unit dummy load corresponding to the desired displacement and then find the sum of the products of internal distortions times the stresses produced by the dummy load, $\Delta = \Sigma u \delta$

Third - In beams the equation $\Delta = \Sigma u \delta$ takes the form $\Delta = \Sigma m \phi$

Fourth - A corrolary of this is that the deflection from the original position due to slightly bending any line can be found as the bending moment on the chord as a simple beam due to the angle changes in the arc considered as loads, provided the ends do not move.

Fifth - In structural engineering these angle changes may be $m \frac{ds}{EI}$, $f \frac{ds}{Ey}$ or may be the angle changes occurring between adjacent chord members in a truss.

Sixth - From this follows very simply that

(a) The total change in slope along a beam equals

$$\int d\phi = \int mw = \int \frac{mds}{EI} = \int \frac{fds}{Ey}$$

(b) The deflection of a point on a beam away from a tangent to the beam is the statical moment about the point of the area under the curve of $d\phi$ or of mw or of m/EI or of f/EY

(c) The slope at any point of a flexed beam with reference to its line of supports is the shear at that point due to the m/EI curve (or its equivalent) as a load figured on a simple beam.

(d) The deflection at any point of a flexed beam with reference to its line of supports is the bending moment at that point due to the curve of angle changes (or its equivalent) as a load - which, of course, is the original theorem.

Seventh - Virtual work has nothing to do with the true internal work of the structure.

Eighth - The principle of least work - that the redundants so adjust themselves that the total internal work is a minimum consistent with statics - leads to exactly the same results as virtual work, but the two should not be confused.

Ninth - The fundamental relations here stated are entirely geometrical. Differences in stating them result from a search for convenience of application. The relations of external displacements to internal distortions are, however, not subject to dispute; what these internal distortions are is subject to dispute; what the significance of these distortions may be is even more subject to dispute.

The relations here presented could be elaborated historically, philosophically and mathematically. The history would be interesting and extensive and would include many famous names, Claperon, Menabrae, Castigliano, Clerk Maxwell, Mueller-Breslau, Fraenkel, Otto Mohr and a long list of others; certain philosophical aspects are evident; the mathematical elaboration may be - has been - very extensive. The important fundamentals are clear, the elaboration found in the literature results from efforts to restate these fundamentals in such a way as to reduce somewhat the tediousness of the computations involved. The value to the engineer of such modification is to be judged almost entirely on this basis.

CHAPTER IV
INFLUENCE LINES

The General Principle. The general principle of virtual work together with the reciprocal theorem furnishes a direct and convenient solution of influence lines.

Let it be required to draw the influence line for any stress function - shear, moment in a beam, stress in a bar of a truss, recation. Let the function produce freely a unit distortion, the structure being otherwise restrained as originally. In other words, assume for vertical shear, a unit vertical displacement along sliding grooves rigidly attached to the structure, for a moment a unit relative rotation on the two sides of an imaginary hinge, let the point of reaction move one unit, let the bar lengthen or shorten one unit. Then will the displacements of points on the load line be influence ordinates for corresponding loads at those points, rotation corresponding to applied moment loads, vertical displacements to vertical loads, inclined displacements to inclined loads, etc. For if we assume a displacement at A, the point at which the stress function acts, to be produced freely -- i.e. without any resistance to that particular displacement at A -- by a unit force at B, then to bring the structure back to its original continuity a resistance at A must be applied.

$$F = \frac{\Delta^a_{P_b=1}}{\Delta^a_{P_a=1}}$$

But, by the reciprocal theorems,

$$\Delta^a_{P_b=1} = \Delta^b_{P_a=1} \qquad \text{Hence } F = \frac{\Delta^b_{P_a=1}}{\Delta^a_{P_a=1}}$$

Also the displacement at A due to a unit load at A equals the reciprocal of the load required to produce a unit displacement, since the displacement is proportional to the load. Hence,

$$F = \Delta^b_{P_a=1} \times \text{(force necessary to produce a unit displacement at A)}$$

or F = displacement at B due to a load at A necessary to produce a unit displacement at A.

Correspondingly, if it is required to draw an influence line for displacements, we apply a unit load at the given point and take as influence ordinates the displacement, in the directions of the actual applied loads, of the other points of the structure. Again at any given point on the structure slope and moment, load and deflection correspond.

The principle may be summarized with a general statement. If any stress function of an indeterminate structure - reaction, thrust, tension, bending moment, shear, fibre stress (for an assumed position of the neutral axis) produce freely and alone a unit corresponding displacement, the structure remaining otherwise as before, then the displacements of the deflected structure are, to scale, ordinates for loads in the direction of these displacements, of an influence line for the stress function. Rotations either across or around the axis of the structure (change of slope or torsional rotation) at various points represent ordinates to influence lines for moments or torques applied at these points.

That is any stress function whatever in any structure, whether simple or complex, will draw to some scale, its own influence line, this being the deflected load line due to a unit distortion corresponding to this stress function.

The sign of the influence area can be determined in most cases by inspection. It is, however, easy to generalize that the influence on a function of a load at any point in a given direction is positive if the displacement of this point due to a displacement corresponding to a positive value of the function is in the direction of the load.

The principle has had an extended application to test of models, the influence lines being usually drawn for external reactions rather than for internal forces, however.

The principle of reciprocality in influence lines is applicable equally to statically determinate and to statically indeterminate structures, to structures in a plane and to structures in space. Obvious difficulties arise, however, in its application either analytically or by use of models to such structures as domes, slabs, etc.

The theorem enables us to sketch directly the general shape of an influence line. Quite commonly this is all that is required; in some cases it is necessary to determine exactly the influence curve or some part of it.

While the influence line may be drawn to scale and the ordinates scaled from it, it is possible either to find the area under the influence line with a planimeter and multiply by the equivalent uniform load or to use the influence line simply to give the load divides and then determine the stress by separate analysis.

Equivalent Uniform Load. The equivalent uniform load is the load which would give the same stress at the point as would be given by the train of concentrated loads if the influence line were triangular. Steinman* has worked out the equivalent uniform loads with various triangles for E-60 and M-60 loading. For other loadings the equivalents are easily determined, but the accuracy of the assumed equivalence evidently depends on how closely the influence line approaches a triangle.

The equivalent uniform load is determined as follows:-

Assume a triangular influence line with segments a - b, Fig. 50. Place the loads to give maximum at A as indicated by this influence line neglecting loads which lie off the diagram and compute the maximum. Then determine what uniform load will give the same maximum, - in other words, divide the maximum obtained for the moving load system by the area of the triangle.

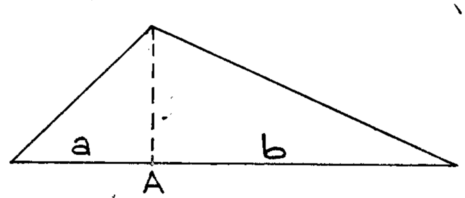


Fig. 50

*Trans. A.S.C.E. 1923.

If now we have a curved influence line, Fig. 51, having segments a - b, we place this uniform load from A to B and compute the stress, - in other words, we multiply the uniform load by the area of the influence line AB. This process involves three assumptions:

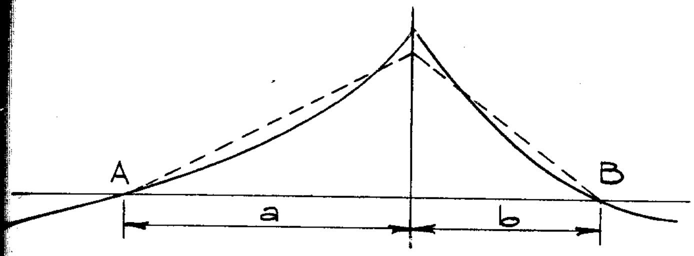


FIG. 51

(a) That the position of the loads for maximum is the same for the curved influence line as it is for the triangular influence line;

(b) That the ordinates of the curved influence line everywhere have the same ratio to those of the triangle as do the areas.

These two together are evidently equivalent to assuming the curved influence line to be a triangle (shown dotted) of the same area.

(c) That the effect of loads outside the length A B is negligible.

Equivalent loads, then, are applicable if and when they are equivalent. For railway bridges where the segments a - b are long relative to the wheel spacing they are probably quite exact. For trolley loading their utility is not so great, and in such cases an equivalent concentrated load might be quite as serviceable.

Their applicability depends on whether the influence line approaches a triangle in shape and whether - as is usually true - the effect of loads outside the influence area is negligible.

This whole discussion loses force if we are thinking only of the safety of the design and not merely of the requirements of a given specification, because in any case the assumed train of loads is necessarily more or less conventional and there can be little objection to treating it by a conventional analysis. We secure uniformity in method of analysis without any real loss of precision in representing facts.

Qualitative Studies. A few cases will illustrate the value of the principle in qualitative analytical investigations. The influence lines indicated will, by inspection, be as shown in Fig. 52 being sketched in each case as the deflected load line due to the application of a unit distortion at the point. For moment the distortion will be a unit rotation; for shear a unit vertical displacement.

In drawing approximate influence lines, it is important to consider all geometrically possible types and then eliminate those which are statically incorrect. The influence line is a deflected structure and so must obey the laws of statics and of continuity. Thus at a corner A of a structure, Fig. 53 at which there is no external moment, statics requires that the corner be deformed as shown in (b) or (c). It cannot be as shown in (d) as ΣM is unbalanced in this condition.

In general the influence line gives some indication as to the moment diagrams for certain loadings and a consideration of these will eliminate lines that are otherwise geometrically possible.

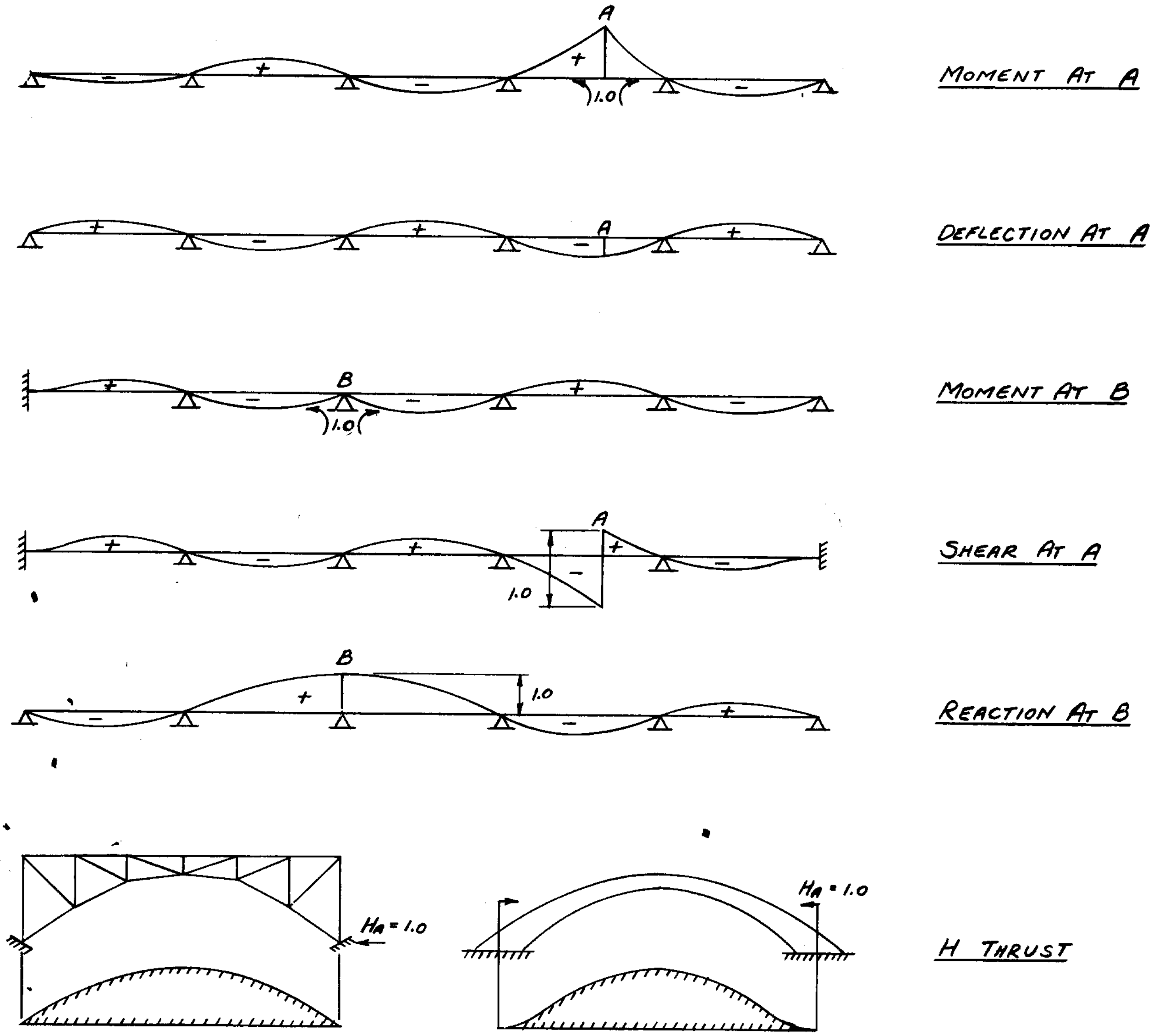

A
+
−
1.0
Moment At A
Deflection At A
B
Moment At B
Shear At A
Reaction At B
$H_A = 1.0$
H Thrust

Fig. 52.

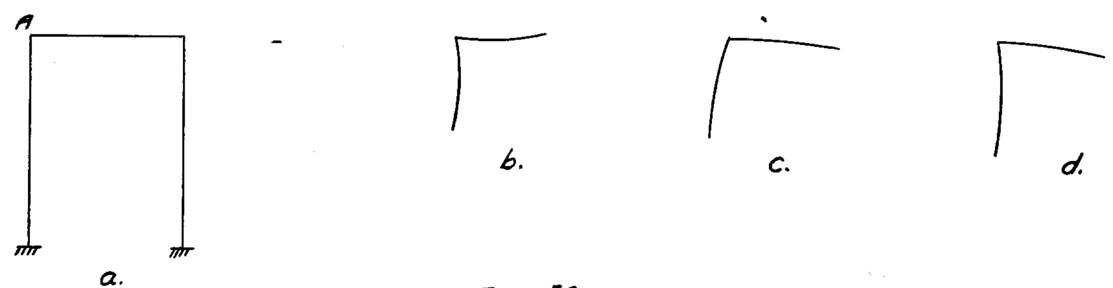

A
a.
b.
c.
d.

Fig. 53.

For instance, in Fig. 54 the four lines shown in 1, 2, 3, and 4 are geometrically possible influence lines for moment at A. But line 4 can at once be eliminated because while the moment curve indicated in (a) may be conceivable, (it is really impossible as shown below) the curve of moments producing the deflections which the influence line represents, (b), is not possible, such a curve requiring an inflection point both in the end span and in that next to the end. But there can be no moment at the free end. Moreover, line 2 can be eliminated because simultaneous loads at B and C will not produce zero moment at A irrespective of the relative value of the loads.

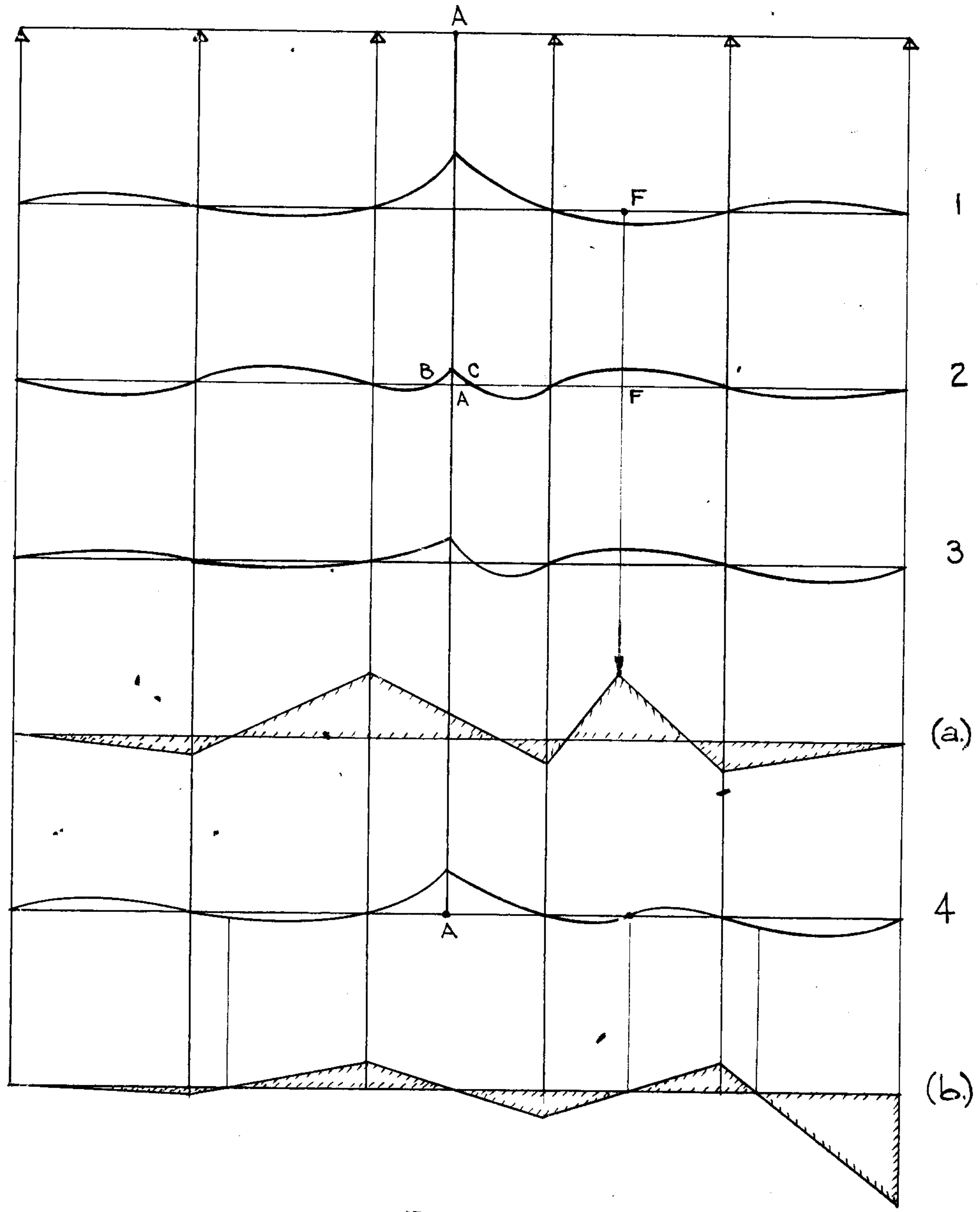


FIG. 54

If the point A is near the center of the span, we may also eliminate line 3 because the moment diagram in (a) indicated as possible by this influence line is actually not possible; M_a cannot be positive for a load at F since this would require M_d greater than M_e. Obviously for a load at F the moment at D cannot for prismatic beams be over 50 per cent that at E. Line 1 is the correct influence line then, if point A is near the center.

If point A is near the support, line 1 is not possible because a load at F cannot produce a negative moment clear up to the support - it must change to positive at some point between the center and the support, (Fig. 55).

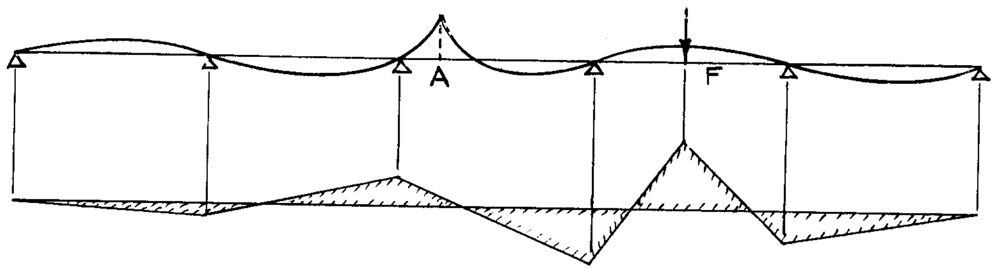


FIG. 55

These four curves, then, should be considered together:

(1) Influence line as a deflected structure.
(2) Moment curve which would produce such deflections.
(3) Possible moment diagrams for actual loads as indicated by the influence line.
(4) Deflected structure indicated by such moment diagrams.

Quantitative Analysis. Quantitative results may be obtained by applying a unit displacement and computing the shape of the deflected structure. Often, however, influence lines drawn to give the general shape may be used as approximate quantitative influence lines because the value of one of the ordinates is known or is readily determined by inspection. Thus, if the shape of the influence line for a vertical reaction is sketched, it is known that the ordinate at the reaction is unity. Similarly, if an influence line for moment at the center of a fixed-ended prismatic beam is sketched, the ordinate at the center is known to be L/8 because unit load at the center will produce this moment at the center. In other cases the area under the influence line is known because we know the effect of a uniformly distributed load. From this we may by knowing - or assuming - the shape of the influence line, determine its area. Thus, if the influence line for crown thrust in a two-hinged arch be assumed to be an ordinary parabola, the center ordinate, Fig. 56, is $\frac{3}{16}\frac{L}{h}$; for full loading the crown moment is zero - rib-shortening neglected - or,

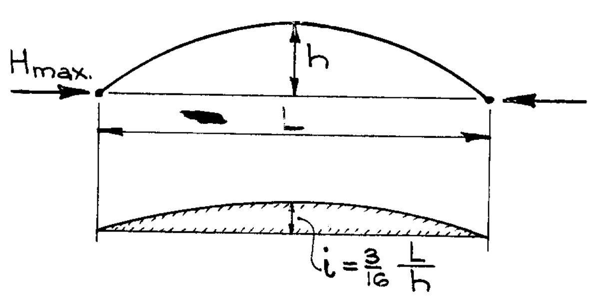


FIG. 56

$M = \frac{1}{8}wL^2 - hx$ (horizontal thrust for full loading) = 0. Hence, H for full loading $= \frac{1}{8}\frac{wL^2}{h}$

Area of parabolic segment =

$\frac{2}{3}iL = \frac{1}{8}\frac{wL^2}{h}$ $\qquad i = \frac{3}{16}\frac{L}{h}$

An illustration of how much may be found out from general studies of influence lines is given by the influence line for horizontal reaction at A in the symmetrical framed bent shown in Fig. 57. Here inspection gives two values of the influence ordinates, unity at the foot of the bent and 0.5 at the top.

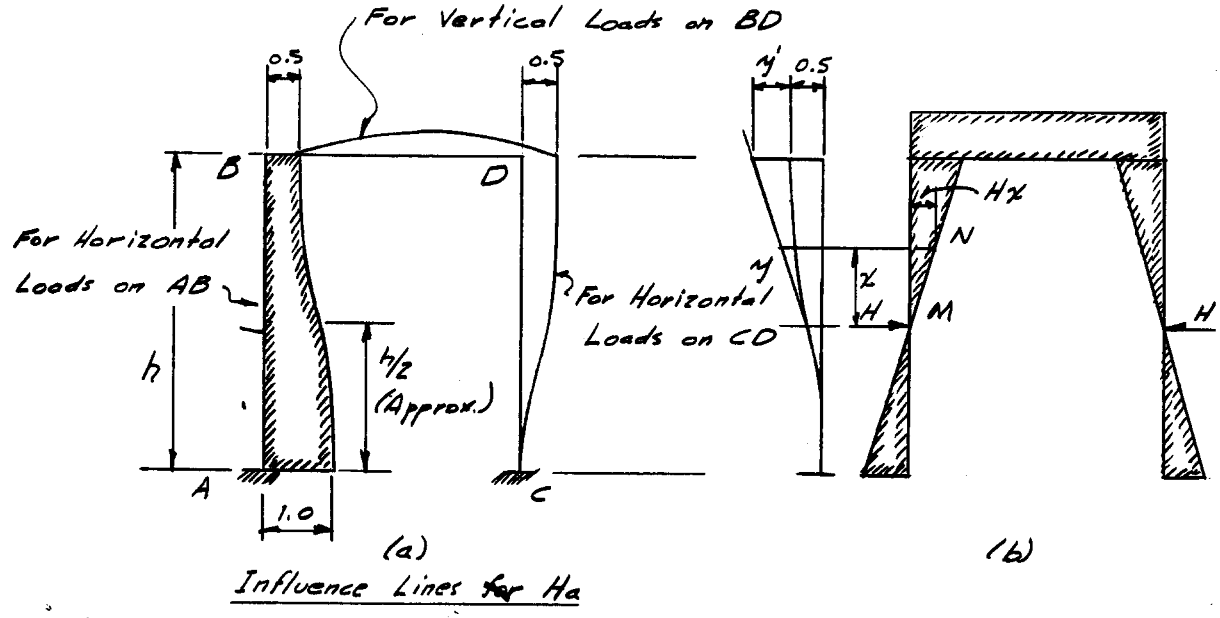


Fig. 57

In view of the usual uncertainties as to the fixation of the base of the column, this diagram probably furnishes about as much information as will ever be obtained on this point. Let us, however, for purposes of illustration, investigate the problem further. Consider, instead of one support remaining fixed, that both are moved relative to each other, both being free to move horizontally - but not to rotate; the stress condition is evidently the same. The bent is now symmetrically loaded and hence the two equal and opposite forces acting on it are horizontal forces acting through the points of contraflexure of the columns. If we assume the point of contraflexure at the half-height (girder very stiff) then the curve of moments will be as shown. The equation of flexure of the column, then, is:-

$$y = (\text{Statical moment of the area of moment curve between M and N about N}) \times \frac{1}{EI_c}$$

$$= Hx \cdot \frac{X}{2} \cdot \frac{X}{3} \cdot \frac{1}{EI_c} = \frac{H}{6EI_c} X^3$$

But H has such a value that the deflection at the top of the column away from the tangent at the bottom equals 1/2 and hence,

$$H \cdot \frac{h}{2} \cdot \frac{h}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} h \cdot \frac{1}{EI_c} = 1/2 \qquad H = \frac{6}{h^3} EI_c$$

$$y = \frac{X^3}{h^3} \qquad y' = 1/8$$

The influence line for vertical loads on the girder is the bending moment curve for the moment curve considered as a load times $\frac{1}{EI_g}$. This is evidently a parabola with a center ordinate of,

$$\frac{1}{8}\frac{h}{2} \cdot HL^2 \frac{1}{EI_g} = \frac{1}{16}\frac{hL^2}{EI_g} \qquad H = \frac{1}{16} \cdot \frac{6}{h^3}\frac{EI_c}{EI_g} \cdot hL^2 = \frac{3}{8}\frac{L/I_g}{h/I_c} \cdot \frac{L}{h}$$

The point of contraflexure, however, is above the half-height of the column since the girder is flexible and we can, if we wish, compute its location and revise the above equations. Let it be clearly understood that this problem has been used as an illustration of method and not from any fancied practical importance attached to it. If the horizontal member is a truss instead of a girder, we can easily take account of horizontal and vertical loads on it either by drawing a Williot diagram or by applying virtual work.

Laboratory Investigations - Use of Models. Another interesting field for the application of this principle is in laboratory investigation of influence surfaces for such complex structures as flat slabs, domes or groined arches. Certain difficulties arise from the fact that relations derived by considerations of virtual work assume very small distortions and hence it is necessary to measure very small movements. Some interesting studies have been made along this line.

It should be evident that in some cases quite precise work is required in using such models. Consider the simple illustration of the bent shown in Fig. 58. Assume that it is desired to draw an influence line for shear at A, due to vertical loads on the girder by cutting the model and producing a horizontal shearing movement. Assume, also, that a load P at B produces at A a horizontal shear P/100. It also produces at A a vertical force P/2. Now suppose that the movement of the model at A is not quite horizontal but slopes 1 to 50. The displacement at B then will be due partly to the horizontal movement at A $(\Delta^b = \Delta^a_h \frac{1}{100})$ and partly to the vertical movement at A $(\Delta^b = \Delta^a_h \frac{1}{2})$

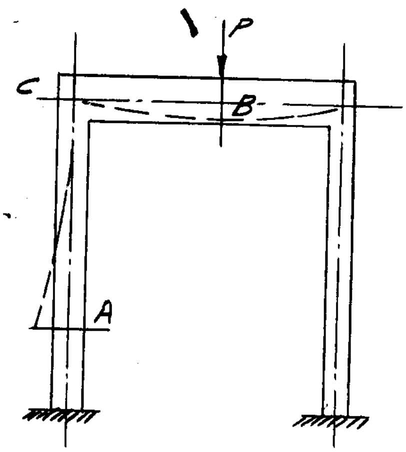


Fig. 58

The deflection at B would then be 100 per cent in error. In this case the error could be detected by observing whether or not there was a vertical movement of C.

Assume, further, that it is desired to draw an influence line for axial moment at A. Cut the model and rotate it about the centroidal point of the section. Assume that a load at P produces an axial moment at A of P_b, where b is the width of the column. Now suppose the rotation is by accident not about the centroid o (Fig. 59) but about some point a having an eccentricity e. The vertical movement of B will then be θb due to axial rotation and θe/2 due to eccentricity. The error is e/2b and may easily be quite large. In this case again we may observe whether or not there is any vertical movement of point C, since it is evident that a central load on the column at C can produce no axial moment at A.

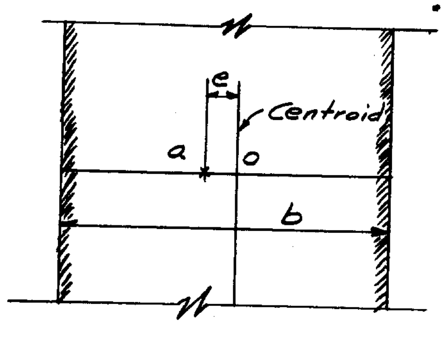


Fig. 59

Further, the use of models becomes impracticable in such cases as secondary stresses. It has been proposed to draw influence lines for secondary moments in truss members by producing an axial rotation at one end of the member in a model and observing the deflection of the panel points. But the panel points of a truss cannot deflect unless there is an axial deformation of the bars and the axial stress in the bars produced by a secondary moment in a member is negligible.

The last case presents the additional difficulty that secondary stresses in a truss are dependent both on the area and the stiffness of the members of the truss, and it is clearly impossible in a uniplanar model to duplicate the variations both in area and in the moment of inertia of the constituent members.

Temperature stresses apparently cannot be determined at all by the use of models unless provision is made for measuring both deflections and forces.

Combined Influence Lines. It will frequently be found convenient to construct influence lines by combining the separate effects of the simple elements that constitute the function. One illustration of this is given in the chapter on Concrete Arches where the influence line for moment is constructed by considering separately the moment as on a fixed-ended beam and the moment due to horizontal thrust acting alone. In some cases the solution is greatly simplified by this expedient. This same principle is utilized in the design of two-hinged arches where it is found most convenient to construct the H-influence line separately, combining it with that for a simple beam and using a multiplication factor to determine the fibre stress in the member. Thus, one solution of the indeterminate element is sufficient being combined in each case with that for the statically determinate element.

Numerical Examples - Determinate Structures. The influence line for any function may be drawn to scale by applying a unit displacement corresponding to the function, computing the angle changes thus produced, and computing the deflections due to these angle changes and combining these with the additional deflection, if any, due to the unit displacement. In general, this assumes that the effect of shear and direct stress is negligible in beams.

If the structure is statically determinate, this procedure is simple, though not especially convenient. It may serve as a good illustration of method.

Simple Beam - Moment. Draw an influence line for moment at A on the simple beam shown in Fig. 60. Produce at A a unit rotation. This now is the only angle change. The deflection diagram is the moment curve due to unit load at A - the familiar triangle as shown.

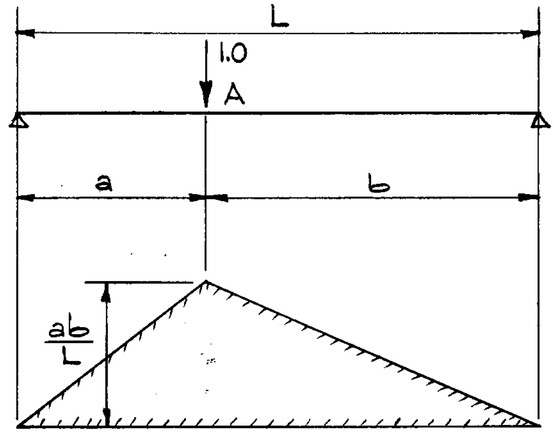


FIG. 60

Simple Truss - Chord Stress. Draw an influence line for stress in bar x for the truss shown. Fig. 61. Unit deformation of x produces at A an angle change 1/h. The deflection diagram is the curve of moments due to this load 1/h. The triangle shown is the influence line, as is known already.

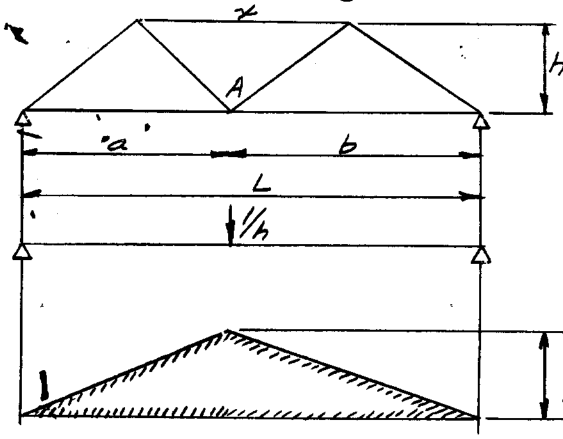


FIG. 61

Simple Truss - Diagonal. Draw an influence line for stress in bar x for the truss shown. Fig. 62. Unit deformation in x produces at A an angle change 1/r. This may be considered as a load applied to the beam through a bracket arm giving the moment curve shown. It is evident that the two branches of the influence line intersect on a vertical through A, the center of moments for x.

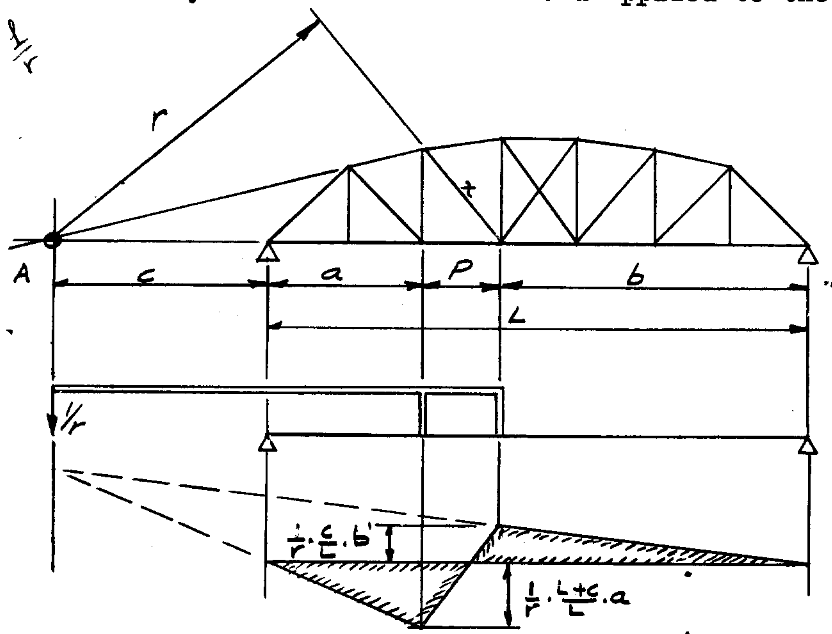


FIG. 62

Simple Beam - Reaction. Draw an influence line for reaction A on the simple beam shown. Fig. 63. The line is as shown with end ordinate unity by inspection.

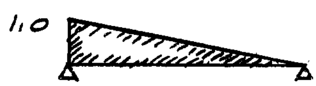


FIG. 63

Simple Beam - Deflection. Draw an influence line for deflection at A on the simple beam shown. Fig. 64. As unit force corresponds to unit displacement, so unit displacement corresponds to unit force. Apply at A a unit force and draw the curve of deflections. This will evidently be to scale.

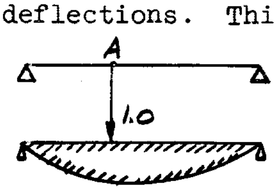


FIG. 64

Simple Beam - Shear. Draw an influence line for shear at A on the simple beam shown. Fig. 65. Produce at A a unit shearing displacement. But deflection curves may be computed as moment diagrams if rotations are treated as loads and displacements as moments. (See Chapter III) Hence, apply a unit moment at A and get the familiar influence line for shear as shown.

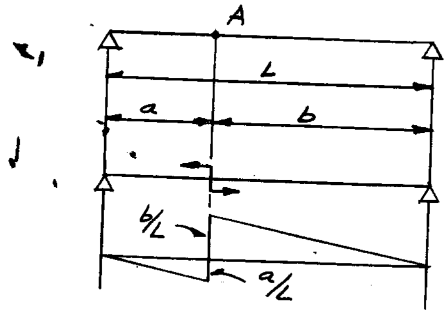


Fig. 65

Application to Indeterminate Structures. The problem of influence lines in indeterminate structures differs from that in applying this method to determinate structures only as the rotations produced by indetermination need to be determined. These rotations - angle changes, angle weights, m $\frac{ds}{EI}$, mw - are found directly in the case of fixed-ended beams and solid arches by the column analogy. If the beams or arches are continuous, the fixed-ended moment must be distributed and the mw values found for the indeterminate moments.

In truss problems the column analogy is also applicable, but it is not convenient for this purpose, because indeterminate trusses which require influence lines are commonly indeterminate in the first degree - two-hinged arches, swing bridges, two-span continuous trusses, many suspension bridges - and less often in the second degree - three-span continuous trusses, the semi-continuous cantilever at Blackwell's Island - and rarely in the third degree - fixed-ended trussed arches such as the Eads Bridge and some suspension bridges.

For single indetermination it is most convenient to draw the influence line for the effect of the indeterminate element and combine this with influence lines for the statically determinate structure produced by omitting the redundant.

This is explained further under two-hinged arches. There we apply a unit horizontal reaction at the hinges, compute all stresses and from these the angle changes along the top chord. The whole change at any panel point may be computed at once by virtual work (See Virtual Work - Angle Weights) or the changes in each of the angles at any panel point may be computed (See Virtual Work - Angle Changes in Trusses) and the results added.

Two-Hinged Arch - H Reaction. The influence ordinate for horizontal reaction due to vertical load is the ratio of the vertical movement at a point on the load line to horizontal movement of the hinges producing it. With the angle changes along the top chord known, the deflections with reference to the top of the end posts is found as the moment curve due to the angle changes as loads along the chord on the simple beam CC. Fig. 66. To this must be added the slight shortening of the end posts, CD. Call the result Δ_v. The horizontal movement of the hinges is

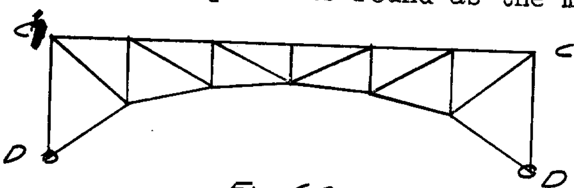


Fig. 66

the statical moment about DD of the angle changes in the top chord. Call this Δ_h. The influence ordinate then, is $\frac{\Delta_v}{\Delta_h}$.

General Method - Masonry Arches. Variations in the details of the process are a matter of choice or of convenience as will appear later, but the fundamental method remains:

Either apply a unit displacement and find for this the deflected load line or apply a unit force and find the ratios of the deflections of the load line to the displacement of the unit force.

In computing influence lines for moment at a point on the axis or at a kern point of a fixed beam or arch, we first determine the indeterminate moments by the column analogy, the unit rotation at the point considered being treated as a unit load on the analogous column. If beam or arch is continuous, the fixed-ended forces will need to be distributed through the structure. These indeterminate moments produce angle changes mw and from these, and the original rotation, the influence ordinates may be computed. For vertical deflections the influence curve is conveniently found as a series of moment curves on simple beams, the requirement of continuity at the supports giving a check on the work; slopes along this curve are influence coefficients for moment loads. These moment curves may be drawn graphically as string polygons or may be conveniently computed by shear increments.

Influence lines for horizontal reactions are rarely wanted. For fixed-ended arches they could conveniently be computed as deflections of a cantilever from either abutment.

Shear Influence Lines - Continuous Beams. Where shear influence lines are wanted, the distortion applied is a unit vertical displacement. By the column analogy this is equivalent to a localized unit couple about a vertical axis at the point, perpendicular to the plane of the paper. From it the angle changes as loads and the deflected structure may be obtained.

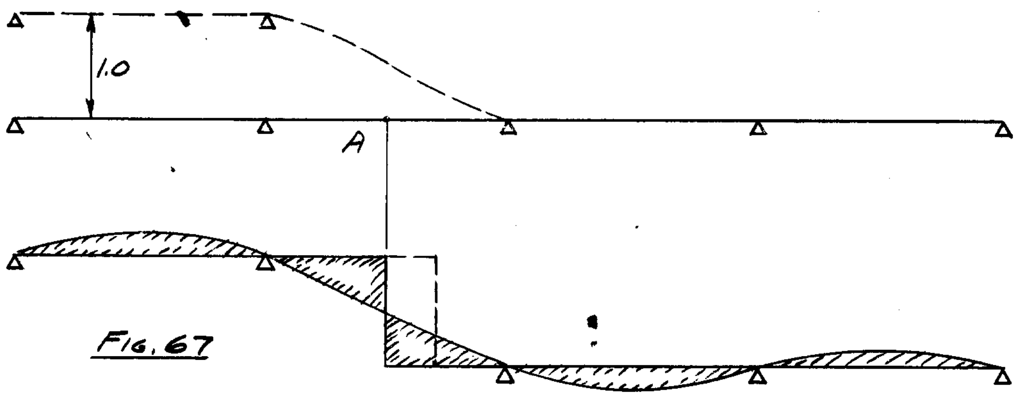


FIG. 67

Or since the Shear at A, Fig. 67 is equal to the sum of the reactions to the left, minus the load when it lies to the left, these reactions may be displaced one unit. The influence line for shear will then be as sketched and all influence lines for shear on this span may be drawn on this one diagram by drawing verticals through the proper sections.

This application of the column analogy is mathematically a case of zero load applied at infinity. The same construction may be performed by considering the separate effects of a unit vertical displacement of the reactions on the left and a unit displacement at the point. This is illustrated in Fig. 68.

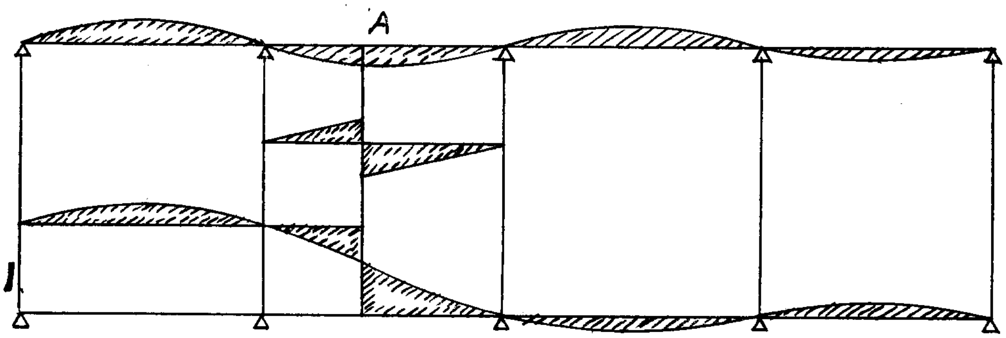


FIG. 68

Dimensions in Influence Lines. Influence coefficients are ratios. Hence influence ordinates for the forces produced by loads are absolute and have no dimension; the influence ordinates for moments due to loads have a dimension L; the slopes of influence lines for force which are influence coefficients for force due to moment loads have a dimension $1/L$; the slopes of influence lines for moment are absolute. For influence ordinates for fibre stress the dimension is $1/L^2$ and the slope of such influence lines is or dimension $1/L^3$. The areas under influence lines, then, may have a dimension of L (force due to load or moment due to moment or L^2 (moment due to force). It will be seen that this is consistent with the idea that the effect of a uniform load is (area under influence line) x load per foot.

If the areas are measured in square inches with a planimeter, they will need to be multiplied by scale of influence ordinates (a units = 1 inch or a ft. = 1 inch according to whether the influence line is for force - shear, reaction, stress - or for moment) times the scale of distance of abscissas (b ft. = 1 inch). Effect of uniform load = w lb./ft. x A x ab.

FOUR SPANS – MOMENT AT A:

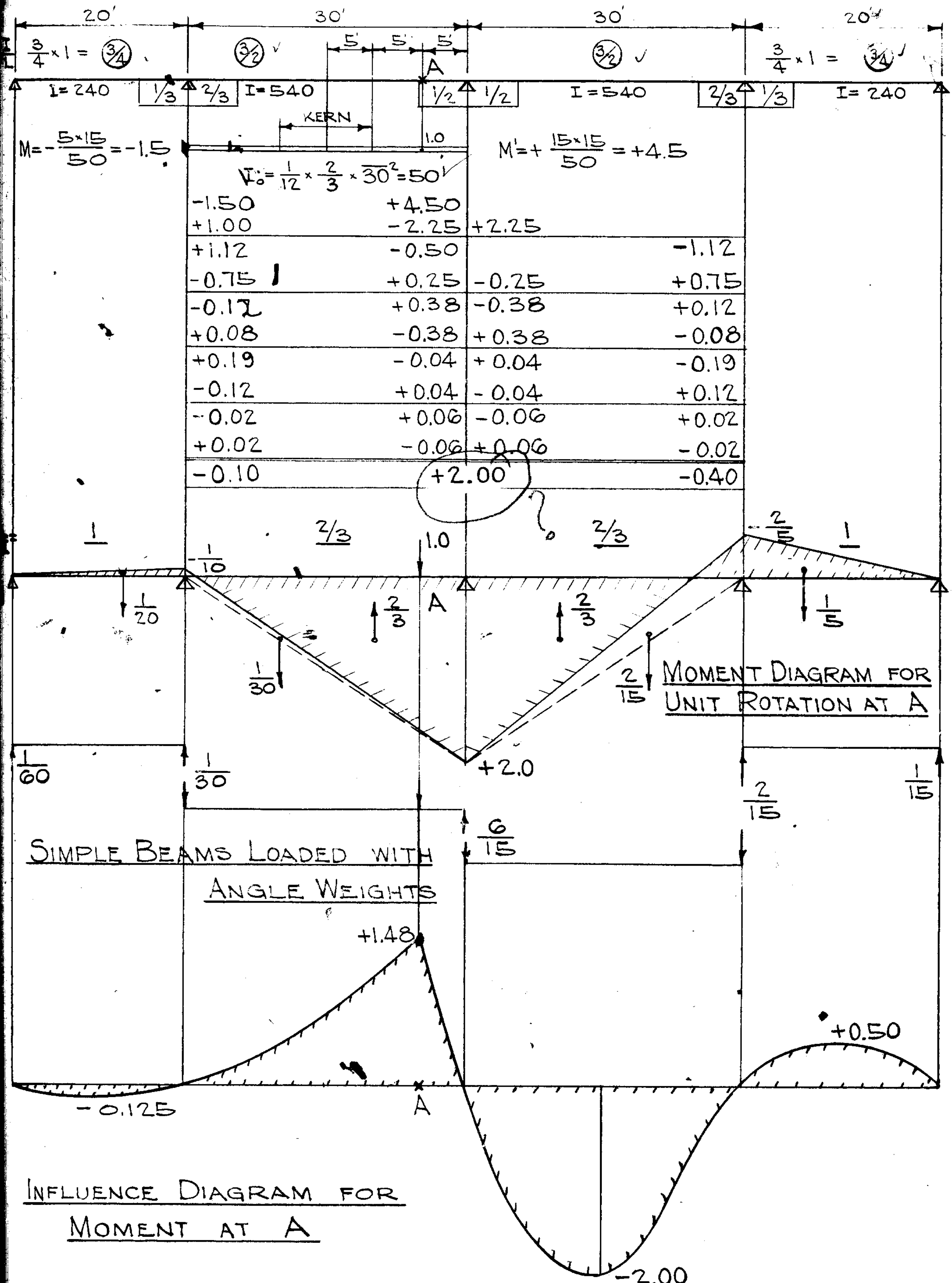
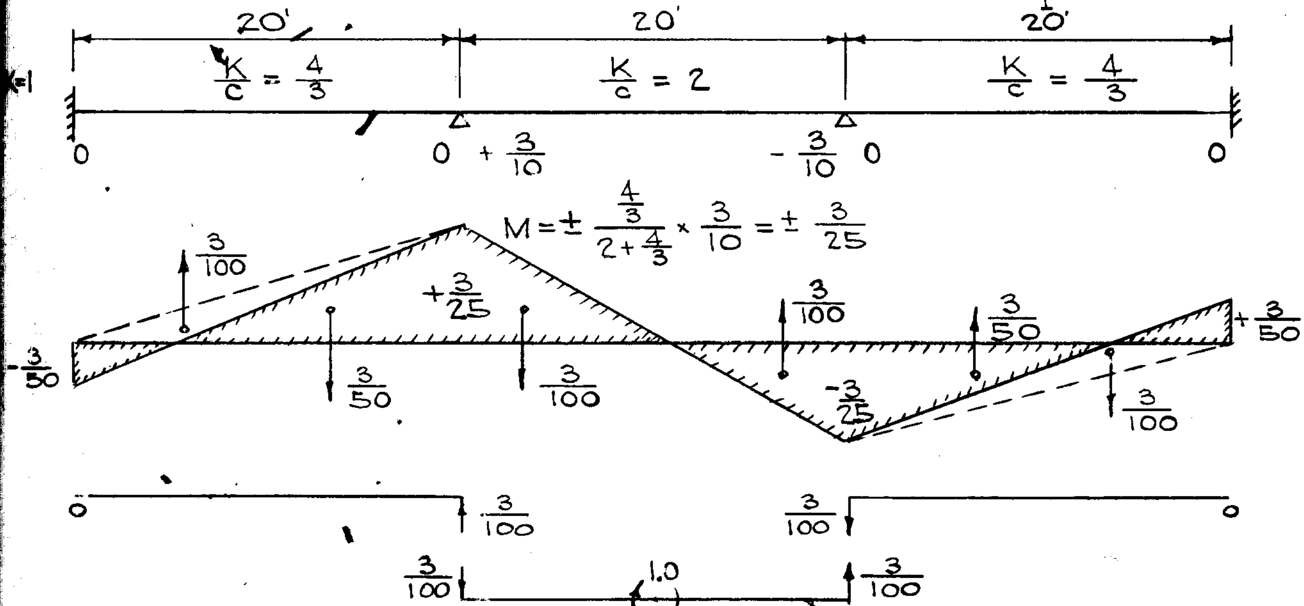


FIG. 69

Three Spans - Shear in Center Span - Fixed Ends

Apply a unit vertical displacement to any point in the span. i.e. a unit moment on the analogous column. Then $m' = \pm \frac{6M}{Ad} = \pm \frac{6 \times 1}{\frac{L}{I} \cdot L} = \pm \frac{6K}{d} = \pm \frac{6 \times 1}{20} = \pm \frac{3}{10}$



At ℄:-

$$-\frac{3}{100} \times \frac{10}{20} \times \frac{2}{3} \cdot 10 = -\frac{1}{10}$$

$$+\frac{3}{200} \times \frac{10}{20} \times \frac{1}{3} \cdot 10 = +\frac{1}{40}$$

$$-0.075 = M_{℄} = -\frac{3}{40}$$

$$\frac{3}{100}\left(10 + \frac{2}{3} \cdot 10\right) = \frac{3}{100} \times \frac{50}{3}$$

$$\pm 0.5 = M_{℄} = \pm \frac{1}{2}$$

$$+0.075 = M_{℄} = +\frac{3}{40}$$

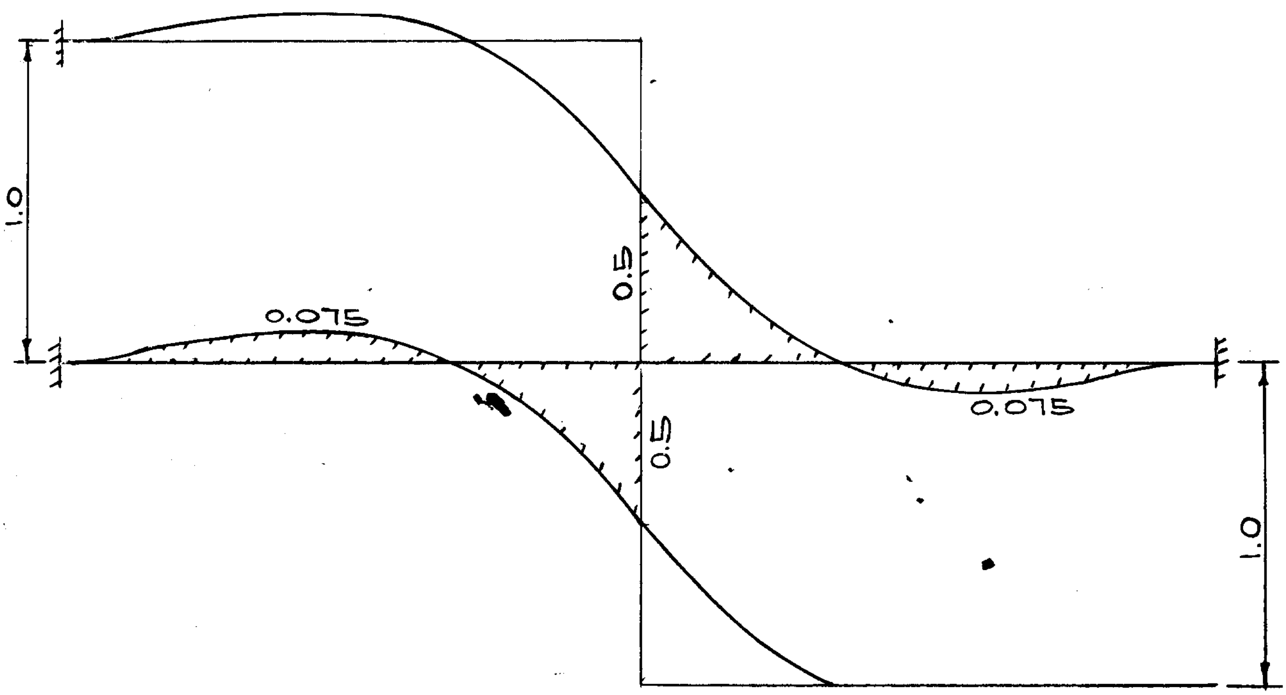


Fig. 70

CHAPTER V
TRUE DEFLECTIONS

Nature of the Problem. It is the purpose of this chapter to discuss the deflections which actually occur in structures rather than to discuss deflection computations as a tool of analysis in indeterminate structures. The two problems have often been confused whereas it is important to distinguish them. In analyzing indeterminate structures the deflections needed are relative and not absolute and are of interest only as a tool of thought. Actual deflections are in inches, but it is not usually necessary to compute them with great precision. True slopes are rarely wanted.

True deflections may be needed for any one of the following purposes:

(a) For camber computations to predetermine the shape of the floor-line of a structure when unloaded with either dead or live load (the structure or false-work) so that under dead load and, let us say, half live load the floor line will be level.

(b) For erection adjustments so that two parts of a structure will be properly joined in the field. For example, where a two-hinged arch is erected by cantilevering from the bluffs, backstays must be adjusted to give an accurate closure at the center. Similar is the design of wedges in swing bridges.

(c) As a measure of the flexibility of the structure, either,

To compare its deflection with that of another design for the same purpose, as where the relative advantages of competitive simple-span and continuous-span designs are to be considered for a bridge crossing.

To limit deformations in architectural work so that, for example, plastered ceilings will not later crack under dead or live load.

(d) To predetermine longitudinal movements at the end of a structure in order to set expansion rollers, or provide for the gap in the rails in a railway bridge, or otherwise to provide for expansion.

(e) To determine the deflection of shallow girders that have been designed for limiting deflection.

(f) To give a basis for comparison of measured and computed deflections.

(g) Very occasionally in long suspension bridges in studying variations in grade.

Methods of Computation - Beams and Girders. Deflections in beams and girders are most readily computed as bending moments due to the angle changes as loads on a simple span. This is true whether or not the structure is actually continuous at the supports.

If the beam is of uniform section the angle changes are conveniently computed as $M \frac{ds}{EI}$ and the deflection is the bending moment due to these angle weights as loads.

Usually with beams and girders only the maximum deflection is wanted. But this differs very little from the deflection at the center-line. In the extreme case which might normally occur of a triangular moment curve, Fig. 71. Δ_{max}. (max. moment due to loading shown) occurs where the shear is zero which is easily shown to be 0.577 L from one end.

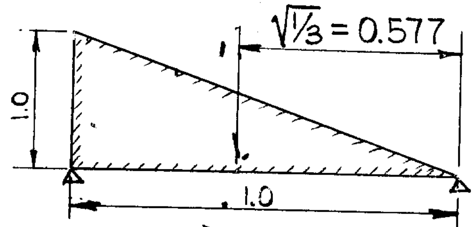


FIG. 71

$$\Delta_{max.} = 1/3 \cdot 0.577 \cdot 1/2\ (1 - 0.577^2)$$

$$\Delta \text{ at center line} = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}\ (1 - \frac{1}{2}^2)$$

$$\frac{\Delta_{max.}}{\Delta_{℄}} = \frac{0.577\ (1 - 0.577^2)}{0.5\ (1 - .5^2)} = \frac{0.386}{0.375} = 1.03$$

As this is an extreme case the absolute maximum deflection of a beam, then, cannot exceed the deflection at the center by more than 3 per cent and will usually be equal to it within less than 1 per cent.

The general expression for deflection is,

$$\Delta = K \frac{PL^3}{EI}$$

and the following values of K are perhaps worth noting:

Simple beam loaded at center - $K' = 1/48$

Simple beam uniformly loaded - $K'' = 5/384 = 5/8\ K'$

Cantilever beam loaded at end - $K = 1/3 = 16\ K'$

Cantilever beam uniformly loaded - $K = 1/8 = 6K'$

Fixed-ended beam loaded at center - $K = 1/4\ K'$

Fixed-ended beam uniformly loaded - $K = 1/5\ K''$

The deflection can in general be estimated well enough from these controlling cases. Note that uniform load produces 62-1/2 per cent as much deflection as center load and that fixing the ends reduces the deflection 75 per cent to 80 per cent.

If the deflection is desired under the design load, it is usually more convenient to consider the angle changes as $\frac{f}{Ey}$ ds. The deflection then, is the bending moment due to the f/y curve as a load times 1/E or, if y is constant, the bending moment due to the f curve as a load times 1/Ey.

This leads to an expression for deflection $\Delta = K_f \frac{f_{max.} L^2}{Ey}$

From these considerations it follows that,

(a) For a given variation in fibre stresses the loading is irrelevant, except as it corresponds to these fibre stresses.

(b) For given deflection of a given beam the maximum fibre stress is inversely proportional to the depth.

(c) For given deflection, depth, and length, the fibre stress is constant and cannot be varied by adding more material.

The last two principles are important in considering such cases as bending stresses produced in the columns of continuous girder viaducts by expansion of the girders. The matter is discussed further in Chapter VII.

Beams of Constant Section. If the section of the beam is constant, we have the following values for K_f in the expression,

$$\Delta = K_f \frac{f_{max}.L^2}{Ey}$$

Simple beam centrally loaded - $K_f = 1/12$

Simple beam uniformly loaded - $K_f = 5/48 = 1/10$ approx.

Cantilever beam load at end - $K_f = 1/3$

Cantilever beam uniform load - $K_f = 1/4$

This gives a convenient basis for computing maximum deflection for given maximum fibre stress where the section is constant. It is often specified in architectural work that the deflection shall not exceed 1/360 of the span. Then,

$$\frac{L}{360} = \frac{1}{12}\frac{fL^2}{Ey} = \frac{1}{6}\frac{f}{E}\frac{L^2}{d}$$

$$\frac{L}{d} = 60\frac{f}{E}$$

For steel with a working fibre stress of 16,000 lb. per sq. in.,

$$\frac{f}{E} = \frac{16,000}{30,000,000} = \frac{0.533}{1,000}$$

For timber - $\frac{f}{E} = \frac{1,200}{1,500,000} = \frac{0.8}{1,000}$

For concrete - $\frac{f}{E} = \frac{800}{2,000,000} = \frac{0.4}{1,000}$

Except for steel, these figures are only rough approximations. Roughly, then, for this limit on deflections,

$L/d = \frac{60 \times 0.6}{1,000} = 1/24$ nearly, which gives the convenient working rule that the depth in inches shall not be less than half the span in feet -- not less than a 10" beam on a twenty foot span, for example. Where live loads are important, however, and vibration is undesirable, such shallow beams are very objectionable and they will preferably be twice as deep.

Deflection as a Function of Stress. If the depth of the beam is constant but the section varies, the f curve will not follow the moment curve. In the theoretically ideal plate girder the flange stress is constant for uniform load. Then, $\Delta = \frac{1}{8}\frac{fL^2}{Ey}$. Actually it is, of course, not possible to build such a girder, but this shows that the maximum deflection of a well designed plate girder is between $\frac{1}{8}\frac{fL^2}{Ey}$ and $\frac{1}{10}\frac{fL^2}{Ey}$. Probably all necessary accuracy is obtained by taking $\Delta = \frac{1}{9}\frac{fL^2}{Ey}$ where f is the fibre stress for dead load, live load, or total load as the case may be. If the depth is 1/8 to 1/12 the span, this indicates a total deflection for 50' girder of from 1/2 in. to 3/4 in.

It is often specified that if clearance requirements call for girders having a depth less than standard (say 1/12 the span) then the fibre stresses shall be reduced until the computed deflection does not exceed that of a standard span. For simple spans it will be seen that this is equivalent to allowing a fibre stress of $f_c \frac{d'}{d_s}$, where f_c is the allowable stress in ordinary design, d_s is the standard minimum depth and d' is the allowable design depth. The specification as ordinarily worded is somewhat ambiguous, but is commonly taken to apply to the total deflection for live and dead load. This is discussed further in Chapter VII.

Designing for Deflection. Another point in this connection concerns the economics of design of such girders. Evidently in such girders, cutting off cover plates results in less economy than it does in girders designed for stress only, since cutting covers increases the center deflection and therefore reduces the allowable fibre stress.

If the flange is constant,

$$\Delta = \frac{5}{48}\frac{f_cL^2}{Ey} \qquad \text{Volume of flange} = A_cL$$

$$\frac{1}{f_c} = \frac{5}{48} \text{ x constant}$$

$$A_c = \frac{M_c}{2f_cy} = \frac{\text{constant}}{f_c}$$

Volume of flange = 5/48 x constant.

If the fibre stress is constant (ideal plate girder)

$$\Delta = \frac{1}{8}\frac{f_cL^2}{Ey} \qquad \text{Area of flange} = \frac{2}{3}A_cL$$

$$\frac{1}{f_c} = \frac{1}{8} \text{ x constant}$$

$$\text{Volume of flange} = \frac{2}{3}\frac{1}{8} \text{ x constant}$$

$$= \frac{\text{Volume of flange in ideal girder}}{\text{Volume of flange in girder of constant section}} = \frac{4}{5}$$

In the case of a girder designed for stress rather than deflection, this ratio is 2/3. The relative economy of cutting off cover plates is even less in an actual girder.

Deflection of Plate Girders. If it is decided that it is necessary to compute deflection using the actual design sections, we may use either the M/I curve as a load and multiply by 1/E or the f curve as a load and multiply by 1/Ey or the M/A curve as a load and multiply by,

$\frac{1}{d}\frac{1}{Ey}\left(f = \frac{M}{Ad}; \frac{M}{I} = \frac{M}{Ady}\right)$

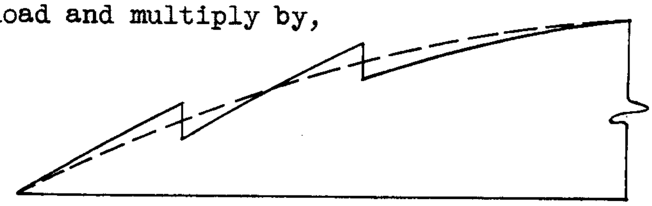


FIG. 72

The I will be for the gross section and the A will be the average gross area of two flanges and one-sixth the area of the web. Note here that deflections are being computed, that it is not a problem in design and that there is more gross area in action than net area. Note also that to use a broken curve breaking suddenly at the ends of the cover plates, such as shown in Fig. 72, may result in a false notion of precision, since the flange plates do not suddenly pick up their stress.

The important point here is that the deflection may be computed as the bending moment of the angle changes and that these latter may be determined in several forms. When the angle changes are determined, the computation of deflection is a matter of geometry.

Uncertainty of Moment of Inertia. There are, however, several disturbing factors in computing the angle changes. In plate girders the cover plates do not immediately come into action; even in rolled beams the application of the ordinary beam theory is imperfect (See Basquin - Eng. News Record). In reinforced concrete beams E cannot be very accurately known, tension in the concrete exists, there is some bond slip (a reinforced-concrete beam cannot have cracks in the concrete around the rods unless the rods slip and yet the concrete must crack by the nature of the stress conditions), lineal stress-strain relations do not obtain in the compression area. Since the computation of deflections from angle changes is a matter of geometry, the comparison of measured with computed deflections gives an interesting measure of the general accuracy of the theory of flexure, provided we can determine E, exactly as strain-gage measurements on a beam give a measure of the accuracy of our stress analysis when E can be determined.

Trusses - Four Methods. Deflections of trusses may be computed in one of four ways,

Virtual Work.--By the formula $\Delta = \sum \delta u$ of virtual work. Here δ is the change of length of any bar and u is the stress in that bar due to a hypothetical unit load at the point of deflection. Usually δ is due to stress and equals $\frac{Sl}{AE}$ and, $\Delta = \sum \frac{Sl}{AE} u$

This is convenient where the deflection of a single point is wanted but is not convenient where a complete camber-blocking diagram is needed, since it requires a separate tabulation for each point.

Angle Weights.--By computing the angle changes along any path connecting the two abutments (immovable points) and then taking these angle changes as loads just as was done in beams. Usually the angle changes along the lower chord are computed. This gives a complete diagram of vertical deflections for this chord.

Treating as a Beam.--By the methods used in computing deflections of beams, neglecting the effect of the web system just as the effect of shear is neglected in beams. It is quick and convenient, but not very accurate, since the influence of web distortions is relatively great in a truss, amounting to perhaps 20 per cent to 30 per cent of the total deflection. In using this method we may, very roughly, consider the truss as a beam of uniform section, or as having a variable,

$$I\ (= A_{av}\frac{h^2}{2})$$

or may compute the elastic weights, $(w = \frac{L}{Ar^2})$ for each chord bar and then find the bending moments due to the mw values.

Williot Diagram.--By the geometrical construction known as a Williot diagram. Evidently the simplest method of computing the shape of a truss after straining is to lay it out to a very large scale. This is clearly impracticable. If, however, only the changes in length (SL/AE values), are laid out, it is possible to use a large scale at the same time keeping the diagram within the dimensions of an ordinary drawing. In Fig. 73 let the lengths of the three sides of triangle, A B C change as shown by the Δ values. It is required to redraw the triangle assuming some side, such as A C, and some point on this line such as A, to remain unchanged in position. The obvious construction is to swing from C' an arc with radius C B - Δ_{cb} and from A an arc with radius A B + Δ_{ab}. The new triangle is A B' C'.

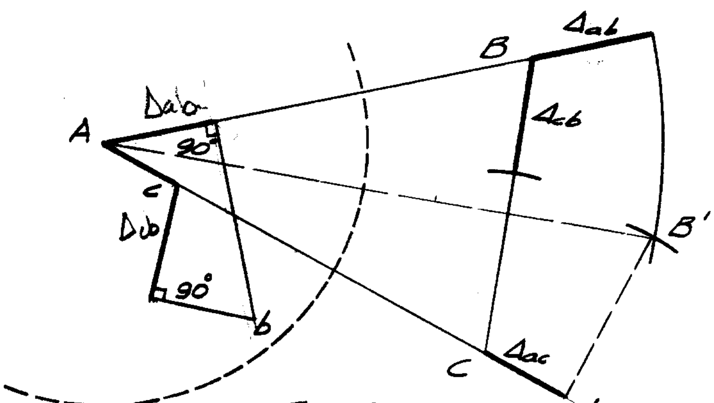


FIG. 73

But the changes of length in structures are so minute that on an ordinary scale for the triangle they could not be found. We may, however, lay off the changes in length and omit the original lengths. Hence, we draw only the small diagram shown encircled at point A.

From A in the triangle (or any other point A) lay off Δ_{ac} and Δ_{ab} parallel to A C and A B and in the direction of the deformation. Then if point A is fixed and the direction of A C is used as a line of reference, point c in the diagram will locate point C' relatively. Point B' is not yet located, however, because of the shortening of side B C. Δ_{cb} may then be laid off from c in the direction of its deformation. Point B' will have two relative movements - one on an arc virtually normal to Δ_{ab} and another on an arc virtually normal to Δ_{cb}. The intersection of two perpendiculars drawn from the ends of Δ_{ab} and Δ_{cb} will, therefore, locate point b establishing the true location of B' with a high degree of precision. Obviously this can be extended indefinitely through any system of triangles making up a truss, since from two points obtained in each case, we locate a third. Subdivided trusses and other special cases may be handled with slight modification.

The method may be further illustrated and outlined as follows, referring to Fig. 74 in which unsymmetrical deformations are assumed.

(a) Select some convenient point and direction of reference. Preferably this will be either end of a bar the rotation of which is slight such as the center vertical. U_3L_3 was selected in this case.

(b) Assuming either end fixed, lay off the movement of the other end with respect to it in the direction of that movement. This locates points U_3 and L_3 in relative position, and makes it possible to locate either point U_2 or point U_4.

(c) $\Delta\ U_2U_3$ a shortening, will be laid off from U_3 to the right, - i.e., point U_2 moves toward U_3. $\Delta\ U_2L_3$ a lengthening, will be laid off from L_3 upward, - i.e., point U_2 moves away from L_3. The intersection of the perpendiculars from the ends of these two deformations locates point U_2.

(d) From U_2 and L_3 point L_2 is located by erecting perpendiculars from the ends of $\Delta\ L_2U_2$ and $\Delta\ L_2L_3$.

(e) The remaining points are plotted in a similar manner giving a deformation diagram which gives the displacement of the joints relative to the assumed position of U_3L_3. In the structure point L_o is fixed and the direction of L_oL_6 is horizontal. Hence, the vertical displacement of L_oL_6 shown by the diagram needs to be annulled by a counter rotation of each joint of the structure in proportion to its radius arm.

(f) This is readily done by drawing on L_oL_6 a rotation diagram of the truss, each member of which is respectively perpendicular to its original position in the structure. Such a diagram is shown dotted. The absolute

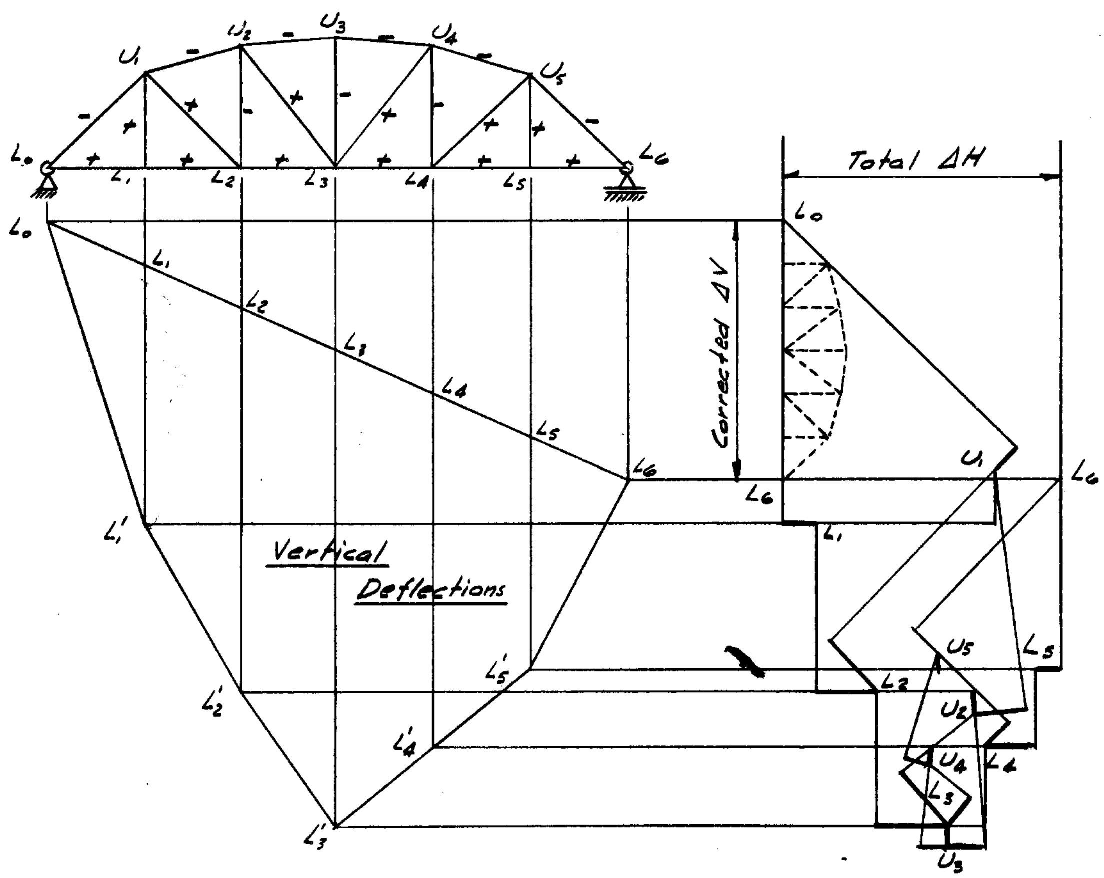


FIG. 74

displacement of any joint is then given by its position in the Williot diagram with reference to the position of the corresponding point in the rotation diagram.

(g) Construction of the rotation diagram is seldom necessary. In analyzing indeterminate structures relative displacements only are required. In computing true deflections the vertical components are given directly by the simple construction shown on the left and the horizontal components may be obtained directly from the Williot diagram.

Relative Rigidity of Types. The method of analogy to a beam gives an interesting basis for study of the relative rigidity of types of structures. Thus, consider two proposed designs as shown in Fig. 75, the problem being to

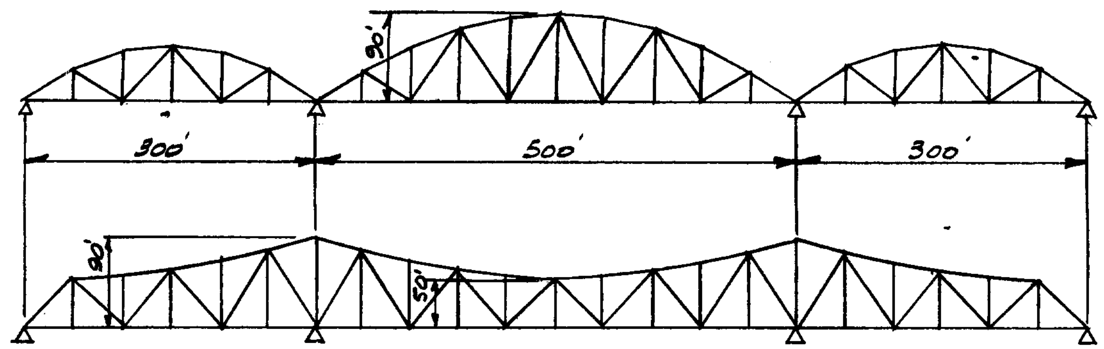


FIG. 75

determine the relative downward deflections at the center of the bridge due to live load.

Inspection of the continuous span shows that the critical loading for maximum deflection is the same as for maximum stress in the simple span and in the center portion of the continuous span. For this loading near the ends of the continuous span the chords will be understressed since the side spans are not loaded. Treating the bridge as a continuous girder of uniform section, the negative moment is,

$$M_n = \frac{5}{5+4} \cdot \frac{1}{12} \cdot w \cdot \overline{500}^2$$

with the center span only loaded and for the center and left end span loaded we have, approximately,

K = ⑤ ③ ⑤

0 $\frac{1}{8} w \overline{300}^2$	$\frac{1}{12} w \overline{500}^2$ $\frac{1}{12} w \overline{500}^2$	0 0
−1.12	−2.08 −2.08	0

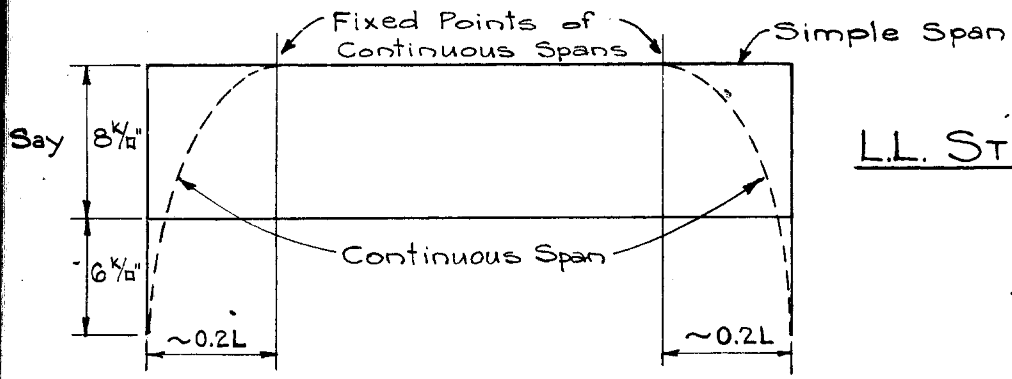
K/c = 5   K/c = 2 (Approx.)

$\frac{2}{7} \times 0.96 = +.28$

−1.80

$$\text{And}: \frac{5/7}{1.80/2.08} = 82\% \text{ of Maximum}$$

Throughout the simple span, then, and near the center of the continuous span, the chord stress equals the design chord stress for live load; near the end of the continuous span the stress is about 80 per cent of the design chord stress for live load, see Fig. 76.



L.L. STRESS DIAGRAM

FIG. 76

Between the fixed points the critical design load is full load in the center span, and the chord stress under load in the center span is the design stress and is constant. Beyond these points the chord stress for uniform load throughout the center span is less than the design stress and so the chord stress begins to drop off at the fixed point passing through zero between there and the support. This is shown in Fig. 77.

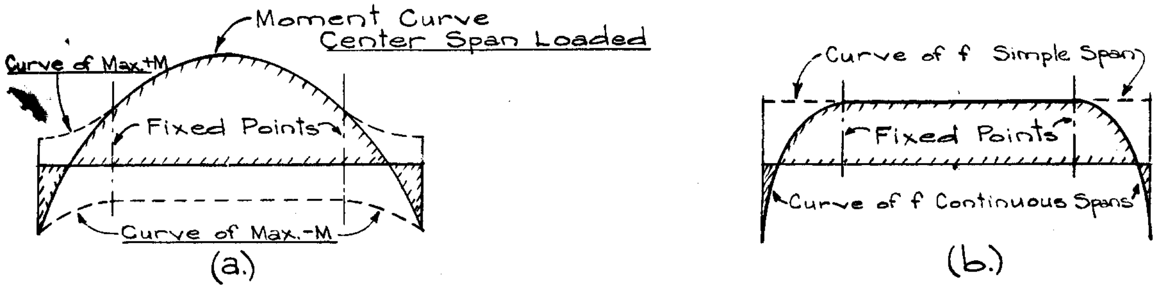


FIG. 77

For variable depth the curves will usually intersect near the fixed point but not at it unless the depths of the trusses are the same at that point.

The curves of f/y then, are approximately as shown in Fig. 78. Inspection shows at once that the bending moment due to the f/y curve as a load is greater for the continuous span.

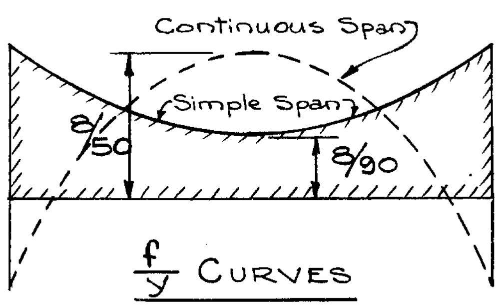


FIG. 78

The whole analysis is approximate and furthermore, neglects the effect of the web, which may be supposed - in the absence of exact comparative data - to be about the same in the two cases. The whole subject will well repay detailed study based on economic proportions of the types involved and the variations in economy due to departing from these proportions. The three span case has here been chosen for simplicity. As an interesting problem, the relative stiffness of the Sciotoville Bridge should be compared with that of two simple spans using the proportions of the Metropolis Bridge of the Municipal Bridge of St. Louis.

It should be noted here that the continuous span has an upward deflection from the dead load position as well as a downward deflection, so that the total range of deflection is greater than the downward deflection indicated above.

In simply supported spans maximum deflection evidently occurs at the center of the span. In continuous girders the maximum deflection also occurs at or near the center. This may be seen from the fact that influence lines for all points in the center portion ofthe span indicate complete loading of that span for downward deflection and of adjacent spans for upward deflection, and with such loading the maximum deflection is practically the same as that at the center.

In arches and suspension bridges, however, maximum deflection is not at the center. The deflection of such structures is discussed in connection with those subjects.

<u>Comparison of Types Where Depth is the Same</u>. In comparing the deflections of simple beams and continuous girders, where the depth is the same, conditions are different from those discussed above. There the continuous structure is more flexible, owing to reduced depth at center dictated by economy. With the same depth in both cases, the continuous girder is somewhat more rigid as regards downward deflection only, though the total range of deflection is still somewhat greater than for the simple span. The reduction in the downward deflection is not so great as might be supposed.

In order to compare the cases, consider the three-span girder shown in Fig. 79. If the simple span were an ideal girder of constant depth and uniform strength the f/y curve would be of constant height as shown in (a). In the actual girder the fibre stress will decrease to zero towards the end as shown in (b). The f curve for the continuous girder will vary as discussed

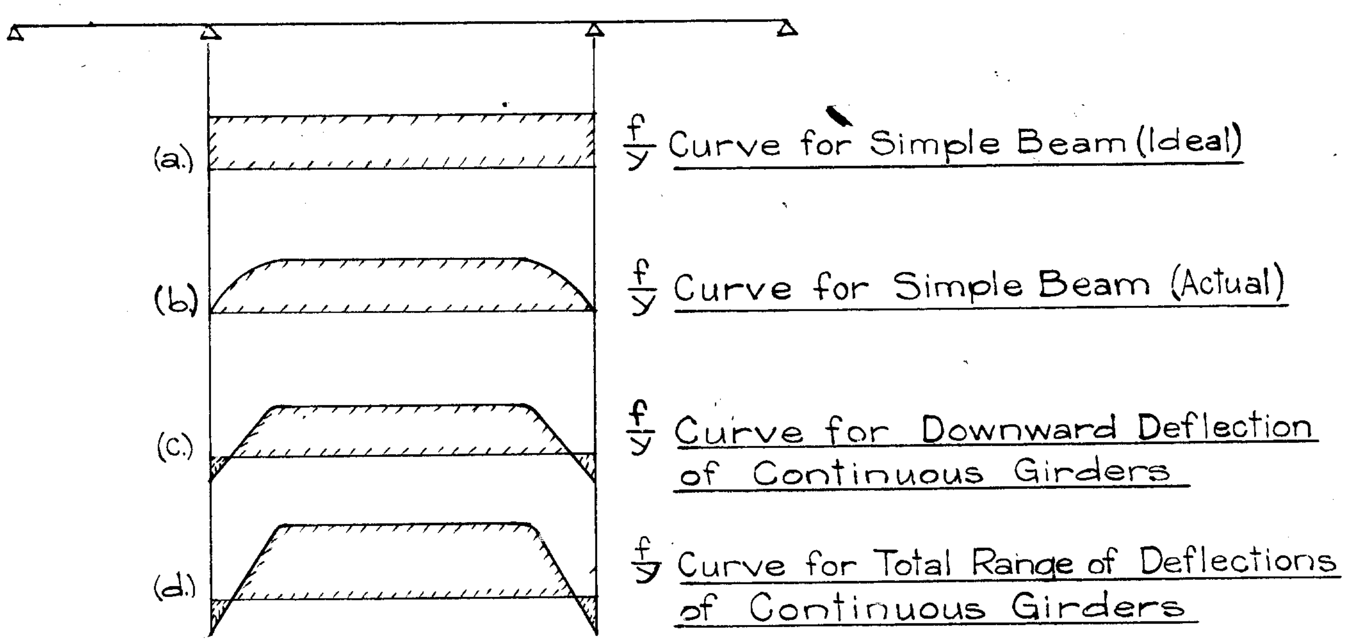


FIG. 79

above for the truss for loads in center span only and gives the curve of f/y for downward deflection as shown in (c). For upward deflection the side spans are loaded and the moment curve in the center span is a horizontal straight line and so also is the f/y curve. For total range of deflection we add the upward to the downward deflection and hence add the curves of f/y. This curve is shown in (d).

Moments at the center computed for the four loadings shown, then, give a measure of the relative deflections respectively of,

(a) The ideal simple span.
(b) The simple span as actually designed.
(c) The downward deflection of the continuous girder.
(d) The total range of deflection of the continuous girder.

Inspection shows that the total range of deflection is greater for the continuous girder, that the downward deflection of the continuous girder is only slightly less than for the ideal simple span and may even be greater than for the simple span as it would actually be designed. Here again, study of actual cases is profitable.

Longitudinal Movements. Erection. Longitudinal movements due to stress in ordinary trusses is readily computed as the total stretch of the chord if this is straight.

Temperature distortions are equal to the temperature change times the thermal coefficient times the horizontal distance. This is evidently true whether or not the chords are straight, as may be seen by considering that, where temperature changes do not produce internal stress, they are simply equivalent to a change of scale in the structure and all lineal dimensions change accordingly. Of course, these temperature distortions take place in a structure whether or not it is statically determinate. In indeterminate structures we add to the direct temperature distortions, the distortions due to the stress produced by the temperature change.

Deflection of Cantilevers. At some stage in the erection of a cantilever bridge it acts as an indeterminate structure. Thus in the cantilever shown, Fig. 80, assume that the bridge is to be erected by cantilevering and that expansion and elastic freedom is to be provided as indicated in the diagram. Assume that closure is, as is usual - to be made at the center of

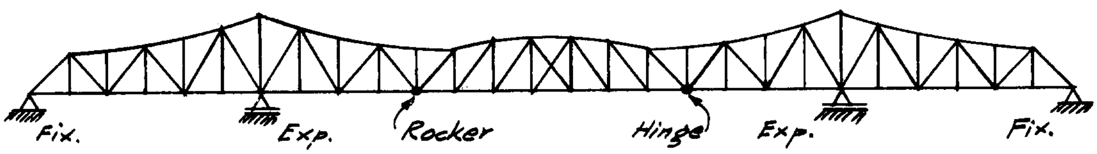


FIG. 80

the simple span. Just before closure, then, the bridge is as shown, movement being prevented at both hinge and rocker. Now when top and bottom chord are closed the bridge is subject to temperature stresses due to the two fixed bearings.

Indeed, the cantilever bridge because of its close kinship to the continuous bridge and of the questions of comparative economy and stiffness involved, is a proper subject for study in connection with indeterminate structures in general, and will be considered in its proper place.

Evidently careful analysis is required for camber in the case of cantilever bridges since during cantilever erection there is a sag toward the center which must be entirely corrected by cambering.

In computing the deflections of cantilevers it is necessary to include three entire spans in the computations. Of course, three or more spans are also necessarily included in the computations for the deflections of a continuous span in finding the moments and stresses, but when the moment curve has once been determined, the deflection computations are restricted to a single span. It is not possible to apply the method of angle weights to a single span only, because at the hinges, AA', Fig. 81, the angle change is not determinate from consideration of a single span (mathematically we may say that the product of zero moment times the infinite elastic weight of the hinge is finite but indeterminate). Perhaps the most convenient procedure is

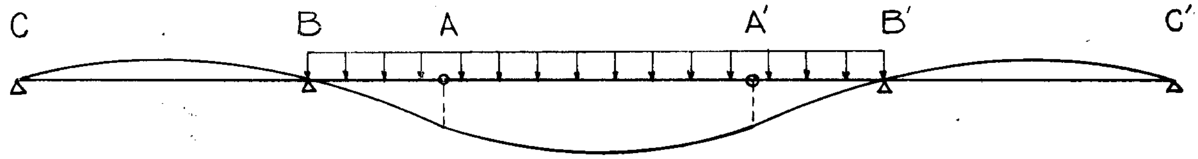


FIG. 81

to find the deflection of B (B') from a line through CA (C'A') by angle weights and then to find the deflections of the points on the suspended span from the line, A A'.

In general the cantilever bridge will be somewhat more flexible than is the continuous span. This question and its bearing on design is discussed further elsewhere.

Summary and General Considerations. A student of indeterminate structures should be quite familiar with the methods of angle weights, deflection diagram and virtual work. These tools are simple enough even though tedious to apply. Quite often simple short-cuts using the curves of f/y instead of M/I give answers with all needed precision.

By studying the f/y curves interesting comparative data as to the action of structures under load is easily obtained. The method of analysis here presented is elsewhere extended to other deflection studies particularly arches and suspension bridges.

After the results are computed, their significance often presents the first real difficulty. There is little doubt that there is an intimate relation between flexibility, vibration, impact and durability. It is important to realize that a structure is a machine absorbing and giving out work and that this fact is intimately connected with its durability.

But such statements as these are very vague and not immediately helpful. There is doubtless sound engineering judgment back of the standard acceptance test common in England of limited deflection under a live test load - a practice sometimes employed also in this country. Deflections exceeding those computed hang up a pretty definite danger sign that somewhere the structure is not acting as design theory would indicate.

In a sense, as previously mentioned, deflection measurements are equivalent to strain gauge measurements of the structure as a whole - rather than of its constituent parts. Like strain gauge measurements also, they are frightfully difficult to interpret - and may be quite misleading in structures of reinforced concrete. But a lack of proportionality between deflection and load is cause for serious study.

Here is a field of study almost untouched, the importance of which should be clearly recognized even though the principles are obscure.

CHAPTER VI
CONCRETE ARCHES

General Discussion. A concrete arch is essentially a closed ring and may be conveniently analyzed by the direct application of the column analogy. Because of their beauty, rigidity and economy, masonry arches are of peculiar interest and importance to the bridge engineer. Their field of usefulness in highway work is becoming increasingly extensive. The common mathematical theory of design is adequate for any set of assumptions as to physical conditions. The attention of investigators is centered, therefore, rather on the physical limitations affecting the assumed continuity of the arch axis than on any further elaboration of the mathematical theory. Except in rare cases the use of influence lines in a quantitative sense is scarcely justified. Usually dead load stresses govern quite largely in the design. In spandrel-filled arches the relative importance of the dead load together with the uncertainty of the live load distribution render refined stress calculations unjustifiable. The distribution of any concentrated wheel load in such cases involves a variant spread in two directions, one transverse and one parallel to the arch axes. In general, then, some equivalent vertical load of nearly uniform intensity will be sufficient. Except in arches of very sharp curvature, horizontal components of earth pressure will usually be of negligible effect in the design and can properly be omitted.

The Pressure Line Theorem. Winkler's Theorem, commonly known as the Pressure Line Theory in arch analysis, is sufficient for all ordinary cases of vertical load in spandrel-filled arches. It may be stated somewhat as follows:

For vertical loads, neglecting rib-shortening, and assuming a uniform depth of ring, the true pressure line is that string polygon for the given loads which lies nearest the arch axis as determined by least squares - or by eye.

But we are almost invariably dealing only with vertical loads - except in sewers and culverts, where it is to be especially noted that the theorem needs modification to be useable. Rib-shortening can be computed by the approximate formula with all needed accuracy, and study of curves, Fig. 99, shows that ordinary variations in depth of arch ring (variations of m) are not very important.

Proof.

By Least Work

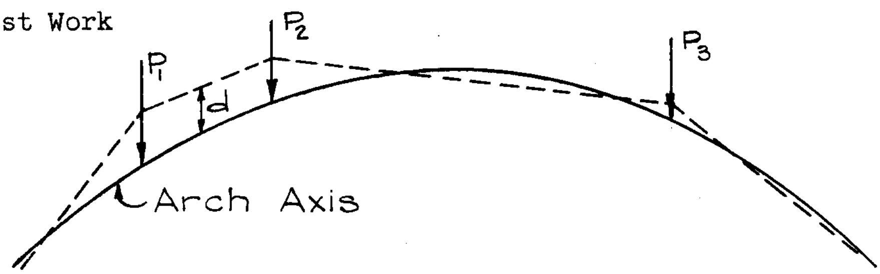


FIG. 82

Let Fig. 82 show any string polygon for the loads
H is constant
M on any short length = Hd

Rotation $\phi = \frac{Mds}{I} = Hd \frac{ds}{I}$

Work $= 1/2\ M\phi = 1/2 H^2 d^2 \frac{ds}{I}$

Total work $= 1/2\ H^2 \int d^2 \frac{ds}{I}$

Obviously for minimum $\int d^2 \frac{ds}{I} = 0$ since by inspection H is not zero.

By Column Analogy

If we can draw a string polygon exactly fitting the arch axis, we shall have zero elastic load on analogous column and hence no change by column analogy.

Evidently the nearer the string polygon fits the axis the smaller will be the values m_i in the column analogy and hence we can draw one polygon which will give zero values for m_i - this is the true polygon.

Winkler's original statement dealt with voussoir arches and stated that if it is possible to draw a string polygon within the middle third for all conditions of loading, the true line of pressure (rib-shortening neglected) will be within the middle third - a necessary condition for compressive stresses over the entire section. The theorem has, however, just as useful an application to monolithic arches.

This theorem is a fundamental conception and should have wider recognition and use in arch analysis. While it neglects rib-shortening, which may be treated separately if the structure warrants, it probably tells us all we can know, or at least all we need to know, about dead load stresses. For earth filled arches of ordinary span the Pressure Line Theorem therefore furnishes virtually the complete solution. Nothing could be more truly the elastic theory than this simple application of graphic statics. In it the principle of least work and the method of least squares is applied in a convenient and simple manner. The elastic theory involves nothing more than continuity coupled with statics, and should not be thought of as a method in itself.

Live Load Analysis. In spandrel braced arches where it is necessary to consider the effect of live load concentrations applied directly to the rib, the extent and exact position of the loads becomes important. Influence lines will accurately determine the point of load divide, but generally they are not necessary. Standard sets of curves are not always available. Inspection of such curves shows that the 3/8 point is very close to the load divide on any ordinary symmetrical arch. For important cases or any unusual conditions it will simply be necessary to compute a small section of the influence line near this point if the exact point of change from positive to negative is desired. Sharp load divides, however, are not really very significant. Their value is more that they permit exact checks on computations than that they have great physical exactness. Where more than one critical point occurs, as in the crown moment influence line, the improbability of the loading also needs consideration.

Load divide for fibre stress is not the same as for the moment on the axis; the effect of the thrust needs to be added to the effect of the axial moment. If influence lines are used it is therefore desirable to construct them for kern moment, giving the simultaneous effect of moment and thrust. That is; $f = \frac{M_k y}{I}$ instead of $f = \frac{P}{A} \pm \frac{My}{I}$ will be found the more convenient expression especially as the kern moment follows directly by the column analogy just as readily as does the moment on the axis or at any other point. If, then, the kern moments have been determined for full LL and again for

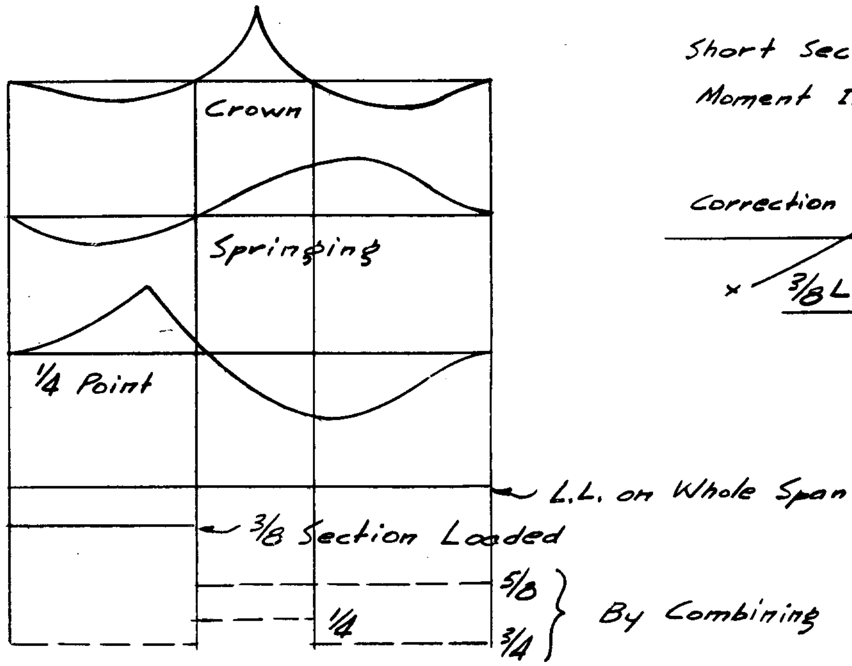


Fig. 83

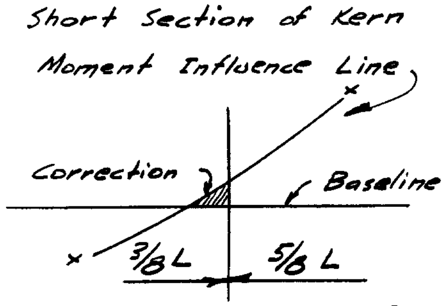


Fig. 84

the 3/8 segment loaded, the moments and stresses will follow by subtraction for the 5/8 section, the center 1/4, or the outside 3/4 as may be required. This is obvious from an inspection of Fig. 83 and covers all the ordinary critical conditions. If more accurate results are warranted, a correction may be made as shown in Fig. 84 by constructing a very short length of influence line for kern moment adjacent to the 3/8 point.

The column analogy provides a convenient tool for computing the live load stresses. The half arch ring may be divided into any convenient number of sections. Equal horizontal projections, Δx, are probably more convenient and more accurate than equal $\Delta s/I$ values, - the chief objection to the latter being the long section at the springing. The $\Delta s/I$ values thus obtained may be treated as elastic areas making up the analogous column section having the shape of the arch axis. Its neutral point may then be determined and the moments of inertia, I_x and I_y about the principal axes found. If this column section is then loaded with the angle changes due to any curve of static moments for the given loading, the indeterminate moments m_i may be found as analogous to fibre stresses on the column. For full live load and for the 3/8 loading, cantilever diagrams are most convenient. A positive value of m_i represents compression on the section and is, therefore, of opposite sign to the static moment load and will be subtracted from it.

Influence Line for Kern Moment. An influence line for moment at the kern point may be drawn by applying at the kern point a unit rotation. By the column analogy this may now be treated as a unit load on the arch and the resulting moments in the arch ring computed. From these the angle changes (mw) are found. Treating these as loads, the influence ordinates may be computed as deflections of the load line. For vertical loading this will be the bending moment curve for the angle changes - and the original rotation - as vertical loads on a simple beam supported at the abutments. For horizontal loads the influence ordinates are found as horizontal deflections of a cantilever by Greene's second theorem (see page 49) as statical moments about the point where the influence ordinate is to be found of all rotations between that point and either abutment. Influence ordinates for moments applied to the arch ring will be found as total rotation between either abutment and the point where the ordinate is to be found or as a slope in either of the two influence lines referred to. The curve of moments for the unit rotation at the kern point is most readily found by drawing the neutral axis of the column for a unit load at the kern point. Two points of zero moment will locate this line. The scale is determined from the moment at the neutral point ($= \frac{1}{\text{Total elastic weight}}$). There is not much occasion for performing this operation quantitatively but it is a powerful tool of analysis.

Influence lines may be constructed directly by applying unit loads at various points and computing the resulting moments. This is perhaps the most convenient method where short sections of line are desired for critical load points. If a series of influence lines are to be drawn, a convenient method is to construct first the influence lines for H, V, and M at the neutral point and from these obtain the other values by statics.

The Column Analogy as Applied to Temperature, Rib-Shortening, Dissymmetry, Etc. Linear displacement in the column analogy is equivalent to a bending moment about the parallel principal axis. Thrust is then $H = \frac{M_h}{I_h}$. Temperature change would produce a horizontal displacement etL if the ends were not restrained and $H_t = \frac{etL}{I_h/E}$. Rib shortening similarly would produce a change in span length equal to f_cL and $H_s = \frac{f_cL}{I_h}$ (E = 1 in computing I_h). The moment at any point is $M = H \cdot y$. Or, we may write directly from the column analogy: $m_i = f = \frac{My}{I}$; And $M_t = \frac{etL \cdot yE}{I_h}$, $M_s = \frac{f_cL \cdot y}{I_h}$

(The factor 1/E should be included in the denominator in each case if omitted in the I_h computations.)

Where e is the temperature expansion coefficient (approximately .000006), t is the temperature change in degrees and f_c is the average unit compressive stress and may be found as $f_c = H/A'$ where A' is approximately the average vertical projection of the depths of section, Fig. 85.

A more logical method for correcting for rib-shortening is to include this factor automatically in the computations. Going back to the fundamental proof of the column analogy, the horizontal movement of the cut ends of the ring due to a unit horizontal force is given as, $\sum \frac{ds}{I} y^2$. This neglects the effect of direct compression. If the latter is included this term will be $\sum \frac{ds}{I} y^2 +$ (shortening of rib due to direct compression from a unit horizontal force).

We may say then, that in the third term of the expression,

$$m_1 = \frac{P}{A} \pm \frac{M_x X}{I_v} \pm \frac{M_y Y}{I_h + \frac{1}{A'_{av.}} \cdot L}$$

the m_y term in the numerator represents the horizontal spread of the abutments if on rollers (no settlement or rotation) and the denominator represents the horizontal shortening due to a unit H. Evidently the first term in the denominator takes account of distortions from moment, and the second takes account of rib shortening. There is no rib shortening term in the numerator because there is no H force so long as the abutment is on rollers. The distortion of the block (Fig. 85) is due to moment (the effect of which is already accounted for), to transverse shear (the effect of which is negligible) and to thrust, which produces a horizontal shortening $\frac{H}{A'} \Delta X$ if the shear is neglected. Neglect of the shear forces (as well as of the shear distortions) produces no error in symmetrical cases and is of no importance in other cases. The total shortening of span, then, will be $H \sum \frac{\Delta X}{A'} = H \left(\frac{L}{Av.A'}\right) = \frac{HL}{Av.A'}$ (approximately).

This method is exact. The usual practice is to correct separately for the effect of rib-shortening by computing the shortening of span and then treating this as we would a change of span due to temperature. By this method the rib-shortening term would appear as a correction in the numerator instead of in the denominator of the third term of the expression used in the column analogy. This method is approximate, since it neglects the rib-shortening effect of rib-shortening, but the error is of no importance. Mathematically it is equivalent to assuming $\frac{1}{1+a} = 1-a$ which is a close approximation if a is small.

Evidently there is also a rib shortening correction in the second term also for unsymmetrical vertical loading, but it is entirely negligible, partly because the vertical shortening is small in amount, and also because I_v is relatively large (the arch is more flexible vertically than horizontally). If this rib-shortening were included the correction (for vertical loading) would be entirely in the numerator because of the load and not at all in the denominator if the arch is symmetrical.

For a suggestion as to the extension of the column analogy to give a perfectly general expression, including rib-shortening and shear distortions, see Chapter . Such an extension, however, is probably, largely academic.

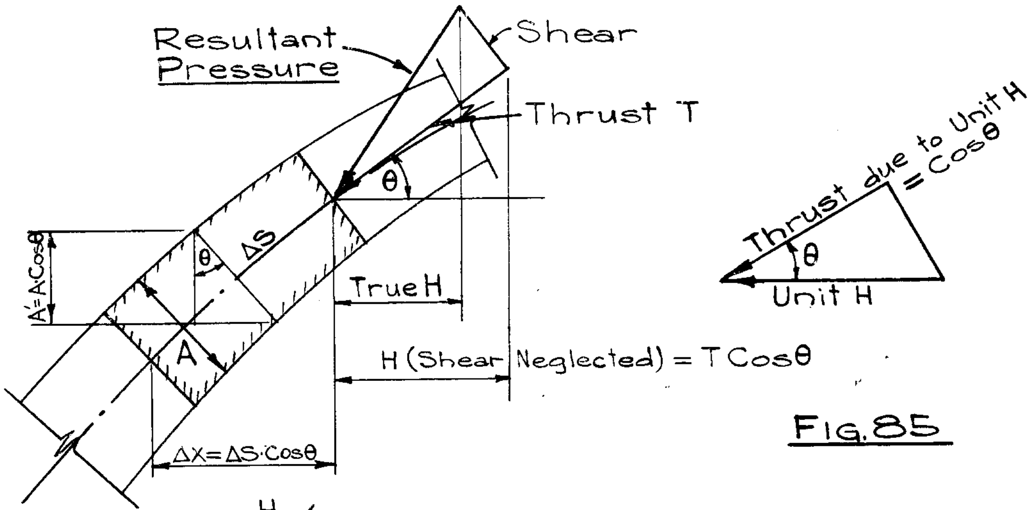


FIG. 85

$$\frac{T}{A}\cos\theta\,\Delta S = \frac{\frac{H}{\cos\theta}}{\frac{A'}{\cos\theta}}\cos\theta\cdot\frac{\Delta X}{\cos\theta} = \frac{H}{A'}\Delta X = \frac{H}{A}\Delta S$$

In a similar manner it may be seen that shear and settlement of the abutments represent vertical movements and may therefore be expressed as bending moments about the vertical axis. Rotation of an abutment may be taken care of in the column analogy as a load on the section applied at the center of rotation.

Unsymmetrical arches may be solved by the column analogy using the same general method of attack. Find the total elastic weight of the section and its neutral point, the values I_h, I_v and the product of inertia I_{hv}. From these the construction of the circle of inertia gives the principal axes and the principal moments of inertia. Working from these axes as a reference the kern moments and fibre stresses may be found exactly as in the symmetrical case. Temperature changes for the unsymmetrical arch would cause a movement of the unrestrained structure having components parallel to each of the two principal axes. While this represents bending moments on the column section in two directions, the magnitude in one direction is apt to be small and the corresponding moment of inertia large so that its effect may be negligible.

Generally the lack of symmetry in an arch will be slight and the 3/8, 5/8 load divide may still be sufficiently accurate. For greater accuracy a small section of the influence line may be drawn. When the springings are at different levels the structure may have an unsymmetrical appearance when it really is symmetrical except for load.

The Arch Ring. It seems reasonable to assert as a general proposition that true economy in design lies in securing a minimum crown thickness. In any span it is economical in material to throw the dead load toward the abutments. From the standpoint of construction, advantages are evident in savings in falsework and labor in placing material. In general, thinner crowns will result in lower temperature stresses, and should tend toward more artistic lines - a matter of great importance.

In considering the shape of the arch ring from the mathematical standpoint, two variations should receive our attention. (1) The shape of the axis. (2) The thickness along the axis. It will be found that a slight change in shape greatly affects the stresses while variations in the elastic weights (ds/I values) make surprisingly little difference. A fixed-ended arch is in reality a fixed-ended curved beam with H added. The arithmetical control for a beam is high in comparison to that for an arch. A large part of the error enters in scaling the ordinates to the axis. In the ordinary method of dividing the ring into twenty divisions the ultimate accuracy is probably not closer than 10 per cent for flat arches. It is probable that the arch-axis is seldom constructed accurate within 1/4" or perhaps 1/2" throughout the greater part of its length. For small flat arches, small inaccuracies in construction amounting to, say 1/4" vertically, may give errors fully as high as those ordinarily involved in the computations.

String Polygons and Arch Axes. It is important for the student of arches to be able to visualize clearly the relations of equilibrium polygons to load systems. The curve of equilibrium for loads uniformly distributed horizontally is, of course, a parabola. Hence, a parabolic arch with full uniform load is subject only to compressive stresses if rib-shortening is neglected.

The rate of change of slope of the curve per horizontal foot is, of course, proportional to the intensity of vertical load per foot. An elliptical arch, then, is appropriate for relatively heavy loads near the abutments, a triangular arch (a frame) for central concentrated load. The geostatic arch of Rankine is of interest in this connection, being the curve of equilibrium for active earth pressure as given by his theory and hence appropriate for tunnel arches.

For any condition of loading the appropriate curve of equilibrium can be determined either directly or by a few simple sketches. The use of a shape of arch axis not suited to the loading is structurally unsound and aesthetically unsatisfactory; but such arches are often seen in buildings.

Proper Shape for Arch Axis.

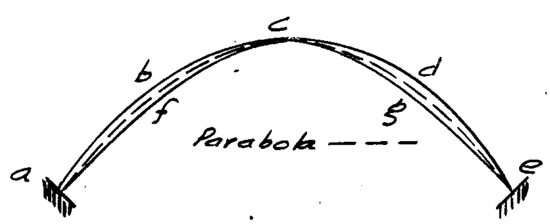


Fig. 86

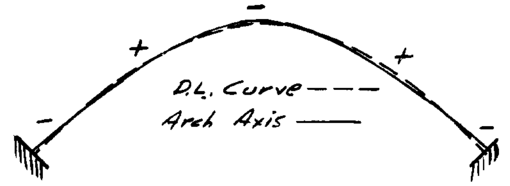


Fig. 87

Due to the relatively greater dead load at the springings, the string polygon for dead load will usually fall along a-b-c-d-e outside the parabola. (See Fig. 86). If economy lies in a minimum crown thickness, the positive effect of the rib-shortening moment at the crown may be counteracted by making the arch axis steeper in the haunches (Fig. 87), thus producing an initial negative DL moment at the crown. Then, as the DL curve fits the arch axis closely to satisfy the Pressure Line Theorem, it will cut the axis in two points each side of the center, producing negative moment also at the springing.

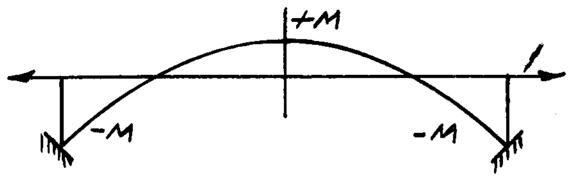


Fig. 88

This will add to the rib-shortening moment at the springing. Fig. 88 shows the effect of rib-shortening alone on the crown and springing moments in the rib.

The arch axis then, should lie between the usual equilibrium polygon for dead load and a parabola. This may be obtained by giving the axis the shape of the string polygon for dead load plus live load of half intensity over the entire arch. In some cases it is probably worth while to make more extensive studies of the effect of raising and lowering the haunch (the crown and springing remaining fixed).

Evidently rib-shortening cannot be eliminated by changing the shape of the axis. Its effect may be counteracted at crown or at springing, but not at both.

Temperature and Crown Deflection. Temperature* and rib-shortening stresses become of particular importance in arches of low rise ratios. The position of the horizontal neutral axis in flat arches is sensitive and the thrust is large. A 40 deg. temperature range either way from an arbitrary mean is customary. However, this may be altered to advantage in some cases by closing at a predetermined temperature.

Plastic flow or time yield in the concrete becomes of importance in considering temperature effects as it bears directly on the value of E. It is necessary to distinguish somewhat between changes that occur within, say a month, and those that are slow or seasonable. The Central States have a monthly variation of air temperature of about 60 deg. F.--perhaps 40 deg. F. in the concrete--superimposed on a slow seasonal variation of about 60 deg. F. in both air and concrete. The monthly variation is probably too rapid for the plastic yield of the concrete to affect the results materially; and no good reason appears why theoretical analysis is not a reasonable guide. For the seasonal changes, however, plastic yield equivalent to a reduction in elastic modulus to perhaps one-third its normal value is not improbable.

It seems almost certain that temperature stresses are not as serious as computations indicate - otherwise some arches would show signs of distress. On the other hand, it may be well to note that time yield of concrete may be as significant an element in failure as is overstress, though there seems to be no evidence for or against such a theory.

Crown deflection furnishes a convenient measure of deformation due to temperature moisture and shrinkage. It may be considered as made up of two parts, a portion CC' (Fig. 89) which would occur if the arch were free at the ends and a portion C'C" due to end restraint. Then we may write,

$$\Delta h = CC' + C'C'' = (h/L)\,\Delta L + i_h\,\Delta L$$

Or, $\Delta h = \Delta L\,(i_h + h/L)$

where i_h is the influence ordinate

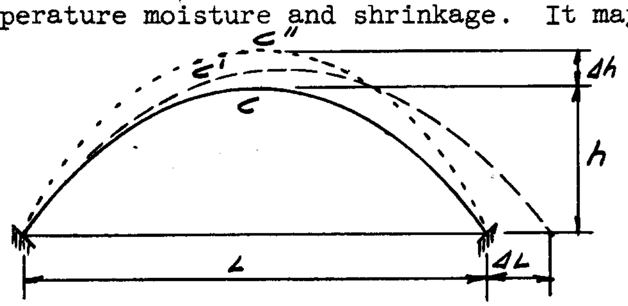


Fig. 89

*See Hardy Cross in (Trans. A.S.C.E. 1925 - Discussion. Eng. Contracting (Oct. 28, 1925. Eng. News Rec. Feb. 4, 1926.

for crown thrust due to a load at the crown. The portion CC', the rise due to the expansion of the free arch, sometimes has been overlooked. Approximately $i_h = 1/4 \frac{L}{H}$ . . And, $\Delta h = \Delta L \left(1/4 \frac{L}{h} + \frac{h}{L}\right) = 1/4 \frac{L}{h} \Delta L \left(1 + \frac{4h^2}{L^2}\right)$. In the case of a full centered arch omission of the second term might lead to an error of 100 per cent.

The usual temperature determination has been to calculate the change in span length for a free arch. Then, for an assumed E the stresses due to end restraint are computed. Since, however, the change in span length is affected by shrinkage and moisture content, and the actual change is what is wanted regardless of the cause, it would seem more logical to measure the crown deflection directly in the field as a source of very useful data in the design of similar arches. That is, if L is the span of a measured arch and L' one to be designed,

$$\Delta L' = \frac{L'}{L} \Delta L = \frac{L'}{L} \frac{\Delta h}{i_h + h/L}$$

<u>Design of Hingeless Concrete Arches</u>. <u>Influence Lines</u>. The general shape of the influence lines for moment may be visualized as the difference of the influence lines for bending moment on a fixed-ended beam and the influence line for moment due to the horizontal thrust.

The influence lines for moment in a fixed-ended beam are readily drawn as follows:

<u>Moment at the Crown</u> - Apply a unit load to the column at this point and draw the deflected structure (moment of the angle weights).

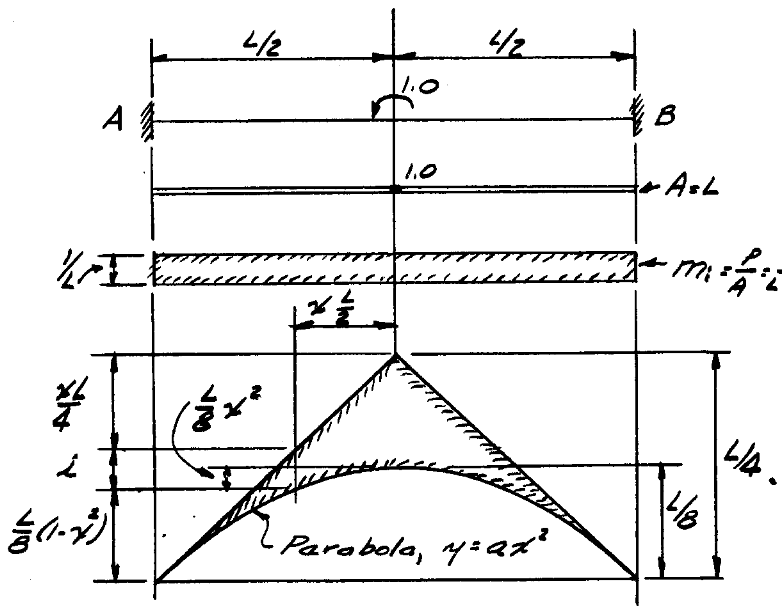


FIG. 90

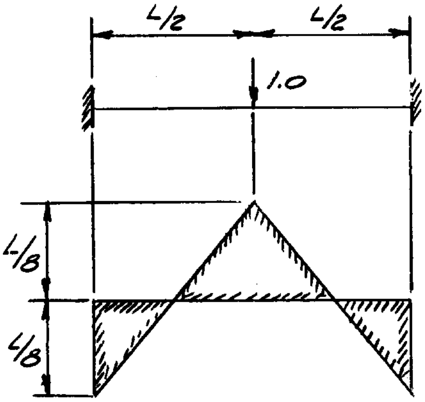


FIG. 91

If ds/Idx is a constant in the arch the influence line will be as shown, Fig. 90. And any ordinate $i = L/4\ (1-x) - L/8\ (1-x^2)$. Evidently the center ordinate is L/8 since a load at the center of such a beam produces a moment curve as shown in Fig. 91.

Moment at the Springing - This is best drawn from the general expression for fixed-ended moments on a beam with constant elastic weight. (Fig. 92).

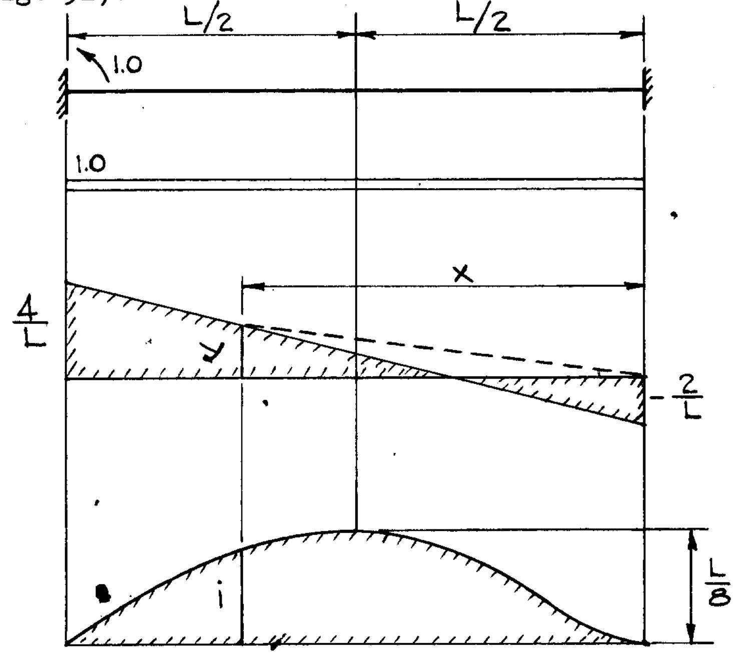


FIG. 92

At any point x the influence ordinate i will then be,

$$y = \frac{x^2}{L^2}(L-x)$$

By inspection its general shape is as shown and the center ordinate is L/8.

Moment in the Arch due to Horizontal Thrust - To produce a unit shortening of the arch apply a thrust along the centroidal axis. The moment curve will be as shown shaded. Fig. 93. For a parabolic arch with ds/Idx a constant, the centroidal axis is as shown and the point of zero moment (inflection point of the deflection curve) is at 0.289 L from the center line. Generally the centroid is higher than this (about 1/4 L from the top), and the point of inflection is about at the quarter point of the span. The influence line, then, is two curves ab and bc, horizontal at a and c and tangent at b. While their equations are evidently complicated, it cannot be much in error to treat them as parabolas - for simplicity as quadratic parabolas.

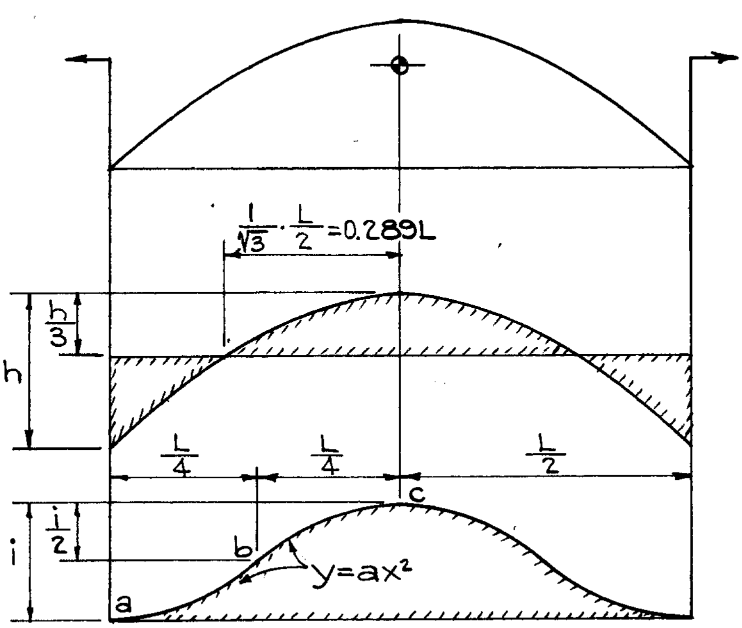
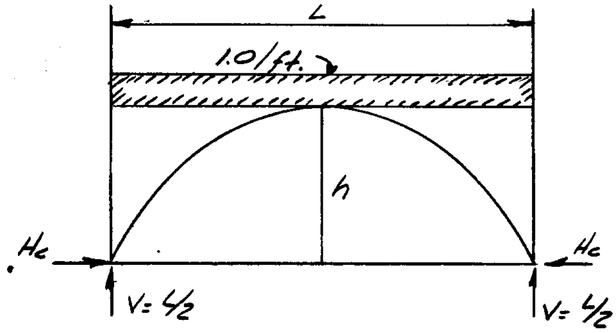


FIG. 93

The total area under the influence line, then is 1/2 iL. But in a parabolic arch uniformly loaded there is no moment at the crown or springing (rib-shortening neglected) and if the load per foot is unity, taking moments about the crown, Fig. 94.



$$H_c = \frac{L/2 \cdot L/4}{h} = \frac{L^2}{8h} = 1/2\ iL$$

$$\text{And, } i = 1/4\ \frac{L}{h}$$

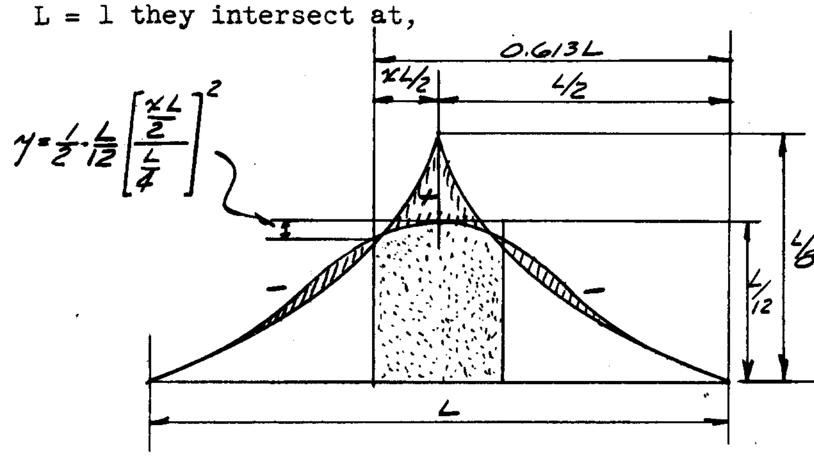
FIG. 94

If the centroidal axis is at the third point of the height this produces a moment.

$$\text{At the crown} \quad 1/4\ \frac{L}{h} \times h/3 = L/12$$

$$\text{At the springing} \quad 1/4\ \frac{L}{h} \times 2h/3 = L/6$$

<u>Combined Influence Lines for Moment at Crown and Springing</u> - These influence lines may be combined as shown. At the crown, Fig. 95, taking L = 1 they intersect at,



$$\frac{1}{4}(1-x) - \frac{1}{8}(1-x^2) = \frac{1}{12}\left(1 - \frac{2x^2}{2}\right)$$

$$7x^2 - 6x + 1 = 0 \qquad x = 0.226$$

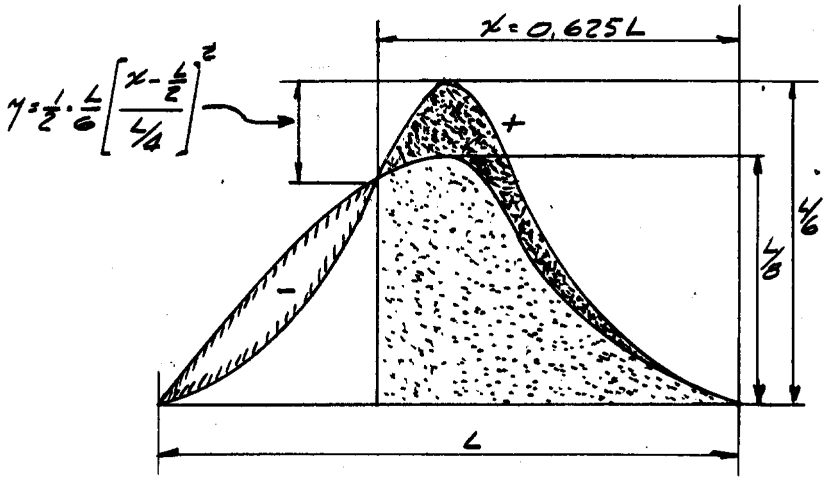
$$\frac{xL}{2} + \frac{L}{2} = 0.613$$

Positive Shaded Area - $0.004\ L^2$

Dotted Area - $0.018\ L^2$

FIG. 95

$$H = \frac{0.018\ L^2}{h/3} = 0.054\ \frac{L^2}{h}$$



At the Springing, Fig. 96,

$$x^2(1-x) = 1/6\left[1 - \frac{(x - 1/2)^2}{L/8}\right]$$

$$6x^3 - 14x^2 + 8x - 1 = 0 \qquad x = 0.625$$

Positive Shaded Area = $0.019\ L^2$

Dotted Area = $0.062\ L^2$

$$H = \frac{0.062\ L^2}{2/3h} = 0.093\ \frac{L^2}{h}$$

FIG. 96

From this it may be seen that the load divides for moment at crown and springing lie near the five-eighths point of the span. These vary very little for ordinary arches and it is evident from the above figures that such is the case.

If the arch ribs are parabolic, the moments for uniform load over the whole span are zero and the positive and negative areas in Fig. 95 and 96 are equal.

Rib-shortening will decrease the horizontal thrust, thus decreasing the negative moment at the crown, increasing it at the springing and throwing the load divide nearer the abutment.

The values for maximum moment computed above - $\pm$ 1/250 wL^2 at the crown and $\pm$ 1/50 wL^2 at the springing - are, of course, only rough approximations. Fig. 99 shows these coefficients for arches not parabolic. More nearly average values are seen to be $\pm$ 1/45 wL^2 at the springing and $\pm$ 1/225 wL^2 at the crown. In general the live moment at the springing is about five times that at the crown.

The dotted areas divided by the lever arms to the centroidal axis give the horizontal thrust accompanying maximum moment 0.054 w L^2/h at the crown and 0.093 w L^2/h at the springing. More nearly average values are 0.06 w L^2/h at the crown and 0.10 w L^2/h at the springing. It will then be about right at the crown to take one-half the maximum crown thrust as accompanying maximum moment, but at the springing this value should be increased by 50 per cent. (Evidently if we accept the same load divide for crown and springing, if $H = 1/2\ H_{max}$ for maximum crown moment then $H = 3/4\ H_{max}$ for maximum springing moment.)

Approximate Formulas Based on the Parabolic Case. The parabolic arch with ds/I constant and equal to dx/I_c submits to simple mathematical analysis and is useful in studying the properties of other arches. For convenience in reference, N may be defined as the ratio of the drop of the arch axis at the quarter point to the rise of the arch h. Also the ratio,

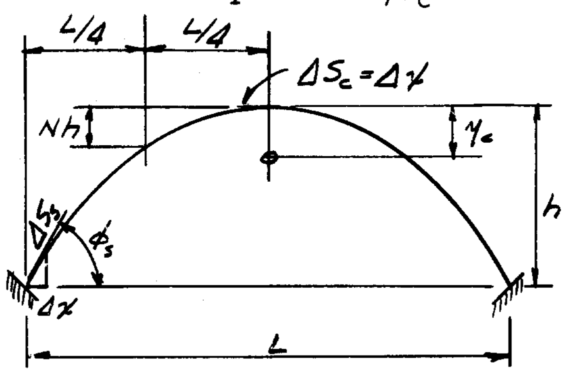


Fig. 97

$$m = \frac{\Delta S/I \text{ at the springing}}{\Delta S/I \text{ at the crown}}$$

for the same ΔX.
Referring to Fig. 97,

$$m = \frac{\Delta S_s/I_s}{\Delta S_c/I_c} = \frac{\Delta X/\cos\phi_s}{I_s} \cdot \frac{I_c}{\Delta X} = \frac{I_c}{I_s \cos\phi_s}$$

y_c - distance from the crown to the horizontal neutral axis. The following expressions may then be written for the total elastic weight, the position of the horizontal neutral axis, and the two moments of inertia of elastic weights about the neutral axes. They are derived directly from the case of the symmetrical parabolic arch having a variation of I according to the secant. For each of these we may write a multiplication factor in terms of m making them reasonably applicable to all ordinary arches.* (Fig. 98)

*For the case of W and I_y these values are given by Whitney in Trans. A.S.C.E. 1925. For the other two cases the multipliers are approximate and based on the ordinary value of N = 0.2.

$$W = \int \frac{ds}{I} = 2\int_0^{L/2} \frac{dx}{I_c} = \frac{L}{I_c} \quad \left(\text{Multiplier } \frac{1+m}{2}\right)$$

$$y_c = \frac{2\int \frac{ds}{I}\cdot y}{W} = \frac{2\int_0^{L/2} \frac{dx}{I_c} h\left(\frac{x}{L/2}\right)^2}{L/I_c} = \frac{8h}{L^3}\cdot\frac{L^3}{24} = \frac{h}{3} \quad \left(\text{Multiplier } \frac{1+m}{2}\right)$$

$$I_v = \int \frac{ds}{I}\cdot x^2 = 2\int_0^{L/2} \frac{dx}{I_c}\cdot x^2 = \frac{1}{12}\cdot\frac{L^3}{I_c} = \frac{1}{12} W L^2 \quad \left(\text{Multiplier } \frac{1+3m}{4}\right)$$

$$I_h = \int \frac{ds}{I}\cdot y^2 = 2\int_0^{L/2} \frac{dx}{I_c} h^2\left(\frac{x}{L/4}\right)^4 - W\frac{h^2}{9} = \frac{4}{45} W h^2 = \text{Approx. } \frac{1}{12} W h^2 \quad \left(\text{Multiplier } \frac{1+3m}{4}\right)$$

TEMPERATURE

$$H_t = \frac{\epsilon E t L}{I_h} = \frac{\epsilon E t L}{\frac{4}{45}\cdot\frac{L}{I_c} h^2} = \frac{45}{4}\epsilon E t \frac{I_c}{h^2} = \text{Approx. } 135 t \frac{I_c}{h^2} \quad \left(\text{Multiplier } \frac{4}{1+3m}\right)$$

$$M_c = \frac{H_t \cdot y}{I} = \frac{45}{4}\epsilon E t \frac{I_c}{h^2}\cdot\frac{h}{3} = \frac{15}{4}\epsilon E t \frac{I_c}{h} = \text{Approx. } 45 t \frac{I_c}{h} \quad \left(\text{Multiplier } 8\frac{1+m}{1+3m}\right)$$

(Approximately — Reinforcement Neglected)

$$f_c = \frac{M_c \cdot y}{I} = \frac{15}{4}\epsilon E t \frac{I_c}{h}\cdot\frac{d}{2 I_c} = \frac{15}{8}\epsilon E t \cdot \frac{d}{h}$$

$$= \text{Approx. } 22\tfrac{1}{2}\, t \frac{d}{h} \quad \left(\text{Multiplier } 4\frac{1+m}{1+3m}\right)$$

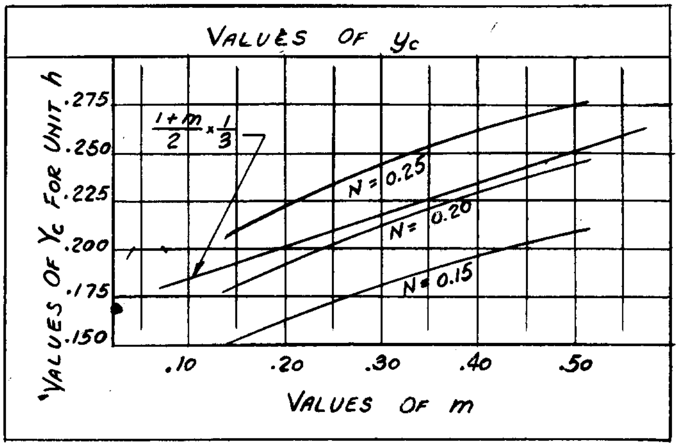
$$= \text{Approx. } 1000 \frac{d}{h} \text{ for } 45^\circ \text{ change in } t$$

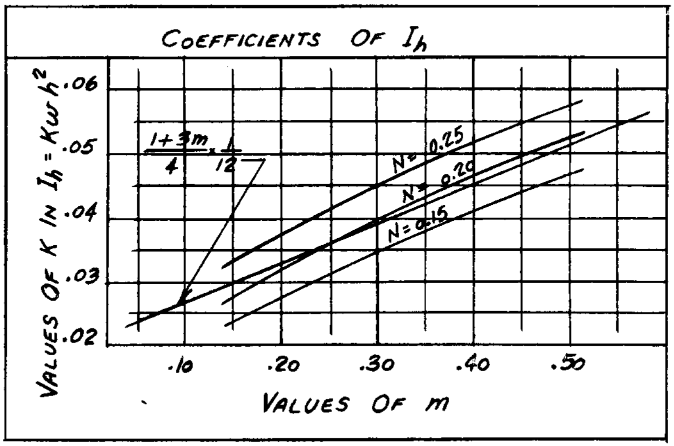
FIG. 98

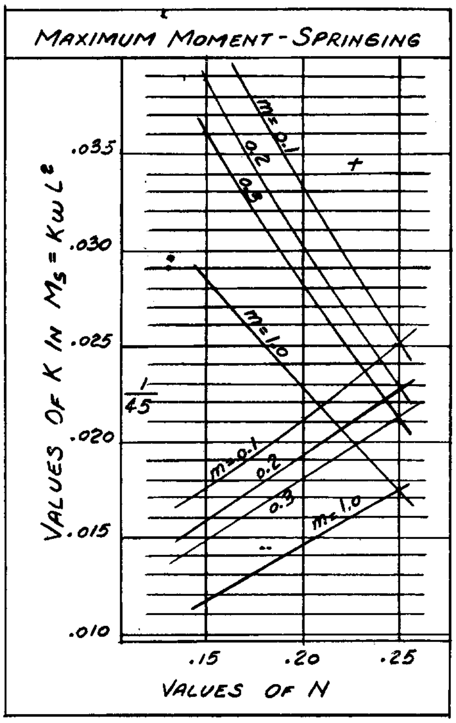
Rib-shortening, if computed separately, and shrinkage stresses, if included at all, are perhaps most conveniently considered in terms of an equivalent change of temperature. In a way, the three effects are the same and there is some advantage in considering them together. Some uncertainties are common to the three. If the arch ring is poured at (or near) the extreme temperature, all three effects may be counteracted at the crown by adjusting the shape of the arch axis, and by use of certain expedients of construction, they may be entirely counteracted at both crown and springing.

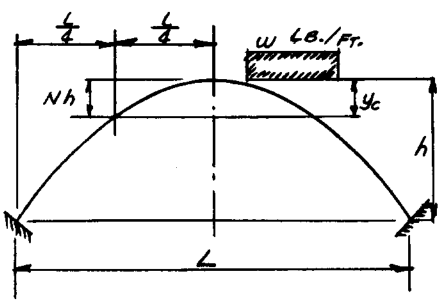
SYMMETRICAL CONCRETE ARCHES - PRELIMINARY DESIGN

General and Approximate Formulas









$W = \frac{1+m}{2} \times \frac{L}{EI_c}$

$y_c = \frac{1+m}{2} \times \frac{h}{3}$ (APPROX.)

$I_v = \frac{1+3m}{4} \times \frac{1}{12} WL^2$

$I_h = \frac{1+3m}{4} \times \frac{1}{12} Wh^2$ (APPROX.)

APPROXIMATELY –

$M_s = \pm \frac{1}{45} wL^2$

$M_c = \pm \frac{1}{22.5} wL^2$

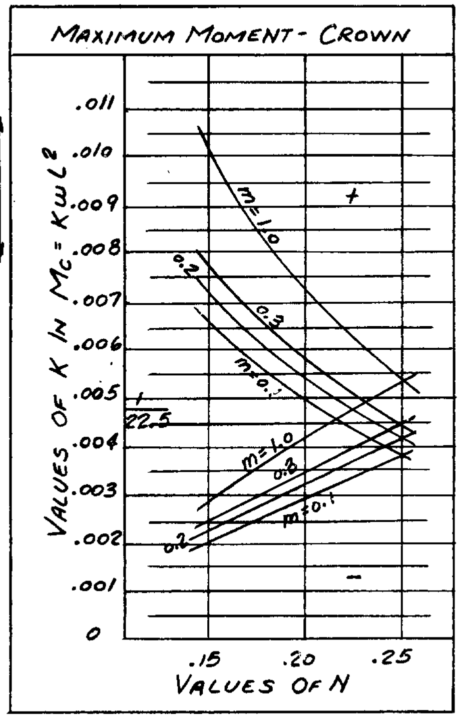
$M_s = .5 M_c$

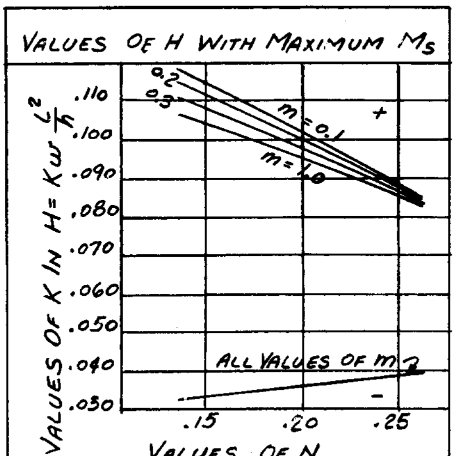
FOR MAX. M_s, – $H = 0.10 w \frac{L^2}{h^2}$

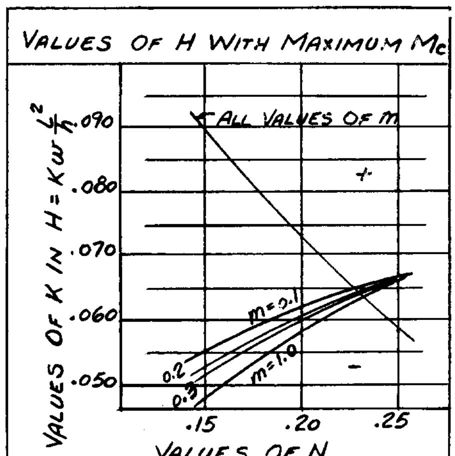
FOR MAX. M_c, – $H = 0.06 \frac{L^2}{h^2}$

$m = \frac{\Delta S/I \text{ AT SPRINGING}}{\Delta S/I \text{ AT CROWN}}$

FOR SAME ΔX







Other Methods of Arch Analysis. Someone has said that every structural engineer has three inalienable rights - to use his own method of stirrup spacing in a concrete girder, to develop his own theory of earth pressure, and to modify the mathematical theory of the elastic arch. The last has been very freely exercised, giving rise to a large number of variations of technique. Of course, the theory of elasticity as applied to arches simply states that the angle changes vary directly as the moments times ds/I and that the abutments do not rotate or spread or settle relatively to each other. These may or may not be facts - they are discussed elsewhere - and are profitable subjects of study by a structural engineer. Beyond these the geometry of the relations is susceptible of statement in an unlimited number of ways and these are valuable according as they enable one to estimate the effect of variations in the physical elements involved or save time and mental effort in evaluating these stresses.

The relations of several of these methods to the method presented above will be evident. In fact, all the geometrical and algebraic relations which have been introduced into the literature of unsymmetrically loaded columns have a corresponding application in the study of indeterminate structures. The ellipse of inertia has recently had a prominent place in structural literature under the name of the ellipse of elasticity. The circle of inertia is almost as useful a tool in such studies, and is often more convenient. Indeed, the whole literature of neutral axes, antipoles of rotation and kerns bears on this subject.

The so-called neutral point method of analysis of Ritter and Müller-Breslau is directly implied in the column analogy since a rotation of the neutral point corresponds to no moment about any gravity axis of the column and hence to no translation of its ends. Similarly, a translation of the neutral point without rotation corresponds to a moment in the column about an axis parallel to this translation, and the neutral axis for this will be conjugate to the direction of displacement.

A brief review of some of these methods is here presented in order that their interrelations may be apparent. The leading methods may be divided into three groups:

(a) Algebraic and graphic methods using influence lines;
(b) Graphical or semi-graphical methods of determining the true equilibrium polygon;
(c) Methods of reaction loci and envelopes;
(d) Purely mathematical analyses assuming integrable functions for arch axis and for I;
(e) Method of least work.

Methods of Influence Lines. In these methods the arch ring is cut at some point and the influence lines for the three redundants at the cut ends are then constructed either,

1. by use of the theorem that an influence line is a deflected load line;
2. by placing loads in various positions and determining the redundants.

The ring may be cut at the crown or at the springing line, as shown in Fig. 100, in which case the redundants H, V and M are interdependent, and simultaneous equations are necessary to evaluate them. Or, the redundants

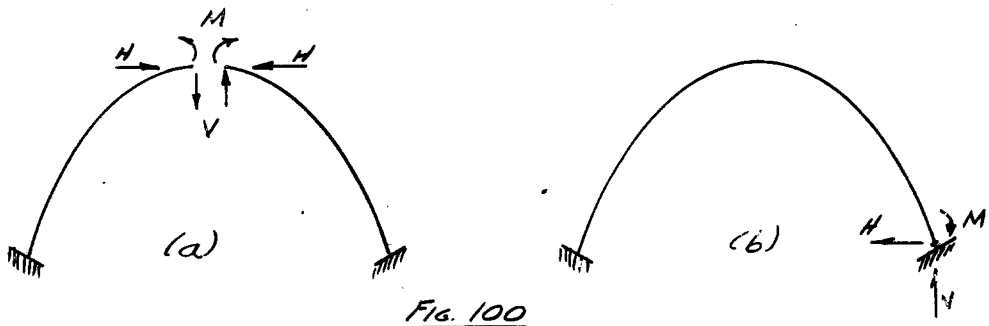


Fig. 100

may be applied through rigid bracket arms at such a point and in such directions that the application of one produces no effect on the others. This point is, of course, the neutral point as used in the column analogy and the axes are the principal axes. The redundants are now independent variables and influence lines for them may be drawn independently.

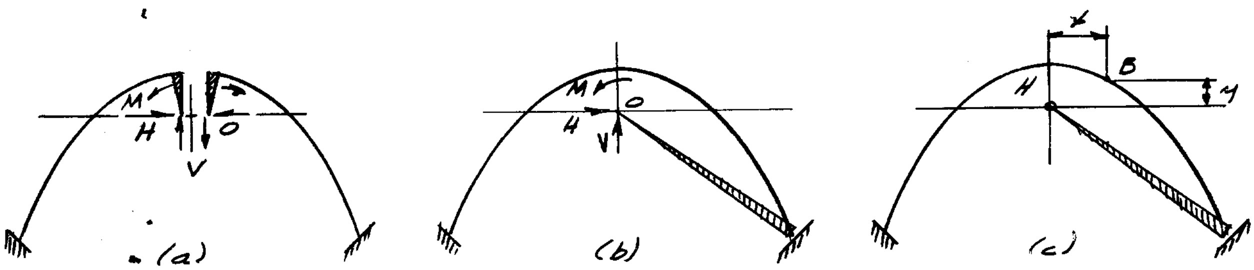


Fig. 101

If now, in Fig. 101, we produce a unit horizontal displacement of O, the vertical deflections of the load line will be influence ordinates. But the vertical displacement of B caused by a horizontal force at O and due to any single short length ($w = ds/I$) is H w xy (by the geometry of angle weights or Greene's theorem, or Fraenkel's equation or virtual work - if one likes to distinguish methods). H is the force necessary to produce horizontal movement at O (considering this particular w alone $H = 1/wy^2$).

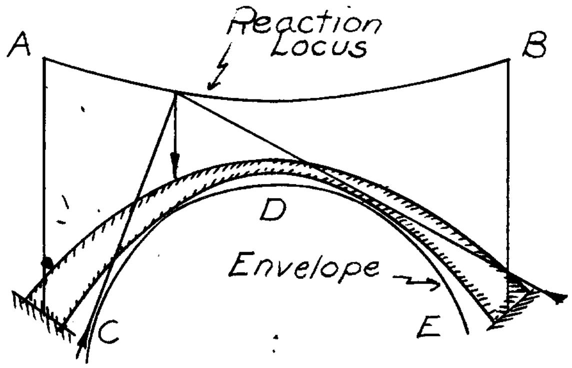
Now, $\Sigma Hwxy = \Sigma wxy / \Sigma wy^2$ for any point due to the effect of all elastic weights between B and the fixed end. Numerator and denominator are at once recognized as products and moments of inertia of the elastic weights about axes V_B and H_o and axes H_o and H_o. Hence, the influence lines for V_o and H_o are curves of products of inertia. That for M_o can similarly be shown to be a curve of statical moments.

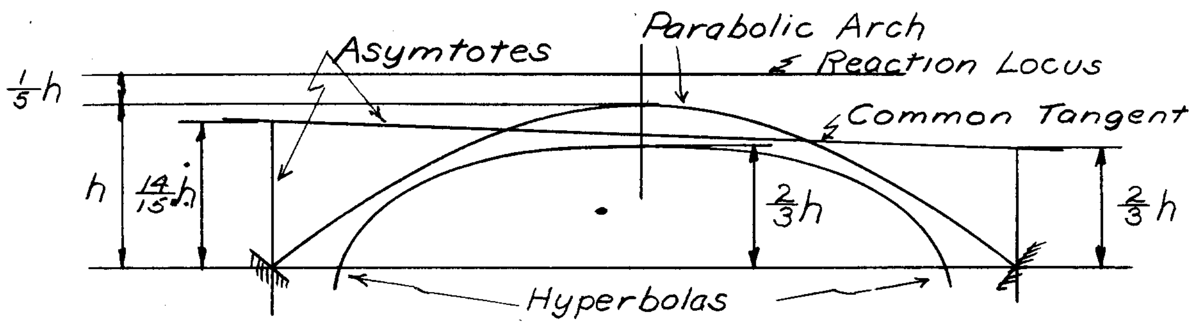
But curves of statical moments and products of inertia may be drawn graphically by well-known properties of the equilibrium polygon. Hence the varied graphical constructions.

The ellipse of inertia is an ingenious mathematical device. If the moments and product of inertia of an area about any two axes are known, we can draw the ellipse of inertia and then by geometrical constructions using poles, foci and antipoles of this ellipse we can find the product of inertia about any other axes. When the ellipse of inertia is applied to evaluating the above expressions, it is called the ellipse of elasticity.

Methods of Determining the True Equilibrium Polygon. In these a trial equilibrium polygon is drawn, and this is then corrected so that, $\sum mw = 0$, $\sum m+w = 0$, and $\sum myw = 0$. The general construction was introduced into America by Eddy, was applied to arches by Cain, and was later used by Baker.

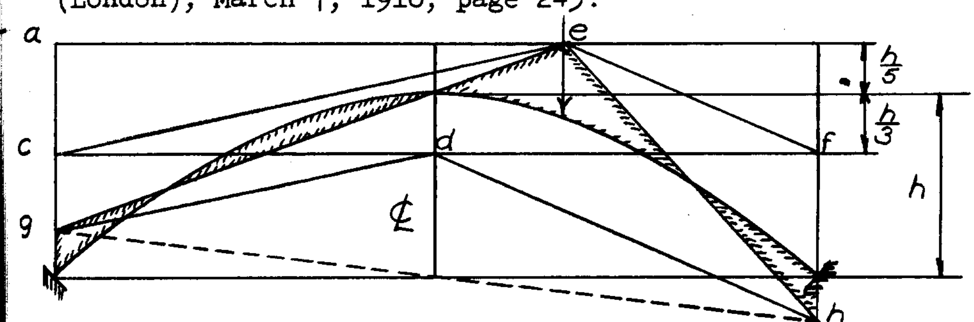
Method of Reaction Loci. In this method the locus of the intersection of reactions (AB, Fig. 102) for a single load are drawn and also the envelope of these reactions (C D E). The equations of these lines may be determined mathematically for an arch axis having a known equation and a known mathematical law of variation of I along this axis. It has been chiefly developed for a parabolic axis in which $ds/I - dx/I_c$. It has been extensively used in France, its popularity there being to some extent a reflection of the large amount of attention which they give to analytical geometry in the schools, and was introduced into American literature by Merriman.





For parabolic arches of the type mentioned, the construction is as shown in Fig. 103. It is very beautiful but not very useful.

A simpler and more useful construction similar to this is given in Fig. 104. It is due to Mesnager and is quoted from Le Genie Civil by Engineering (London), March 7, 1916, page 245.



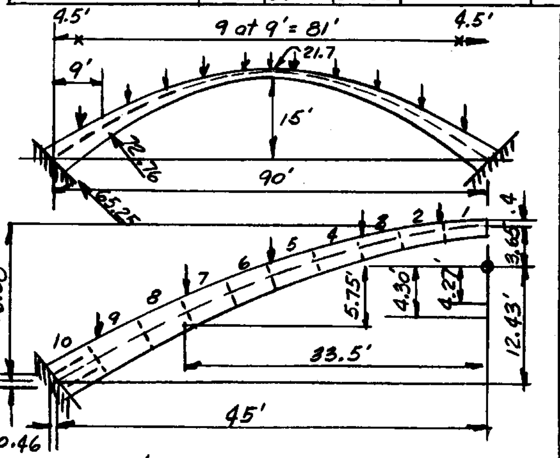
Scale of moments as given by intercepts between axis and reaction is same as scale of Δ egh as representing bending moment on simple beam gh.

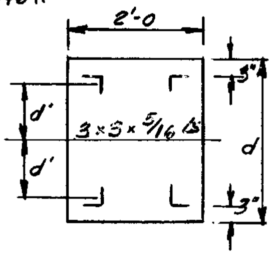
Fixed-ended arch - Parabolic Axis - I∝ Sec. θ

1. Draw ae and cd horizontally.
2. Draw ec and ef for load shown.
3. Draw dy and dh parallel to ec and ef.
4. Draw reactions as shown.

Analysis of Concrete Arch

Point	d	I	ΔS	$\frac{\Delta S}{I}=\omega$	$\frac{\Delta S}{d}$	x	y	ωx^2	ωy	Wy^2	Dead Load: Load	Dead Load: m	Dead Load: mw	Dead Load: mwy	Full Live Load: m	Full Live Load: mw	Full Live Load: mwy	3/8 Live Load: m	3/8 Live Load: mw	3/8 Live Load: mwx	3/8 Live Load: mwy
1	2.18	2.19	4.51	2.06	2.07	2.25	0.09	10	0.19	0.0	27.2	0	0	0	0	0	0	0	0	0	0
2	2.22	2.30	4.53	1.97	2.04	6.75	0.37	90	0.73	0.3		61	120	40	2.25	4.4	2	0.07	0.1	1	0
3	2.28	2.49	4.56	1.83	2.00	11.25	0.95	231	1.74	1.7	27.6	183	335	320	6.75	12.4	12	0.20	0.4	4	1
4	2.35	2.71	4.61	1.70	1.97	15.75	1.90	422	3.23	6.1		368	626	1190	13.50	23.0	44	1.95	3.3	52	6
5	2.45	3.07	4.68	1.53	1.91	20.25	3.15	627	4.81	15.2	28.1	615	941	2970	22.50	34.4	108	5.33	8.2	166	26
6	2.56	3.49	4.78	1.37	1.87	24.75	4.65	839	6.37	29.6		931	1276	5940	33.75	46.2	215	10.95	15.0	371	70
7	2.68	3.99	4.92	1.23	1.84	29.25	6.60	1051	8.13	53.7	29.2	1304	1605	10600	47.25	58.1	384	18.83	23.2	680	153
8	2.85	4.76	5.11	1.07	1.79	33.75	8.95	1219	9.60	85.9		1743	1865	16700	63.00	67.4	604	29.95	32.0	1080	287
9	3.20	6.67	5.35	0.80	1.67	38.25	11.65	1171	9.35	108.9	33.9	2250	1800	21000	81.00	64.9	755	42.35	33.9	1300	395
10	3.85	11.36	5.65	0.50	1.47	42.75	14.60	915	7.26	106.0		2830	1415	20700	101.25	50.6	739	56.98	28.5	1220	416
			48.70	14.06	18.63	$y_c = 3.65$		6575	51.41	407.4	y = 7.95		9983	79460	y = 7.92	361.4	2863		144.6	4874	1354
			97.40	28.12	37.26			13150		814.8			19966			722.8			x = 33.5		y = 9.4





W = 28.12
$I_h = 814.8 - 28.12 \times \overline{3.65}^2$
$+ 37.3 = 487$
$I_v = 13150$
LL = 10.4 per Floor Beam
Temperature ±40°
$\Sigma E = 12$ lb./in.²

FIG. 105

		Dead Load	Full Live Load	3/8 Live Load	5/8	1/4	3/8 ; 3/8
Crown	Upper Kern	$m_i = \frac{19966}{28.12} - \frac{19966 \times 4.3 \times 4.05}{487} = -3$ $f_L = \frac{3 \times 1.08}{2.14 \times 1.44} =$ **-11**	$m_i = \frac{722.8}{28.12} - \frac{722.8 \times 4.27 \times 4.05}{487} = -0.2$ $f_L = \frac{-0.2 \times 10.4}{.285} =$ **-7**	$m_i = \frac{144.6}{28.12} - \frac{144.6 \times 5.75 \times 4.05}{487} = -1.7$ $f_L = -1.7 \times 36.5$ **-62**	+55	+117	-124
	Lower Kern	$m_i = 710 - 176 \times 3.25 = +137$ $f_U = 137 \times 3.50$ **-480**	$m_i = 25.7 - 6.39 \times 3.25 = +4.9$ $f_U = 4.9 \times 36.5 =$ **-179**	$m_i = 5.2 - 1.71 \times 3.25 = -0.4$ $f_U = -0.4 \times 36.5$ **+15**	-194	-209	+30
Springing	Upper Kern	$m_i = 710 + 176 \times 11.77 = +2780$ $m_s = -3226$ $M = -446$ $f_L = \frac{446 \times 2.20}{16.75 \times 1.44} =$ **-406**	$m_i = 25.7 + 6.39 \times 11.77 = +100.8$ $m_s = -114.8$ $M = -14.0$ $f_L = \frac{-14 \times 10.4}{1097} =$ **-133**	$m_i = 5.2 + 1.71 \times 11.77 + \frac{144.6 \times 33.5 \times 45.46}{13150}$ $M = m_i + m_s = 42.1 - 67.1 = -25.0$ $f_L = -25.0 \times 9.48 =$ **-237**	+104	-	-
	Lower Kern	$m_i = 710 + 176 \times 13.09 = +3010$ $m_s = -3092$ $M = -82$ $f_U = 82 \times 0.91 =$ **+75**	$m_i = 25.7 + 6.39 \times 13.09 = +109.2$ $m_s = -110.2$ $M = -1.0$ $f_L = 1.0 \times 9.48 =$ **+9**	$m_i = 5.2 + 1.71 \times 13.09 + 0.37 \times 44.54 = 44.0$ $m_s = -63.7$ $M = -19.7$ $f_U = 19.7 \times 9.48 =$ **+187**	-178	-	-

Temperature

$m_t = \frac{\Sigma E t I_s}{I_h} y = Hy = \frac{480 \times 90 \times 1.44}{487} y = 12.8y$

Crown		$f_L = 12.8 \times 4.05 \times 3.50 = 182$
		$f_U = 12.8 \times 3.25 \times 3.50 = 146$
Springing		$f_L = 12.8 \times 11.77 \times 0.91 = 137$
		$f_U = 12.8 \times 13.09 \times 0.91 = 153$

Maximum Stresses

	Crown Upper +	Crown Upper −	Crown Lower +	Crown Lower −	Springing Upper +	Springing Upper −	Springing Lower +	Springing Lower −
DL	0	480	0	11	75	0	0	406
LL	30	209	117	124	187	178	104	237
T	146	146	182	182	153	153	137	137
Total	0	835	288	317	415	256	0	780

Purely Algebraic Analyses. This has been a favorite field for the use of the calculus. Usually, however, the work has been confined to the parabolic arch with $ds/I = dx/I_c$.

The tables given by Cochrane, Strassner and Whitney are based on the use of integrable functions but with more practical arch forms.

Analysis by Least Work. The equations for the redundants may be evaluated by least work. Using this method, Spofford has derived the same equations as are used in other methods.

Comparison of Methods. These various methods may be further modified in minor details. In nearly all of them the effect of rib-shortening may be included directly in the analysis or may be corrected for, and the separate correction may be made approximately - as is usual - or exactly according to whether the effect of compression due to the rib-shortening correction is included or not. Also the values of ds/I may be made constant, which has some advantage in saving computation combined with the disadvantage that the construction for determining constant ds/I is not very convenient and the more important disadvantage that error results from the long end section; or dx may be made constant.

These methods are all, of course, exactly the same method. Most of them will give the same equations and many will give exactly the same numerical work but in different order. Some have the defect that the same computations are repeated in different parts of the analysis though the fact is not at first evident.

Three other factors determine the relative advantages of the methods. In analyses of indeterminate structures, the determination of signs should be made as nearly automatic as possible, the use of simultaneous equations is to be avoided, the procedure should be of a familiar type and should be easy to remember.

This subject has been elaborated somewhat here because it presents certain characteristics very often found in the literature of indeterminate structures. The beginner is likely to hunt among these different methods for new facts, when there are no new facts to be had, though some methods tend to illuminate and others to obscure the relations which do exist.

Analysis of the Concrete Arch Problem. Outline of method following the problem shown in Fig. 105. The problem illustrated is a symmetrical open ribbed arch of 90 ft. span and 15 ft. rise. The rib has a constant thickness of two feet and is reinforced with 4 - 3 x 3 x 5/16 L^s Fig. 105 is a complete stress analysis covering dead load, live load temperature and rib shortening. The loads in this case are all applied at floorbeams spaced 9 feet on centers as indicated.

The method of analysis follows directly from the column analogy. The upper table is divided into four parts:

(1) Arch Properties. The half axis is divided into 10 divisions Δs having equal Δx values, 4.5'. The I of the rib section is computed at the

center of each of these divisions. The $\Delta s/I$ values (E being taken as unity) thus obtained are the elastic weights constituting the analogous column, the cross-section of which has the general shape of the elevation of the arch axis. As in column design it is then necessary to find the centroid of this column section (—◆— in the figure) and the moments of inertia I_h and I_v. The separate columns in the table may be described as follows-

Points--Centers of divisions of the elastic section numbering from the crown.

d--Scaled depth of rib in feet normal to the axis at the center of divisions.

I--Moment of inertia in ft^4 of the cross-section of the rib at the centers of divisions. This might be taken with fair accuracy as $1/12\ bd^3$. The figures shown, however, include also the I of the steel angles figured as equivalent concrete (n = 15) that is,- adding an additional $\frac{14 \times Ad'^2 \times 4}{144} = 0.692\ d'^2$, holding to foot units. Then $I = d^3/6 + 0.692\ d'^2$.

Δs--Scaled (stepped-off) lengths of axis in feet for equal horizontal lengths of 4.5 feet. The total length, 48.70' should be checked by independent scaling.

$\Delta s/I$--Elastic weights or w values, E being constant and omitted.

$\Delta s/d$--Equal also to $\Delta x/A'$ - the rib shortening factor to be added to I_h for automatic correction.

x--Distance from center arch to centers of divisions.

y--Distance from crown point of axis to centers of divisions.

wx^2--Moment of inertia of the column section about the vertical axis. It neglects the I of the elastic weights about their own axes.

wy--Static moment about the crown point. From it the centroid may be located as $y_c = \Sigma wy/\Sigma w = 3.65'$ from the crown.

wy^2--Moment of inertia about the horizontal axis at the crown. I_h may then be found by transferring to the parallel axis and adding $\Sigma \frac{\Delta s}{d}$.

(2) Dead Load. The angle changes due to dead load may be found by considering any static moment curve for these loads. Working with the half arch, a cantilever extending from the support was used.

Load--computed dead loads at the floorbeam concentrations in kip units.

m--Static cantilever moment due to these loads.

mw--Angle changes or angle weights on the column section.

mwy--Static moment of these angle weights about the crown, from which the centroid may be determined, as $y = \Sigma mwy/\Sigma mw - 7.95'$ from the crown.

(3) Full Live Load. The angle weights are found exactly as for DL working with unit values at the load points. This assumes either a uniform LL or an equivalent uniform load. The method, however, is applicable to any sort of load system.

(4) 3/8 Live Load. The bridge is loaded with a uniform LL extending out 3/8 of the span, or 33.75' from the springing. This happens to coincide with Point 3 in this problem giving a slight load (0.03) at the floorbeam nearest the center. The moments and angle weights are computed exactly as before. The load being unsymmetrical, in order to locate the centroid, static moments are computed about the crown and the center vertical. This is done in the last two columns and in the table establishing the center of angle weight at 33.5' from the ℄ and 5.75' below the neutral point.

The middle table follows in which are computed the upper and lower kern moments corresponding to the lower and upper fibre stresses respectively at both crown and springing. Moment at the kern point makes a separate computation for the thrust unnecessary. The indeterminate moment is found exactly as fibre stress on a column where $M_c = f = P/A \pm My/I_h \pm M'x'/I_v$. The last term appears only in the 3/8 loading at the springing. The signs follow automatically without any new convention or study. The static moment m_s is negative, being the moment of a cantilever. The resulting moment $M = m_s - m_i$ (numerically, = the difference with the sign of the greater). A positive value of M represents compression in the upper fibres and tension in the lower fibres as in other beams.

From the fibre stresses thus found for full LL and the 3/8 loading, values may be obtained by subtraction for load on the 5/8 section, the center 1/4 and the two 3/8 end sections. This follows from the sketch of the influence line, writing fibre stresses for total areas. Thus, at the crown - upper kern - we have -62 for the two negative areas and -7 for the total area, leaving +117 for the center positive area.

Temperature stresses follow directly from the data already obtained. It will be noted that various multiplication factors thoughout the moment and fibre stress computations are used repeatedly and should therefore be recorded about as indicated in order to avoid repetition.

The final combination of stresses needs no explanation. Rib-shortening stresses are included automatically throughout. Stresses, at the 1/4 point, if desired, may be found from the same combination of M without additional calculation in the upper table.

If it is desired to check the load-divide all that is needed is a small section of the influence line near the 3/8 point. From it an exact correction is readily applied. While this is seldom necessary, and clearly unnecessary in this case, it will be done by way of illustration. Attention should be called to the fact that as far as load-divide is concerned, an influence line does not mean very much. At least six may be drawn for both crown and springing moment giving a different load-divide in each case. Obviously, unless the critical one is drawn for the particular fibre being investigated, the significance of a precise load-divide is entirely lost.

Fig. 106 shows these six cases. The critical one in this problem at the crown, happens to be that for the upper fibre (lower kern moment) with the rib-shortening correction, and at the springing, that for the lower fibre (upper kern moment) with the rib-shortening correction. The correction, then, involves the addition of the shaded areas multiplied by the load per lineal foot. At the crown this amounts to 11 lb./sq. in., (835 + 11 = 846) an increase of 1.3 per cent. At the springing the correction is 13 lb./sq. in. or 1.6 per cent.

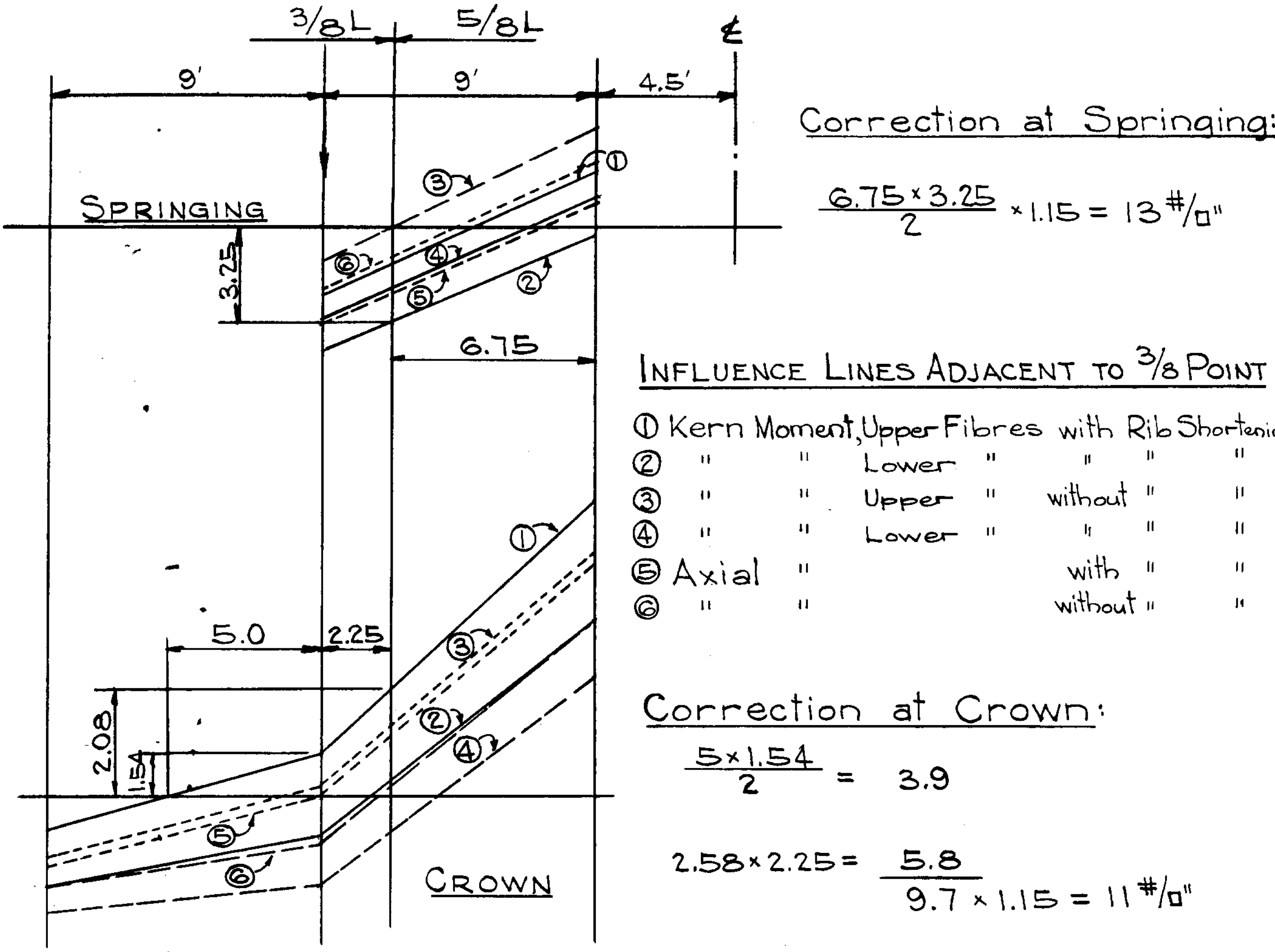


FIG. 106

Clearly, then, influence lines other than free-hand sketches are not necessary. If used in a quantitative sense they lose their significance unless drawn for kern moment with or without rib-shortening according to the governing condition in the fibre investigated. If desired for purposes of research or special study they are most readily obtained by means of the column analogy as previously explained.

Refinements in Rib Proportionment. In connection with extreme refinement in the theoretical proportionment of the arch ring it should be remembered that saving in concrete in the rib represents a much smaller proportionate saving in the bridge as a whole. The plant layout and equipment is largely a fixed charge - certainly independent of minor variations in the thickness of the rib. Foundations are apt to be relatively expensive and piers, abutments and wing walls run heavily into yardage. While this does not justify waste of material in any portion of the structure, it does, as a practical consideration, make excessive refinement in rib proportion unnecessary. Again, it may be stated that aesthetic considerations should be the controlling factor, both in the shape of the ring and its final proportions.

The following data taken from some of the arch bridges recently built by the Illinois State Highway Department verify the fact that rib variations represent a small proportion of the total cost of structure:

(1) 3 - 70 Ft. spans

Abutments	760 cu. yds.	$21,300.
Piers	315 cu. yds.	6,000.
Arch Ribs	200 cu. yds.	6,600.
Superstructure above ribs		14,500.
Total Estimated Cost - - -		$48,400.

(2) 3 - 70 Ft. spans

Abutments	550 cu. yds.	$15,300.
Piers	200 cu. yds.	3,200.
Arch Ribs	265 cu. yds.	9,000.
Superstructure above ribs		14,300.
Total Estimated Cost - - -		$41,800.

(3) 8 - 65 Ft. spans

Abutments	610 cu. yds.	$17,600.
Piers	1250 cu. yds.	34,800.
Arch Ribs	555 cu. yds.	18,700.
Superstructure above ribs		29,200.
Total Estimated Cost - - -		$100,300.

Estimated value of one cu. yd. added to or deducted from the arch ring - $12 to $15.

Special Problems in Arches. The material presented in this chapter is intended to give a clear understanding of the theory of arches as commonly used in design. Many special problems - of the action of continuous arches, of the effect of the spandrel columns, of erection methods - are discussed in a separate chapter.

CHAPTER VII
CONTINUOUS GIRDER VIADUCTS

Advantages and Limitations. In this chapter it is evidently possible only to point out some of the more important considerations bearing on the design of these structures and to indicate the general methods of design.

Various reasons have at times been assigned for the use of continuous viaducts. They have been supposed to be less subject to vibration than simple spans, to have a much smaller deflection away from the dead load position, and to permit more satisfactory architectural treatment. As to the relative vibration, no accurate data exist; the continuous bridge is almost certainly more steady than the cantilever but probably has no great advantage in this respect over simple spans. Relative deflections are discussed elsewhere. The advantage of the continuous girder is less pronounced than is commonly supposed. Desire for a uniform fascia treatment, as freely as possible from joints, has been the most important consideration in many cases. The continuous girder bridge shows no uniform economy over other types. Continuity of girders and columns as stiffening against traction forces becomes an important consideration where longitudinal bracing cannot be used. In steel viaducts the use of continuous girders sometimes makes possible the use of a narrower column, sometimes an important consideration in the design of over-crossings at railway terminals.

Obvious disadvantages of continuous steel viaducts are complicated connections at the columns, heavy flange splices in the field, more difficult erection and liability to overstress from settlement. In reinforced concrete these objections are replaced by the difficulty of providing for expansion moments.

Importance of Details. These remarks indicate the great importance of the study in these structures of certain critical details at an early stage of the investigation. Expansion joints are expensive and otherwise objectionable in highway bridges and if a continuous bridge is selected, it is of advantage to reduce their number. The design of intermediate columns to permit this expansion may determine their spacing. It is desirable in steel to have continuity between girder and column but a detail satisfactory for fabrication and erection is sometimes difficult. In concrete construction expansion becomes extremely important. Roller bearings are cumberous and not very satisfactory here. It is well for several reasons to reduce expansion joints to a minimum but the length of bridge for which such expansion may be taken up in the columns seems to be limited to about 200 ft. The simplest method of providing for expansion at joints is by the double column, though this is sometimes prohibited by horizontal clearance limitations.

Design Considerations. Various aspects of design are referred to throughout the chapter. Special consideration, however, should be given to the effect of:

(a) Clearance, both horizontal and vertical; and approach grades.
(b) Appearance, especially as regards expansion joints.
(c) Details of connecting girders at columns. Advisability of using a cross-girder.
(d) Economic advantages of haunching concrete girders and deepening steel girders.
(e) Deflections if shallow girders are used.

Methods of Analysis. The object of analysis is, of course, to obtain curves of maximum positive and maximum negative moments in the girders, curves of maximum shears in the girders, and absolute maximum moments in the columns. Stress-producing conditions are dead load, live load, traction, temperature, and possibly wind, centrifugal force, and other lateral forces.

Dead load moments and shears are readily obtained on the usual assumptions. Details of technique are discussed later.

Approximate influence lines for live load moments and shears are easily drawn, as discussed in Chapter IV.

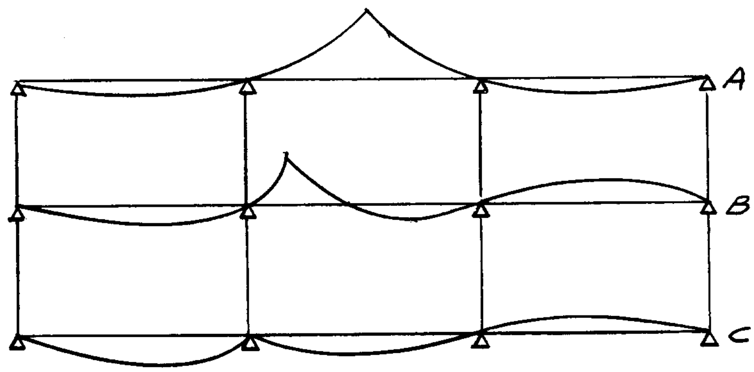


Fig. 107

A little study of such influence lines for moment will show that there are two possible types within any span - A and B, Fig. 107, and that the influence line for moment at the support is of type C.

With uniformly distributed live load, then, for points near the center of the span and for the supports, maximum positive and maximum negative bending moments occur with full loads in several spans, and with partial loads in no spans. For points near the support, however, partial loading of the span under consideration is indicated.

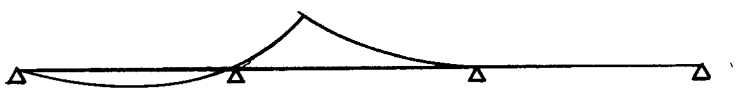


Fig. 108

For the critical point at which the influence line changes from type B to type A the influence line is as shown, Fig. 108. Hence, loads in spans to the right of this span produce no moments at this critical point and this then is the fixed-point of the span as defined elsewhere.

Curves of Maxima. It is now possible to determine for uniform live load the general shape of the curves of maximum live load moments for points between the fixed points in any span. Also, numerical values may be determined for these maxima and for the maximum moments at supports without drawing any influence lines to scale and without any partial loading of spans. Thus, for maximum positive moment between B and C, Fig. 109, we load as in (b) and the curve of maximum positive live moments is a parabola. For maximum negative moment between B and C we load as in (a) and the curve of maxima is a straight line. For maximum positive live moment at A, load as in (d); for maximum negative moment, as in (c).

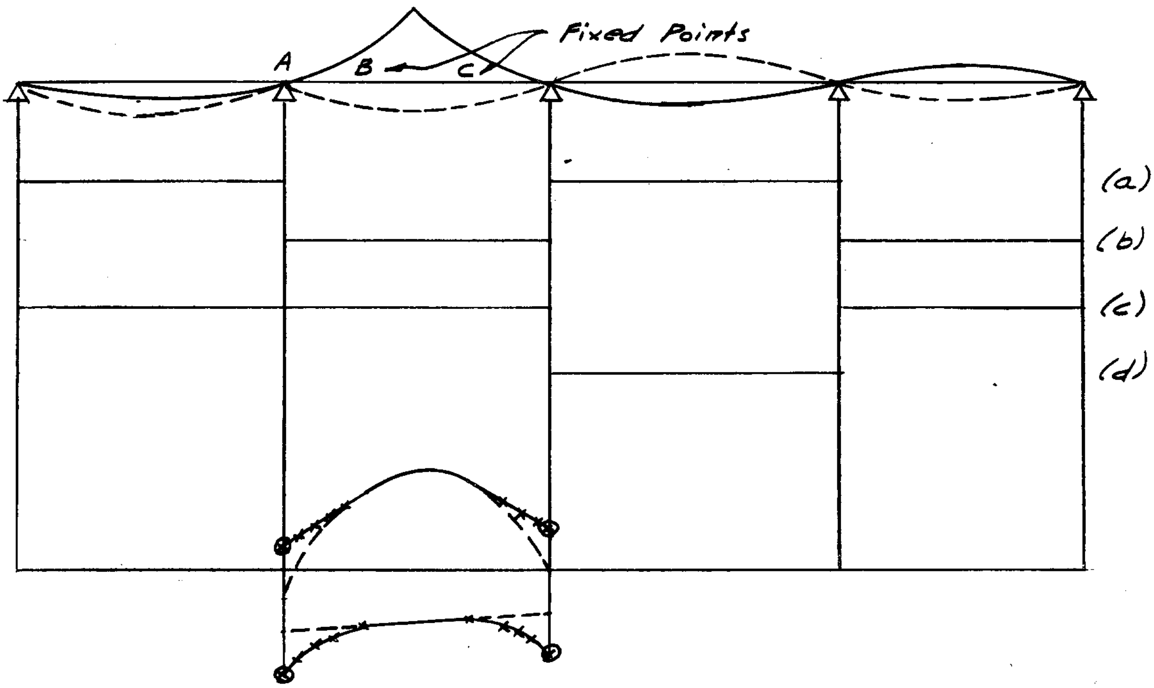


FIG. 109

In the figure the parabola for maximum positive at center and the straight line for maximum negative at center are shown dotted to the support. Beyond the fixed points the curves of true maxima (shown -x-x-) lie outside the dotted lines and pass through the circled points shown for maximum moment at the supports.

Usually the curves of maxima for points inside the fixed points (which ordinarily lie near the quarter points) do not need to be determined very accurately. They do not determine maximum design sections but are needed to determine cut-offs for cover plates and bend points for reinforcement.

Points on this part of the curve may be determined either from influence lines or, more conveniently for uniform loads, by loading a portion - one-quarter, say - of the span, drawing the moment curve, and then determining the point for which this loading gives maximum. This may be done by applying a concentrated load at the quarter point of the span (the end of the uniform load) and finding the point at which this gives zero moment.

In the above, influence lines have been used pictorially rather than quantitatively, which is the best procedure where uniform loads are under consideration. If railway train loads are under consideration, the method of equivalent uniform loads will probably give as much accuracy as is required. For trolley car loading and truck loads, the quantitative influence line may be used, though it is probably simpler and possibly quicker to place the truck or car in several positions on the span, draw the moment curve and combine these curves for maxima. It is worth noting that the shape of the curves of maximum positive moment and maximum negative moment cannot be very different whether the live load is a single concentrated load or a train of loads or uniform load. Influence lines and maxima for the supports, the fixed points and the center should, then, be sufficient in each span.

Influence Lines from Moments at Supports. Another aspect of the matter which may sometimes be useful is that all influence lines are readily constructed from those for moments at the supports though such complete studies probably will not often be needed. Thus, suppose the moment at A is wanted, Fig. 110,

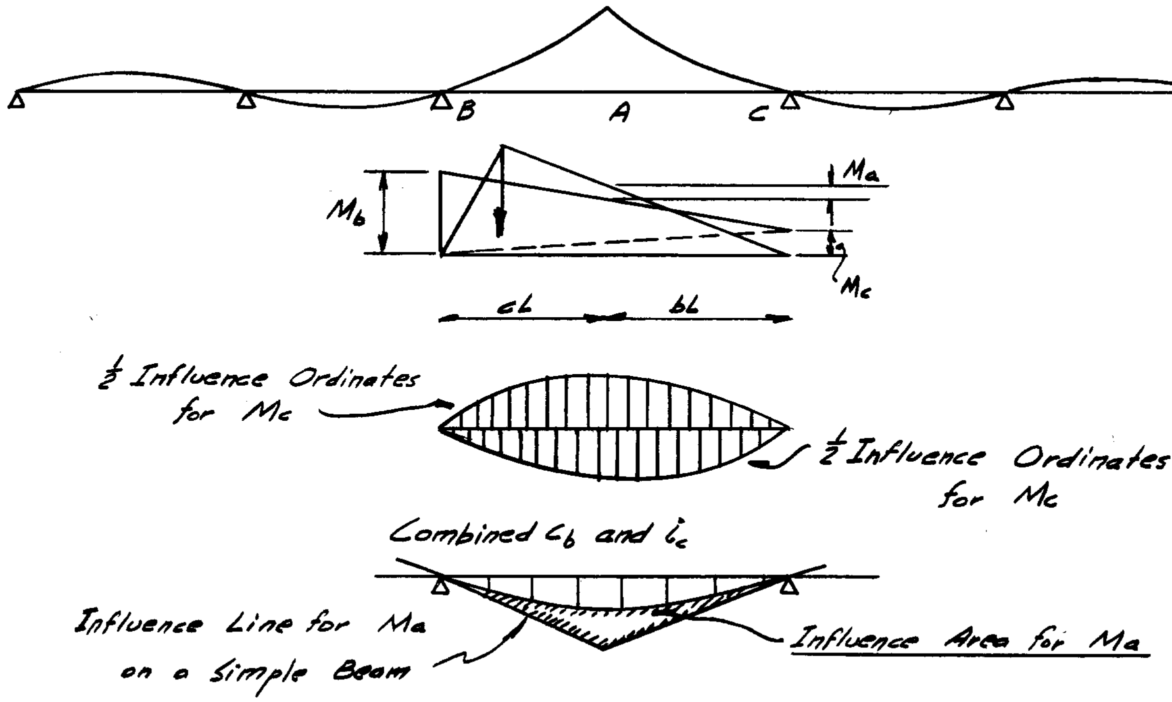


FIG. 110

Then, $M_a = -M'_a$ (due to continuity at B and C) + M_a'' (due to the effect of loads on a simple beam) = $-(cM_b + bM_c) + M_a''$. By combining the influence lines for moment at B and C in the proper ratio and then combining this with the triangular influence line for M_a'', then, we get the influence line for M_a. The method evidently requires precision in plotting.

Several different methods of combining these curves will at once suggest themselves. For the three span girders, the influence lines for reactions may also be conveniently used in a similar way.

Influence lines for shear will be as previously shown, Chapter IV, and all influence lines in one span may be drawn from one curve. Evidently the load divides are the same as for a simple span and the maximum shears do not ordinarily differ very much from those in a simple span. Exact shear computations should not be made until it is known that they will affect the design (thickness of web plate, flange pitch, web reinforcement).

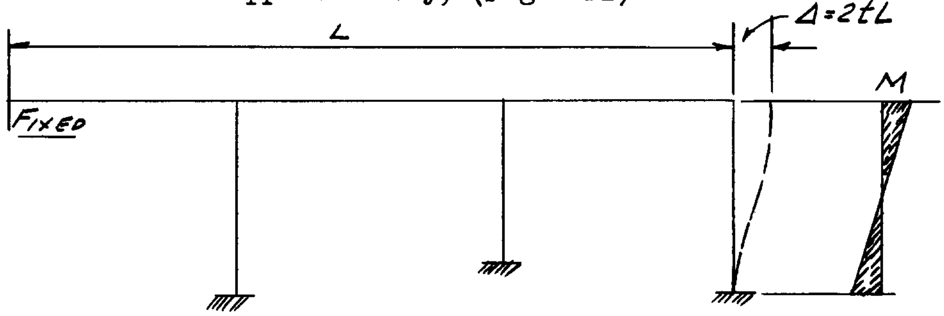
If it is considered necessary to draw these influence lines, they can also be obtained from the influence lines for moments at the supports.

Thus, $V_a = \frac{M_b - M_c}{L} - V'_a$ (due to load on a simple span). The basic curved line for shear then, is the difference of influence ordinates for M_b and M_c divided by L.

All influence lines for moment and shear in girders, then, may be got from combining the influence lines for moments at interior supports (n - 1, where n is the number of spans).

<u>Column</u> <u>Moments</u> <u>and</u> <u>Stresses</u>. Expansion and contraction due to temperature and shrinkage of concrete girders also produces moments in the girders due to the column resistance. These are usually small and approximate results are readily got from simple moment distributions.

Maximum column stresses occur either when the load is maximum or when the moment is maximum. The design of the columns for traction, temperature and live and dead loads may be a critical point in the design. For temperature it will be true approximately, (Fig. 111)



$$M = \frac{6\Delta}{h\,\frac{h}{EI}} = \frac{6\Delta K}{h} \cdot E$$

FIG. 111

$$M = \frac{6\,\Delta\,KE}{h} = (6\xi\ E\ t^{o}\ K)\ \frac{L}{h}$$

$$f = \frac{My}{I} = (6\xi\ E\ t^{o})\ \frac{L}{h^2}\ y$$

The fibre stress, for a given expansion, then, is definitely fixed by the width of the column. If the girder is supported by columns only, with no point (abutment) fixed horizontally, (Fig. 112) then traction produces a moment,

$$M = \frac{Hh}{2(n+1)}$$

$$f = \frac{Hhy}{2(n+1)I}$$

where n is the number of spans. (approximately) which stress may be high for a narrow column.

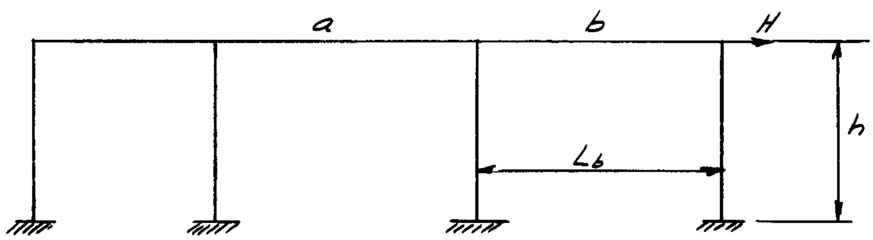


FIG. 112

Due to live load,

$$M = \text{(roughly) } 1/12\ w_L\ L_b^2 \frac{K_c}{K_a + K_b + K_c}$$

$$f = 1/12\ w_L\ L_b^2 \frac{y/h}{K_a + K_b + K_c}$$

which again fixes the column stress as roughly proportional to the column width.

A wide column, then, may be inevitably overstressed whereas a narrow column may have too small a section modulus to properly resist traction.

When spacing of columns and of expansion joints and girder and column sections have been tentatively decided on, the analysis of stresses is not difficult, though it still contains some elements of uncertainty. Dead load stresses are figured directly, live load stresses directly by moment distribution and temperature and traction stresses indirectly as explained elsewhere. It is misleading, however, to figure column stresses from a line diagram along the axes of the members. The stiffness of the columns should be determined from the column analogy considering the moment of inertia infinite above the bottom of the girder. (This refinement is probably unnecessary in determining girder moments.) This gives the moment at the center line of girder which must be reduced to that existing at the bottom of the girder. If the girder is haunched, it is probably sufficient to estimate the free length of column as somewhat greater than to the bottom of the haunch.

If the column is fixed or partly fixed at top and bottom, reduction in working stresses for column action should be made for direct stresses only, since the bending is localized near the ends. With column fixed at either top or bottom, but not at both, allowance should be made by the use of some modified column formula which provides for the effect of bending.

Columns may be pin connected in steel, but this device does not seem very satisfactory in concrete. In steel it is an expensive detail and should be used only after careful consideration. In general, of course, the column fixed at both ends has higher stresses, but it is simpler, more rigid and more dependable.

Investigation should also be made of column stresses due to lateral forces, but these are not critical and the investigation need only be carried far enough to show this. If the columns have abnormal batter, Fig. 113, the relative vertical displacement of the ends of the girder is of theoretical interest (see Chapter) but it is rarely, if ever, of practical importance.

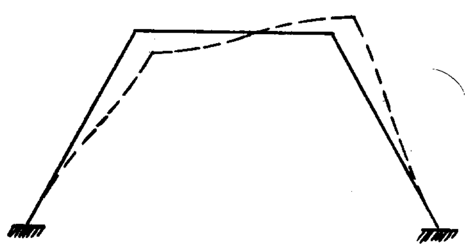


Fig. 113

Questions Arising in Design. Precedent, tradition and experience in American practice are not plentiful in the design of indeterminate structures. It is, therefore, not possible to comfortably accept existing methods of design and we are often forced, more than in familiar types, to a consideration of ultimate failure of the structure in selecting specifications and interpreting analyses. The following questions arise:

(a) Shall split loading conditions be required to carry the same factor of safety as continuous loads?
(b) What allowance, if any, shall be made for reversal of stress and fatigue?
(c) How shall the reduction in safety factor be allowed for when dead load stresses are reversed?
(d) What allowances shall be made for impact?
(e) What value of E, and what temperature variation shall be provided for in concrete structures?
(f) What allowance, if any, shall be made for shrinkage?
(g) To what extent can we depend on the shifting of stress from critical to less critical sections as failure approaches?
(h) What allowance, if any, shall be made for settlement of supports?
(i) What condition of fixation shall be assumed at the base of the column?

Specifications for Design Conditions.

(A) Steel.

1. Basic allowable stress, 16000 lb. per sq. in.
2. Allow an increase of 25 per cent for either of the following:
   (a) Split loading
   (b) Dead load, live load and traction combined.
) Increase to 33 per cent with lateral forces.
3. Increase live load 50 per cent where it tends to reverse the dead load stress.
4. No allowance for reversal of stress.
5. Usual allowances for impact.
6. In temperature computations E = 30,000,000, t^o + - 50^o F.
7. Assume I varies as (gross area of flanges + 1/3 area web) times (distance b. to b. of angles)^2.
8. Provide for 1/2 in. settlement at any support combined with dead and live loads. Allowable stress 30,000 lb. per sq. in.
9. Allow for the stiffening effect in the column of the large moment of inertia above the bottom of the girder.
10. Traction, 20 per cent of live load.
11. Total range of deflection (upward plus downward) for live load shall not exceed the downward deflection of a simply supported plate girder having a depth back to back of angles equal to 1/12 the span. If shallower girders are used, the allowed live load stresses shall be decreased accordingly.
12. Otherwise according to standards for simple spans.

(B) Concrete.

1. Basic allowable compression for 2000 lb. concrete - 800 lb. per sq. in. thoughout.
2, 3, 4, 5. As for steel.

6. $E = 2{,}000{,}000$ lb. per sq. in. for 2000 lb. concrete. $t^o = \pm 40^o$ F. Shrinkage equivalent to $t^o = 20^o$ F. unless provided for by erection methods used.
7. Assume I varies as bd^3 in girders and columns. In girders b is the width of stem.
8, 9, 10, 11, 12. As for steel.

Such supplementary specifications are, of course, only tentative. They serve, however, to call attention to several matters which are often neglected. A few explanations are needed.

In clause 4 for steel. The use of reversal formulas of the Launhardt type in the design of members seems to be a heritage not at all justified by the results of modern investigations. Provision for reversal of stress in connections is, however, debatable.

In clause 7 for steel. Indeterminate stresses are evaluated from the geometrical relations of the angle changes. These angle changes depend on the prevailing section, which is gross and not net. At any section,

I = moment of inertia about centroidal axis of flanges + web = $I_f + I_w$.

Assuming centroidal axis at the center,

$I_f = A_{top}(h/2)^2 + A_{bottom}(h/2)^2 = (\text{total flange area}) \times (h/2)^2$

h is the distance to center of gravity of the flange, which for present purposes may be taken back to back of angles.

$$I_w = 1/12\, A_w h^2 = 1/3\, A_w\, (h/2)^2$$

$$I = (\text{area both flanges} + 1/3 \text{ web})\, (h/2)^2$$

$$I \text{ is proportional to } (A_f + 1/3\, A_w)\, h^2$$

In clause 8 for steel. Some investigation should be made of the probable effect of settlement unless it is absolutely certain that it cannot occur. A settlement of more than 1/2 in. would probably cause the bridge to be practically abandoned irrespective of the stresses and so represents failure in any case. It will be noted that for this case the allowable stress is pushed practically up to the yield point.

In clause 11 for steel. Such clauses are common. There does not seem to be very definite data to show what limiting deflection is objectionable. Ordinary girders are not built of depths less than 1/12 span and as these are ordinarily satisfactory in this respect, the same limiting deflections are to be provided in other cases. Such is the usual argument. It does not seem, however, that dead load deflection is involved in such reasoning, since it can be provided for by camber. The clause as ordinarily written is not specific as to whether the limit is on live load deflection or on total deflection but is commonly interpreted to mean total deflection. The clause as here written seems to give a fairer measure of rigidity and certainly may be expected to result in a bridge as free from vibration and impact as the simple span would be.

In clause 1 for concrete. Note that an allowable stress of 800 lb. per sq. in. is recommended throughout. Practice in this regard, as outlined by codes and committees, used 950 lb. per sq. in. near the supports. This practice, however, originated in connection with building frames and seems to have been based on two ideas - first, that the computed maximum negative moment did not actually exist on the critical section at the column face and, second, that at this point failure was delayed by the lateral restraint offered by the column face and also -- which is a part of the same idea -- that the negative moment was more or less localized. It seems very doubtful whether any of these conceptions have any application to large girders in which the true absolute maximum moments have been determined at every section.

In clause 6 for concrete. The acceptance of a value of E of 2,000,000 lb. per sq. in. for 2000 lb. concrete does not at first seem justifiable. The value may vary from 1,500,000 lb. per sq. in. to 5,000,000 lb. per sq. in. But there seems fairly good evidence that the value of E and of ultimate strength as determined in the laboratory are closely related and that the ratio of the two is about ten to one. If, then, the value of E is greater than 2,000,000 the concrete will probably be stronger than 2,000 and the factor of safety greater, rather than less, than if conditions were as designed for.

The selection of a 40° range of temperature -- less than for steel -- is probably justified partly -- only slightly -- by the lag of the concrete with reference to that of the air and partly by the action of the time yield in the column in reducing the fibre stresses (time yield is, in this respect, equivalent to a reduction in the value of E).

In clause 7 for concrete. The assumption that I varies as d^3 has simply the justification that it is simple and obvious and probably as well supported by facts as any. The assumption that the I of the girder varies as the width of stem has the same justification in comparing different girders. As regards the relative values of K of girder and column, however, it gives probably too low a relative value for the stiffness of the girder. This is on the safe side as regards the column moments and otherwise it is not especially important.

Outline of Analysis.

1. Make preliminary study of columns and fix spacing of columns and expansion joints, depth of girders.
2. Compute dead load and maximum live load shears and moments in girders assuming constant I and neglecting columns.
3. Design girders and columns, study detail of connection. In concrete haunch if possible and study economy of haunching.
4. Make final computation of stresses.

CHAPTER VIII
JOINT DISPLACEMENTS

Distribution of Joint Forces. Where lineal movement of the joints, without rotation, is to be dealt with, it is possible to distribute unbalanced joint forces just as unbalanced joint moments were distributed above. The unbalanced force at a joint is allowed to move that joint while all other joints are held fixed, all joints being held against rotation. The unbalanced force is thus distributed among connecting members in proportion to the force needed to produce unit movement of that member (I/I_o, of the analogous column). All of this force is carried over to the next joint and there again distributed and so on to an exact solution. The direction of positive forces may be arbitrarily assumed.

This method may be illustrated by the approximate analysis of a five-span arch series using the example given by Charles S. Whitney, Trans. A.S.C.E. 1926. Mr. Whitney assumes that the rotation of the top of the piers has an inappreciable effect on the arches, and that the pier rotates as a cantilever acted on by a force acting through the arch centroid. He computes the horizontal force necessary to produce unit horizontal deflection of the top of the pier as follows:

For an arch $1/2974 = 0.000337$

For the pier $1/880 = 0.001240$

Since the center span is symmetrical, only one-half of it is effective in producing movement of one end, and hence,

For center arch - $2 \times .000337 = .000674$

Hence, a thrust at A, Fig. 114, is distributed;

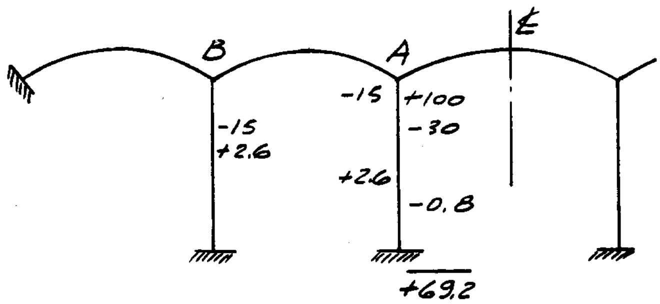


FIG. 114

$\frac{674}{2251}$ = 30 per cent to center arch

$\frac{337}{2251}$ = 15 per cent to side span

55 per cent to pier

while at B it is distributed,

$\frac{337}{1914}$ = 17.6 per cent to either arch

64.8 per cent to pier.

Hence, the thrust at A is 69.2 per cent of its value in a fixed-ended arch.

The General Equation of Displacements and Slope-Deflection. The general equations of continuity at any joint of a structure when adjoining spans are unloaded may be written as follows, provided, as is usually true, the principal axes of the members are parallel and normal to each other.

$$M_a = \sum \phi_a N_a - \sum \phi_b r_a N_a + \sum (\Delta_a - \Delta_b) d_a/I_o$$

$$F_a = \sum \frac{\Delta a}{I_o} - \sum \frac{\Delta b}{I_o} + \sum (\phi_a - \phi_b) d_a/I_o$$

M_a = total external moment at any joint a

F_a = total external force at any joint a

(note that in general there will be two equations of force at any joint, one for horizontal and one for vertical forces).

ϕ_a, Δ_a are respectively rotation and displacement at the joint considered.

ϕ_b, Δ_b are corresponding quantities at the other end of each member successively.

N_a is the moment at joint a corresponding to a unit rotation of this joint, the other end being fixed.

r_a is the carry-over factor at a (the ratio of the moment at b due to a unit rotation at a to the moment at a due to such a rotation).

I_o is the moment of inertia of the elastic weights about their centroid for any member.

d is the distance of a or b from the proper centroidal axis ($d_a = d_b$ for a symmetrical member).

For signs we may adopt the following convention:

M and ϕ are positive when clockwise.
F and Δ are positive to the right horizontally and downward vertically.
d is to be taken vectorially as the distance from the neutral axis to the joint, positive downward and to the right.

It is not difficult from these general forms to write the equations for every joint in a complex structure such as a continuous arch series or a Vierendeel truss. These equations may then be solved simultaneously for displacements and from these the moments, shears and thrusts may be determined - preferably by the column analogy.

It may be noted above that it was specified that these equations are applicable when adjoining members are unloaded. This condition is easily realized by computing the end restraints on the loaded members and then annulling them. M and F, then, may also be taken as the unbalanced internal force or moment at the joint.

The use of such equations, involving possibility of error in signs and the necessity of simultaneous solution, is to be thought of as a research tool and rarely as a tool of design.

The equations may also be written:

$$\phi_a = \frac{M_a}{\Sigma N_a} + \frac{\Sigma \phi_b \, r_a N_a}{\Sigma N_a} - \frac{\Sigma (\Delta_a - \Delta_b) \frac{d}{I_o}}{\Sigma N_a}$$

$$\Delta_a = \frac{F_a}{\Sigma \frac{1}{I_o}} + \frac{\Sigma \frac{\Delta_b}{I}}{\Sigma \frac{1}{I_o}} - \frac{\Sigma (\phi_a - \phi_b) \frac{d}{I_o}}{\Sigma \frac{1}{I_o}}$$

These equations may in many cases be solved by successive approximation or by successive convergence as will be further explained

The method of successive approximation is as follows: Asume for ϕ_a and Δ_a their most probable values. Then repeatedly apply the equation, substituting the values of the other ϕ's and Δ's last found until the same value is found twice in succession. This method has an advantage over the method of convergence in that it is sometimes possible to estimate closely the true values.

The method of successive convergence writes the first approximate values as the first term on the right, then writes in the next terms using the values just found, then writes in corrections for these corrections until the correction terms are negligible, and finally adds up all terms. It deals with smaller figures but does not permit hastening the result by estimating.

The method as a whole has an advantage over that of distributing moments and forces in that there are fewer quantities to deal with.

If there are no joint displacements, or if, as is discussed elsewhere, it is convenient to make separate allowance for such displacements, the process is much simplified. We then have,

$$\phi_a = \frac{M_a}{\Sigma N_a} + \frac{\Sigma \phi_b \, r_a \, N_a}{\Sigma N_a}$$

This, perhaps, is more conveniently written,

$$\phi_a = \frac{M_a}{\Sigma N_a} + \Sigma \phi_b \frac{r_a \, N_a}{\Sigma N_a}$$

which is readily evaluated by successive approximation or convergence. This procedure is very convenient in evaluating secondary stress in bridge trusses. If connecting members are treated as prismatic,

$$N_a = 4 \frac{I}{L} \qquad r_b = -1/2$$

$$\phi_a = \frac{M_a}{4 \Sigma \frac{I}{L}} - \Sigma \phi_b \frac{I/L}{2 \Sigma \frac{I}{L}}$$

If further M is due to a known rotation of the bar,

$$M = \frac{6 \Delta}{L \frac{L}{I}} = 6 \psi \frac{I}{L}$$

$$\phi_a = \Sigma 3 \psi \frac{K}{2 \Sigma K} - \Sigma \phi_b \frac{K}{2 \Sigma K}$$

which is the equation used in evaluating secondary stresses. Or this may be derived from the fundamental equation above

$$\phi_a = \frac{M_a}{\Sigma N_a} + \frac{\Sigma \phi_a \, r_b \, N_a}{\Sigma N_a} - \frac{\Sigma (\Delta_a - \Delta_b) \frac{d}{I}}{\Sigma K_a}$$

$$M_a = 0, N_a = 4 \frac{I}{L}, r_a = -1/2 \quad \frac{\Delta_a - \Delta_b}{L} = -\psi \frac{d}{I} \qquad = 6 \frac{I}{L} \cdot \frac{1}{L}$$

$$\phi_a = \frac{3 \Sigma K \psi}{2 \Sigma K} - \frac{\Sigma K \phi_b}{2 \Sigma K}$$

The above discussions include any combination of bars of any shape or form provided the axes are parallel or normal to each other and it is not very difficult to extend it to include skewed axes. An important special case is indicated in what is known as the slope-deflection equation for prismatic beams,

$$M = \frac{2EI}{L}\left(2\phi_a + \phi_b - 3\psi\right)$$

which may be derived in a number of ways and follows from our first general equation when,

$$N_a = 4K, \quad r_a = -1/2, \quad \psi = \frac{\Delta a - \Delta b}{2d} \quad d^2/I = \frac{L^2/4}{L^2/12 \cdot L/I} = 3K$$

Bents Subject to Side-Sway. The method of moment distribution is primarily a method for solving continuous structures in which the joints suffer no lineal displacement or where the displacement is definitely known. Obviously such bents as those shown in Fig. 115 cannot be analyzed directly as continuous beams. Distributing the fixed-ended moments will not be sufficient because the bent will deflect sidewise. There are three methods of making allowance for the side-sway:

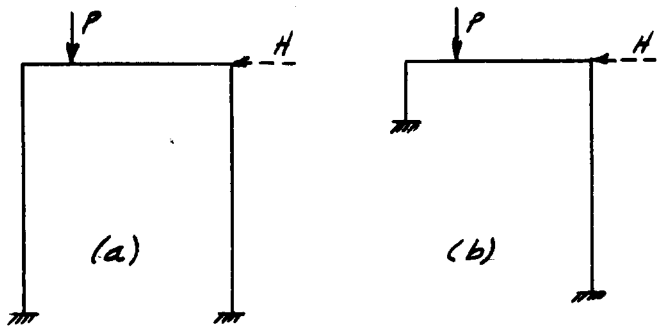


FIG. 115

(1) We can consider a force H sufficient to prevent side lurch and equal to the unbalanced shear in the two legs. This we can neutralize by adding an opposite force and analyze again - adding the results.

(2) We can assume any value for H and find the moments. This method is to be recommended as having great advantage in this and similar cases.

(3) We can assume a side lurch and find H, attacking the problem indirectly.

If the girder is relatively very stiff, (as is usually the case), and its flexure negligible, the point of contraflexure of the column will be in the middle.

$$\text{For deflection due to a force, } \Delta \propto \frac{PL^3}{EI}; \; P \propto \frac{I}{L^3} \propto \frac{K}{L^2}$$

$$\text{For deflection due to a moment, } \Delta \propto \frac{mL^2}{EI}; \; m \propto \frac{I}{L^2} \propto \frac{K}{L}$$

The moments throughout will be proportional to the deflection which in turn is proportional to the total shear in the columns. We may then assume any convenient value for the total shears in the columns in the beginning and distribut this shear among the columns in proportion to their values, K/h^2, where h = the height of the columns. Compute the end moments in all columns on the assumption that the joints do not rotate. Redistribute these moments as in any other case. Finally compute the total shear in all columns (this will be different from that first assumed, because releasing the joints will cause the bent to sway sideways) and multiply all moments by the ratio of the total shear desired to the total shear found.

Multiple Frames and Viaducts Longitudinally Unsupported. A girder frame such as shown in Fig. 116 may be thus analyzed. The moments due to vertical loads may be found in the usual manner, assuming fixed ends and distributing.

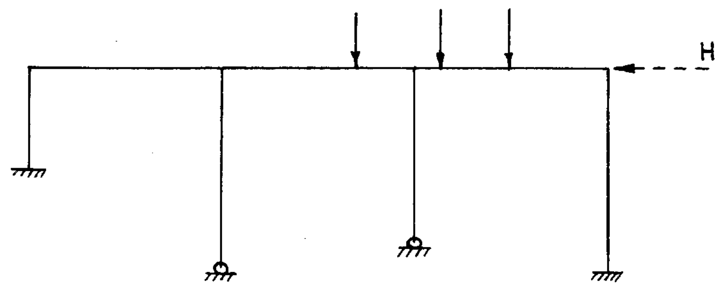


FIG. 116

In so doing, the structure is considered held against longitudinal movement.

To find the effect of the end lurch, and of horizontal forces along the girder, assume any horizontal force H and distribute the shears due to it among the columns in proportion to their values of K/h^2. With these known, assuming points of contraflexure at the mid-heights of the columns, the end moments are computed and distributed in the usual manner. From these moments the shears are re-computed. The true moments will then be in proportion to the computed moments as the true total shear is to the assumed value of H.

In other words, the moments due to end lurch or to traction vary as the displacements and while the assumed value of H does not give the correct moments, it does give the correct ratio of shears to moments. If the shear is corrected to satisfy statics, the moments corresponding to it will be of correct value.

Multi-storied Bents. Bents of more than one story, subject to side-sway, either as a result of unbalanced loading or due to horizontal forces, may be solved by similar methods. It is understood that such problems are not commonly of great interest. Where they do occur it is rather the approximate effect of abnormal or unusual conditions that is desired than exact analysis.

Consider a two-story bent shown in Fig. 117 and enough dissymmetry to make it worth figuring. As a problem in analysis it will be necessary to make two configurations - one for each story. From each of these, - i.e., from the assumed shear in each story (producing, of course, shears in the other stories) a set of moment values may be obtained. These may be combined to obtain the true shears, and from the true shears the true mome follow in proportion.

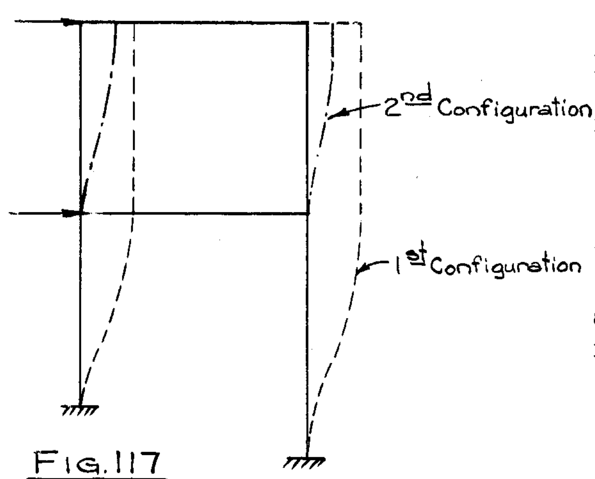


FIG. 117

Indirect Analysis of Multiple Frames. The problem of analyzing multiple frames such as Vierendeel girders or wind stresses in steel buildings has from time to time occupied a prominent place in the literature of structural engineering. Sometimes uncertainties of the problem justify an attempt at exact analysis; usually they do not. In complex cases, where a large number of bars meet at a joint, the method of successive convergence of rotations is perhaps to be preferred, but the method of distributing moments is very satisfactory.

Analysis of the Vierendeel Girder. In such cases as the Vierendeel girder, a displacement may be assumed in any one panel. Thus in Fig. 118 it is assumed that the two parts of the truss AB and CD are moved as shown. Any value for this movement may be assumed. Find now the moments, or the values of ϕ, for this condition. Note that after the first displacement, no further lineal movement of the joints is permitted, the movements being only of rotation. This makes the above methods applicable.

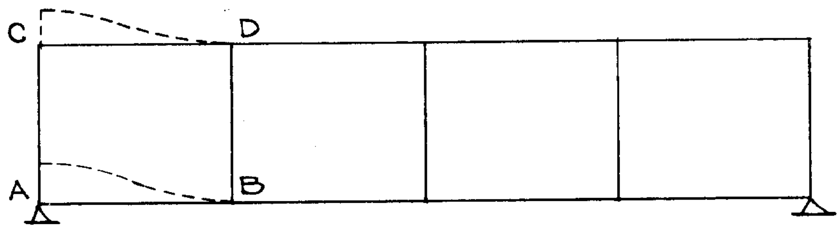


FIG. 118

For the moments (or values of ϕ and I thus found) find the shears in all panels. We now know a set of shears and the moments which correspond to them.

Repeat the process for each panel. We now have as many sets of moment values and shear values corresponding thereto as there are panels and by a series of simultaneous equations these moments are combined to give the desired shears.

In such cases there may be an unbalanced shear along a horizontal plane through the truss, which may or may not be large enough to be important. This may be corrected for by producing a relative displacement of upper and lower chords, finding the moments and shears, thus furnishing data for another simultaneous equation.

General Method of Indirect Analysis of Multiple Frames. In general, then, produce as many independent arbitrary configurations of the frame as there are degrees of independent movement of the joints of the frame, thus finding a number of sets of corresponding moments and shears. Combine these sets to give any set of desired shears, thus finding the final moment values.

The method is perfectly general and has the special advantage that it permits separation of the most complex problems into a series of short simple steps. However, it will not often be used, because simple approximate methods which assume points of contraflexure and shear distribution are usually sufficiently exact.

The method is useful when applied to the study of Vierendeel girders with abnormal proportions. In such cases it is convenient to determine the rotations by successive convergence of the values of ϕ. The end moments in the members are finally determined as proportional to K $(2\phi_a + \phi_b)$ due to the ϕ values and $-3K\psi$ due to ψ. Hence, the shear in a member due to the ϕ values is,

$$\frac{M_a + M_b}{L} = \frac{3(\phi_a + \phi_b)}{L} 2K = 6\frac{K}{L}(\phi_a + \phi_b),$$ and that due to ψ is,

$$-\frac{6\Delta}{\frac{I}{K}L}\frac{1}{L/2} = -\frac{12K\psi}{L} = 6\frac{K}{L}(-2\psi).$$

It should be noted that throughout the solution only relative values are wanted down to the final adjustment, when a multiplier is used for all values in order to give the correct shears.

The effect of distortions due to direct stress are readily included in this solution, either directly or as a final correction. The effect is not pronounced.

Usefulness of Vierendeel Girders. The principal field of usefulness of the Vierendeel girder would seem to be as an occasional expedient in building construction where diagonals interfere with window space and passageways. In some cases an ordinary truss with an occasional open panel may be required. It is such unusual conditions rather than the straight symmetrical type of truss that may call for an exact analysis.

On the face of it, deflections will be relatively high in quandrangular frames. Both moment and direct stress are contributing factors in deflection - moment in a relatively high degree, and since bending stresses are such important factors in trusses without diagonals we would expect relatively high deflections.

No economy is apparent in the general use of Vierendeel girders. The unit force in steel is probably between that of the girder and the truss.

For highway bridges, Vierendeel girders may have some advantage in appearance. The common through girder is particularly objectionable as a highway bridge and the through truss is inherently unsightly but the Vierendeel girder, either in concrete or steel, with its smooth lines and open panels, undoubtedly has some architectural possibilities.

Vierendeel girders of normal proportions can be analyzed with all needed accuracy by assuming the shear in any panel distributed between the chords in proportion to their moments of inertia and assuming contraflexure at the middle of each chord panel.

Multiple Arch Problems. The method here indicated is also applicable to the problem of arched bents, either single or multiple, to arches on elastic piers and to similar problems. The procedure is to determine the fixed-ended moments and thrusts for immovable and non-rotating joints, to distribute these moments throughout the frame and from them determine the unbalanced forces which prevent movement of the joints. Then one joint is displaced, or some other configuration of the frame is assumed, and for this a set of moments corresponding to a set of joint forces is determined. As many such sets are determined as there are degrees of freedom of lineal motion for these joints. These sets are now combined with the set of moments and joint forces due to the loads as first determined in such ratios as to give no unbalanced joint forces. In those problems which are directly associated with design, other methods are probably more useful.

Wind Stresses in the Frames of Office Buildings. The wind load on the sides of a steel-frame building may be carried to the foundations either by sway bracing or by portal bracing. The former is stiffer and cheaper, but is not practicable except along the walls and sometimes around elevator shafts and other wells. Often a combination of the two systems is used, but all bracing is often of the portal type.

A wind frame with several rows of portal bracing is indeterminate first in the distribution of the load between the rows and second as regards the stresses in one row.

Approximate Method. Several approximate methods have been in use for computing stresses. Of these exact analyses seem to show the following method to be the most satisfactory:

(a) Assume the shear in any story to be distributed among the columns so that all interior columns carry the same amount and an exterior column carries one-half as much as an interior column.

(b) Assume points of inflection at the centers of all columns and girders.

The structure is now determinate by statics. An analysis by this method is given in Fig.

Semi-Rational Method. It is evident that in some cases this method leads to absurd results, as where some interior columns do not go through all stories and are of lighter section. In general, of course, the point of contraflexure in column or girder tends to crowd toward the more flexible end and the shear tends to be greater in the stiffer columns. The following formulas try to take these factors into account. They are only partly rational, but comparison with the results of more exact analyses shows them to be dependable. For data on this point see, "Wind Stresses in Steel Buildings Having Discontinuous Members," J. E Keranen, Master's Thesis, University of Illinois, 1925.

Let $S = \dfrac{K \text{ of all the members at any joint}}{K \text{ of all the girders at any joint}}$

S_t for any column is the S value at the top

S_b for any column is the S value at the bottom

$\Sigma = S_t + S_b$ for any column

$\Delta = S_t - S_b$ for any column

$e = \dfrac{\Delta}{4 + \Sigma}$ = factor of eccentricity of the point of contraflexure of any column in terms of h/2, the half story height.

$H \propto \dfrac{K_c}{\Sigma}$, where H is the shear in any column.

The general procedure in applying this method is as follows:

1. Determine the S for each joint. It is evident that this need be done for only one-half of the structure in ordinary cases.

2. Find the difference between the S at the top and that at the bottom of each column (Δ).

3. Find the sum of the S at the top and that at the bottom for each column (Σ).

Find the location of the point of contraflexure in each column (e). The falue of "e" is a percentage of h/2 and is in the direction of the larger S (which can be indicated by an arrow on the figure of the building). The location of the point can best be figured as a percentage from top and bottom as: 0.55/0.45.

5. Find the ratio K_c/Σ for each column. The per cent of the total shear for that story taken by any one column is obtained by dividing the K_c/Σ ratio for that column by the sum of these ratios for all the columns in the story.

6. Find the moments in the columns by considering the shears acting at the respective contraflexure points. This gives the end moments for outside girders directly.

7. Find the girder moments as for a continuous beam either by computation or estimate.

The method has real value sometimes in studying the effect of special construction. Its application is illustrated in Fig.

CHAPTER IX
SECONDARY STRESSES

Definition of Secondary Stress. The term, "secondary stress" is used in somewhat different sense by different writers. All agree that those stresses which occur in riveted trusses due to the bending of the members which must accompany deflection of the truss should be called secondary stresses. Perhaps this is the only standard use of the term, but it is often extended to include somewhat vaguely stresses which result from deformations of the structure. Evidently such stresses, where the structure is statically indeterminate, may become of primary importance.

Perhaps as useful a definition as any would restrict the term to those stresses which result from deformations of the structure and yet do not appreciably relieve the primary stresses. Thus defined, secondary stresses become necessary evils. On the other hand they are evidently less significant in threatening collapse than are the primary stresses and hence demand a lower factory of safety. Several aspects of this question are discussed below.

We are interested in computing, with reasonable accuracy, any stresses of whatever nature that affect the life and service of the structure we are designing. In analyzing trusses the first (and generally the only) problem in stress analysis is to compute the direct axial stresses in the bars. The effect of these stresses is to change the length of the bars, producing deflection of the structure as a whole and relative displacement of the joints. In riveted trusses (and likewise in pin spans to the extent of the friction on the pins) the angles between the bars at the joints are held against change. Clearly, displacement of the joints cannot take place, the angle between the bars remaining constant, without bending of the members somewhat as indicated in Fig. 119. This bending of the bars due to deflection of the structure may produce stresses of considerable magnitude.

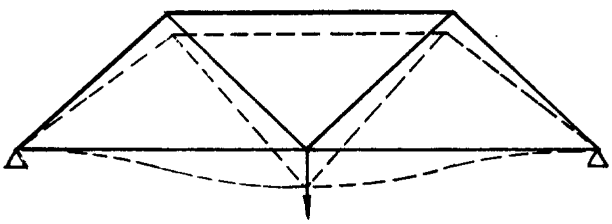


Fig. 119

General Method of Analysis. The method of procedure is analogous to that used for all other problems in continuous frames. Since the bending at any one joint affects the bending at all other joints in the truss, the joints cannot be solved independently. We can set up equations between the moments and rotations at the two ends of the bars and their physical properties, and by methods of successive approximation or of successive convergence, satisfy these equations. The solution of an ordinary truss treating the bars as prismatic - of constant I from joint to joint - is only a matter of a few hour's time. The effect of the gussets on the resistance of the bars to bending - particularly in heavy trusses where the gussets may be as much as one-fourth or one-fifth the length of the member - is considerable, and should be taken into consideration if fairly close results are desired. The additional labor taking account of this variation is slight.

In continuous beams the fixed-ended moments were found and the unbalanced values distributed directly. Where many bars meet at a joint, it is often simpler to find the values of the joint rotations and from these deduce the moment values. This seems advisable in the case of secondary stresses, Vierendeel girders, and multi-story bents (wind stresses in buildings). Mohr used the device in his solution of secondary stresses and it has been elaborated in the literature of the so-called slope-deflection method. Maney also has indicated the advantages of solution by successive approximation.

The treatment below is presented first for prismatic beams and is later extended to beams of variable moment of inertia - that is, to include the effect of gussets and changes of section at the joints.

Solution of the Prismatic Case. Consider any loaded truss and consider that the joints are held against rotation during deflection. The bars will assume new positions. If, then, we release the joints, allowing them to rotate, one at a time, we may find the moments due to the rotations carried over to each joint in turn.

Assume any number of prismatic beams meeting at a point A as shown, Fig. 120, the other ends B of all beams being temporarily fixed.

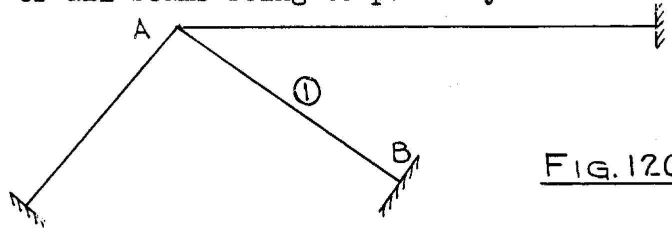


FIG. 120

Assume an unbalanced moment Σm at A.

Then, $$\phi_a = \frac{1}{4}\frac{m_1}{K_1} = \frac{1}{4}\frac{m_2}{K_2} \; . \; . \; . \; \text{etc.}$$

$$m_1 = 4\,\phi_a\,K_1$$

$$m_2 = 4\,\phi_a\,K_2$$

etc.

$$\Sigma m = 4\,\phi_a\,\Sigma K \qquad \phi_a = \frac{1}{4}\frac{\Sigma m}{\Sigma K}$$

Now let the other end of any beam rotate.

$$\text{Moment in 1 at B} = 4\phi_b\,K_1$$

$$\text{Moment in 1 at A} = -1/2 \cdot 4\phi_b \cdot K_1$$

There will then exist at A an additional unbalanced moment equal to

$-1/2 \cdot 4 \cdot \Sigma\,\phi_b\,K$ which will produce at A a rotation

$$\phi_a{}' = 1/4\,\frac{\Sigma\,(-1/2 \cdot 4\phi_b \cdot K)}{\Sigma K} = -\,\frac{\Sigma\,K\phi_b}{2\,\Sigma K}$$

Hence total $\phi_a = 1/2 \cdot \frac{\Sigma m}{2\,\Sigma K} - \frac{K\phi_b}{2\,\Sigma K}$ where Σm is the unbalanced moment at A and ϕ_b is successively the ϕ value of each adjacent point.

$\sum m$ may be due to any cause, - loads on the beams, eccentricity of the bars at the joint, displacement of one of the B ends of the bars. Fixed-ended moments from loads on the bars have already been discussed. Moment due to eccentricity is computed by statics. Moments due to displacement of the joints follow from the column analogy

$$m' = \frac{6d}{\frac{L}{EI} L} = 6K \cdot \psi$$

for the latter alone,

$$\phi_a = 1/2 \frac{\Sigma m}{2\Sigma K} - \frac{\Sigma K\phi_b}{2\Sigma K} = \frac{3\Sigma K\psi}{2\Sigma K} - \frac{\Sigma K\phi_b}{2\Sigma K} = \sum \frac{k}{2\Sigma K} \cdot 3\psi - \sum \frac{k}{2\Sigma K} \cdot \phi_b$$

This may be evaluated either,

(a) By successive approximation, substituting successive values of ϕ_b until the values of ϕ_a are the same for successive approximations at all joints.

(b) By successive convergence, substituting first the approximate values of ϕ_b and then successively the corrections in these values until no corrections remain.

ϕ may be defined in these equations as the rotation of any joint.

ψ represents the rotation of the chord of the bent axis of the member from an assumed reference - that reference being usually the original position of some one bar in the structure.

<u>Computations for Rotation of Bars</u>. The ψ values may be found by summing up the angle changes beginning with any bar as a reference (preferably one, near the center, of small rotation). Signs will need no special consideration if rotation in either direction is adhered to as positive. The direction of bending, if needed, - as it generally will not be - may be readily determined by inspection after the moments are found. The angle changes are found as explained in Chapter III

$$E\Delta\alpha = \left[f_a - \left(f_b \frac{a_b}{a} + f_c \frac{a_c}{a}\right)\right] \frac{a}{r}$$

or by the Williot Diagram. The values of a_b, a_c and r are most conveniently scaled or they may be computed from the relations (See Fig. 121).

$$a_b = \frac{a^2 + b^2 - c^2}{2a} \qquad r = \sqrt{b^2 - a_b^2}$$

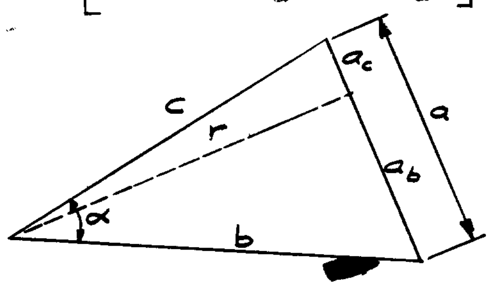


FIG. 121

With the ϕ and ψ values known the moments are found as also explained in Chapter III.

$$M_a = 2\ EK\ (2\phi_a + \phi_b - 3\psi)$$

(the value of E may evidently be omitted throughout).

Or, the fibre stress may be found directly as,

$$f_a = \frac{M_a y}{I} = \frac{2y}{L}(2\phi_a + \phi_b - 3\psi)$$ where y is the distance to the extreme fibre of bar of length L.

The angle changes in 4-segment triangles such as occur in sub-divided or K-trusses may be found by a process similar to that employed in Chapter III, as follows: Referring to Fig. 122 the change $E\Delta\alpha$ may be found by applying unit moments to AB and AC. Then $u_{a_c} = u_{a_b} = 1/P$,

$$u_b = -\frac{a'_b}{b}\cdot\frac{1}{P};\; u_c = -\frac{a-a_b}{c}\cdot\frac{1}{P}$$

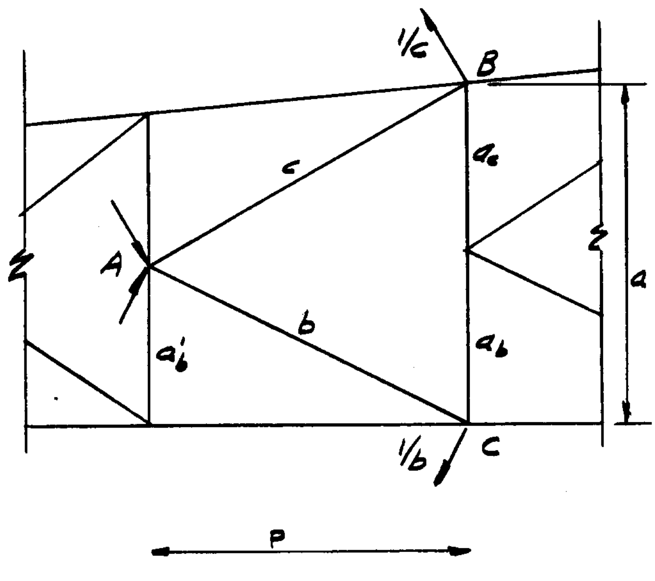


Fig. 122a

Then as before,

$$\Delta\alpha = \sum \delta u = \sum \frac{f\cdot L_n}{E}$$

$$E\,\Delta\alpha = f_{a_c}\cdot\frac{a_c}{P} + f_{a_b}\cdot\frac{a_b}{P} - f_b\,\frac{a'_b}{bP}\cdot b - f_c\,\frac{a-a_b}{cP}\cdot c$$

For parallel chord trusses $a_b = a_c = a_b'$ and,

$$E\Delta\alpha = f_{a_c}\cdot\frac{a_c}{P} + f_{a_b}\frac{a_b}{P} - f_b\frac{a_b}{P} - f_c\frac{a_c}{P} = (f_{a_c} - f_c)\frac{a_c}{P} + (f_{a_b} - f_b)\frac{a_b}{P}$$

Similar methods may be used to find the angle changes in the quadrilateral abcd of Fig. 122a. The ψ values may also be conveniently found by drawing a Williot diagram and so finding the Δ values for the members. Then $\Delta/L = \psi$. The diagram may be drawn assuming any bar as fixed -- preferably a bar of small actual rotation -- and it is not necessary to rotate the diagram.

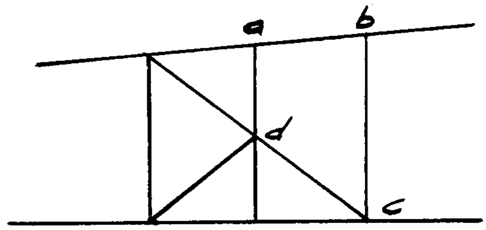


Fig. 122a

Solution for Variable I. If the members be taken as having varying section -- that is, if the effect of the gussets be taken into account -- we may write the general equation of displacements (Chapter VIII) for each joint. Referring to Fig. 123,

$$\phi_a = \frac{M_a}{\Sigma N_a} + \frac{\Sigma\phi_b\, r_a\, N_a}{\Sigma N_a} + \frac{\Sigma\,\psi\, L\,\frac{X_a}{I_o}}{\Sigma N_a}$$

(Note the change in sign in the last term, due to a change in the convention of signs used for Δ).

Where $N_a = 1/A + x_a^2/I_o$

Also, $Lx_a/I_o = (1 - r_a)\,N$

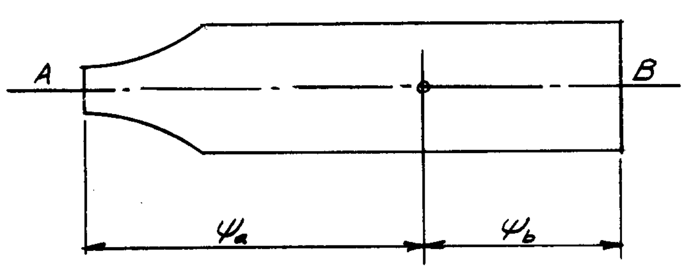


Fig. 123

Hence, where there are no external moments,

$$\phi_a = \sum \frac{(1 - r_a)\, N_a}{\Sigma N_a} \psi + \sum \frac{r_a\, N_a}{\Sigma N_a} \phi_b$$

Finally,

$$M_a = N_a \left[\phi_a - r_a\, \phi_b - (1 - r_a)\psi\right]$$

In all of these expressions, r is taken algebraically and is always negative for straight beams. It may, then, be better to write,

$$\phi_a = \sum \frac{(1 + r_a)\, N_a}{\Sigma N_a} \quad - \sum \frac{r_a\, N_a}{\Sigma N_a} \phi_b \quad \text{and,}$$

$$M_a = N_a \left[\phi_a + r_a \phi_b - (1 + r_a)\psi\right]$$

If we assume $\phi_b = \phi_a$ for a first approximation and all r_a values the same,

$$\phi_a = \sum \frac{N_a}{\Sigma\, N_a} \quad \text{approximately.}$$

If the cross-section is constant,

$$N_a = 4\,K \qquad r_a = 1/2$$

$$\phi_a = \sum \frac{3K}{2\,\Sigma K} \psi - \sum \frac{K}{2\,\Sigma K} \phi_b$$

$$M_a = 2K\,(2\phi_a + \phi_b - 3\psi)$$

$$\phi_a = \sum \frac{K}{\Sigma K} \psi \quad \text{approximately.}$$

This, then, reduces the solution for members of varying section to the same routine as for constant section, and the effect of gussets may be included on any convenient assumption as to their effect in increasing the moment of inertia of any section.

Numerical Example - Unsymmetrical Loading - Prismatic Sections. An illustration of the method of computing secondary stresses is given in Fig. 124 which represents the complete computations for the ϕ and ψ values for a load of 1.5 at Joint 9. Secondary fibre stresses in all bars meeting at Joints 8 and 9 are also computed. The truss is one used in von Abo's article on Secondary Stresses in Trans. A.S.C.E. 1926. The K or I/L values of the members are taken from the structure as there recorded.

It will be noted that the computations are most conveniently made directly on the truss diagram and immediately surrounding the joints. The method of procedure is as follows:-

1. Write on the bars the value of K (= I/L) and the primary fibre stresses f. The values of I are for the gross sections and may be computed as Ar^2 using approximate values of r as given in any handbook. I may be computed in inch units and L in feet, the decimal being adjusted as found convenient, as only relative values of K are involved.

2. From the values of K write at each joint the value of ΣK and at the end of each member the value of $\frac{K}{2\,\Sigma K}$.

$$F\Delta = f_a \frac{a}{r} - f_b \frac{a_b}{r} - f_c \frac{a_c}{r} \qquad a_b = \frac{a^2+b^2+c^2}{2a} \qquad r = \frac{300}{477.9} \times 372 = 233.5 \qquad \frac{a}{r} = \frac{477.9}{233.5} = 2.045 \qquad \frac{a_b}{r} = \frac{300}{377} = 0.808$$

$$\phi_a = \Sigma \frac{K}{2\Sigma K} \cdot 3\psi - \Sigma \frac{K}{2\Sigma K} \cdot \phi_b$$

$\frac{2}{3}\frac{\Sigma K\psi}{\Sigma K}$ Used for first approximation of ϕ_b except joints ⑥ and ⑦. ψ Values begin at Joint ⑦. Solution begins at Joint ①.

$\frac{K}{2\Sigma K}\phi_b$ (rows 1–4); $\sum \frac{K}{2\Sigma K} \cdot 3\psi =$ (row 5); $= \phi_a$ (row 6)

+72.9	+20	+20.4	+ 0.1
+ 11.0	+ 3	+ 3.3	—
+ 1.9	+ 1	+ 1.1	—
+45.0	+16	+15.3	—
+130.8	+40	+40.1	+ 0.1
	+90.8	+90.7	+90.6

+44.5	+13	+12.7	+0.5
+ 0.6	—	+ 0.4	—
+69.9	+23	+22.4	—
+115.0	+36	+35.5	+0.5
	+79.0	+79.5	+79.0

+36.2	-15	-16.9	+0.9
- 0.5	- 3	- 2.7	—
0	+ 1	+ 0.4	—
+ 4.8	+ 3	+ 2.8	—
+12.8	+16	+16.0	-0.1
+53.3	+ 2	- 0.4	+0.8
	+51.3	+53.7	+52.9

-125.0	-39	-43.7	-0.6
- 0.9	—	- 0.4	—
+ 15.7	+ 13	+13.3	-0.2
-110.7	- 26	-30.8	-0.8
	-84.2	-79.4	-78.6

-79.4	-28	-26.8	+1.0
- 3.1	- 2	-2.0	—
-18.6	- 3	- 3.2	+0.1
-130.8	-22	-20.5	+0.2
-231.9	-55	-52.5	+1.3
	-176.9	-179.4	-180.7

Bar 6-7 assumed fixed in direction.

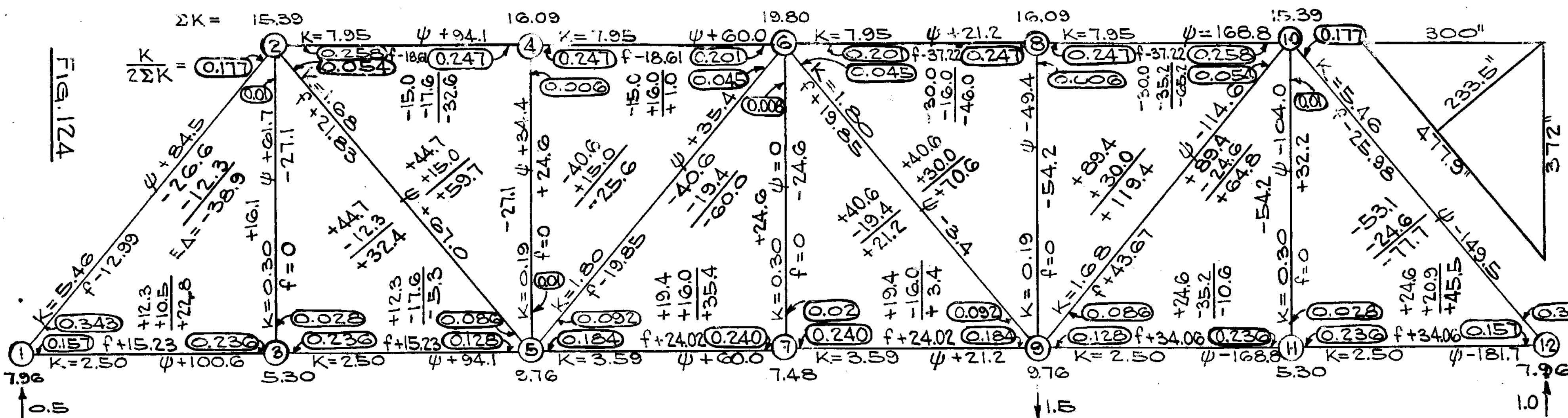


FIG. 124

+87.0	+30	+31.2	—
+47.4	+ 15	+16.6	—
+134.4	+45	+47.8	—
	+89.4	+86.6	+86.6

+71.1	+ 21	+20.5	—
+ 5.2	+ 2	+ 2.5	—
+66.6	+14	+14.1	—
+142.9	+37	+37.1	—
	+105.9	+105.8	+105.8

+36.2	+12	+13.6	—
+ 17.3	+ 8	+ 7.8	—
+ 1.0	+ 1	+ 0.8	—
+9.8	+ 5	+4.9	—
+33.1	+10	+10.4	—
+97.4	+36	+37.5	—
	+61.4	+59.9	+59.9

+43.2	+16	+14.7	-0.3
0	+ 1	+ 1.1	—
+15.3	-14	-13.9	—
+58.5	+ 3	+ 1.9	-0.3
	+55.5	+56.6	+56.9

+11.7	+ 11	+10.2	+0.2
- 0.9	+ 5	+ 4.9	—
- 1.5	- 1	- 0.8	—
-29.6	-15	-15.5	—
-64.9	-26	-26.1	—
-85.2	-26	-27.3	+0.2
	-59.2	-57.9	-58.1

-119.6	- 13	-14.0	+ 0.3
- 8.7	- 5	- 5.0	—
-128.6	-36	-34.4	+0.1
-256.9	-54	-53.4	+0.4
	-202.9	-203.5	-203.9

-85.6	-27	-31.9	-0.1
-153.9	-61	-61.6	-0.4
-239.5	-88	-93.5	-0.5
	-151.5	-146.0	-145.5

$$f = \pm \frac{2Y}{L}(2\phi_a + \phi_b - 3\psi)$$

JOINT 8
- 8-10 0.063(-157.2 - 180.7 + 506.4) = 10.62
- 8-9 0.028(-157.2 - 58.1 + 148.2) = 1.87
- 8-6 0.063(-157.2 + 52.9 - 63.6) = 10.58

JOINT 9
- 9-7 0.50(-116.2 + 56.9 - 63.6) = 6.13
- 9-6 0.31(-116.2 + 52.9 + 10.2) = 1.65
- 9-8 0.28(-116.2 - 78.6 + 148.2) = 1.30
- 9-10 0.31(-116.2 - 180.7 + 343.8) = 1.45
- 9-11 0.50(-116.2 - 203.9 + 506.4) = 9.35

3. Opposite each angle within each triangle sum up the value of the angle change $\Delta\alpha = f_a \frac{a}{r} - f_b \frac{a_b}{r} - f_c \frac{a_c}{r}$. a_b and r may be scaled or computed as previously described.

4. Using the center vertical as a reference, $\psi = 0$ and beginning at Joint 7, the ψ values are written by summing up the angle changes, thus,

7-9	$\psi = 0 + 21.2 = +21.2$
9-6	$\psi = +21.2 - 24.6 = -3.4$
9-8	$\psi = -3.4 - 46.0 = 49.4$, etc.

Checks are readily observed in the process - finally closing at zero at the starting point.

5. At each joint (in the tables in this case) the value of $\frac{K}{2\Sigma K} \cdot 3\psi$ is written for each bar and the summation $\sum \frac{K}{2\Sigma K} \cdot 3\psi$ obtained.

6. The solution of the equations $\phi_a = \sum \frac{K}{2\Sigma K} \cdot 3\psi - \frac{K}{2\Sigma K} \cdot \phi_b$ for each end of each bar by successive approximation is begun by using for the first approximation of ϕ_b the value $2/3 \sum \frac{K}{2\Sigma K} \cdot 3\psi$ except at the center joints (6) and (7) where the full value $\sum \frac{K}{2\Sigma K} \cdot 3\psi$ was used. Any value for ϕ_b dictated by judgment or experience may be used, sometimes hastening the convergence remarkabley, or a strictly mechanical process may be employed using the last full value previously obtained at the joints.

At Joint (1)

1-2	$\frac{k}{2\Sigma k}\phi_b = 2/3 \times 0.343 \times 130.8 = +30$
1-3	$\frac{k}{2\Sigma k}\phi_b = 2/3 \times 0.157 \times 142.9 = \underline{+15}$
	$\sum \frac{k}{2\Sigma k}\phi_b = +45$

$$\phi_a = 134.4 - 45 = +89.4$$

7. The process is continued in the order (2), (4), (6), etc., using the last value of ϕ_a obtained as the trial value of ϕ_b at the next joint.

At Joint (2)

2-4	$\frac{k}{2\Sigma k}\phi_b = 2/3 \times 0.358 \times 0.115 = +20$
2-6	$2/3 \times 0.054 \times 97.4 = +3$
2-3	$2/3 \times 0.01 \times 142.9 = +1$
2-1	$0.117 \times 89.4 = \underline{+16}$
	$\sum \frac{k}{2\Sigma k}\phi_b = +40$

$$\phi_a = 130.8 - 40 = +90.8$$

After the process has been continued the second time for each joint, the magnitude of the changes in the ϕ_a values will be at once apparent, making it possible to decide whether a repetition of the process is worth while. Three trials will usually give results that are more accurate than the underlying physical assumptions, unless variable moment of inertia has been taken into consideration.

Approximate Solutions. A study of the method above indicates that in general the effect at any joint of other joints not connected to this point by a bar are negligible. Thus, it is possible to isolate two or three suspected panels and confine the computations to these only. Studies can be made quite rapidly in this way. Thus, at the support of the continuous truss in Fig. 124a, study of a few panels as shown in full lines would probably tell the story.

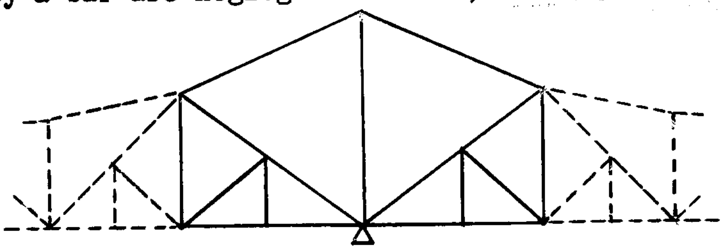


Fig. 124a

Effect of Secondary Stresses on Primary Stresses. In the first solution of secondary stresses to be published, that of Manderla (1879), the effect of the secondary stresses on the primary stresses was included throughout the solution. This effect may be easily included by computing the primaries for a statically determinate structure and from these the secondaries and for these secondaries the changes in the primaries by statics and for these new primaries the secondaries. For ordinary trusses the effect of the secondaries on the primaries is absolutely insignificant. The method, however, is worth noting and can be used to advantage in some problems.

Effect of Pins on Secondary Stresses. In case some bars of a structure are pin-ended while the remainder of the structure is riveted, these pins may be treated as an infinite elastic weight in the bars and the usual methods followed.

This, however, is not quite correct, because the members will not turn freely on the pins, but will turn until the line of pressure falls within the circle of friction for the pin. This circle of friction is normally so small that the moment thus represented is also small.

Sometimes, after the secondary stresses have been computed some are found to be so high that pins are put into some of the members. It is then desired to estimate the effect of such pins on the secondaries in the other members. This may be done by applying at the pin a rotation such as will produce a moment at the pin on a structure without a pin equal to the moment originally computed at the pin. This is done by applying at the pin a unit rotation, figuring the fixed-ended moments in the bar in which the pin occurs and distributing these moments through the truss - preferably directly by moment distribution, since they soon fade out. These distributed moments should then be multiplied by the ratio of the moment found at the pin in the original structure to that just found for unit rotation. These moments are then added to the secondary stresses previously found.

Effect of Ductility of Metal on Secondary Stresses. Conditions Beyond the Elastic Limit. The statement has sometimes been made that secondary stresses are "absorbed by the ductility of the metal." The idea is perhaps that in case the total stress at any point exceeds the elastic limit the plastic deformation will relieve the secondary stress and that therefore a dangerous situation results only when the primary stress exceeds the elastic limit.

It is, of course, true that plastic yield reduces the secondary but it is dangerously misleading to assume that it eliminates it. The phenomena which would accompany stress beyond the yield point may be studied as follows:

Assume that all of the structure except one end of one bar acts elastically, the stress at this end exceeding the elastic limit. Consider the plastic deformation at this end as a localized rotation, or series of rotations, superimposed on a structure all of which acts elastically. We now restrict our study to the effect on the elastic structure of these localized excess deformations.

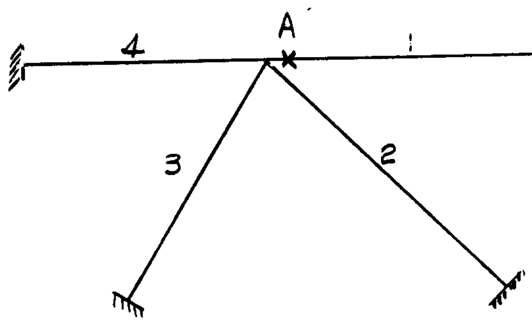


Fig. 125

If we treat all bars as fixed at ends, a rotation at A in bar 1 (Fig. 125) will produce at A a moment $4\phi_a EK_1 \frac{\Sigma K - K_1}{\Sigma K}$

$\phi_a = \frac{f_e}{E_e y}$ ds, where f_e is the excess of the figured stress over the elastic limit.

E_e is slope of the tangent to the stress-strain curve with reference to the initial modulus of elasticity of the steel. Evidently such a quantity is neither very definite nor very accurate, but it may serve our purpose here.

y is the distance from centroidal axis to outer fibre.

ds is the length of bar 1 over which the excessive stress exists.

Evidently if the length ds extends for some distance from the end of the bar, the coefficient 4 will be somewhat modified.

This moment will be a reduction in moment in the structure at A and will produce a fibre stress,

$$f_a = \frac{m_a y}{I_1} = \frac{4\ \phi_a\ EK_1 y}{I_1} \cdot \frac{\Sigma K - K_1}{\Sigma K} = 4\ \frac{f_e}{E_e}\frac{E}{y}\frac{I_1}{L_1}\frac{y}{I_1}\ ds\ \frac{\Sigma K - K_1}{\Sigma K}$$

$$= f_e \frac{E}{E_e} \cdot \frac{4ds}{L_1}\frac{\Sigma K - K_1}{\Sigma K}$$

Or, $\frac{f_a}{f_e} = \frac{E}{E_e}\frac{4ds}{L_1}\ \frac{\Sigma K - K_1}{\Sigma K}$

The reduction in stress produced by the plastic deformation of the steel will

Increase as the rate of plastic deformation increases (E/E_e increases)
Increase as the length of bar overstressed increases ($4ds/L$)
Increase as the relative flexibility of the bar with reference to the joint as a whole increases.

These relations are not unsuspected but their definite statement helps to clear the matter up. The problem is not so simple as this, of course, but may be further modified by the effect on overstressed metal, of metal not overstressed. But in any case it seems evident that the overstress is not "absorbed" - that is, does not disappear. On first exceeding the elastic limit the reduction in stress is very small. Later it becomes larger but the metal is reaching the condition where we are not sure whether stress or strain is the surer measure of strength - a field of thought into which few dare venture with assurance of bringing back facts on which the strength of an important structure may be made to depend.

Secondary Stresses in Cross Frames of Bridge Trusses. Transverse bending may occur in the verticals of truss bridges from,

Flexure of the floor-beams due to loads
Flexure due to unequal deflection of the trusses where,
- They are unequally loaded due to,
  - Only one track being loaded in a double-track bridge.
  - The track not being on center due to curvature.
  - Heavy skew of the span.
- They are equally loaded but not alike due to,
  - Use of a three-truss span the other outside truss being either built or proposed.

Flexure due to some of the wind load along the top chord coming down intermediate verticals to the lower lateral system.

All of these cases except that due to wind are readily figured as continuous beams. It might at first seem that in the unsymmetrical cases, since the cross shears in the posts are unequal, a correction should be made for side-sway - or else a solution should be made by the column analogy. The lateral systems in the planes of the upper and lower chords prevent side-sway, however, except to the small extent that this is permitted by the flexibility of bracing trusses and portals. Evidently the unbalanced shears produce loads on the lateral systems, which in the case of heavy bridges of wide skew may be of appreciable magnitude.

Fixed-ended moments due to loads will be found as usual; those due to differential deflection are $6K\Delta/L$ where Δ is the differential deflection and L the distance center to center of trusses.

The wind stresses carried by intermediate verticals are normally small as may be seen by considering the relative stiffness of the paths along the upper lateral system and portals and along the intermediate verticals and lower lateral system. This theory of relative stiffness of stress paths - its meaninglessness except by analogy and its real practical value as an analogy - is discussed elsewhere.

Significance of Secondary Stresses. Whatever is done, the exact computation of secondary stresses will remain a strenuous pursuit and at the end of the chase one faces the problem of what to do with the quarry. The cases will probably not be numerous in which forcible measures of elimination are justified. In other cases, evaluation gives some idea of possible points of weakness. High secondary stresses may then be reduced by revising the make-up of the members or by such devices as adjusting hangers and idle struts. It does not seem practicable at present to effect a better distribution of metal by designing for the total stress (using generally increased basic stresses) rather than for the primary stress. The following uncertainties bear on the question:

(a) Secondary stresses in tension members are probably less serious than in compression members;

(b) Secondary stresses in compression members are less serious when there is double flexure (with point of contraflexure in the member) than otherwise and they may even be beneficial in such cases in that they retard buckling;

(c) Secondary stresses are often less dangerous than primary stresses because of their localization, this being especially true when secondary moments in two planes are combined;

(d) Present knowledge of impact effects on secondary stresses is much less than in regard to primary impact, inadequate as are the data on the latter; perhaps secondary stresses tend to dampen vibrations and an unformulated knowledge of this supports the structural engineer's preference for "stiff" structures in spite of their secondary stresses.

These statements, however, should not be taken to indicate any doubt as to the growing importance of a thorough understanding by bridge engineers of the nature, approximate theoretical value, and general theory of secondary stresses in bridge trusses; but the same importance attaches to the study of secondary stresses due to the deformation in the frames of steel buildings and in other cases. There is, at present, a decided swing toward more elaborate analyses in both steel and concrete structures; the requirement in the new specifications for concrete of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete* that bending stresses shall be computed in the columns of concrete buildings, even in the face of uncertainties more pronounced than those outlined herein, is an instance in point. All this is very well as long as it is understood that such analyses merely indicate danger points. Elaborate analyses alone, however, will not justify a general increase in fibre stress; the relation of secondary stresses to the factor of safety is a much more intricate problem than their mathematical formularization or evaluation.

Comparisons of Secondary Stresses. It is usually dangerous to reason in general terms as to the relative values of secondary stresses. If at any joint of a chord we determine the deflection relative to two adjacent panel points, we may distribute the unbalanced moments caused by this deflection if we assume values of c for the connecting member -- that is, if we make assumptions as to the moments at the other ends of the members. This neglects any moments due to ψ values of the web members. Often it can be seen that the rotation of the web members is not a very important element, but estimates as to moments at the ends of main members may be very misleading.

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*Proceedings, Am. Soc. C.E., October, 1924, Papers and Discussions, pp. 1204-1206.

Nevertheless, it is possible from general considerations, to see the cause of secondary stresses and to form some judgment as to when they are to be feared.

Consider the Warren truss shown in Fig. 126(a). The relative deflection of A with reference to BB' may be found by virtual work by applying at A a unit load with reactions at B and B'. The deflection is relatively great and may cause high secondaries at A The same is evidently true of the deflection of point c with reference to points D and E. Evidently hangers and idle struts may cause heavy secondaries.

We have shown that the secondaries are functions of the ψ values and also of d/L. Hence, the depth of the member relative to its length is a clear indication that secondaries may be high. Ordinarily, where the depth is greater than ten per cent of the length, the secondaries should be studied.

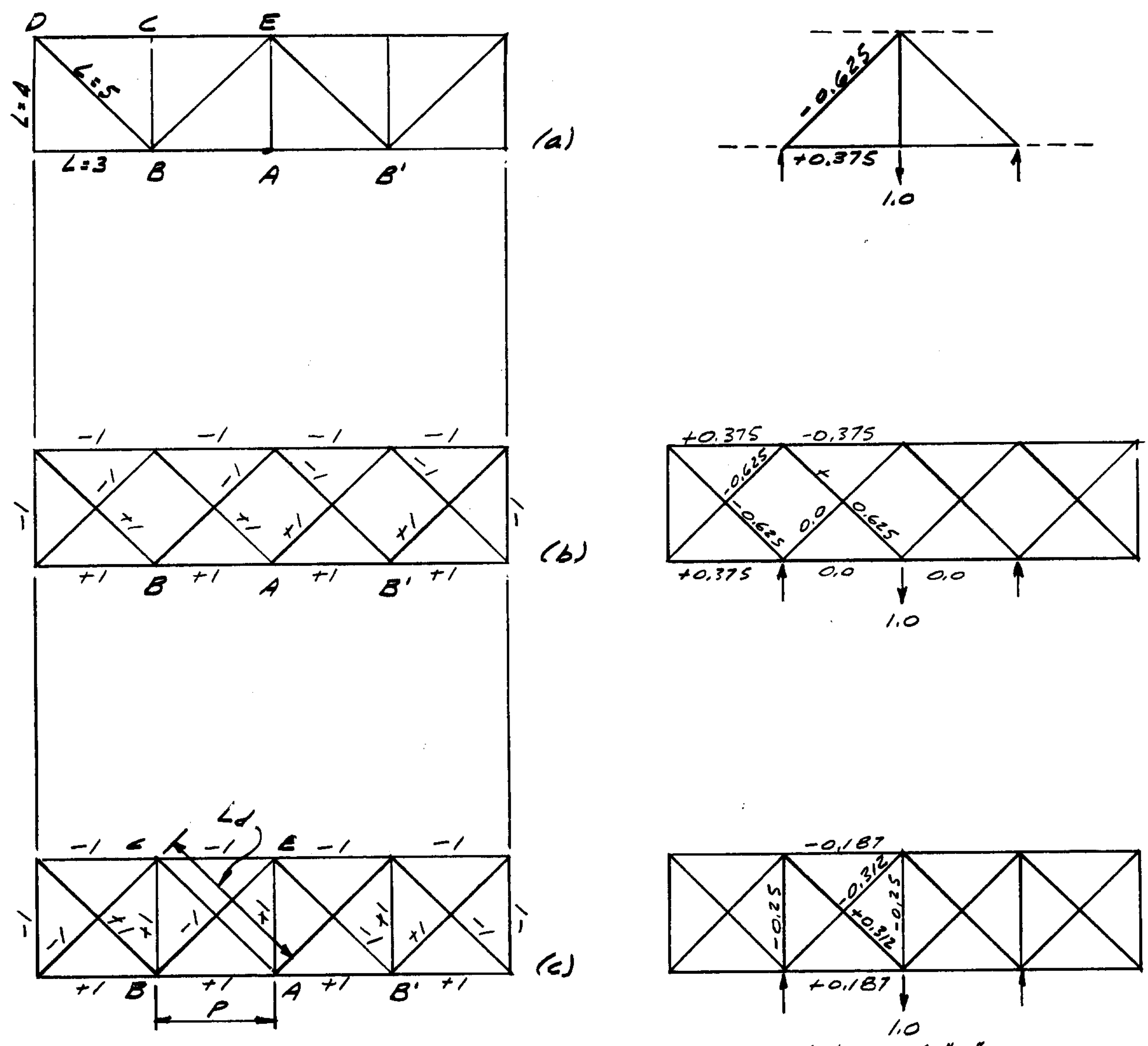


FIG. 126

Compare the three types of truss shown in Fig. 126. Assume full loading and the same fibre stress in all members. Thinking in terms of the equation of virtual work, it will be seen that the relative deflection is the same for trusses (a) and (b) except for the effect of the hanger in (a) and the end post in (b). In (a) the effect of the hanger is to be added and in (b) that of the post is to be subtracted. The deflection in (b) is less, then, than in (a).

Truss (c) is four-fold indeterminate. The u values, however, would be approximately as shown and it is evident that the relative deflection is about the same as for (b).

Assuming f = 1 and lengths as shown, we have,

$$\text{Truss (a)}\quad -\Delta_A = \begin{aligned} +0.375 \times 6 &= +2.25 \\ +1.0 \times 4 &= +4.00 \\ +0.625 \times 10 &= \underline{+6.25} \\ &\ +12.50 \end{aligned}$$

$$\text{Truss (b)}\quad -\Delta_A = \begin{aligned} +0.375 \times 6 &= +2.25 \\ +0.625 \times 10 &= +6.25 \\ -0.5 \times 8 &= \underline{-4.00} \\ &\ +4.50 \end{aligned}$$

$$\text{Truss (c)}\quad -\Delta_A = \begin{aligned} +0.375 \times 6 &= +2.25 \\ +0.625 \times 10 &= +6.25 \\ -0.25 \times 12 &= \underline{-3.00} \\ &\ +5.50 \end{aligned}$$

The stresses in the diagonals near the center for full loading will actually be small, perhaps not over 1/2, but this will not materially change the relative values. Even if the sag of the hanger is taken out of (a), it will still indicate high secondaries.

Assume zero stress in the web, and that the stresses in the chords are of like sign, as would occur in the bracing truss of a bridge due to live load. Trusses (a) and (b) now give the same values for Δ and truss (c) apparently gives zero. Actually there must be stress in the diagonals of truss (c) as is shown below but the effects in the two diagonals will be of opposite sign.

Suppose the stresses in the chords to be unlike in sign, but no stress in the diagonals. This would be the result of uniform moment and does not alone represent a very probable case. We now have the same total deflections in all three trusses.

Such considerations as these are not conclusive, but they indicate general effects. Hangers and idlers, single bracing or double bracing without cross-struts are or may be objectionable. For an interesting discussion of these and other matters, see Johnson Bryan and Turneaure, Vol. II.

Truss (c) above is the common form of lateral bracing in trusses. It has been shown to be relatively free from secondary stresses due to bending. But the diagonals are evidently subject to stress due to the indetermination of the structure. If chords AB and CE stretch and the verticals CB and EA are relatively stiff, then diagonals CA and EB must stretch also.

The stretch of AB will be $f_c p/E$ and the horizontal stretch of AC will be, $f_d/E \cdot L_d \cdot L_d/p$. Since these are equal, $f_d/f_c = p^2/L_d^2$. Hence the unit stresses in laterals due to live loads on the truss are to those in the chords inversely in the ratio of their lengths. This indicates a live load tension in lower laterals of about 12,000 lb. per sq. in. and a live load compression in upper laterals of about 10,000 lb. per sq. in. The whole subject of secondary effects of lateral systems will repay further study.

Influence Lines for Secondary Stresses. Only a few studies of influence lines for total stress -- primary plus secondary -- have been published. Most notable are those made by Dr. Turneaure for the American Railway Engineering Association in connection with his studies on impact. See A.R.E.A. Bulletin 125, 1910.

Influence lines for a Pratt truss are given in Modern Framed Structures, Volume II, Johnson, Bryan and Turneaure, p 454, and for a small Warren truss in Trans. A S.C.E. Vol. 89 (1926) pp. 116-118. Such studies for a cantilever arch are reported in Trans. A S C.E. Vol. 82 (1918) pp. 1101-1103 by Professor Jacoby.

A valuable study along this line is that of M. N. Quade (Master's thesis, Univ. of Illinois, 1926) on secondary stresses in two types of two-hinged steel arches designed for single-track E-60 loading. These are shown in Fig. 127. Quoting from this thesis:

"To summarize briefly the entire investigation, it may be said that it has included a complete analytical investigation of two steel arch bridges for the purpose of determining secondary stresses in the members of the trusses. To accomplish this, influence lines for primary fibre stress, secondary fibre stress and maximum combined primary and secondary fibre stresses were drawn for each member of each arch. The three important conclusions that may be drawn from the investigation of these two arches are:

"(a) Load Divides - The points of load divide for maximum combined primary and secondary fibre stresses are almost identical with those for primary stress.

"(b) Magnitude of Secondary Stress - The critical members for secondary stress in Arch 400-A are U_0U_1, U_1U_2, U_4U_5, L_3L_4, L_4L_5, M_3M_4, and all verticals. The percentage of secondary stress in terms of the primary stress varies from 22 to 56 per cent.

"The critical members in Arch 400-B are U_4U_5, L_4L_5, U_3U_3, U_4L_4, and U_5L_5. The percentages vary from 28 to 64 per cent.

"(c) Loading - To obtain the secondary stresses acting simultaneously with the primary stresses for maximum combined stress the following loads may be used:

"Arch 400-A - Members U_0U_1, U_1U_2, U_0M_2, U_1M_4, U_2L_2, and M_3M_4; Load panel points 0, 1, 2, 3, 4. Members U_4U_5, L_3L_4, L_4L_5, U_3L_3, U_4L_4, and U_5L_5 load all panel points to the right of and including panel 3.

"Arch 400-B - All members -- load panel points to the right of and including panel 3.

"Conclusions (b) and (c) are shown diagramatically in Fig. 127.

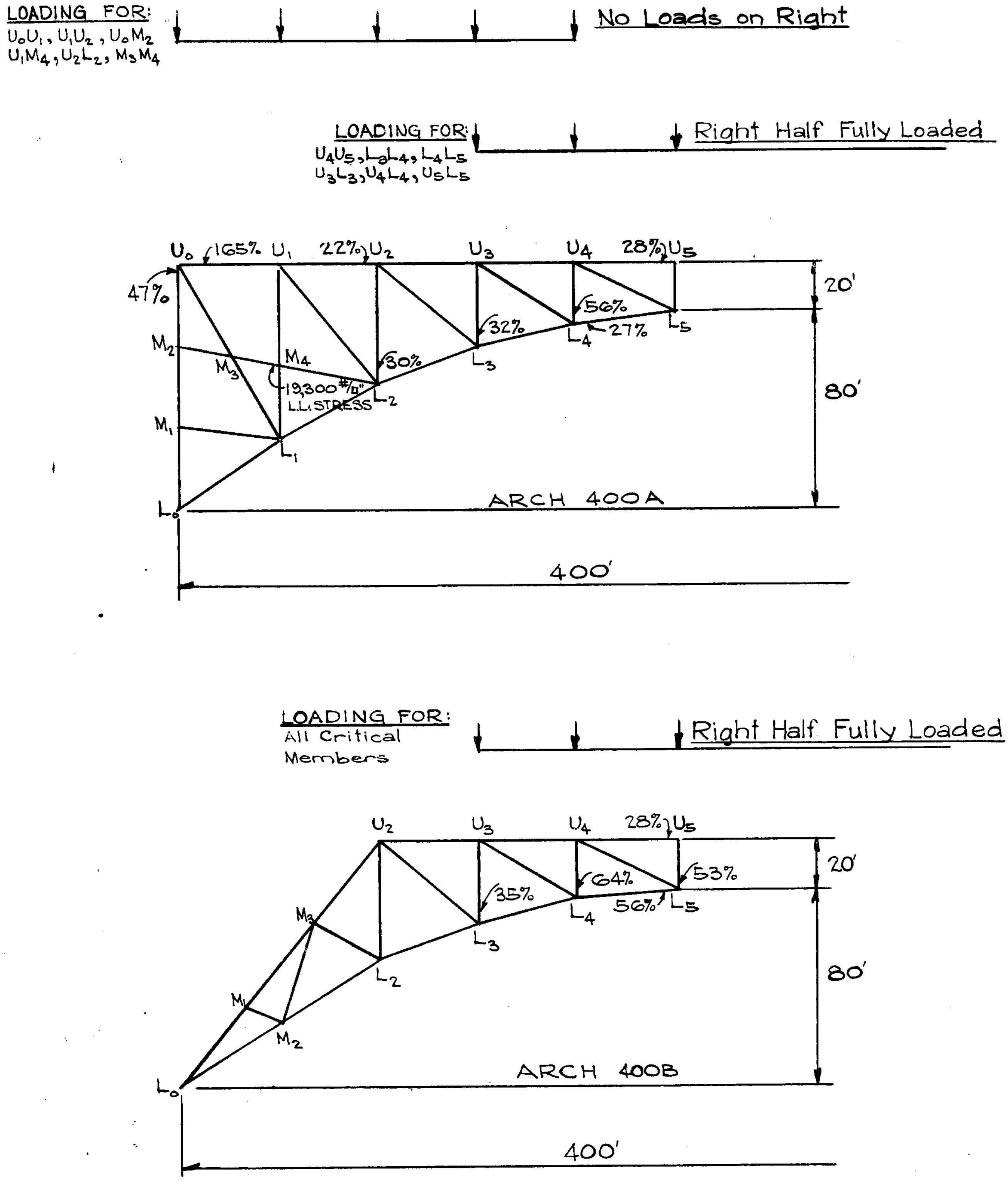


FIG. 127

It is interesting to note that the critical loading for the members near the crown is approximately 5/8 loading on the span. The high secondaries found at the end of Arch 400-A are not very significant since the top chord here almost invariably has a considerable excess of material to provide for erection stresses. The high indicated live load stress in the bracing is, however, of considerable interest.

It is almost impossible to generalize about secondary stresses, but we may note that usually loading for maximum primary will give a good idea of the combined maximum.

Effect of Gussets. It has been shown that the effect of gussets can be readily included in computations for secondary stress if we can make some assumption as to the effect of the gusset in increasing the area and moment of inertia of each section. Important studies along this line were made by Nelson (Precision in Secondary Stress Computations, G. V. Nelson, Master's Thesis, University of Illinois, 1926). He analyzed a small Pratt truss on each of the following assumptions: that the gusset affected neither the area nor the moment of inertia of the member (the usual assumption); that the moment of inertia of the gusset was zero and its area equal to that of the member; that the reciprocal of the moment of inertia of member and gusset varied uniformly from a value equal to that of the member at the edge of the gusset to zero at the intersection of members and that the area of the gusset was successively zero and equal to that of the member; and finally, that the gusset was infinitely stiff and its area successively equal to zero and equal to that of the member.

In this case the gussets were taken large (length of gusset 10 per cent of that of the member) but not uncommon. The results of the six solutions showed variations from 90 per cent to 170 per cent of the standard results, the most probable values varying ± 10 per cent from standard.

Studies were also made using several different assumptions as to the effect of gussets on arch 400-B (Fig. 127) showing variations as high as 40 per cent from the standard solution.

The statement has been made that approximate allowance for the effect of gussets may be made by solving by the standard method and multiplying the stresses thus found by the ratio of the center-to-center length to the member.* Nelson's studies show no basis whatever for this conclusion. Apparently the best way to include the effect of gussets is to base the computations on assumptions as to the action of the gussets. The results may differ ± 20 per cent from those obtained by the usual analysis.

*See Secondary Stresses in Bridges -- O. H Ammann, Eng. News-Record, Oct. 23, 1924.

CHAPTER X
TWO-HINGED STEEL ARCHES

The Nature of the Problem. The two-hinged arch is indeterminate in the first degree. The indeterminate problem is not a complicated one but consists essentially in developing a system, a consideration of great importance in all types of statically indeterminate structures and especially in the larger ones.

In problems of this kind where there is only one indeterminate element, the influence line for H, the best method is to combine the influence line for the indeterminate element with that for the determinate element. By using multiplication factors, a single construction for the indeterminate element will serve for all bars in the structure.

The H-Influence Line - Method of Angle Weights. An exact analysis of stresses due to live load requires the influence line for H. This, in turn, involves the sections of the members and it is therefore necessary to use an approximate value for H in preliminary design. With the properties of the members known the H-influence line is constructed as the displaced load line due to a unit horizontal displacement of the hinge. If the angle changes due to a unit displacement at the hinge are treated as loads at the load points on the structure as a simple beam, the moment curve is the influence line for H.

It should be noted that this gives deflections relative to the tops of the end posts. Since the end posts are in tension and lengthen, all ordinates must be corrected as shown, Fig. 127a.

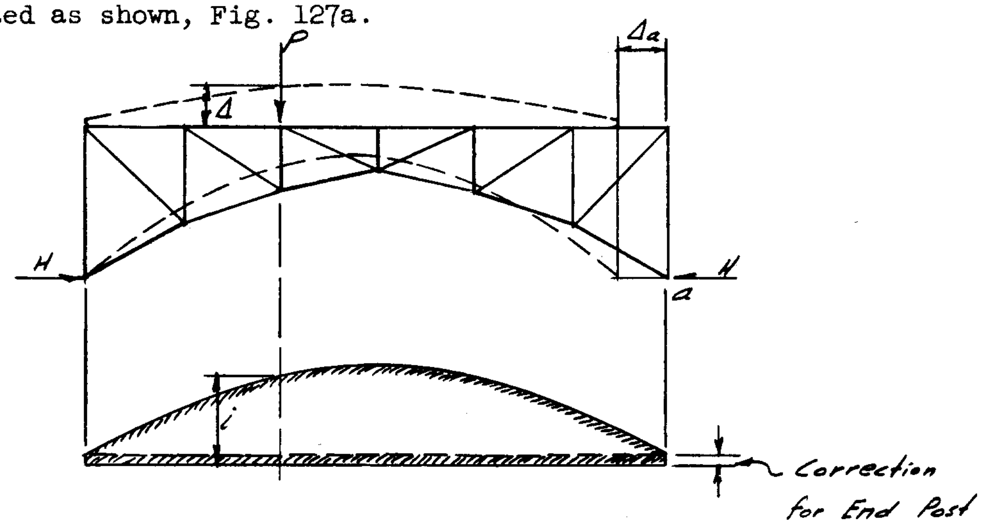


Fig. 127a

For convenience the angle changes along the load line are best found for H = 1 and divided by the value of horizontal deflection due to unit H. In other words, the influence ordinate is the ratio of vertical deflection at the panel point to horizontal deflection at the hinge, both due to H = 1.

Horizontal deflection due to unit H = $\sum \frac{u^2 L}{A}$ if E is taken as unity.

Also, it is evident from the geometry of the figure that it equals the statical moment about the line of hinges of the angle changes already computed. Since these angle changes are known this is more convenient.

The angle changes along the load line may be found either by finding the changes in each angle meeting at any joint, as in secondary stresses, or directly by applying virtual unit moments at the joint as explained in Chapter III. By this method, where the load points do not coincide with the panel points the angle changes may still be computed at the load points. Otherwise, there will be a correction for the deformations in the verticals. In all cases we are interested in the displacement of the load line and it is most convenient, in principle at least, to work with it.

Application of the Williot Diagram. For ordinary arches the quickest method of evaluating the H-influence ordinates is by means of a Williot diagram. If the unit stresses are found due to H = 1 and the deformations (f L) plotted (taking E = 1), a single diagram will give the ratio of the displacements Δ of the load points in the direction of the loads, to the total displacement Δ_a of the hinge. For the usual symmetrical arch it will only be necessary to work from the ℄ to one end - doubling the total displacement of the hinge thus found. Since one diagram gives the effect of both horizontal and vertical loads, its convenience is apparent.

For very long bridges it is difficult to draw a displacement diagram with accuracy. In all important cases it is desirable to have an algebraic check on at least the center ordinate. This can be readily got by virtual work as follows:

Unit Load Method. The general relation,

$$i = \frac{\Delta}{\Delta_a} \text{ due to unit H, may be written,}$$

$$i = \frac{\sum \frac{SuL}{A}}{\sum \frac{u^2L}{A}} \text{ where,}$$

S - stress, statically determined due to a unit load P = 1.
u - stress due to a unit horizontal reaction H = 1.

This relation may be evaluated by applying unit loads at each panel point in turn - a somewhat tedious process as it involves a different set of S calculations for each panel point loaded. For symmetrical arches a saving can be effected by loading symmetrically opposite panel points simultaneously.

Elastic Weights. The influence line for H has been defined as the displacement of the load line due to a unit displacement in the direction of H - that is the moment curve due to the angle changes or $\frac{mds}{EI}$ values as loads. These angle changes are found most conveniently at the load points as previously explained. They may, however, be treated as separate elastic loads for each member acting at the center of moments for that member. They will be computed in this case as $m\frac{ds}{I} = m\frac{L}{Ar^2}$.

The method has already been discussed. It is evidently tedious and inconvenient where web members are considered. It is, however, interesting in principle and indicates an available method in the rare case of framed hingeless arches.

Preliminary Design Considerations. One of the first problems in arch design is, therefore, to get an arch to investigate. The sections should be predetermined with at least reasonable accuracy if much repetition of work is to be avoided. Omission of the web members in preliminary design is not a very close approximation. In the case of beams the error from neglect of web members is small, probably less than 5 per cent; for trusses it may be several times that amount. To assume the area of all sections constant or A = 1, is also a poor approximation.

If in a spandrel-braced arch the bottom chord is parabolic, as it usually is, and if we neglect the effect of direct compression in this chord, then least work indicates that only the lower chord will be stressed for full live load. The lower chord is in reality a structure in unstable equilibrium and the chief function of the web and upper chord is to stay and stiffen it under unbalanced loading. Similarly, the suspension bridge, which is fundamentally an inverted arch, needs a stiffening truss under unbalanced loading but not under full uniform load.

Preliminary Design - H-Influence Line. For preliminary design we are concerned first with the approximate influence line for H. As explained above, it is approximately true that the lower chord alone is stressed under full load if its shape is parabolic. Hence, approximately,

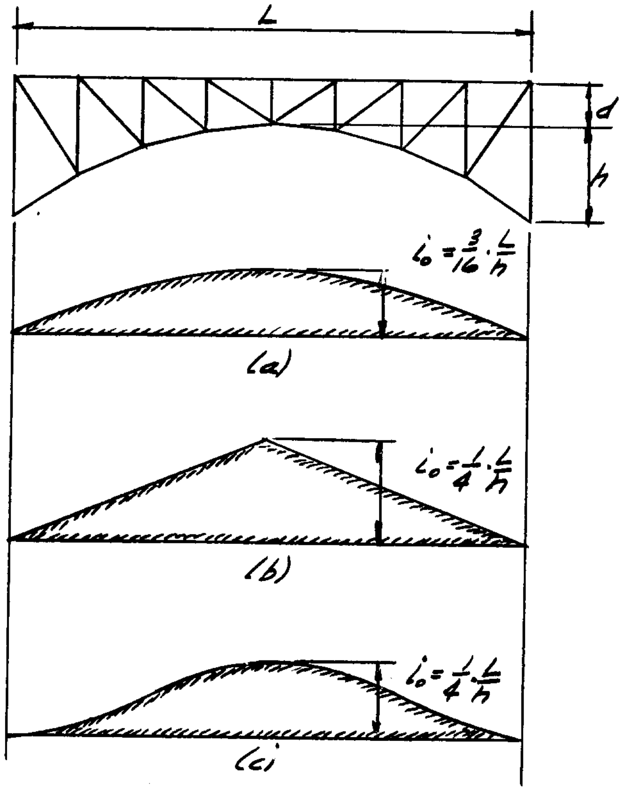


Fig. 128

$$H_{max} = M/y = \frac{1/8 \; w \; L^2}{h} = \frac{wL^2}{8h} .$$

If the H influence line is assumed to be a parabola, the total thrust due to full uniform load (equals the area under the curve, Fig. 128) is for w = 1

$$\frac{2}{3} i_o L = \frac{L^2}{8h} \qquad i_o = \frac{3}{16}\frac{L}{h} .$$

For a 3-hinged arch (and many 2-hinged arches will be erected with temporary center hinge),

$$\frac{1}{2} i_o L = \frac{L^2}{8h} \qquad i_o = \frac{1}{4}\frac{L}{h} .$$

For the hingeless arch, likewise, the mid ordinate has been shown to be approximately, $i_o = \frac{1}{4}\frac{L}{h}$.

The mw curve (mL/Ar^2) forms the loading for the H-influence line. If this is constant or uniform throughout, the H line will be a parabola - the ordinates varying as $y = ax^2$. If the mw loading is relatively heavy in the center and light at the ends (approximating the 3-hinge condition), the H-line, which is the moment diagram for this loading, will approach a triangle, or $y = ax$. For the usual spandrel-braced arch the H-curve will lie between these two extremes, say a semi-cubic parabola, $y = ax^{3/2}$ - the loading for this condition being about as shown in Fig. 129. The area of the curve is,

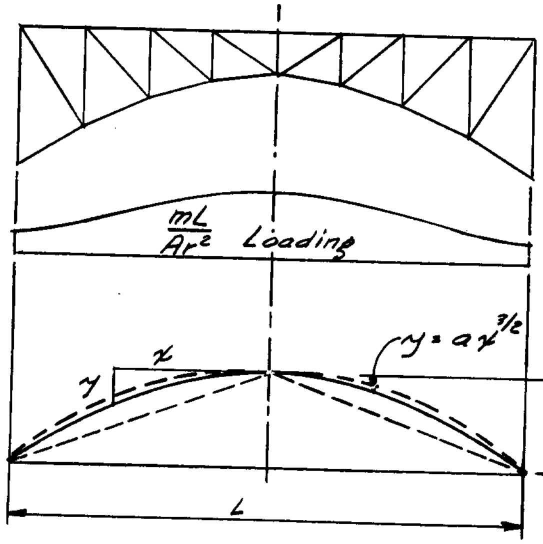


Fig. 129

$$i_oL - 2\int_o^{L/2} ax^{3/2} = 3/5\ i_oL$$

Equating this to the uniform load,

$$\frac{3}{5}\ i_oL = \frac{1}{8}\frac{L^2}{h} \qquad i_o = \frac{5}{24}\frac{L}{h}$$

As there is actually some stress in the upper chord from full load, this value should be reduced slightly. Comparison of a number of spandrel-braced two-hinged arches under a wide range of conditions, indicates that the semi-cubic parabola with center ordinate $\frac{1}{5}\frac{L}{h}$ is remarkable close approximation. This is verified in the table shown in Fig. 130.

Using this value the thrust for full uniform load, w per ft. is found as w times the area under the curve,

$$H_{max} = \frac{3}{5}\ i_o\ Lw = \frac{3}{5} \cdot \frac{1}{5}\frac{L}{h} \cdot L.2 = \frac{3}{25}\frac{L^2}{h}\ w$$

Temperature Stresses. A fairly close approximation for temperature stresses follows by assuming the arch to act as a ribbed arch in which the moment of inertia of the cross-section varies as the secant of the slope of the rib axis - that is, - $I = I_c \sec \phi$ or $\frac{ds}{I} = \frac{dx}{I_c}$. This is equivalent to assuming that the elastic weight is uniform per running foot of span.

H_t due to temperature will be such as to prevent the displacement (etL) which would occur if one end were free to move. The displacement due to a unit H is equal to the internal work done in the girder which is equal to twice the static moment of the parabolic mw diagram about the base, Fig. 131.

Curve of moment due to H = 1
mw-curve for constant w.

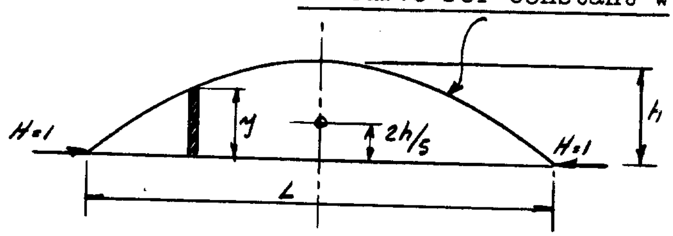
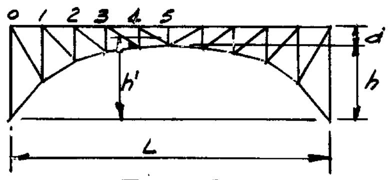


Fig. 131

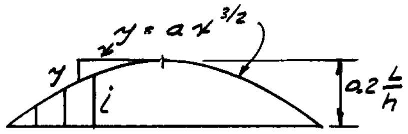
$$\Delta_{H=1} = \frac{1}{EI_c}\left(2 \cdot 2/3\ hL \cdot 2/5\ h\right) = 8/15\ \frac{Lh^2}{EI_c}$$

$$H_t = \frac{etL}{\Delta_{H=1}} = 15/8\ \frac{eEtI_c}{h^2}$$

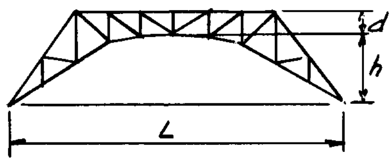
TABLE OF COMPARISONS	ACTUAL AND APPROXIMATE VALUES	$\Delta H_H = 1$ H-INFLUENCE ORDINATES and Ht



TYPE A



APPROXIMATE H-CURVE



TYPE B

Arch		Hell Gate	Niagara-M.C.R.R.	Type A	Type B	Sapinero Colo. Type A
Span		977.5'	640'	400'	400'	180'
Rise		220'	105'	80'	80'	40'
i_0	Actual	.004	.007	.016		.013
	Approx.	0	0	0		0
i_1	Actual	.111	.212	.272		.246
	Approx.	.113	.220	.284		.257
i_2	Actual	.213	.406	.507	.495	.460
	Approx.	.221	.427	.527	.550	.482
i_3	Actual	.311	.596	.721	.691	.652
	Approx.	.324	.618	.751	.760	.672
i_4	Actual	.410	.777	.889	.836	.810
	Approx.	.421	.788	.925	.920	.820
i_5	Actual	.502	.941	.960	.908	.885
	Approx.	.511	.940	1.000	1.000	.900
i_6	Actual	.588	1.082			
	Approx.	.594	1.068			
i_7	Actual	.665	1.186			
	Approx.	.672	1.164			
i_8	Actual	.730	1.230			
	Approx.	.738	1.220			
i_9	Actual	.783				
	Approx.	.799				
i_{10}	Actual	.820				
	Approx.	.846				
i_{11}	Actual	.841				
	Approx.	.880				
H $\Delta H=1$	$\Sigma \frac{u^2 L}{AE}$	$\frac{784}{E}$	$\frac{612}{E}$	$\frac{1190}{E}$		$\frac{3000}{E}$
	$\frac{16}{15}\frac{L}{EA_c}\left[\frac{h'}{d}\right]^2$	$\frac{718}{E}$	$\frac{692}{E}$	$\frac{1080}{E}$		$\frac{2750}{E}$
H_t	Actual	215500 lb.	165000 lb.			10100 lb.
	$170A_c+\left[\frac{d}{h'}\right]^2$	214000 lb.	146000 lb.			10700 lb.

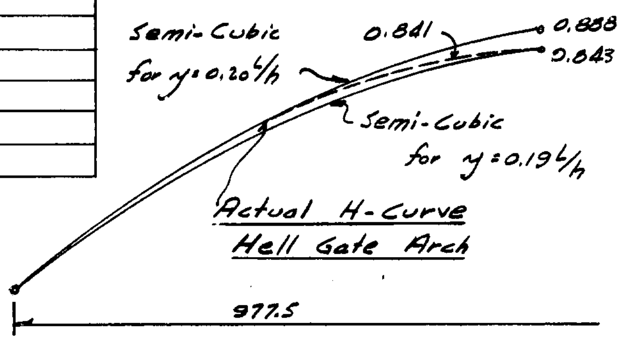


Fig. 130

Taking $eE = 180$ and I_c at the crown as $A_{c(av.)} \times d^2/Z$ we have,

$H_t = 170\ A_c\ t(d/h')^2 = 10{,}000 A_c\ (d/h')^2$ (approximately) where h' is the rise to mid-point between chords and the temperature change is 60 deg.

The table in Fig. 130 shows that the results obtained by this equation are fairly close to those resulting from direct computation.

While more direct values are readily obtained from the data of the final solution it is to be remembered that any results must be subject to a liberal interpretation. There may be a 10 or 20 per cent difference in temperature between the more exposed and the less exposed portions of the steel rib - particularly the top and bottom chord.

Combined Influence Lines. The influence ordinates for stress in any member may be plotted as the difference between the influence line for the effect of the indeterminate element H and the influence line for stress in this bar as a member of a simply supported truss. So long as the load is on one side of the section, the determinate element is the reaction on the other side. Then,

$$S = M/r = \frac{Vx - Hy}{r} = y/r\ (\frac{Vx}{y} - H)$$

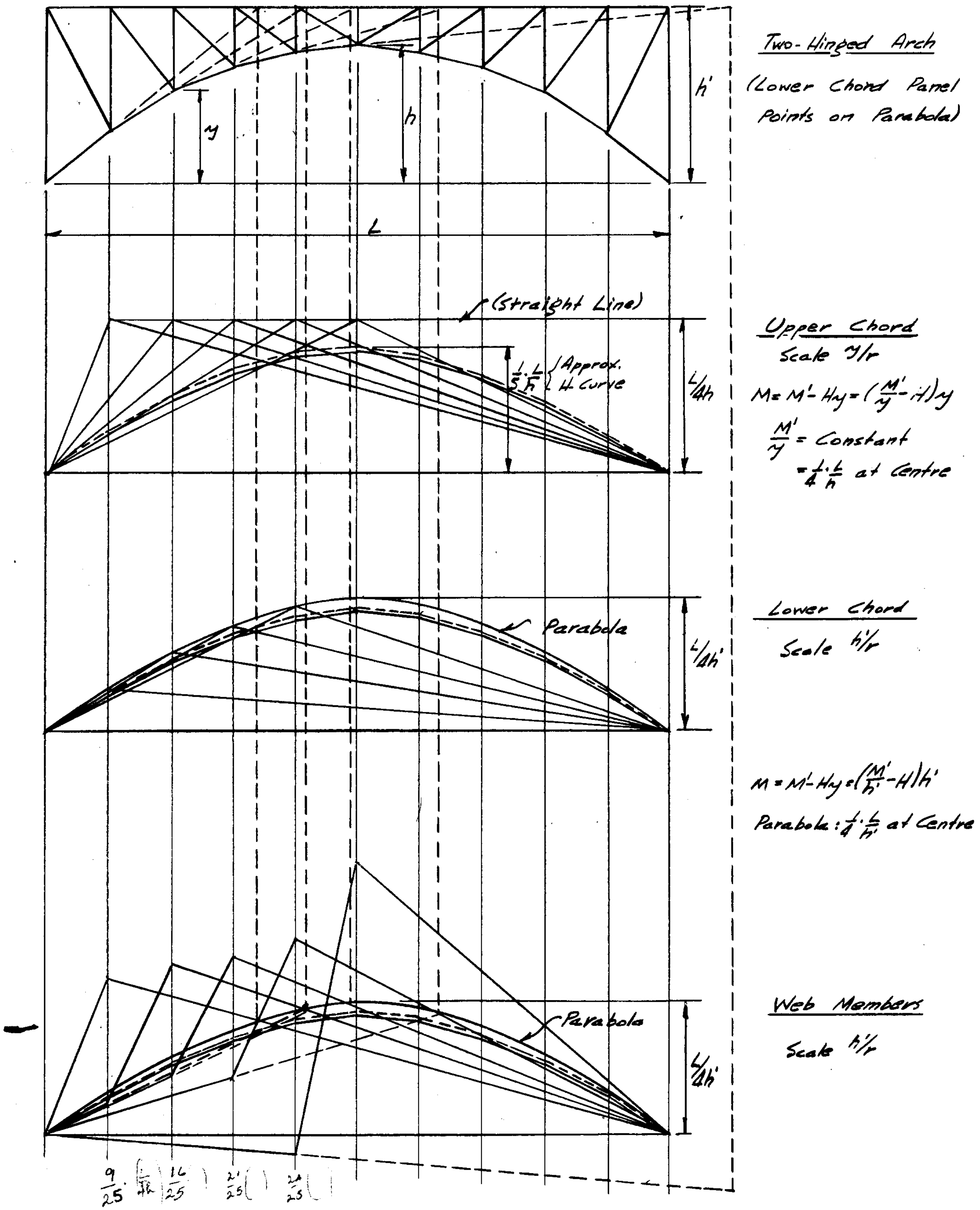
where x, y and r are the lever arms to the center of moments of V, H and the member itself. If the ordinates, Vx/y are plotted directly on the H-curve, a single diagram is sufficient but the multiplier y/r will vary for each member.

The determinate influence line, then, will consist of two parts representing the effect of the reactions, one for loads on the right of the section, the other for loads on the left. These lines will intersect below the moment centers of the bars. They may be constructed readily from the end intercepts (x/y), which are the values of Vx/y when V = 1. For the chord members these diagrams will be simple triangles with vertex under the center of moments. For the web members this same triangle abc forms the basis of the construction as indicated in Fig. 132. The slopes of the two sides will break in the panel in which the section cutting the member is taken, the variation being obviously a straight line de.

Typical influence lines for a common type of spandrel-braced, two-hinged arch are shown in Fig. 132a. The case illustrated is that of a small highway bridge of 180 ft. span at Sapinero, Colorado. The approximate H-curve, based on the semi-cubic parabola with mid-ordinate $1/5\ \frac{L}{h}$, is shown dotted. The actual H-curve is shown full. The ordinates for both curves are given in Fig. 130.

It should be noted that the influence areas for the final H-curve may be applied as corrections to the preliminary values without further construction.

The figure shows a few of the many simple geometrical relations that may be employed in such constructions. For the arch with lower chord panel points on a parabola, the ordinates at the moment centers for the upper



Influence Lines For Two-Hinged Arch

Preliminary H-Curve Shown Dotted

(Semi-Cubic Parabola - Centre Ordinate : $\frac{1}{5}\cdot\frac{L}{h}$)

Final H-Curve Shown in full

FIG. 132a

chord members are constant and equal to $\frac{1}{4}\frac{L}{h}$. For the lower chord and web members these ordinates fall along a parabolic arc having a mid-ordinate $\frac{1}{4}\frac{L}{h'}$. Such relations are of value when they are simple, otherwise they may be cumbersome and less useful than the more direct construction previously indicated.

Final Analysis. With the preliminary H-line drawn as a semi-cubic parabola, (mid-ordinate $\frac{1}{5}\frac{L}{h}$) the stresses and sections may be determined with provision for temperature stresses as explained. Wind stresses - and secondary stress investigations, if made - will generally follow the so-called final analysis. Erection stresses may be predetermined to a large extent. For cantilever erection these stresses are usually critical in the case of the top chord. The layout of typical panel points and other critical details (such as floorbeam connections) should be made in advance of the final analysis as such factors will be intimately related with the choice of sections. In the case of light structures, framing details and facility and economy in fabrication and erection may be more important in design than unit stress.

Provision for Wind Stresses. The upper and lower lateral systems, acting with the intermediate sway bracing form a redundant system. The distribution of wind between upper and lower lateral systems will be governed by the relative rigidity of the two principal paths of travel. In general it is better to use a light system of top laterals, and carry the wind chiefly through the sway bracing and bottom laterals. Where concrete deck slabs are used in the floor, rigidity in the plane of the floor (usually the upper chord) is assured and perhaps an equal division between the two systems is a satisfactory distribution. Any elaborate analysis of the problem is clearly not justified. Wind acting with dead load alone will not usually cause concern and when combined with live load, impact and temperature, the probability factor becomes important.

Wind Stresses - Vertical Trusses. Horizontal Top Chords. With the division of load between the two resisting systems assumed, the top laterals may be designed as a simple truss for the loads w', assuming that the tension members carry all the stress, in the case of a double web system, see Fig. 133. The top chords of the arches will act as chords of the lateral truss and bracing in the plane of the end posts will carry the reaction of this truss to the abutments.

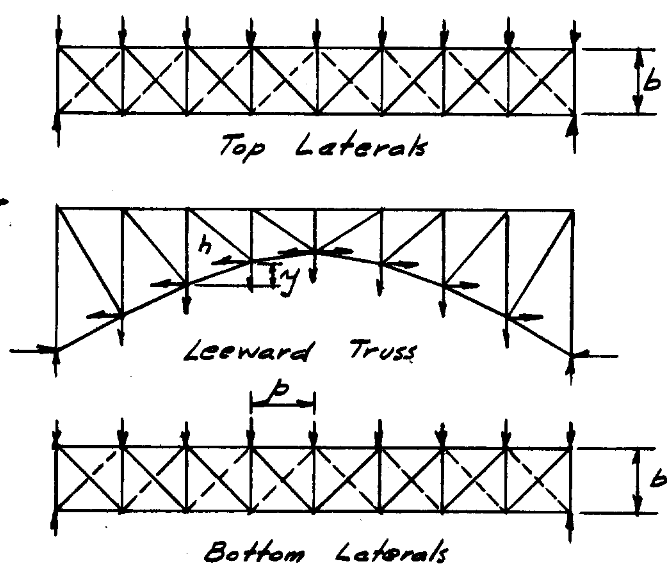


Fig. 133

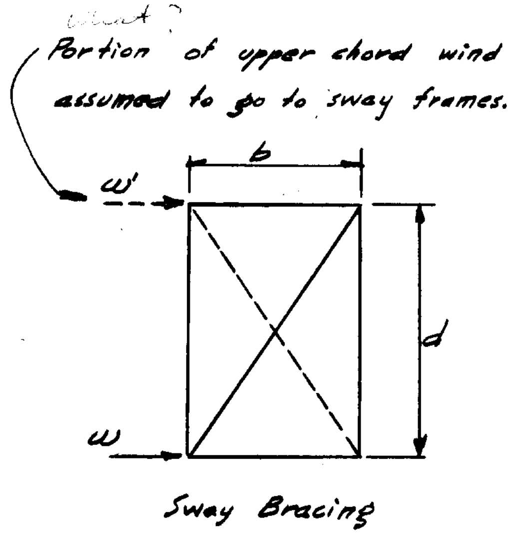
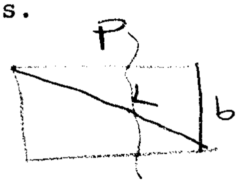


Fig. 134

The horizontal load at any panel point of the lower chord equals the load from wind on this chord plus that part of the wind on the upper chord assumed to come down the sway frame. Since the lower laterals lie in an inclined plane, vertical and horizontal components in a vertical plane are produced at each joint. These produce stresses in the main truss. On the leeward truss there is in addition a vertical force w'd/b (See Fig. 134). On this truss the horizontal forces evidently act outward toward the abutments.

If X is the shear due to the forces w in any lower chord panel, of length p, the stress in any tension diagonal, of length L, will be

$$S = L/b\ X$$ where b is the width between trusses.



If y is the dop in the panel,

$$h = p/b\ X$$

$$V = y/b\ X$$

The stresses in the lower laterals may be found by considering the horizontal projection of the truss system and increasing the stresses by the ratio of the actual to the projected length of any diagonal.

The wind stresses in the main truss may now be obtained by previously outlined methods.

<u>Approximate</u> <u>H</u> <u>for</u> <u>Horizontal</u> <u>Loads</u>. Assuming a parabolic arch rib with I variation according to the secant of the slope of the axis, we have, as before, the relation, that the horizontal displacement at the free hinge is equal to twice the static moment of the parabolic mw diagram about the base.

That is, $\Delta_{H=1} = 8/15\ \frac{L y_c^2}{I_c}$

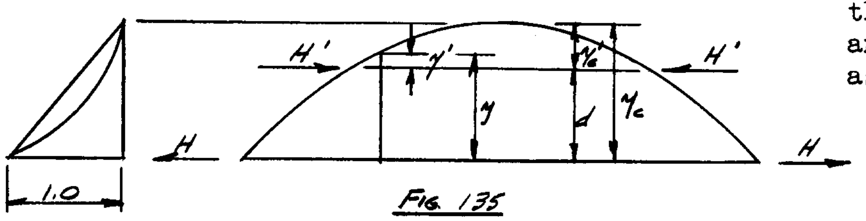


Fig. 135

Referring to Fig. 135, the relation between H and H' may be established as follows:-

$$H/H' = \frac{\int y\, y'\, dw}{\int y^2\, dw} = \frac{\int (y' + d)\, y'\, dw}{\int y^2\, dw} = \frac{\int y'^2\, dw + \int d\, y'\, dw}{\int y^2\, dw}$$

$$= \frac{\frac{8}{15}\frac{L}{I_c} y_c^2 (y'_c/y_c)^{5/2} + \frac{2}{3} d \frac{L}{I_c} y_c (y'_c/y_c)^{3/2}}{\frac{8}{15}\frac{L}{I_c} y^2_c} = (y_c'/y_c)^{5/2} + \frac{5}{4}\frac{d}{y_c}(y_c'/y_c)^{3/2}$$

$$= (y_c'/y_c)^{3/2}\left[\frac{y_c - d}{y_c} + \frac{5}{4}\frac{d}{y_c}\right] = (y_c'/y_c)^{3/2}\left[1 + 1/4\frac{d}{y_c}\right]$$

It has been suggested that H may be obtained with sufficient accuracy for horizontal forces on the basis of a 3-hinged arch. This is equivalent to the straight line variation $H/h = y'_c/y_c$ whereas the above expression is more nearly parabolic as shown in Fig. 135(a). Either is probably sufficiently accurate.

For horizontal forces acting from one side only, as direct wind load or traction, the crown thrust (not the pin reaction) will, by symmetry, be one-half the value given above.

Wind Stresses - Arch Trusses Inclined. Fig. 136.

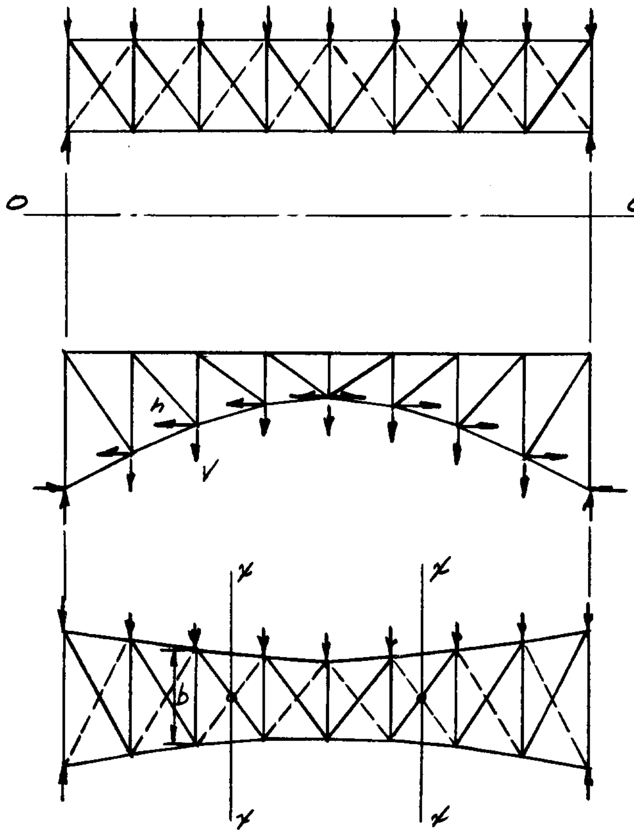


Fig. 136

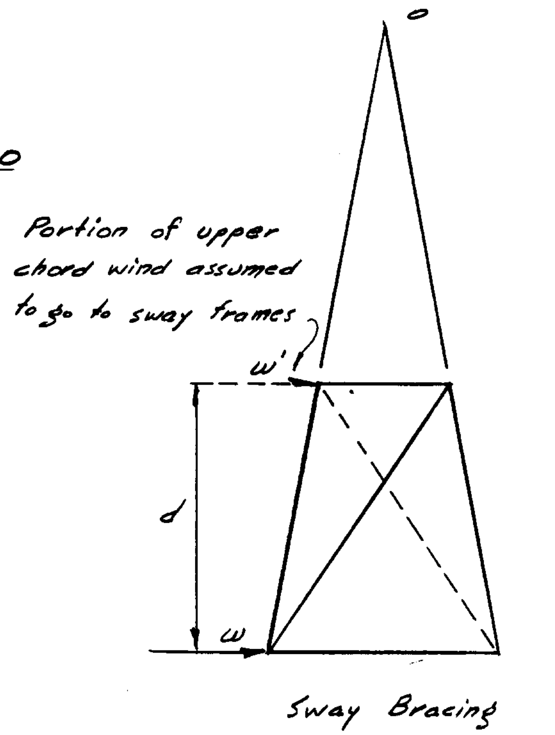


Fig. 137

If the main trusses are inclined to the vertical for greater stability against overturning, the problem is somewhat different. The path of stress down the sway frames is somewhat stiffer than with trusses vertical, but this is not very important. Stresses in the upper lateral system are computed as before.

Stresses in the sway frame and loads on the leeward arch from these stresses present no special difficulty. The batter is usually so small that stresses in the arch from this source and also from vertical loads may as well be figured for trusses vertical.

Stresses in the diagonals of the lower lateral system can no longer be computed directly from the shear, because the chords of the two trusses in any panel are not parallel. The simplest procedure is to pass a vertical plane through the longitudinal center line of the span and take moments about 0-0, the line of intersection of the planes of the trusses. Neglecting upper laterals and sway bracing, we have on the free body cut by this plane and any two planes as x-x, symmetrical about the crown only two forces not in the plane 0-0. These are the shears along planes x-x due to load w + w' on the lower chord and the transverse horizontal components of the stresses in the diagonals where they cut the vertical plane through 0-0. Then,

$$X = V \frac{r_w}{r_r}$$

where V is the transverse shear on vertical planes x-x due to loads w + w' (See Fig. 137).
r_w is the lever arm about 0-0 of the resultant wind.
r_v is the vertical distance to the points where the vertical plane through 0-0 cuts the diagonals.
With one component of the diagonal stress known,

$$S = \frac{L}{b} X \qquad h = \frac{p}{b} X \qquad v = \frac{y}{b} X$$

Note in Fig. 136 that b is not the distance center to center of trusses, but is the horizontal distance from a panel point on one truss to the next panel point on the opposite truss.

With the stresses in the diagonals known, the stresses in the truss may be computed.

General Features of the Arch - The Horizontal Tie. In arches of the spandrel-braced type, the lower chord is primarily the arch proper, the upper chord and web members being essentially bracing in the nature of a stiffening truss. For maximum economy the lower chord should approximate the equilibrium polygon for DL + 1/2 LL. Where abutments can be designed for the thrust there is certainly no economy in a tied arch. For ordinary lengths the economy over a simple span in any arch with horizontal tie is doubtful for in general the economy of any arch lies in substituting for the lower chord the solid earth. In some cases the tied arch may present an advantage in appearance, but it should be remembered that in general every steel structure is inherently ugly unless it is so large that the main members obscure the details and the segmental chord appears as a smooth curve.

Analysis of the Tied Arch. The method of analysis of the arch with horizontal tie is not different from that already outlined for the simple two-hinged arch except that the tie is included in the summation, Fig. 138.

$$i = \frac{\Delta a_{V_{H=1}}}{\Delta H_{H=1}} = \frac{\sum SuL/AE}{\sum u^2L/AE}$$

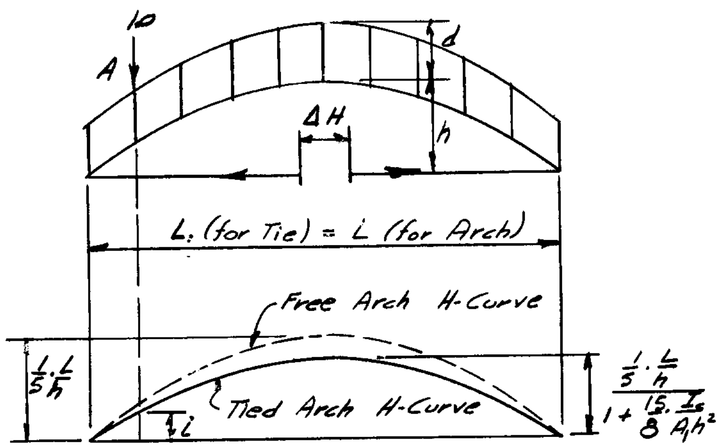


Fig. 138

Since S = 0 for the tie, the numerator is identical with that of the free arch. The denominator will include the value for the tie, where u = 1. Hence, the H-influence ordinates for the free arch will need to be multiplied by the factor,

$$\frac{\sum u^2L/AE}{\sum u^2L/AE + L_1/A_1E} = \frac{1}{1 + \frac{L_1/A_1E}{u^2L/AE}}$$

Tied Arch - Approximate H-Curve. The semi-cubic parabola will closely approximate the H-influence line as it does for the arch without tie. The reduction in the center ordinate is negligible but may be computed on the basis of the parabolic rib of constant section. For this condition,

$$\sum \frac{u^2L}{AE} = \frac{8}{15}\frac{L_1}{I_c} h'^2$$ taking h' to the centroid between chords at the crown. Assuming the area of the tie A, to be equal to the area of the lower chord at the crown, and the area of the upper chord to be 1/2 A, we may write,

$$I_c = \frac{A_1 \cdot 1/2\, A_1}{A_1 + 1/2\, A_1} d^2 = 1/3\, A_1 d^2.$$

Using these values the multiplication factor for the center ordinate of the free arch reduces as follows:

$$\frac{1}{1 + \frac{L_1/A_1}{u_2L/A}} = \frac{1}{1 + \frac{L1/A1}{\frac{8}{15}\frac{L1 \cdot 3}{A_1\, d^2} \cdot h'^2}} = \frac{1}{1 + 5/8\,(d/h')^2}$$

Using an average value of 1/5 for d/h' we have,

$$\frac{1}{1 + 5/8\,(d/h')^2} = \frac{1}{1 + 1/40} = 97.5 \text{ per cent}$$

The semi-cubic parabola with a mid-ordinate of $1/5\,\frac{L}{h}$ or perhaps $0.19\,\frac{L}{h}$ is therefore a good approximation for this case as well as for the arch without tie.

Erection Considerations. That the arch lends itself readily to cantilever erection without falsework is one of its principal advantages. For long spans, erection stresses will be particularly significant in the top chord and these stresses should therefore be investigated as early in the computations as possible. To avoid erection adjustments it will generally be advisable to erect as a three-hinged arch, and then close the center panel of the upper chord. It has also been proposed to close at the center joint of the upper chord during erection and then add the center panel of the lower chord.

Economic Proportions and Preliminary Estimate of Weight. The following data on weights and proportions are given by F. C. Kunz in Chapter XV of his book "Design of Steel Bridges."* They serve to give some idea of normal proportions.

Location of Floor - Preferably above the trusses or framing between the top chords where these follow the grade. Arch chords rising out of the floor cause complicated details and should be avoided.

Spacing of Trusses - For vertical trusses with floor supported above; not less than 1/15 of span length, nor less than 1/3, preferably 1/2, of the total height of floor above the end pins. Where this is impracticable a batter of 1/7 to 1/12 should be considered, remembering that this will also add to shop costs.

For Suspended Floors - Not less than 1/20 of the span length or 1/4 the total height from bearings to crown.

Rise Ratio - The ratio of rise to span length ranges ordinarily between 1/4 and 1/12. The average economical value will usually be between 1/5 and 1/6.

Center Depth - For R. R. bridges 1/25 to 1/20 with slightly greater depth for crescent arches. If the center depth is limited, a solid web may be used in the center panels with a depth as small as 1/40 of the span. For highway bridges these depths may be reduced about 25 per cent.

Panel Length - Controlled largely by economy in the floor system. Diagonals at the quarter points should be approximately at 45 deg.

Preliminary Weights - About the same as for simple trusses, for bridges of economical rise (1/5 to 1/6) and up to about 300 ft. span. The weight of trusses increases for decreasing rise ratios up to 30 or 40 per cent for a rise of 1/12. For spans greater than about 300 ft., and of economical proportions, the weight will be from 10 to 15 per cent less than for simple spans.

*For an extensive and valuable discussion of these and other elements in arch design, see Trans. A.S C.E. 1919-20. "Economics of Steel Arch Bridges," by J. A. L. Waddell and discussion, particularly that of Col. E. Fowler.

CHAPTER XI
CONTINUOUS ARCHES ON ELASTIC PIERS

Nature of the Problem - Questions to be Answered. Continuous concrete arch series on very slender piers have been more common in European than in American practice. Recently they have received more attention in this country, though they are still not very common. Where the piers are of such proportions as usually occur, it is sufficient to treat each arch as the center arch of a series of three arches, (the end arch having an assumed infinitely stiff arch for an abutment). Melan has indicated two approximate treatments for this case;* another approximate treatment is indicated by Whitney.**

The outstanding questions in such cases are those usually occurring in indeterminate structures in the order given,

1. When and where should such structures be built?
2. What are the proper proportions of piers, span, rise?
3. What points are critical in the design of such structures?
4. How are these problems most readily analyzed?

Also, as usual, the order of study is necessarily in reverse order to the questions.

Influence Lines - Qualitative Studies. Considering the general action of arches on slender piers, it is seen that the influence line for crown moment, say, is made up of three parts:

(a) The influence line for the fixed arch.
(b) The influence line for the effect of deflection of the pier tops.
(c) The influence line for the effect of rotation of the pier tops.

The influence of (c) is commonly less pronounced than that of (b). If, then, we combine (a) and (b), we arrive at an influence line for crown moment as shown in Fig. 139(a). Here it is evident that the piers are pulled inwards in all spans on either side of the span considered, causing the arch rings to rise (plotted down for graphic reasons) in the span considered and to drop in all spans on either side (negative influence ordinate). The curve of rise or fall if we assume no rotation of pier tops evidently has the shape of an influence line for crown thrust, but becomes smaller as we go outwards from the pier considered.

Similarly, Fig. 139(b) shows the two important constituent parts of the influence line for moment at the springing, the influence line for fixed-ended condition and that for the effect of the reduction in crown thrust. This again shows the tendency to produce an influence line with all ordinates positive outside the span considered.

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*J. Melan, "Plain and Reinforced Concrete Arches," (Translated by D. B. Steinman). In one solution Melan treats the pier as a cantilever but neglects rotation of its top; in the other, the effect of rotation at the top is included.

**"Analysis of Continuous Concrete Arch Systems," by C. S. Whitney, Proc. Am. Soc. C.E., May, 1926. This differs from Melan's first treatment in applying the unbalanced thrust to arches and piers along the centroidal axes of the arches and not at the pier top.

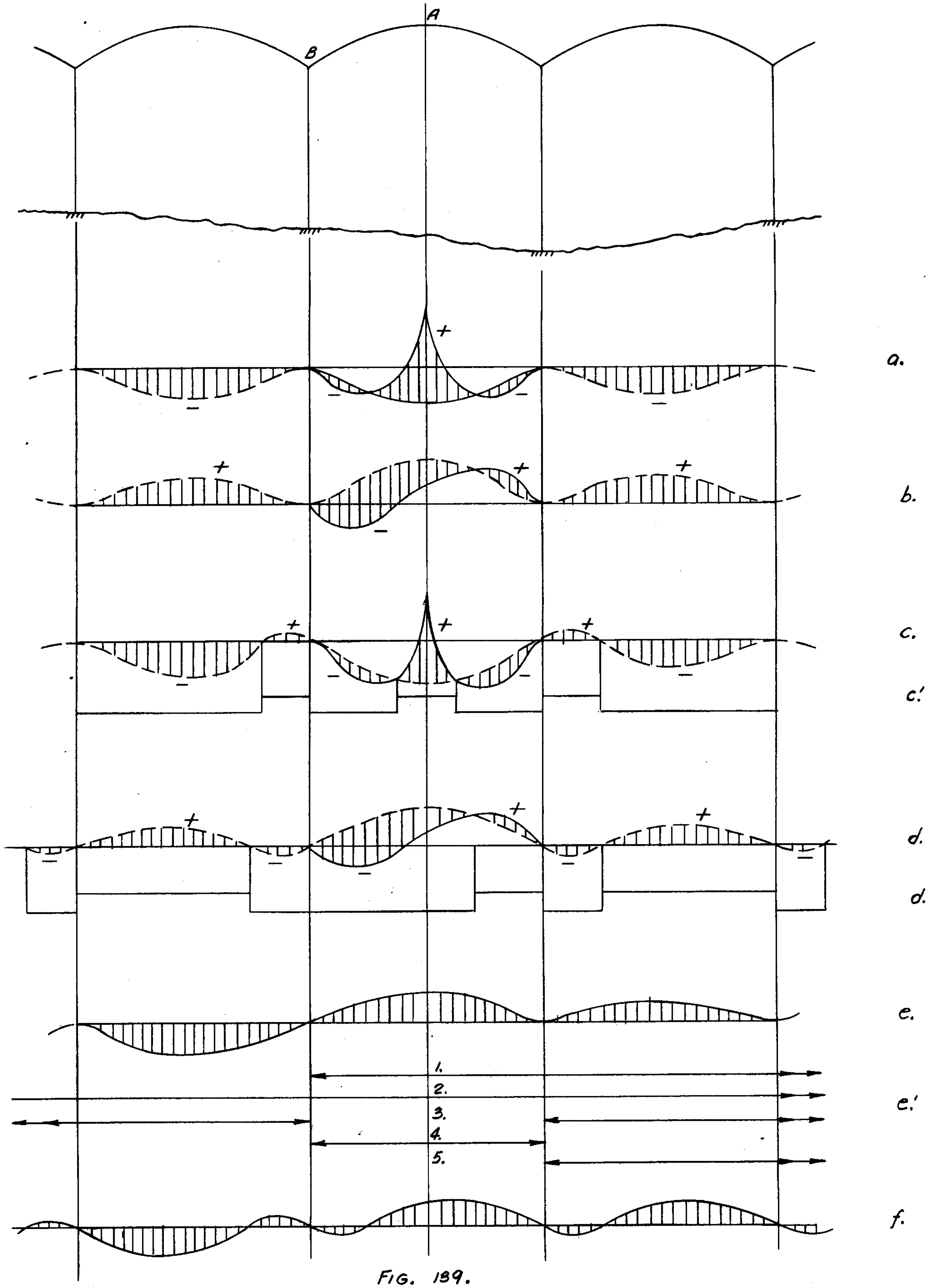


FIG. 139.

Effect of Rotation of Pier Tops. Rotation of the pier tops modifies somewhat this influence line, but not very materially. In an extreme case the error in live load stress was found to be about 25 per cent for crown moment and very small for springing moment. The effect can be qualitatively estimated in Fig. 139 (c) and (d) by rotating the pier tops thus straightening out the H effect on one Side of the pier and aggravating it on the other. The effect of continuity then will give an influence line more like those shown in (c) and (d) than like those shown in (a) and (b).

Moment at the base of the pier is due to the horizontal thrust taken by the pier as the pier tops spread, and slightly to the rotation of the pier tops. The influence line then for pier B will have the general shape shown in (e) (tops of piers not rotating) or in (f) (tops of piers rotating).

Typical influence lines for critical elements in continuous arches are shown in Fig.

Load Conditions - Split Loads. Influence lines shown above indicate a critical condition of loading for crown moment, for example, as shown in Fig. 139 (c') and (d'). As a matter of design such a loading condition is utterly impossible. We accept then as possible working load conditions the following:

(a) For live-load moments in the arch ring, load all or none of the spans other than the span under consideration and partly load that span as indicated by the load divide.

(b) For live-load moments in the pier, load all spans to the right or to the left of the pier under consideration.

From this it follows that the effect on any arch ring of only four conditions of live load need be computed. These are:

(a) Full live load over the whole series of arches;

(b) Full live load over the span under consideration;

(c) Load at the ends of the arch under consideration as indicated by the load divide for stresses at the crown;

(d) Load at one end of the arch under consideration as indicated by the load divide for stresses at the springing.

These may be combined as follows to give maximum moments in any arch ring. The loading a needs to be analyzed only once, but further study is needed to simplify the computations of b, c, and d.

Critical Pier Loadings. The influence line for critical loading for moment at the pier base indicates a load over all spans to the right or to the left of the pier, (Loading (1)-Fig. 139(e'). If the system is symmetrical about the span to the right of the pier it will be true that for crown thrust and moment in this span, loading (5) has one-half the effect of loading (3).

While shear at A for loading (5) cannot be thus found, it is evidently small. Critical conditions at the pier base then will result from the effect of crown thrust at A for loading (5) plus the effect of loading (4).

Dead Load Stresses. It is evidently desirable that the series of spans should nearly balance under dead load, that is, that the pier tops should not move for this loading. Then under this condition every arch will act as a fixed-ended arch.

Such a condition is not always possible. Assuming that successive arches have the same shapes of axes and variations of I, the crown thrusts will vary as L^2/h. If the thrusts are to balance L/h must vary as $1/L$ or the rise ratio must be directly proportional to the span.

The fixed-ended moments at springing for full loading will vary as the crown thrust times the rise (distance to neutral point). If the thrust is constant, these moments will be proportional to the rise and will not balance. This, however, is not very important; it is the spread and not the rotation of the pier tops that affects the stresses seriously.

To proportion the rise ratio to the span we may either,

(a) Camber the whole arch series; a very satisfactory device if not exaggerated;
(b) Spring adjoining spans at slightly different levels; an aesthetic abomination into which the designer is sometimes forced.

This is illustrated in Fig.

Temperature Stresses. If the dead load thrusts are balanced as just indicated, the temperature thrusts will be unbalanced. For arches having the same variations of axis ordinates and I,

$$H_t \propto \frac{L}{\frac{L}{I_C} h^2} \propto \frac{I_c}{h^2}$$

If further we assume $d_c \propto L$ (a fairly good assumption) and h/L^2 constant,

$$H_t \propto I_c/h^2 \propto \frac{L^3}{(h/L^2)^2 L^4} \propto 1/L$$

Actually d_c decreases with the rise and increases with the span. If d_c and h/L^2 are constant,

$$H_t \propto 1/L^4$$

We may expect, then, that the crown thrust will vary inversely perhaps as the square of the span in an ordinary arch series and the pier tops will close in on the longer spans toward the center due to rise of temperature.

The effect of continuity on temperature stresses is relatively less pronounced, however, than on live load stresses because the thrusts are nearly balanced anyway and also the spread of the piers affects the temperature stresses almost directly in proportion to the crown thrust; whereas for loads a decrease in crown thrust affects the value of one quantity (moments due to H forces) which is to be subtracted from another of about the same value (moments due to continuous beam action) and the resulting change is proportionately very great.

The moment at springing $\propto H_c h$. If $H_t \propto 1/L^2$ and h/L^2 is constant, $M_s \propto H_t h \propto 1/L^2 \; L^2 \propto$ constant. Hence, the moments at the pier tops will be in general nearly balanced for temperature.

Analysis for Stresses Due to Dead Load and Full Live Load and Temperature. For these conditions the pier tops are very nearly balanced. A very satisfactory analysis may in most cases be made by successive approximation. This may, for a well designed arch series, be done by distributing the unbalanced thrusts, correcting the moments for these changed thrusts and then distributing the unbalanced moments.

For a series of identical arches, the dead, full live and temperature forces are completely balanced.

Analysis for Maximum Live-Load Stresses. Determination of maximum stress for live load as indicated above requires the analysis of the forces in each arch of the series for loads in that arch alone. This is the problem with which writers on this subject have chiefly been concerned.

Several exact and semi-exact and several approximate methods are suggested. It should be noted that in the case of continuous arches the live load stresses in the arch ring are relatively more important than for fixed arches. In the case of fixed arches we may expect such relative values as dead load stress, 50 per cent, live load stress 25 per cent, temperature stress 25 per cent while in arches on quite slender piers we may have live load stress 50 per cent, dead load and temperature stress 25 per cent each.

Exact Methods. A. Determine the elastic properties of the end span, combine this with the first pier, combine this with the second arch and thus proceed until the arch under consideration is reached. Proceed similarly from the other abutment to the span under consideration. We now have the elastic properties of the arch under consideration, and of two equivalent elastic piers on which it is considered to rest and solution for stresses in the span may be made by the column analogy.

This method gives an exact and definite procedure; the expressions used are not very formidable and may be simplified without appreciable loss of accuracy to give convenient solutions. We get systematically the properties of each arch in the series and from this the stresses in the various arches are readily determined.

B. The general displacement equations are written for all pier tops. If only three arches in series are to be considered, direct solution of the equations is not forbidding. Where it is thought necessary to include the effect of five or seven spans, the solution is rather tedious.

C. The indirect method of solution may be employed. While of some interest, it apparently is not very convenient.

Approximate Methods. A. The piers may be treated as cantilevers loaded along the arch axis with the unbalanced arch thrust and the arch thrusts distributed outwards between fixed arches and cantilever piers, neglecting entirely the effect of the rotation of pier top on the moments in the arches and the effect of the rotational restraint of the arches on the deformation of the pier.

This may be modified by making a final correction for the rotation of the pier tops. Other modifications will suggest themselves.

It is a fascinating problem in mechanics. As a problem in practical design other factors enter. Slight compressibility of the soil will very seriously affect the movement of the pier top produced by unbalanced thrusts. Soil yielding is partly an elastic and partly a plastic deformation, and the relative proportions of these are not determined and are often indeterminable. Both of these deformations are produced by dead load and to some extent by temperature changes, whereas probably only the elastic deformation is produced by live load.

Other elements might here be discussed -- improbability of maximum live load conditions, complication of action of floor and spandrel posts. The conclusion is probably justified that while the effect of each element is a proper subject for research study, the design may be determined safely by including only the effect of pier deflection and that this is most conveniently done by distributing unbalanced thrust between the two arches treated as fixed and the pier treated as a cantilever loaded along the centroidal axis of the arch. Neglect of rotation of pier tops is on the safe side so far as arch moments are concerned.

The method B of treating three - or in some cases five - arches as a complete series is excellent as a method of general study and sometimes as a method for final design.

The method indicated in C is to find the forces produced by a displacement and a rotation of the pier top and then to combine these to equalize the known unbalanced thrusts and moments at the piers. It will not in general, be a convenient procedure.

Forces Acting on the Pier. It has been shown that the influence lines for moment and shear in the pier indicate that all spans to one side or the other of that pier should be fully loaded for maximum. Also, in general, the effect of loading one side will be equal and opposite to that of loading the other side since the spans are nearly balanced for full loading.

But in Fig. 139(e') loading (1) = loading (5) + loading (4). But crown shear and moment for loading (5) = 1/2 the difference of that for loading (2) and that for loading (4). This follows from symmetry, if the series is symmetrical about A.

Hence, $(1) = \frac{(2) - (4)}{2} - (4) = \frac{(2) + (4)}{2}$ so far as thrust and moment at A are concerned.

Hence, we can find thrust and moment at A for the required loading as the average of that on this arch due to full loading in this arch only acting first as a fixed arch and then as an elastic-ended arch in the series.

This argument has depended on the assumption that the pier thrust is zero for full uniform load over the entire series and that the influence lines for H and M at A have the same areas to right and left of A. If these assumptions are thought to be seriously in error, a correction must be made.

The moment and thrust at A will now need to be distributed between the pier and the next arch. Finally the effect of shear from loading (4), 1/2 wL, will need to be added to the pier. It is evident that the shear at the pier from loading (1) cannot be very different from wL/2 in any case.

This method, however, requires the distribution of the thrust and moment found at A between pier and arch. A convenient approximation is as follows:

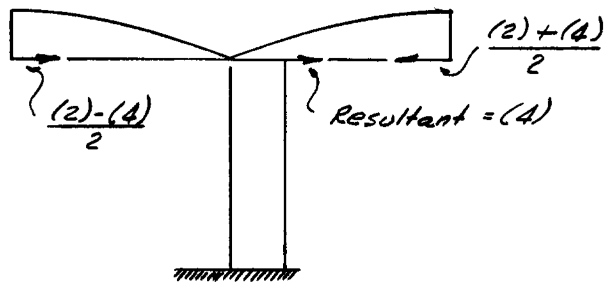


Fig. 140

The required loading (1) will produce at A a thrust and moment due to,

$$\frac{(2) - (4)}{2} + (4) = \frac{(2) + (4)}{2}$$

At B it produces, on the same assumptions, thrust and moment $\frac{(2) - (4)}{2}$ if the flanking arch spans are alike.

The resultant of these forces acts on the pier and is equal to all of the crown thrust at A due to (4) as shown in Fig. 140.

The pier, then, may be approximately analyzed by considering that the pier takes unaided all of the crown thrust and moment produced in one flanking arch by full loading in that arch. This is combined with a vertical load 1/2 wL.

This is readily evaluated by distributing the pier thrusts for loading (4). That the requirements thus imposed on the pier are very severe is evident. They are certainly well on the safe side as regards probable loading conditions.

Theory of Exact Analysis. (a) Reduction Process. The equations here given are developed in Trans. A.S.C.E. 1925, pp. 1198-1202, by using the theory of conjugate axes. Similar but less simple expressions may be deduced directly from the column analogy or the reduction values may themselves be obtained directly by use of the column analogy, but the process is somewhat involved.

The application of the equations is not very tedious. The process, in its barest essentials is as follows: The arch adjacent to one abutment is combined with the first pier to find an equivalent pier having elastic properties equivalent to the combined arch and pier. This equivalent pier is then combined with the next arch to obtain an equivalent arch having the elastic properties of the combined arch, pier and arch. This equivalent arch is then combined with the second pier to obtain a new equivalent pier. This process of reduction is continued from each end until the arch under consideration is reached. This arch is then computed in the same manner as a fixed arch with additional end voussoirs having elastic properties equivalent to those of the combined spans on either side of the arch in question.

The fundamental equations for the reduction process in the exact analysis are:

$$\frac{1}{J_s} = \frac{1}{J_p} + \frac{1}{J'_a} \quad \text{or} \quad J_s = \frac{J'_a \, J_p}{J'_a + J_p} \qquad (1)$$

$$y_p = \frac{J_s}{J'_a} y' \qquad (2)$$

$$\frac{1}{W_s} = \frac{1}{W_a} + \frac{1}{W_p} + \frac{x^2}{I_a} + \frac{(y')^2}{J'_a + J_p} \qquad (3)$$

Where:

J_s = the moment of inertia of the elastic weight of the equivalent pier for the system about a horizontal axis through its neutral point;

J_p = the moment of inertia of the elastic weight of the actual pier about a horizontal axis through its elastic centroid.

J'_a = the moment of inertia of the elastic weight of the equivalent arch for the system about the axis conjugate to the vertical axis through the neutral point of the combined arch and pier, (about H'-H' in Fig. 1). Note that the arch becomes unsymmetrical when the pier on one side is combined with it or when the equivalent piers on either side are not alike;

or $J'_a = J_a - Z_a^2/I_a$, where J_a = the moment of inertia of the elastic weight of the equivalent arch about a horizontal axis through its neutral point;

Z_a = the product of inertia of the equivalent arch about simultaneous horizontal and vertical axes through its neutral point, and;

I_a = the moment of inertia of the elastic weight of the equivalent arch about a vertical axis through its neutral point.

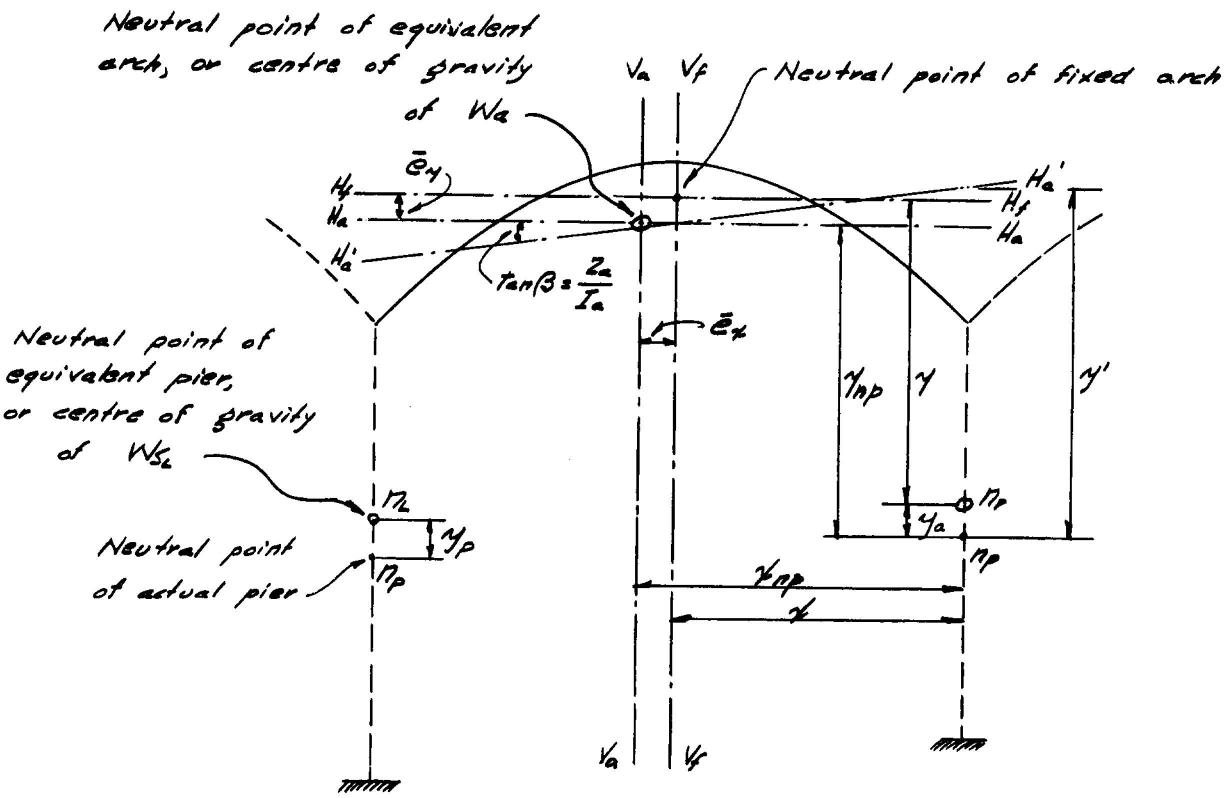


Fig. 141

y_p = the vertical distance from the neutral point of the actual pier to the neutral point of the equivalent pier.

y' = the vertical distance from the neutral point of the actual pier to the conjugage axis $H'_a - H'_a$, (See Fig. 141), or $y' = y_{np} + x_{np} Z_a/I_a$, where

y_{np} = the vertical distance from the neutral point of the actual pier to the neutral point of the equivalent arch;

x_{np} = the horizontal distance from the neutral point of the equivalent arch to the neutral point of the actual pier, and

Z_a and I_a are the same as defined above.

w_s = the total elastic weight of the equivalent pier;

w_a = the total elastic weight, s/EI, of the equivalent arch;

w_p = the total elastic weight, L/EI, of the actual pier.

(b). Arch On Equivalent Elastic Piers. The properties of the arch on equivalent piers may be expressed in terms of I_o, J_o, Z_o, and W_o, where I_o and J_o are the moments of inertia, respectively about the vertical and horizontal axes through the neutral point of the equivalent arch, of all the elastic weights including the equivalent piers on each side of the span in question; Z_o is the product of inertia of all the elastic weights about simultaneous horizontal and vertical axes through the neutral point; and w_o is the total elastic weight of the arch and equivalent piers.

The value of each of these functions may be stated in the following general equations:

$$I_o = I_a + \bar{e}_x^{\,2} \; W_o + \; \Sigma W_s \, x^2 - 2 \, \bar{e}_x \; \Sigma W_s x \text{ - - - - - - - - - (4)}$$

$$W_o = W_a + W_{sL} + W_{sR} \text{ - (5)}$$

$$J_o = J_a + J_{sL} + J_{sR} - \bar{e}_y^{\,2} \; W_o + \; \Sigma \, W_s y^2 - 2 \, \bar{e}_y \; \Sigma \, W_s y \text{ - - - (6)}$$

$$Z_o = \bar{e}_x \, \bar{e}_y \, W_o + W_s xy - \bar{e}_x \; \Sigma \, W_s y - \bar{e}_y \; \Sigma \, W_s x \text{ - - - - - - - (7)}$$

$$\text{Tan } \beta = Z_o / I_o \text{ - (8)}$$

As noted in Fig. 141, $\bar{e}_x$ and $\bar{e}_y$ are the coordinates of the neutral point of the combined system with respect to the neutral point of the fixed arch as the center of coordinates;

β = the angle which the axis conjugate to the vertical axis through the neutral point of the system makes with the horizontal axis through the same netural point;

x and y = the horizontal and vertical distances respectively from the neutral points of the equivalent piers to the neutral point of fixed arch.

The other terms were previously defined.

When the critical arch is the center span of a symmetrical series, the equivalent elastic piers on either side have the same properties, $\bar{e}_x = 0$, and equations (4), (6), (7), and (8) simplify to the following:

$$I_o = I_a + 2W_s \, x^2 \text{ - (9)}$$

$$J_o = J_a + 2J_s + \bar{e}_y^{\,2} \; W_o + 2W_s y^2 - 4W_s \, \bar{e}_y y \text{ - - - - - - - - (10)}$$

$$Z_o = 0 \text{ - (11)}$$

$$\text{Tan } \beta = 0 \text{ - (12)}$$

The influence ordinates for moment (M_o), horizontal thrust (H_o), and vertical shear (V_o) at the neutral point of the arch on elastic piers are then determined as follows for a symmetrical case.

$$M_o = \frac{W_s x' + \; \Sigma \; Wx'}{W_o} \text{ - - - - - - - - - - - - - - - - - - (13)}$$

$$V_o = \frac{W_s xx' + \; \Sigma \; w xx'}{I_o} \text{ - - - - - - - - - - - - - - - - - - (14)}$$

$$H_o = \frac{W_s x' y + \; \Sigma \; w_{x'} y - \bar{e}_y \; (\; \Sigma \; w_{x'} \; + W_s \; x')}{J_o} \text{ - - - - - - - (15)}$$

These are the typical equations for the neutral point method. The notation used above is indicated in Fig. 142.

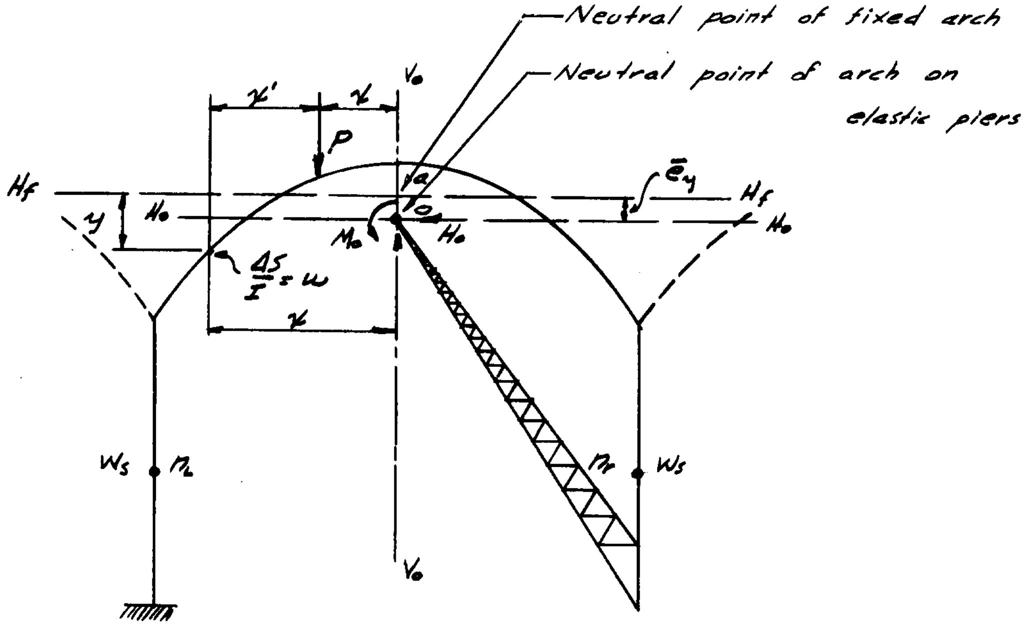


Fig. 142

x = horizontal distance from the neutral point to the elastic weight under consideration.

x' = horizontal distance from the load point to the elastic weight under consideration.

$\bar{x}$ = horizontal distance from load to neutral point.

y = vertical distance from the neutral point of fixed arch to the elastic weight under consideration.

Approximate Method of Analysis. An approximate method of reduction has been developed which considerably simplifies and shortens the process of computation indicated by equations (1), (2), and (3) and yet is exact enough for all practical purposes. In equation (1), $J_a' = J_a$ for the approximate solution. This eliminates the use of the correction factor Z_a^2/I_a, although J_a changes for each equivalent arch. J_a is then computed as the moment of inertia of the elastic weight of the equivalent arch about a horizontal axis through the neutral point of the fixed arch.

In equation (2) y' is made equal to y. This eliminates the correction factor $x\ \frac{Z_a}{I_a}$. y theoretically is the distance from the neutral point of the equivalent arch down to the neutral point of the actual pier. However, in the approximate method y is taken as the distance from the neutral point of the fixed arch to the neutral point of the actual pier. Also, J_a' is made equal to J_a.

The equations for the approximate analysis can then be restated as follows, referring to Fig. 143:

$$\frac{1}{J_s} = \frac{1}{J_p} + \frac{1}{J_a} \quad \text{-----------------} \quad (16)$$

$$y_p = \frac{J_s}{J_a}\, y \quad \text{-----------------} \quad (17)$$

$$\frac{1}{W_s} = \frac{1}{W_a} + \frac{1}{W_p} + \frac{x^2}{I_a} + \frac{y^2}{J_a + J_p} \quad \text{-----------} \quad (18)$$

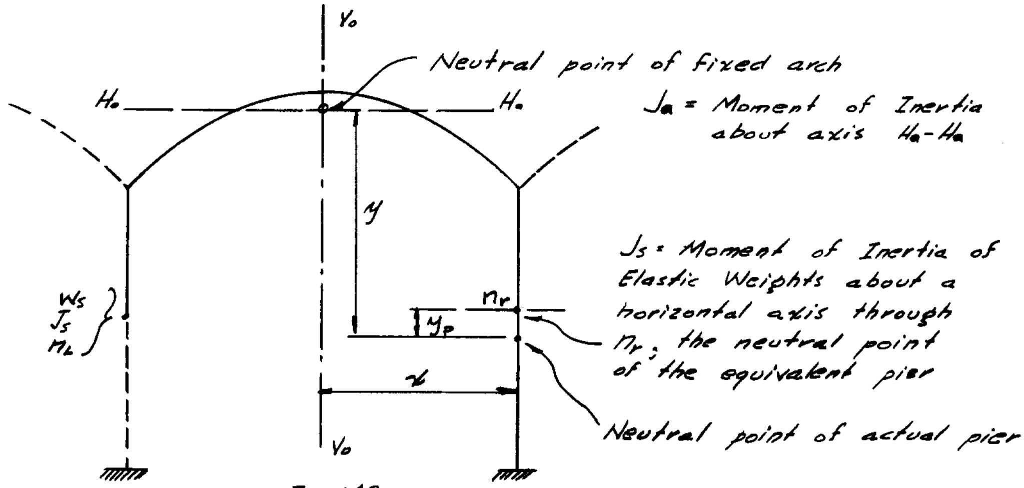


Fig. 143

Extended studies have shown that no considerable error results from use of the more convenient formulas.

<u>Economy in Proportioning in a Continuous Arch Series</u>. Much remains to be done in this field of investigation. The stresses in an arch ring which is continuous differ from those in a fixed arch chiefly -- almost entirely -- in the additional live load stresses produced by movement of the pier tops and the rotation which accompanies it. Economy then indicates the importance of minimizing the thrust due to live load and hence of using as large a rise ratio as is permissible. This also reduces the moments in the piers and the cost of foundations. The sensitiveness of the arch stresses to change of rise-ratio is shown in Fig.

Now the yardage per running foot of floor and to a large extent that of the spandrel columns is independent of length of span. The yardage per running foot of the arch ring is affected by the span length, by the rise-ratio and by the pier height and slenderness. For dead load the kern moments vary directly with the span length; for temperature they are independent of span length; for live load they vary as the square of the span and with the height and slenderness of the pier. Pier yardage per foot of height will be independent of span length for short piers and may vary as the first or higher power of the span for slender piers. From such considerations we may conclude,

Cost of ribs per foot of span = ax for short piers to $ax^{3/2}$ for high piers;

Cost of pier per foot of span = b/x for short piers to $b/x^{1/2}$ for long piers;

Hence, for minimum cost $dc/dx\ (ax + b/x) = 0$.

$$ax = b/x$$

Cost of ribs = cost of pier for short piers.

$$dc/dx \ (ax^{3/2} + \frac{b}{1/2}) = 0$$

$$1.5 \ ax^{1/2} = 2 \frac{b}{x^{3/2}}$$

$$ax^{3/2} = \frac{2}{1.5} \frac{b}{x^{1/2}}$$

Cost of ribs = 1.3 (Cost of piers).

This is all very general and not very valuable, but it suggests that the classical rule for span length in steel bridges may be a good rough guide here. Apparently the rule should exclude arch centering.

The exact adjustment of span will depend on many factors and as in other cases of minima, variation from the optimum is not very important. The exact layout will depend in the final analysis on appearance and topography -- especially foundation conditions.

Influence of the Floor in Continuous Arch Systems. The methods indicated for including the effect of the floor system in fixed arches may evidently be extended to continuous arches on flexible piers. No special complications arise other than those inherent in the individual solutions if there is an expansion joint at the pier. The arches are stiffened against both rotation and displacement by the floor system but this will not change very greatly the distribution of unbalanced thrusts and moments, just as small haunches in all of a series of beams do not materially change the distribution of unbalanced fixed-ended moments among these beams.

If the floor is continuous across the pier, the effect of the horizontal force along the floor line must be included. This may be computed by imagining the floor to be cut at the pier, finding the displacement and also the displacement produced by unit force along the floor and then the value of this force. The displacements at the cut in the floor line may be found directly for temperature effects and for load from the flexure of the pier vertical.

The movement of the floor and the effect of its continuity at the pier will not ordinarily be very important in considering stresses due to loads. The effect on temperature stresses, however, may be very pronounced. In this case it is often possible to determine by inspection the horizontal movement of the floor and then to include this in the computations for column moments.

It is evident that the subject has important design aspects.

CHAPTER XII
SPECIAL PROBLEMS IN CONCRETE ARCHES

Limitations in the Common Theory. This chapter enters into a wide field of investigation. Even a novice soon realizes that the theory of the concrete arch as given in Chapter VI is distinctly "half-baked." It neglects the effect of the elasticity of the piers; it omits the influence of floor and spandrel columns with the added complication of expansion joints and the pronounced effect of merging the floor with arch ring near the crown; it makes no mention of distribution of load by the arch barrel in some cases nor of the effect of skew in others -- all these in addition to special problems there indicated. An effort is made here to show how these problems may be studied and in general to indicate the kind of effect produced. The chapter is much too long in spite of efforts to condense it, and yet it would be very much longer if the problems were discussed in detail.

After all, individual judgment will decide whether it is better to follow a definite and simple analysis and then estimate the effect of other elements -- making sure that none are forgotten; or whether one should seek a precise analysis in the hope of resulting economy.

Effect of Spandrel Posts on Stresses in the Arch Ring. When an arch ring deforms under the influence of loads or temperature changes, the spandrel posts are subjected to bending and direct stress and the stresses in the arch ring are modified. The first important questions in this connection are to what extent this action modifies the results arrived at by the theory as ordinarily applied, whether such action adds to the strength of the structure and whether such additional strength can be economically used in design.

The movements of the spandrel posts may produce on the arch ring vertical forces accompanying shear in the floor -- horizontal forces, accompanying shear in the columns -- and moments, accompanying bending in the columns.

Of these the most important factor as affecting rib stresses is the moments. The vertical forces are not very large because of the relative flexibility of the floor system; we certainly cannot depend on the floor system to hold up the arch ring. The effect of the horizontal forces is usually not very large except where the posts are very short, because the posts are relatively flexible and also because the influence ordinates for kern moments for horizontal loads are small. Moreover, the immediate question is not whether the posts might be stiffened to give larger effects from their shears -- to hold the haunches against lateral movements as horizontal earth pressure does in a tunnel arch -- but whether with posts as ordinarily designed for vertical loading, such phenomena are important. To seek economy by thickening the spandrel posts seems futile.

The following discussion is intended to give an idea of the general action of the spandrel posts and their effect on the arch rib together with some idea of the quantities involved. It is only a general survey of extended studies too long for inclusion with the other material in this volume.

Assume an arch with spandrel posts as shown, Fig. 144. Neglecting the effect of the spandrel posts the reaction lines for a load at the crown will be as shown in full. This will bend the arch rib and thus will produce moments at the bottom of the spandrel posts depending on the slopes and horizontal deflections of the arch axis. These moment loads on the arch

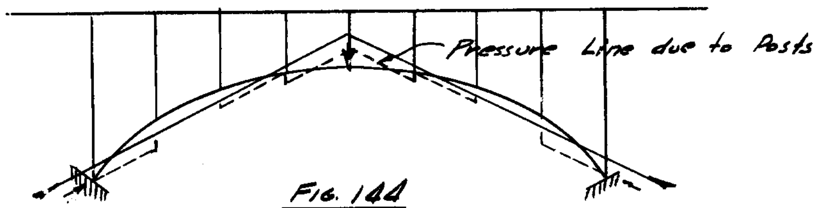


FIG. 144

ring will cause the pressure line to jump up or down at the posts by an amount = $\frac{\text{moment in post}}{\text{thrust of arch}}$. The pressure line will take some such position as shown dotted. This will modify slightly the shape of the axis as first computed. The total area under the moment curve, however, will still be zero.

This presents a general and relatively simple method of computing the effect of spandrel posts to include all factors in the analysis. Compute the pressure lines for the unbraced arch. From this compute the shape of the arch axis. From the displacements thus found, compute H, V, and M at the bottoms of the columns. For these loads and the given load, revise the arch pressure line. Repeat the process to any desired degree of accuracy, including rib shortening if desired.

Simple studies will give an idea of the magnitude of the forces involved. In making such studies it is convenient to think of the line of pressure in the arch ring as a curved rather than as a broken line. This is equivalent to considering a large number of small spandrel posts the total effect of which is the same as that of such spandrel posts as are actually present.

Effect of Posts on Crown Thrust. As will be seen later, the true value of the influence ordinate for crown thrust due to a vertical crown load has special significance in temperature studies. To draw an influence line for crown thrust apply a thrust along the horizontal centroidal axis and compute rotations and horizontal displacements of the deflected structure. This produces moments in the spandrel which produce a curved pressure line. Moreover, since the total change of slope around the arch axis will still be zero, this curved pressure line must balance about the old straight pressure line so that the total moment area will still be zero. This neglects the changes in arch thrust due to column shears.

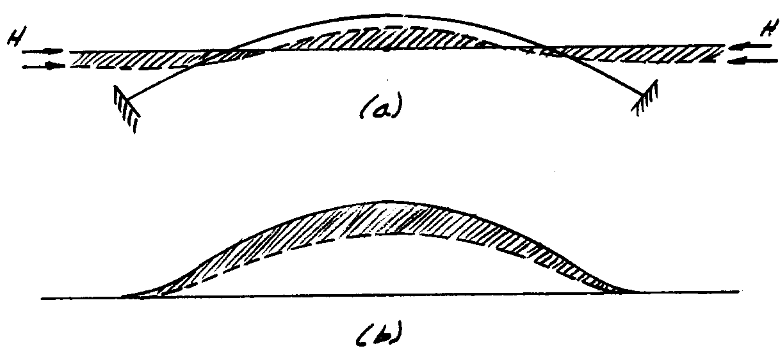


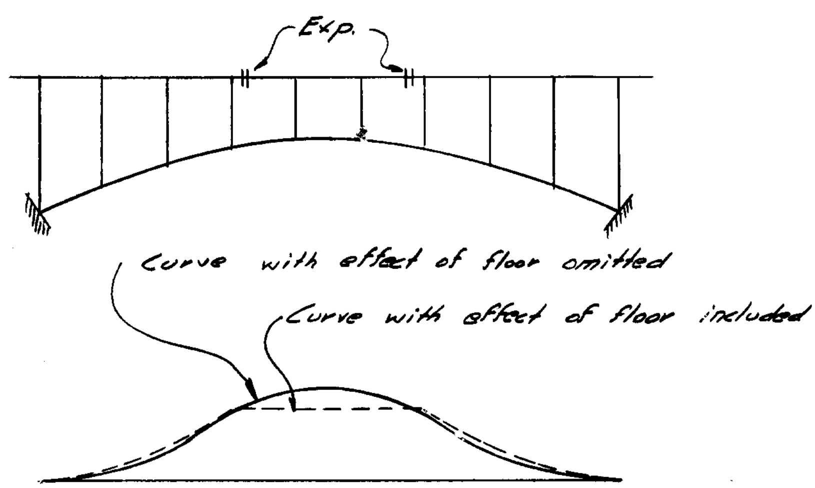
FIG. 145

The new pressure line, then, is perhaps as shown dotted in (a), Fig. 145. Actually it will commonly have a double curvature. This produces a change in the deflected structure as shown dotted in (b). The resulting curve of differences between the full and the dotted line in (b) will evidently have nearly the same shape as the original black line in (b) and hence the influence line for the braced arch will have nearly the same shape as for the unbraced arch, but the H force in (a) or scale of the curve in (b) will be different.

But if the arch is parabolic, except for rib shortening, there is no bending under full load and the area under the influence line is $1/8\ L^2/h$ whether the arch is braced or not, this being the total crown thrust for full load in a parabolic arch. Since, then, neither the area nor the shape of the influence line is changed very much, the center ordinate is not appreciably changed.

Temperature Stresses in Braced Arches. It has just been shown that the deflected shape of the arch axis produced by changing the span is not very different whether or not the action of the spandrel columns is taken into account. From this it follows that the relative values of the angular rotations along the arch axis are about the same in the two cases. But the change in span is due entirely to these angle changes and equals their statical moment about the line of springings. Then, since the change in span which would result from free temperature movement is the same in any case, it follows that the angle changes will not differ greatly in the two cases and hence that fibre stresses due to temperature would not be very much altered if the columns were closely spaced. Wide spacing of the posts concentrates the effect of the change at the foot of the posts. This cannot be very important near the crown because columns near the crown have no moments due to temperature. The moment in the column at the springing may be found directly from the horizontal movement of the floor.

From this, however, it does not follow that the temperature thrusts are unaltered, since they evidently may be changed considerably.

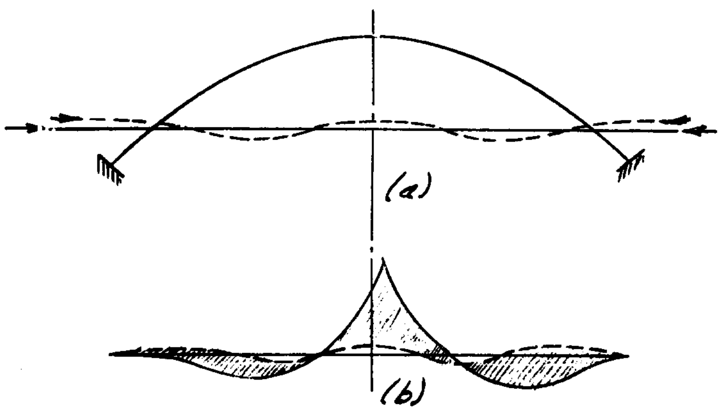


Deflection Curve Showing the Effect of Floor Expansion Joints

Fig. 146

An exception to the general conclusion as to temperature stresses must be made where expansion joints permit free movement in the floor line. If the posts are very rigid in such a case almost all rotation will take place below the expansion joints. Fig. 146 shows how the stresses may be much increased by the action of the posts. The full line is the curve of deflection or thrust influence line with floor omitted. The dotted line shows the effect of including the floor - concentrating rotations below the expansion joints. The effect of posts in this case may be roughly thought of as a stiffening of the arch axis at other points relative to that at the expansion joint.

Effect of the Floor on the Load Divides. Crown Moment. To study an influence line for moment at the crown, assume a parabolic arch. Apply at the crown a unit rotation and draw the usual influence line. The line of pressure is, of course, a horizontal line as shown, Fig. 147. The effect of bending the arch axis is to throw moments into the spandrel posts which will produce a broken pressure line. Since we still maintain unit rotation at the crown there is no added net angle change along the arch axis and hence this new pressure line will closely follow the old pressure line as shown dotted in (a).

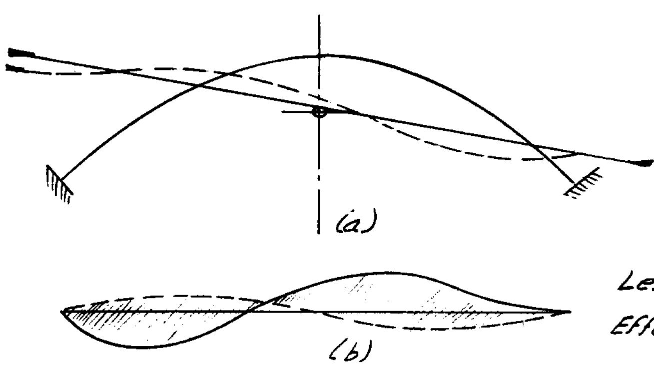


Crown Moment
Effect of Floor on Load Divide

Fig. 147

This change in pressure line will produce additional deflections as shown by the dotted line in (b). But the area under the influence line must be zero whether or not the floor posts are in action, since the arch is parabolic and a parabolic arch has no moment under full load if rib-shortening is neglected. Hence the dotted line must have four points of inflection as shown. From this it would seem that the position of the load divide for crown moment is not very much affected by the action of the spandrel posts.

Springing Moment. Similar reasoning holds for the moment at the springing as shown in Fig. 148. The shapes of the curves here shown will vary with the case in hand. The figure merely illustrates that there cannot ordinarily be any great change in the load divide.



Left Springing Moment,
Effect of Floor on Load Divide

Fig. 148

Analysis of Braced Arch. Thus it is seen that the effect of the spandrel columns on the moments due to temperature is ordinarily not very great except perhaps at the springing, where the change is easily figured and that the load divides are not very much altered. It may also reasonably be assumed that the effect of the posts is negligible under full load. Also, we can with good assurance correct for rib shortening by the usual methods.

Apparently, then, an analysis of the arch for 3/8 loading will, with reasonable accuracy, include the important effects of the posts.

(a) Compute horizontal deflections and rotations for 3/8 loading.
(b) From these compute the moments and shears in the columns. (Column thrusts may usually be neglected).
(c) With the column moments recompute the arch moments. (The effect of the column shears may usually be neglected).
(d) For these new arch moments compute the deflections.
(e) Proceed in this manner as far as warranted.

If expansion joints occur in the floor, correct for the horizontal movement which occurs at them by the indirect method indicated elsewhere. This consists in summing up the shears in all columns between two expansion joints and then applying a force along the floor line to balance them.

This method is further indicated in Problem

General Effect of Floor Posts. The effect of floor posts is generally to be considered as a secondary effect -- dangerous to the posts and only slightly helpful to the arch -- rather than as a source of strength to be utilized in design. To make the strength of an arch rib dependent on the posts requires careful design of the latter; but failure of a single column in an arch as commonly designed, though a serious matter, does not constitute disaster.

Probably the most important thing indicated by this study is the futility of overnice theorizing as to the stresses in arch rings as computed by the usual methods.

Further, studies of the stress effects of temperature on spandrel columns are sometimes important in locating expansion joints. Here, as elsewhere, expansion joints are undesirable but necessary. The objectionable effect of expansion joints on temperature stresses, however, has been noted above.

Effect of Saddles. Arches are often designed with crown of the arch ring lying just below the floor and rigidly connected to it by a saddle. The effect of this construction is to increase very greatly the stiffness at the crown. The obvious effect of this is to lower the horizontal axis of the arch and to increase considerably the thrust from temperature. Fig. 146 compares the stresses in an arch ring with and without a saddle. This serves to indicate the importance of studying the effect of this construction.

The Skew Arch - Problems Arising. The skew arch involves several distinct problems. So far as the action of the arch is concerned, there is no problem if the arch is a narrow rib.

The important problem is in the spandrel-filled barrel arch. Here we reconqize the following elements.

(a) Uncertainty of distribution of load to the arch ring through the fill -- distribution of surcharge through earth fill;
(b) Uncertainty as to the magnitude of horizontal earth pressure against the arch barrel and the spandrel walls;
(c) Integral action in the curved slab which constitutes the arch barrel;
(d) Action of the arch ring under vertical loads symmetrical with respect to the longitudinal axis of the ring;
(e) Action of the arch ring under the horizontal forces from earth pressure acting on the arch barrel and spandrel walls;
(f) Influence of the spandrel walls on the action of the arch ring.

To these we must add the problem of the design of the abutments.

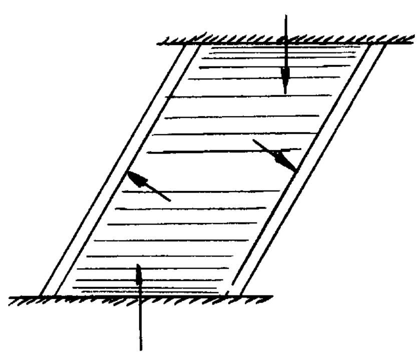
The questions of distribution of loads (a) by the fill, (b) by the barrel, and (f) by the integral action of the barrel with spandrel wall occur in the right as well as in the skew arch.

Space forbids a review here of the problem of distribution of loads by earth fill. Distribution on an angle of 30 deg. seems conservative.

The problem of slab action occurs in any slab. If the slab is plane and rectangular the problem is difficult enough. It is discussed elsewhere and it is there shown that neither tests nor mathematical analyses give complete information from which to predict rupture. If the slab is curved and if, further, it is on a skew, the problem is vastly complicated and no accurate analysis in workable form is to be expected. Even then the whole question is inextricably tied up with the disturbing effect of the spandrel walls and the tendency of the stress to "crowd" to the line of greatest resistance -- least work -- along these spandrel walls with resulting cross-bending in the slab.

Analyses purporting to give the effect of load concentrations have little immediate utility in design unless the elements just mentioned are taken into account.

The problem of horizontal earth pressure is not involved in the same way in the study of the barrel of a right arch as it is in the case of the skew arch. In the skew arch the horizontal forces constitute couples producing a large rotating effect on the whole structure as indicated in Fig. 149.



Plan of Skew Arch

Fig. 149

As to the commonly recognized problem of the skew arch, the action of the arch barrel under symmetrical vertical loading, the question is apparently not very serious. The statement has sometimes been made that the line of pressure for such loads will "~~follow the~~ shortest path" as shown by ABA in plan Fig. 150 and common sense is said to indicate this.

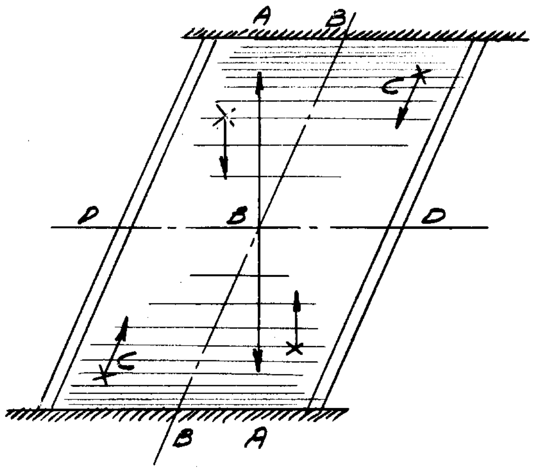


Fig. 150

Apparently this cannot be true, because pressure line for loads C-C cannot be normal to the abutments but must be skewed and even if pressure lines for central loads were square (to the abutments) the line of total pressure would be skewed.

The Six Limitations on Deformation. Assume the thrust line in elevation along DD to be the same as for a right arch and in plan along BBB. This satisfies the laws of statics and of symmetry. There exist six limitations on deformation here, only three of which are significant in right arches. These are that one abutment shall not move with reference to the other along any of three non-concurrent axes nor shall it rotate about these axes.

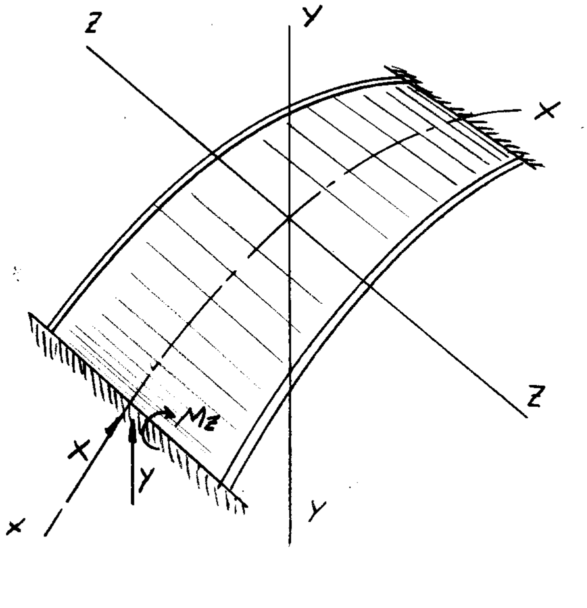


Fig. 151

The condition of no relative movement of abutments along axes X and Y, Fig. 151, and of no relative rotation about axis Z would now be satisfied if the arch were right. Also, the conditions of no relative movement in the Z direction and of no relative rotation about X or Y would be satisfied if the arch were right.

But the forces X, Y, M_Z and the loads acting alone on the skewed cantilever will cause the abutments to move relatively along axis Z due both to horizontal shear on layers parallel to Z and to the relative skew of these layers and the consequent effect of their shortening due to thrust. These factors do not seem to be very great. Additional skew movements are produced by the moment M_Z.

Now these six variables can be tied in by independent equations just as was done for three variables in right arches, but any such analysis will depend on considerations of shear distortion due to shears such as that along Z and due to torsion about axis X.

We know little about shear distortions in any case and practically nothing about torsion in wide thin curved shafts. About the best that can be concluded is that for symmetrical loading the pressure line in elevation is the same as for a right arch and in plan lies between the skew and the square direction. Apparently in plan it more nearly follows the skew; to assume that it does is in general on the safe side so far as the crown is concerned and possibly on the unsafe side as regards stresses at the obtuse corner of the springing.

As regards unsymmetrical loads, it does not seem probable that they control the design except in the case of heavy concentrations -- road rollers, for example. But studies of the effect of these lead back inevitably into the problems outlined above and so do not seem very promising.

As regards the rotating effect of the horizontal loads on arch barrel and spandrel walls, it is evident that these produce a concentration of pressure at the obtuse corner of the springing, and may produce tension at the acute corner of the springing line. It seems on the safe side to consider this moment resisted entirely by bending along the springing planes, using the common formula.

Analysis, based on a pressure line along the skew for vertical loads and on a resistance of rotating moment by two bending moments about vertical axes parallel to the springing lines will give a safe and not excessively expensive design for both arch barrel and abutments.

Temperature and Shrinkage Effects in Concrete Arches. This subject has been briefly discussed elsewhere, but the general argument will bear reexamination. The thesis is that rise and fall of the arch crown gives a clear and readily interpreted measure of those internal readjustments which accompany temperature changes and shrinkage in the concrete and hence that measurements of rise and fall of crown furnish fundamental data for the design of arches.

The rise and fall of crown due to internal deformations results from two causes; the rise and fall which would take place if the arch were not restrained at ends and the rise and fall resulting from such restraint.

As to the first of these elements there appears to be no reason for difference of opinion.

Δ h for the free arch = $h/L \, \Delta L$

As to the second element, it has been indicated elsewhere that Δh due to restraint = (influence ordinate for crown thrust due to a load at the crown) x ΔL. Now, there might at first seem to be some doubt as to the value of this influence ordinate, i_h. A little consideration, however, will show that neither rotation of abutments nor variation in modulus of elasticity of concrete, cracking of concrete, time yield, action of spandrel columns nor effect of saddles and expansion joints or even of hinges can greatly alter the center ordinate of the influence line for crown thrust.

For full uniform load the crown thrust is fixed pretty definitely as $1/8\ L^2/h$. For a parabolic arch this is exact if we neglect rib shortening, because there is no crown moment in a parabolic arch under full load. For arches not parabolic and for parabolic arches when the effect of rib-shortening is included it is very nearly exact, because the full load pressure line never -- in a practical arch -- departs far from the axis at the crown. This is still true even if the effect of the spandrel posts be included, because under full load the spandrel posts can produce only a small effect.

From this it follows that the area under the influence line is $1/8\ L^2/h$. The center ordinate is then dependent on the shape of the influence line. If this shape is triangular the center ordinate is $1/4\ L/h$; if parabolic, it is, $3/16\ L/h$; if of the usual double curvature form for a hingeless arch, it is, $1/4\ L/h$; if zig-zag due to the action of floor columns, it is still about $1/4\ L/h$.

If there is a break in the floor line due to expansion joints and the effect of spandrel post is pronounced, we may expect the influence line for horizontal thrust to be somewhat as shown dotted in Fig. 146, but even this does not materially affect the center ordinate.

If, then, we write,

$$\Delta h = (i_h + h/L)\ \Delta L$$

we may be reasonably sure that,

$i_h = 3/16 + 10$ per cent; h/L is the rise ratio and

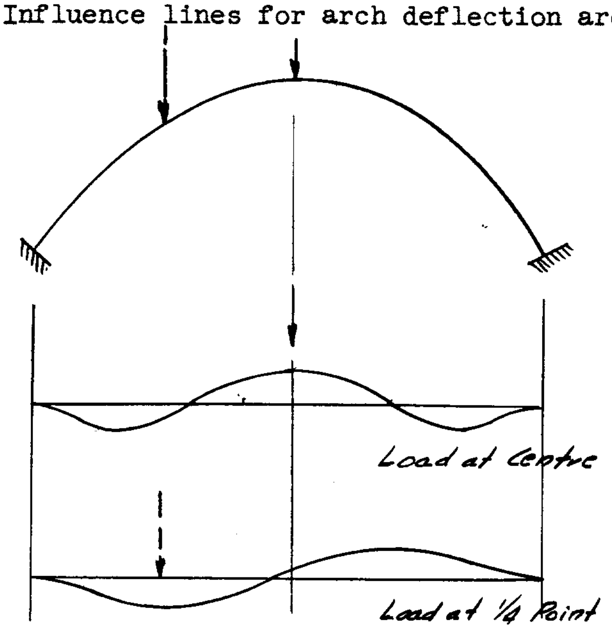
may be, let us say, 1/8 to 1/4.

Hence the factor $(i_h + h/L)$ is known in any case within 5 per cent to 8 per cent; which indicates a remarkably constant factor.

Deflection of Concrete Arches. The dead load deflections of concrete arches are of interest in camber computations. They are, of course, quite small, being due chiefly to the direct compression in the arch rib. In some cases, however, the bending due to dead load is as important a factor as is the direct compression. Either element of deflection is readily computed. Evidently great accuracy is not needed since a one hundred foot span gives a dead load deflection of about one-half an inch. To this is to be added camber for form settlement, which in the case of timber centering is of necessity more or less of an estimate but which can be computed with reasonable accuracy in the case of self-supporting steel centering.

Live load deflections are of only slight interest in the case of concrete arches, though the general pehnomena appearing here are of real importance in the case of steel arches and suspension bridges.

Influence lines for arch deflection are shown in Fig. 152. These may be drawn as deflection diagrams for loads at the center and quarter-point respectively. Evidently the diagrams in all cases show both upward and downward deflection areas, which for a parabolic arch are equal, since we neglect the effect of direct compression, there is no deflection at any point under full uniform load.



Deflection Influence Lines

Fig. 152

For the crown the maximum deflection will be found to be about one-half of one per cent and at the quarter-point about one per cent of that of a simple beam having the same depth at center. Since the center depth of the simple span will be perhaps three or four times that of the arch, the relative maximum deflection of the arch will be perhaps one-fourth to one-half that of the simple span. The total range upward and downward of live load deflection, however, may be almost, if not quite, as great as that of the simple span.

Almost certainly flexibility is a very important element -- perhaps the most important element -- influencing dynamic stress (impact). Impact and vibration certainly contribute largely to the deterioration of steel structures and it is possible that in hastening cracking they accelerate disintegration in concrete structures.

The following formula for deflection of a parabolic arch in which, $ds/Idx_2 = 1/I_c$ may sometimes be of value.

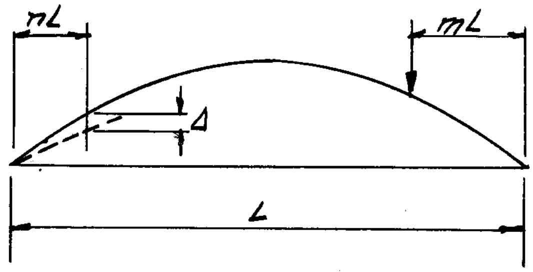


Fig. 153

$$\Delta = \frac{m^2n^2}{6}\left[3\ (1-m)\ (1-n)\ -mn\right]\frac{Pl^3}{EI_c}$$

$$- \frac{m^2n^2}{0.8}\left[(1-m)\ (1-n)\right]^2\frac{Pl^3}{EI_c}$$

In this equation where m and n are as defined in Fig. 153, the first term represents the deflection of the arch as a fixed-ended beam and the second term represents the effect of the arch thrust.

CHAPTER XIII
SWING BRIDGES AND LONG SPAN BRIDGES

General Treatment of Swing Bridges. This subject will be discussed only to show the indeterminate elements involved. The design of the machinery is in itself a specialty. Perhaps no field of structural engineering has had more ingenuity brought to bear on it than that of movable bridges. The more modern types of movable bridges - the rolling lift bridge of the Scherzer type, the trunnion bridge of the Strauss, Chicago, American or other type, the vertical lift bridge of the Waddell type - have largely supplanted the swing bridge; but the latter will probably always have a legitimate place. In all types, details of machinery, locking arrangements and provision for the break in continuity of the floor are of the greatest importance.

The swing bridge may turn on a pivot or on a turntable. In the former case it is simply a two-span girder. In the latter case it does not act as a three-span girder with a short center span over the circular drum of the turntable because it is not possible to develop the high shear which would result in this panel. The type would be used only for heavy trussed bridges and it has been shown that* the shear in the center panel is small no matter what size of bracing is used in this panel. Consequently, the center bracing is usually made nominal and the shear in this panel is assumed to be zero. Hence, $M_b = M_{b'}$, (Fig. 154) and we may treat the span as

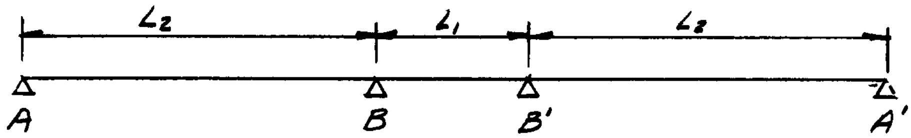


Fig. 154

a continuous two-span girder of spans L_2 having an elastic weight over the center pier equal to L_1/I_1 for the span BB'. Neglect of this added elastic weight is never serious and hence the trusses of a rim-bearing bridge may be satisfactorily designed for a center-bearing bridge having spans equal to the side spans of the rim-bearing bridge.

The ends of a swing bridge are usually lifted when the bridge is closed by means of wedges. These wedges are set to give a vertical movement such as computations show to be needed to produce a given end reaction, which is usually 150 per cent of the computed maximum uplift or negative reaction at the end supports from live load without impact.

For dead load stresses then, the end reactions are supposed to be determined and the stresses follow.

For live load stresses, the span may act either as a continuous girder or, if for some reason the wedges are not thrown, as a simple span.

Analysis of reactions as a continuous girder are satisfactory on the assumption of constant moment of inertia. Influence lines are conveniently constructed by combining those got by assuming the side span to act as a simple span with those for the effect of the end reaction.

*See "Modern Framed Structures," Johnson, Bryan and Turneaure, Vol. II.

For this condition, referring to Fig. 155:-

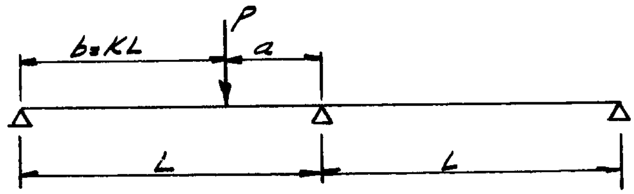


Fig. 155

$$M_a = \left(\frac{P\ a\ b^2}{L^2} + 1/2\ \frac{P\ a^2\ b}{L^2}\right) 1/2 = PL/4\ (K - K^3)$$

From this the end reaction is easily computed.

The technique of determining live-load maxima varies.

Equivalent uniform loading is reasonably satisfactory. If more exact values are wanted, it is well to spot the live loads by judgment using the criteria for ordinary cases applied to the approximate triangles of the influence lines. Several loads should be tried, the computations being perhaps most conveniently made by scaling influence ordinates and multiplying by loads. Criteria for curved influence lines are generally too complicated to be useful.

Critical Load Conditions. The critical load conditions, then, are:

I. Dead load swinging free.
II. Dead load with end reaction as determined.
III. Live load on a continuous beam.
IV. Live load on a simple span -- for end not held down.
V. Live load so placed that far end cannot rise.

There is some question as to the reasonableness of the last case, but its use is conservative.

Possible combinations are:

I alone.
I and III or I and V.
II and IV.

Continuous Turntables. Similar to the swing bridge is the continuous turntable which has come into use for handling long heavy modern locomotives. Ordinarily, however, turntables tip on the center pivot and hence act either as simple spans or as cantilevers when the engine is "spotted."

Rim-Bearing Swing Bridge - Moment at Turntable. The value of M in Fig. 156(a) cannot be found by the ordinary methods for continuous beams on unyielding supports if we assume no shear in span L_2, because the assumption of no shear in the center is impossible unless the turntable "tips"--that is, unless there is a relative movement of the center supports.

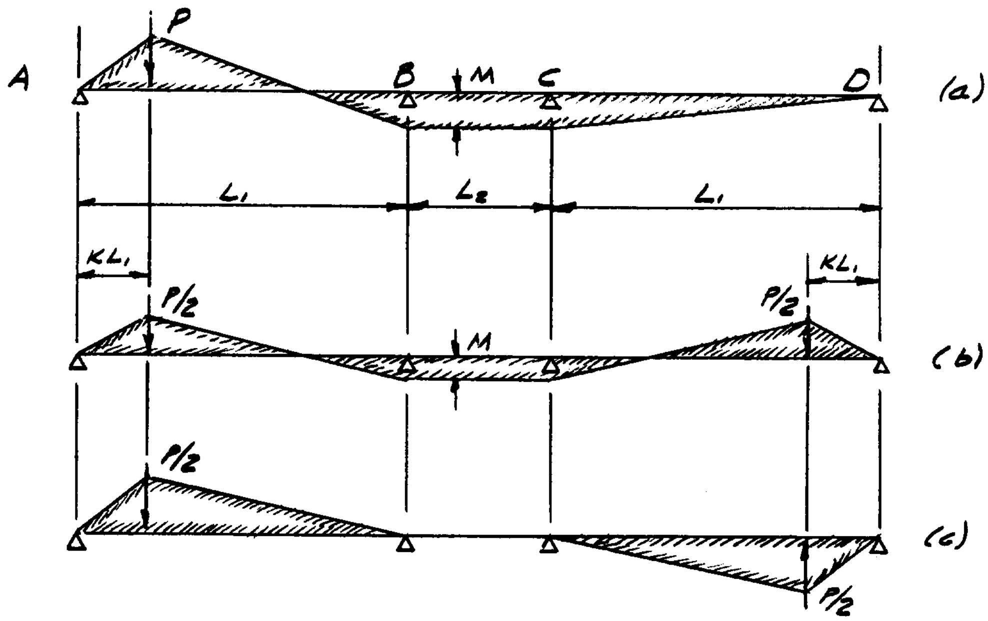


FIG. 156

We may arrive at a general expression for M as follows: Loading (a) = (b) + (c). Now for loading (c), M = o by symmetry. For loading (b) there is, again from considerations of symmetry, no tipping of the turntable and hence the ordinary analyses are applicable.

Hence in (b) K/C for AB = $1/L_1$

K/C for BC = $2/3 \; 1/L_2$

$$\text{Fixed-ended moment at B} = \frac{PL_1}{2} K^2 (1-K) + \frac{PL_1}{2} \frac{K(1-K)^2}{2} = \frac{PL_1}{2} \frac{(K-K^3)}{2}$$

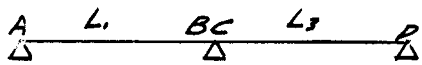
$$M = \frac{PL_1}{2} \frac{K-K^3}{2} \frac{2/3 \frac{1}{L_2}}{2/3 \frac{1}{L} + \frac{1}{L_1}} = PL_1 (K-K^3) \frac{L_1}{4L_1 + 6L_2}$$

For a center-bearing bridge,

$M = PL_1 (K-K^3) \; 1/4$

and since L_2 is usually not very large compared with L_1, it makes not very much difference whether a span is designed as center-bearing or as rim-bearing.

The value of M may be written directly if we consider that beam BCA acts like a two-span continuous beam AB and CD with an elastic weight $3L_2$ at the support. A little consideration,



will show this to be correct so far as rotation at B is concerned. Then directly,

$$M = PL_1 \frac{K-K^3}{2} \frac{L_1}{L_1 - (L_3 - 3L_2)} \qquad \text{If } L_1 = L_3,$$

$$M = PL_1 (K-K^3) \frac{L_1}{4L_1 + 6L_2} \qquad \text{as above.}$$

Double Swing Bridge with Shear Lock. Sometimes, for wide openings, double swing spans have been built, with a shear lock at the center for partial continuity. The problem does not often arise but has some theoretical interest.

The indeterminate element is evidently the shear at the shear lock. For loads in either the end or the middle span, Fig. 157, we can write,

loading a = loading b + loading c.

By symmetry, shear at the lock for loading b is zero. For loading c, the span ABC acts as a two-span beam simply supported at A and C.

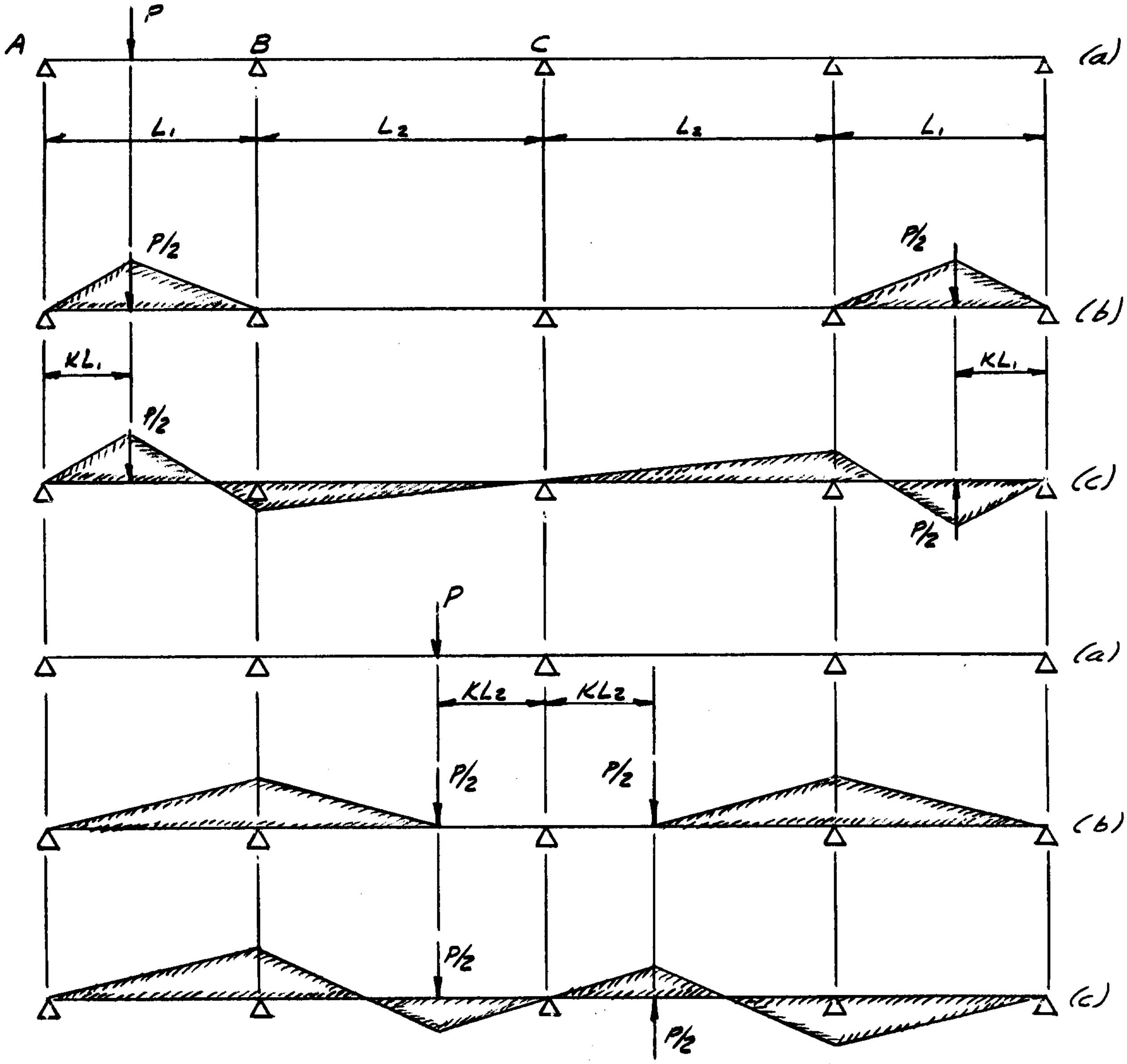


FIG. 157

For loads in L_1 $M_b = \frac{PL_1}{2} \frac{(K-K^3)}{2} \frac{L_1}{L_1 + L_2}$

$$V_c = \frac{P\ (K-K^3)}{4} \frac{L_1}{L_1 + L_2} \frac{L_1}{L_2}$$

For loads in L_2 $M_b = \frac{P}{2} L_2 \frac{K-K^3}{2} \frac{L_2}{L_1 + L_2}$

$$V_c = P \frac{(1-K)}{2} - P \frac{K-K^3}{4} \frac{L_2}{L_1 + L_2} = \frac{1-K}{2} \left[1 - \frac{K-K^2}{2} \frac{L_2}{L_1 + L_2} \right]$$

Problems Arising in Long-Span Bridges. The important and difficult problems in the design of long-span bridges result from the magnitude of the structure and the resulting importance of otherwise simple details rather than from statical indetermination. Long-span bridges may be simple spans, arches, semi-continuous, continuous, cantilever, or suspension bridges.

The following elements are of especial importance in the design of long spans: selection of the type of loading, and of the probable and possible combinations of congested loads with multiple lines of traffic; provision for future increase in live load; selection of kind of steel to be used; determination of working stresses; study of erection methods; proportions of structure as affected both by economy and rigidity; selection of span length to satisfy the engineers of the National Government (War Department); estimates of magnitude and distribution of dead load as a preliminary to design.

There is, of course, no standardization in this field and the combination of elements makes comparison of factors of safety and even of economy prohibitive unless the design specifications are carefully considered. There has been much difference of opinion among the best experts as to the relative economy of the various types and generalization in this field is unwise. Rigidities can be compared on the basis of working stresses, as shown elsewhere and will in general follow the order -- arch, simple span, continuous, semi-continuous, cantilever, suspension. But the relative rigidities may be changed by change in proportions.

The problems of stress analysis in simple spans and in cantilevers do not properly belong here, since these structures are statically determinate. Arches and suspension bridges are treated elsewhere.

Continuous trusses are, in modern types, relatively new in this country, the most notable examples being Sciotoville, Allegheny River and Nelson River. Sciotoville has two spans continuous, the other two have three spans continuous. Queensboro Bridge is semi-continuous for five spans, the second and fourth spans being hinged. This presents a slight complication in analysis but one which does not justify much detailed discussion.

For special information on long span bridges, the reader is referred to "Design of Steel Bridges," Kunz, and to articles dealing with the individual structures.

Stress Analysis in Continuous Trusses. Continuous trusses may be conveniently analyzed by the use of elastic weights, because in general the curvature of the chords makes these lie within or just outside of the span. In the end spans influence lines are most readily drawn by combining those for simple beam action with those for the end reaction. In an intermediate span simple beam influence lines may be conveniently combined with those for moments at the piers. If the elastic weights of the web members are brought in at the proper panels the procedure will be similar to that for drawing influence lines for continuous girders of variable moment of inertia.

A problem arises as to the elastic weight of the pier vertical, Fig. 158.

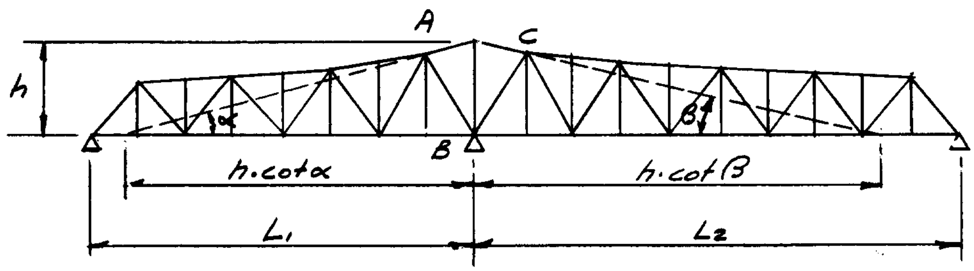


Fig. 158

The effect of stress in this vertical is to change the angle ABC and its elastic weight lies at B. It may be found by virtual work, Sul/AE.

$$S = M_b/h\ (\tan\alpha + \tan\beta)$$

$$u = 1/h\ (\tan\alpha + \tan\beta)$$

$$Sul/AE = M_b/h^2\ (\tan\alpha + \tan\beta)^2\ h/AE$$

If $M_b = 1$

$$w = \frac{h}{(h\cot\alpha + h\cot\beta)^2\ AE}$$

Evidently the elastic weight is small. This also indicates the general procedure for finding elastic weights of truss bars.

Another method of analyzing continuous trusses which is in some ways simpler than the direct use of elastic weights is the computation of angle weights along the loaded chord as was done in the study of two-hinged arches. Continuous trusses are simple in that they are of only two or three spans, but the analyses are tedious and hence a simple routine becomes important.

It is sometimes suggested that in studying continuous trusses and swing bridges closer preliminary analyses result from assuming a trapezoidal variation of the moment of inertia or from taking the elastic weights of chord members only. Experience seems to show that the assumption of constant moment of inertia gives results that are closer to the true values than any of the more complicated methods.

The effect of various assumptions on the influence ordinates is shown by comparative data for Sciotoville as given by Steinman, "Long Span and Movable Bridges," Hool and Kinne, p. 235. The errors from variation of truss depth and that from neglect of web seem to be more or less compensating, perhaps because the inclined chord brings the web into play in resisting moment. Note that the fact that neglect of shear (web) does not very much affect the influence ordinates does not mean that the effect of the web on the deflections is negligible, but only that this effect is similar in both numerator and denominator of a fraction.

Semi-Continuous Trusses - Queensboro Bridge. For purposes of general study these may be treated on the assumption of constant moment of inertia. Fig. 159 shows the arrangement of spans in the case of the Queensboro Bridge over the East River at New York.

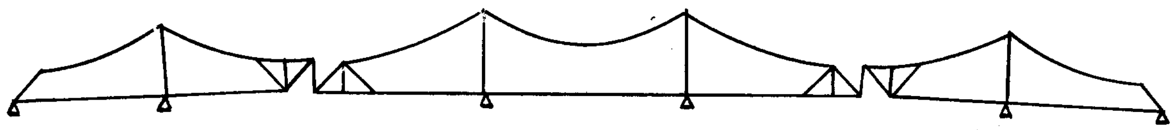


Fig. 159

Here, influence lines may be drawn for moments and shears in the beam using the usual methods except in finding the properties of the spans containing hinges. In these, the end moments, rotation values, and carry-over factors are readily found by the column analogy, the elastic weight of the hinge being infinite. Hence the centroid of elastic weights lies at the hinge and the total elastic weight is infinite. The expression K/C here is indeterminate.

1. The problem may also be studied by treating it first as a continuous beam and then neutralizing the moment at the hinge, as suggested in secondary stresses. Another method, applicable in symmetrical cases, is indicated under swing bridges. For final analysis the stresses in the hangers may be treated as redundants and found by simultaneous equations of displacement.

Suspension Bridges - The General Theory. It is intended here to use the general theory of the suspension bridge as an excellent illustration of the application of general methods of stress analysis in indeterminate structure. The important problems of fabrication and of erection are not discussed and the more important phases which enter into the design of a large suspension bridge are omitted. In special cases the suspension bridge has a proper place even for quite short spans and its general theory should be available to all bridge engineers.

The suspension bridge may have a stiffening truss with one or two or three hinges and the flanking spans may or may not be supported. It is essentially an inverted arch, as will appear, and if the cable is used as a top chord member in the stiffening truss, it is obviously an inverted arch.

The function of the stiffening truss is to restrain local deformations of the cable under concentrated live loads. The hangers are adjusted so that under dead load at normal -- that is, closing -- temperature the truss is not stressed. Since the dead load is chiefly that of truss and floor, the cable is practically a parabola under dead load. The function of the stiffening truss is to keep it parabolic.

<u>Influence Line for Cable Pull - Approximate Method</u>. As in the two-hinged arch influence lines are built on that for thrust, so here influence lines may be built on that for pull in the cable at center.

The shape of the influence line for H may be determined by considering the cable shortened. This throws a uniform load on the stiffening truss since the cable remains a parabola and the shape of the resulting deflection curve can be determined.

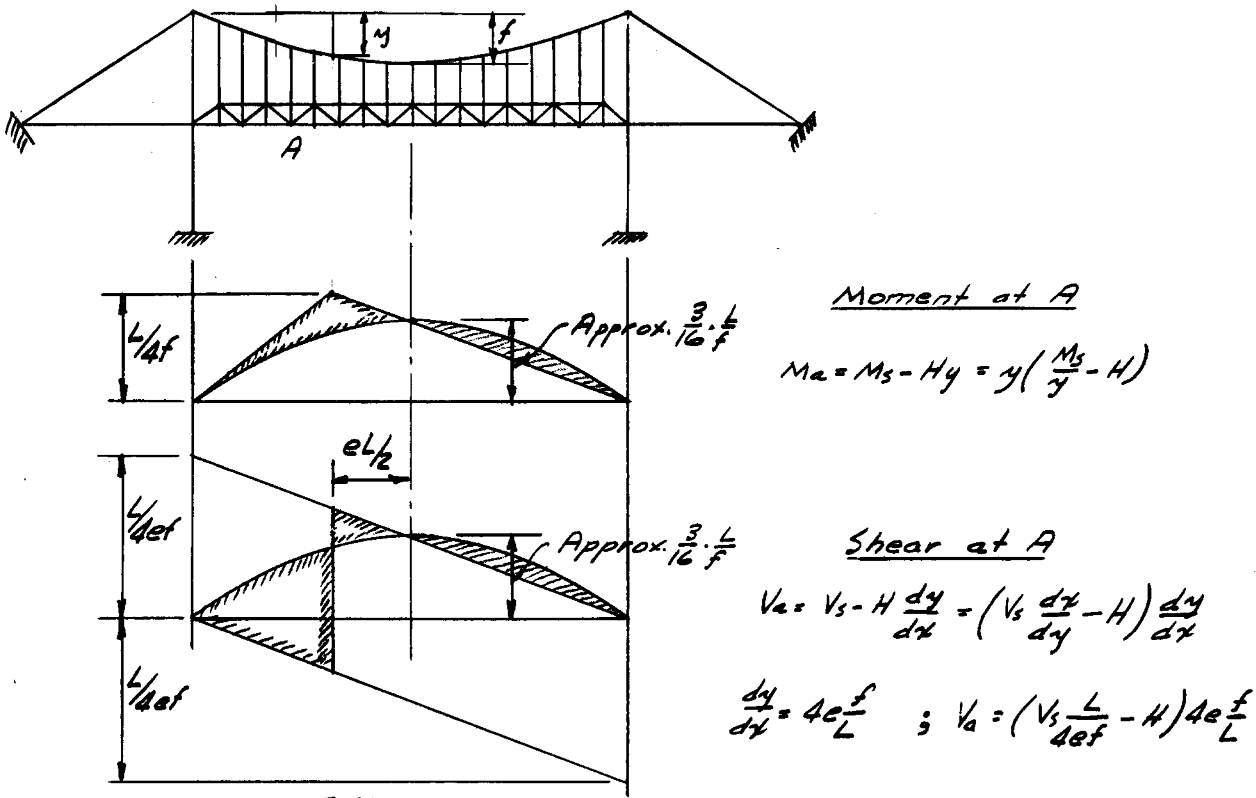
The area of the influence line for H is determined by the condition that for full uniform load there is no bending in the truss. The assumption that there is no bending in the truss under full live load is evidently not quite correct for if the hangers are adjusted without live load, then under live load the towers will shorten, the cable will stretch and the hangers will elongate and the truss will be bent. But these results of direct stress are small considering the flexibility of the truss just as the effect of shortening the columns is small considering the flexibility of the girder in a simple bent. Apparently the total error resulting will not exceed 5 per cent.

<u>Effect of Cable Stretch - Exact Theory</u>. This summarizes what is known as the approximate theory of the suspension bridge. It neglects an element which in the case of long bridges may be quite important. When the live load causes the cable to stretch it changes its sag. The previously computed stress in the cable from dead load is now more than sufficient to sustain the dead load and should be reduced. Evidently the same result follows from decreasing the H for live load by this amount. Hence, under full live load not only is there a bending in the truss resulting from the deformation of the cable, hangers and towers, but there is also a bending due to the change in dead load cable stress produced by this sag. The result in long bridges is to reduce considerably the maximum moments in the stiffening girder. This is the so-called exact method. The effect may be evaluated by successive approximation. The phenomenon of correction of computed stress due to change in shape of the structure under stress is always theoretically present. Apparently nowhere else is it of any significance, however, its importance here being due to the great flexibility of the construction.

<u>Typical Influence Lines for Shear and Moment</u>. In the case of the two-hinged girder if we assume constant moment of inertia in the stiffening truss, the equation of the deflected structure is that for a beam of constant section uniformly loaded. If desired, this curve may be revised for variation of section in the stiffening truss. The effect is very small, as our studies in continuous girders would lead us to suspect.

If the side spans are suspended, the three-hinged span is evidently not affected, since it is statically determinate. In the case of the two-hinged and hingeless girders, the total area of the influence diagram for both center and side spans is $1/8 \; L^2/f$ and the center ordinate is reduced accordingly depending on the sag and relative stiffness in the side spans. The hanger pull per foot of span in the side spans is, of course, f/f_1 (where f is the vertical sag of cable in the main span and f_1 is the vertical sag of cable in the side span) times that in the center span in order that equilibrium may exist in the cable.

The effect of these influence lines for cable pull may now be combined with those for moments and shears in the stiffening girder considered as an unsuspended span -- simple if hinged, and continuous if hingeless.



Influence Diagrams Fig. 160

Fig. 160 shows typical influence lines for shear and moment in a two-hinged suspension bridge with side spans free. From such lines maximum shears and moments may be obtained.

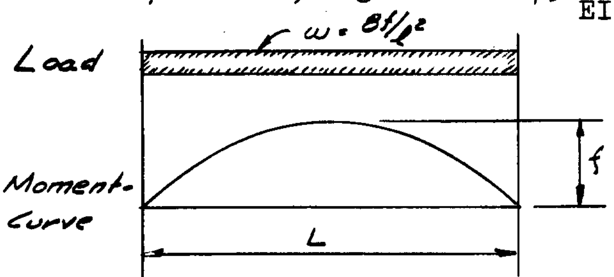
For a valuable collection of formulas and curves for the design of suspension bridges, see "Suspension Bridges," Steinman.

Stresses occur in the stiffening girders, if two-hinged or hingeless, just as in arches and the same approximate methods of determining H are applicable. H may also be determined as follows:

In a two-hinged truss with side spans free assume a unit value of H.

If H = 1, hanger pull = $8f/L^2$.

Work in girder due to uniform load $8f/L^2$ is twice statical moment about base of M/EI curve, Fig. 161 = $2\,2/3\ \frac{fL}{EI}\ 2/5\ f = 8/15\ \frac{f^2L}{EI}$



$$8/15\ \frac{f^2L}{EI}\ H = \epsilon\, t^oL$$

$$H = 15/8\ \frac{E\,\epsilon\, t^oI}{f^2}$$

This is the same expression as for two-hinged arches. It may be corrected for work in the cable if desired.

Effect of Hinges in the Stiffening Truss and Suspension in End Spans. Fig. 162 (a), (b), (c), and (a'), (b'), (c') shows the shapes taken by the three types of girder when the center span only is loaded with uniform load -- side spans not suspended -- and when all spans are loaded with uniform load -- side spans suspended. Where the side spans are not suspended, there is now no difficulty in writing the equations of the deflection curves in each case. Here the total area in the center span is $1/8\ L^2/f$, or if we allow 5 per cent for the work done in the cable -- the towers and hangers add less than 1/4 per cent -- the total area is $1/8\ 0.95\ L^2/f$. It is not difficult to compute the total work done in trusses, cables, towers, and hangers by a uniform hanger pull and check or revise this allowance of 5 per cent. With both shape of curve and area known, the center ordinate may be computed.

Thus, with suspended side spans, we have,

For three hinges -- $1/8\ L^2/f = 1/2\ i_o L \qquad i_o = 1/4\ L/f$

For two hinges -- $1/8\ L^2/f = 2/3\ i_o L \qquad i_o = 3/16\ L/f$

assuming the deflection curve to be a parabola. For no hinges, $i_o > 3/16\ L/f$ $< 1/4\ L/f$ depending on the stiffness of side spans. For absolutely rigid side spans the curve would approximate that for a fixed arch.

Deflection of Suspension Bridges. The suspension bridge is ordinarily a very flexible structure because of the small depth of the stiffening truss, the depth being determined by the small sections consequent on the small moments. Ordinarily the stiffening truss relieves the cable of not over ten per cent of its live load stress. If the truss is made deeper, the temperature stresses are increased somewhat, the truss carries a somewhat larger proportion of the load and details may be less satisfactory. Evidently the deflection for given working stresses is determined by the depth as we have seen in other cases. Brooklyn Bridge has a truss depth of 17' in a span of 1600', Manhattan 24' in 1470', Williamsburg 40' in 1600'; compare the Queensboro cantilever with a truss depth from 45' to 185' in 1200' span.

Data on Manhattan Bridge. For a masterly analysis of special problems in the design of long-span suspension bridges, see "Modern Framed Structures," Johnson, Bryan and Turneaure, Volume II. The following data from the study there made of the Manhattan Bridge gives some idea of the importance of some of the factors:

Manhattan Suspension bridge,
  Main span, 1447' c to c pins
  Side spans, 714' c to c pins -- suspended
  Sag ratio 1/10
  Depth of stiffening truss 24'
  Computed deflections - down 15' due to live load,
    up 8' due to live load,
    ± 2' due to temperature.

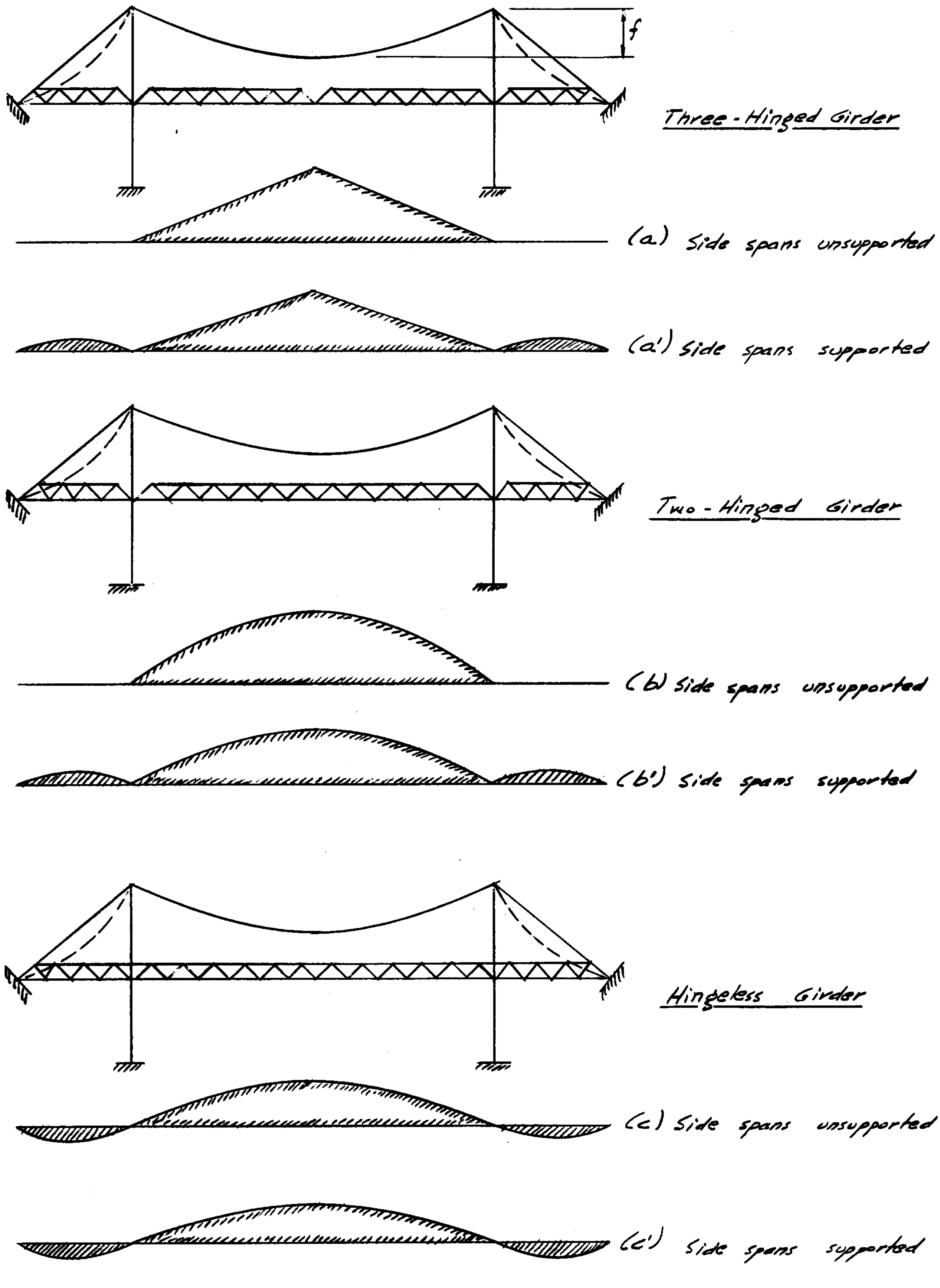

f
Three-Hinged Girder
(a) Side spans unsupported
(a') Side spans supported
Two-Hinged Girder
(b) Side spans unsupported
(b') Side spans supported
Hingeless Girder
(c) Side spans unsupported
(c') Side spans supported

FIG. 162

Stiffening truss relieves cable of 8.4 per cent of stress due to live load.

Variable moment of inertia in the stiffening truss reduces the maximum moment at quarter-point by 3 per cent compared with assumption of constant depth.

Effect of variable hanger pull is negligible.

Exact analysis based on cable deflections reduces the positive moment at the quarter-point by 45 per cent and that at the center by 34 per cent.

References and Comparison of Types. Provision for movement at the top of the tower, by saddles, or hinging the base of the tower, or in the elastic deformation of the tower itself can only be mentioned. For interesting studies of erection see,

"Suspension Bridges," Steinman;
"Movable and Long Span Bridges," Hool and Kinne, (Chapter on Suspension Bridges by Steinman);
"Erection Methods for Delaware River Bridge," R. G. Cone, Eng. and Cont., July 1926;
"Construction of Parallel Wire Cables for Suspension Bridges," a pamphlet especially describing Bear Mountain Bridge, issued by John A. Roebling and Sons Co., 1925.

The table in Fig. 163 shows a comparison of types and is compiled from data given by Turneaure in the book referred to above. It will be noted that continuity increases the moments. Continuous trusses, however, give a more rigid structure.

Comparative Data on Suspension Bridge Types.

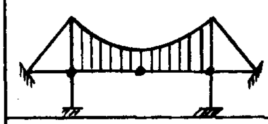
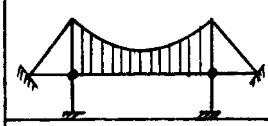
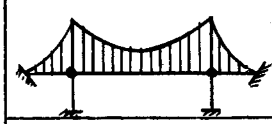
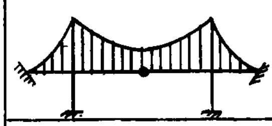
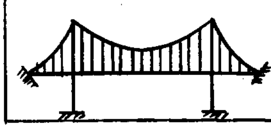
TYPE OF BRIDGE (HINGES AT •)	H LL and Temp. (%)	M_c LL and Temp. (%)	M_d LL and Temp. (%)
	100 ± 0	0 + 0	± 100 ± 0
	96.6 ± 1.2 = 97.8	+47.7 + 8.2 = + 55.9 -29.6 - 8.2 = - 37.8	+ 93.1 + 6.1 = + 99.2 - 77.5 - 6.1 = - 83.6
	96.5 ± 1.2 = 97.7	+68.3 + 7.9 = + 76.2 -48.5 - 7.9 = - 54.4	+103.8 + 5.9 = +109.7 - 87.0 - 5.9 = - 92.9
	107.0 ± 1.8 = 108.8	0 ± 0	+107.2 + 3.0 = +110.2 -114.1 - 3.0 = -117.1
	95.8 ± 4.4 = 100.2	+96.0 + 10.5 = +106.5 -23.2 - 10.5 = - 33.7	+ 83.1 + 5.1 = + 88.2 - 44.1 - 5.1 = - 49.2

Fig. 163

CHAPTER XIV
INTERNAL INDETERMINATION

Significance of Internal Indetermination. Thus far we have discussed cases in which the indetermination resulted from redundant reactions. Internal indetermination results in the case of trusses from there being more bars in the truss system than are required to satisfy statics. Such trusses were formerly quite common and are still sometimes built.

All beams are, of course, indeterminate by statics as regards internal stresses, and are analyzed only by the use of convenient approximations.

It is intended here to point out some general facts and relations which bear on problems in internal indetermination.

In the first place, indeterminate structures cannot, in theory be accurately designed by ordinary methods for definite fibre stresses. This follows from the fact that when the section of all bars but one have been determined, the fibre stress in this bar is determined by the theory of deflections and may be largely independent of its area. In the more important cases, where any individual member or section is only a small part of the whole, this fact is of little or no practical importance, but it is sometimes significant.

From this it follows that there is a tendency to self-interference with efficiency in a structure which is statically indeterminate and the argument that such is the case did much to discourage the use of such structures in this country. In most cases the argument, while theoretically valid, is not practically a control on economy.

The King-Post Truss - Problem. Assume the problem of reinforcing a 24" I 80 lb. by sag rods to carry at a working stress of 16,000 lb. per sq. in. in bending a central load of 40,000 lb. on a 40 ft. span as shown in Fig. 164.

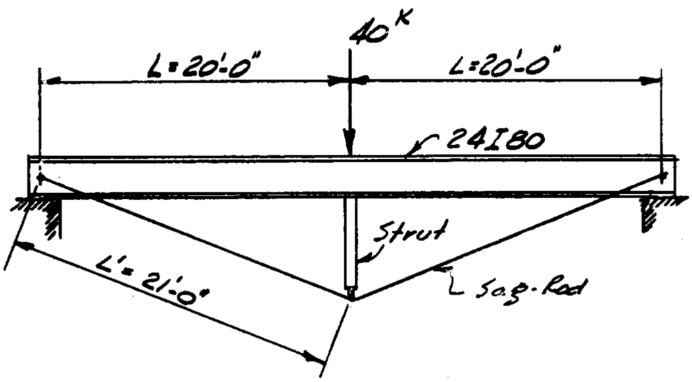


Fig. 164

Assume the strut so large that its deformation may be neglected. Maximum fibre stress in beam = 16,000 lb. per sq. in.

Moment at center = $\frac{16,000 \times 174.0}{12}$ = 232,000' lb. Net load at center = $\frac{4 \times 232}{40}$ = 23.2K Up thrust of strut = 40 - 23.2 = 16.8 K

Deflection of I at center = 1/12 $\frac{16,000 \times 40 \times 40 \times 12 \times 12}{Ec \times 12}$

$= \frac{25,600,000}{E}$

Let fibre stress in rods be f

Deflection of point A due to stretch of rods (shortening of strut and beam neglected) $2\frac{f}{E}$ u l = $\frac{f}{E} \times \frac{21}{6}\, 21 \times 12 = 882\,\frac{f}{E}$

$$882\ f = 25{,}600{,}000$$
$$f = 29{,}000 \text{ lb. per sq. in.}$$

$$\text{Area required} = \frac{16.8}{2} \times \frac{21}{6}\,\frac{1}{29} = 1.01 \text{ sq. in.}$$

The stress in the beam due to direct thrust is,

$$\frac{L}{L'}\,\frac{A_r}{A_b} = \frac{20}{21}\,\frac{A_r}{A_b} = \frac{20}{21}\,\frac{1.01}{23.32} = 4{,}200 \text{ lb. per sq. in.}$$

Total maximum unit stress in beam = 20,200 lb. per sq. in.
Total maximum unit stress in rods = 29,000 lb. per sq. in.

From this analysis it is evident that the fibre stresses cannot be chosen at random but will depend on the proportions of the structure This condition occurs in other cases; thus in a reinforced concrete beam the ratio of the fibre stresses in compression and in tension is definitely fixed by the percentage of steel and in a homogeneous beam the ratio of these stresses is determined by the shape of the beam.

Another interesting and important case occurs in the rim-bearing swing bridge, where the fibre stresses in the bracing over the turntable is independent of the size of this bracing and so small as to be negligible.

Indetermination in The King-Post Truss. The analysis of the King-Post truss above is not of great practical importance. This analysis tacitly assumed that there was no initial stress in beam or rods. Actually by turning up or loosening the nuts at the ends of the rods, the maximum stress in either beam or rod may be varied anywhere between the limits of no bending in beam and no stress in rod.

This illustrates the very great importance of considering in an indeterminate structure whether the precision of fabrication and method of erection used assures no initial stress or a known condition of stress or else whether the uncertainty is important. The solution of the problem is often to erect with most of the dead load as a determinate structure and then close at a predetermined temperature, the structure acting as an indeterminate structure for live load and perhaps for some of the dead load and for temperature changes from that at closure. In the case of important structures such procedure is worth while, but in smaller structures it is not done and the indeterminate elements may be determined by chance rather than by elastic distortion.

Distribution of the Load in a King-Post Truss. The purpose of this section is to try to give definite meaning to several terms which are carelessly used in structural engineering, often with seriously misleading results. We speak of the "distribution" of load between two members, of the "path" that a stress follows, of the "readjustment" of load distribution, of "carrying" a load in a certain way. Now loads are not distributed or carried nor do stresses travel along certain paths. The term stress is, after all, a rather vague picture of a complex molecular phenomenon of which we know little and acquires meaning only in terms of the statics of particles. We may, then, speak of loads as being carried and of stresses as traveling only by analogy -- unless we wish to go much further in our thinking than the structural engineer expects to go.

If used with due caution by one accustomed to thinking in terms of deformations, however, the pictures which these terms represent are very useful. Thus, in the King-Post truss above, we may say that of the total load of 40K, 58 per cent is carried by the beam and 42 per cent by the truss and that these ratios are determined by the condition that the deflections of beam and truss shall be the same.

Flitched Beams. Another illustration is that of a beam made up of two I beams bolted onto the sides of a timber beam. Thus consider the beam of Fig. 165, made up of two 18" x 55 lb. I beams and an 18" x 18" timber stick bolted together. Then at all points the deflection of the two is the same. But, $\Delta \propto PL^3/EI$. Since the spans are the same, we may conveniently say that the load is distributed between the parts of the beam in proportion to the values EI, provided we understand that we are using a convenient figure of speech.

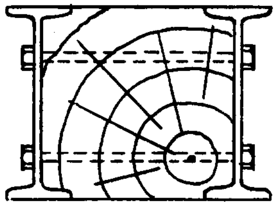
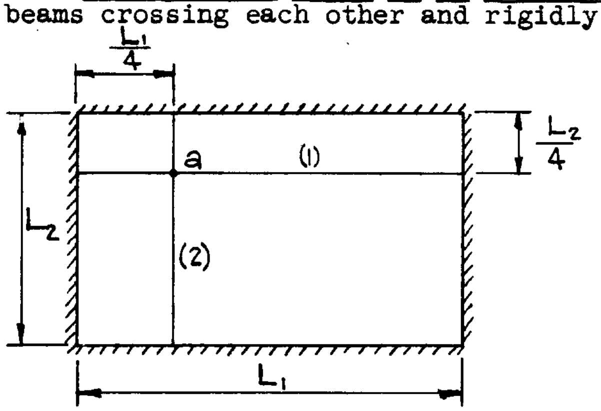


FIG.165

Intersecting Beams as an Indeterminate Problem. Again, assume two beams crossing each other and rigidly attached as shown in plan in Fig. 166. A load applied at a will produce the same deflection in the two beams. $\Delta \propto L_1^3/EI$ and if both beams are of the same material the load is distributed between them in proportion to I/L^3, provided the ratio of the segments is the same in the two beams. If the segment ratio is different, the ratio of the deflection coefficients must also be included.



Plan
FIG.166

If, in this case the load is not at a but at some other point on beam (1), we may find the reaction from beam (1) at a as a continuous beam with no settlement at a. This reaction may then be "distributed" between the two beams.

Rigidity of "Stress Paths." In all of the above cases we may find the ratio of stresses. Thus, in the King-Post truss we found that the ratio of stress due to bending in beam to stress in bars was 16:29. In the flitched beam of Fig. 165, $\Delta \propto fL_1^2/Ey$ and the stresses are proportional to the values of E, that in the steel being about 20 times as great as that in the wood. In the case of the crossing beams with load at a the stress varies as y/L^2.

In all of these cases it is possible to say that the load -- or the stress -- tends to follow the stiffer path and is distributed between the paths in proportion to their stiffness or rigidity.

It is easier to estimate the stiffness of a "path of stress" than it is to compute it. Actually the stiffness is the reciprocal of the deflection at the point where the force acts due to a unit value of the force. We are already familiar with it in distributing moments -- we distribute unbalanced moments between two beams in proportion to their stiffness or in inverse ratio to the rotations at their ends due to a unit moment.

These deflections may often be approximately visualized and will guide the judgment in some very complicated cases. The principle, of course, is often recognized vaguely without accurate statement. An interesting case of its neglect sometimes occurs in concrete construction where a designer attempts to frame short beams into a long girder. The actual condition is that the reaction is distributed at the junction between beam and girder in proportion to their stiffness (reciprocal of deflection at junction due to unit load there) and most of it may go to the beam; thus, the beam may "carry" the girder with serious results. Of course, this redistribution of reactions always occurs to some extent, but is not ordinarily important because of the much greater stiffness of the girder.

The Path of Stress in Lug Angles. As another application of the conception of "path of stress" where we can qualitatively visualize the effect but cannot evaluate it, consider the distribution of stress in rivets of a single angle connected with a lug, Fig. 167. It is evident that a unit force acting at a will produce less total deformation if it follows the path a b c, c, c into the plate than if it follows the path a b d e, d e, d e. If we think the total stretch will be very much less along a b c we conclude that the lug is nearly useless, which seems to be the case.

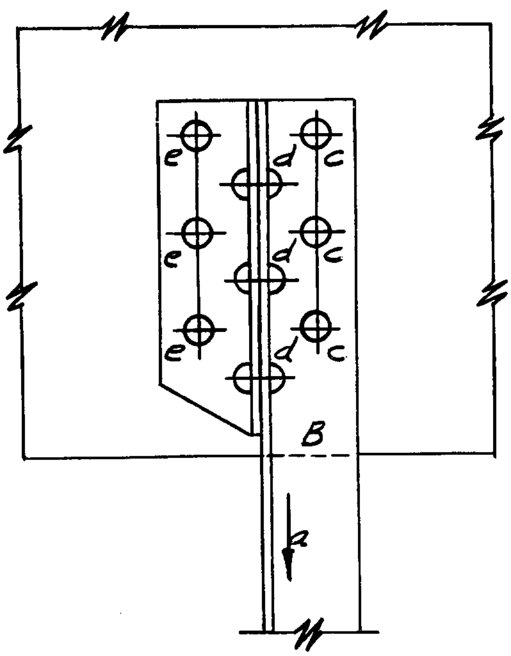


Fig. 167

The argument from "rigidity of stress paths" is rigidly accurate in so far as we can compute or estimate those rigidities without omitting some important element. It is, of course, obvious from the fact that the total deformation is the same by whichever path; and it is also derivable from -- or rather is identical with -- the principle of least work. It is one of the most useful conceptions in structures; and it may be one of the most dangerous.

Combinations of "Stress Paths." The use of lug angles is a result of a fallacy which is common in structural thinking and has the implied support of high authority. This is the idea that if sufficient total resistance to failure is provided, failure is thus prevented. This may or may not be true depending not only on the ultimate resistance of the elements which resist failure but also on their relative stiffness. If the resistances have the same ratio as the stiffnesses, then it is safe to consider only the total resistance; otherwise not. Thus timber and steel do not work well in tandem, because the resistance of steel is about ten times that of timber and its stiffness about twenty times as great. Hence the steel will fail without utilizing the full strength of the wood.

Numerous other illustrations will suggest themselves. For an interesting discussion of the false notion that it is immaterial what distribution of moment between support and center is assumed in designing concrete beams, see "Principles of Reinforced Concrete Construction," Turneaure and Maurer, p. 314.

Trusses with Redundant Members. If a truss has more bars than are needed to ensure stability it is internally indeterminate. The classical method of finding the degree of indetermination is to say that for every joint we have two equations of statics. But the equations of statics are not independent, for we have already fixed one end joint in two directions and the other in one direction in finding reactions by statics. This gives 2J - 3 equations and b unknowns. If these are equal and the bars are not misplaced -- two diagonals in one panel and none in another -- the structure is internally determinate; if we have too many bars, the truss is indeterminate, if all bars can take stress of either kind -- that is, provided we do not have a case of counters. If the frame is not in a plane but in three dimensions, we have 3j - 6 equations to find the stress in b bars.

More practically the indeterminate truss is usually a modification of a simple determinate form and the convenient method is to identify this basic determinate truss and then add joints and bars until we get the truss being studied. If we have added two bars for each joint the truss is still determinate. Otherwise the number of additional bars determines the degree of redundancy.

Method of Analyzing Redundancy. Single redundancy is readily solved on the basic principle of continuity. This, in Fig. 168, the relative movement of A and B is the same whether figured along the path AB or along any other path. Hence, $\sum \frac{Sl}{AE} u$ is the same whether the summation includes AB or includes all other bars. For other bars than AB, S is the total stress = S' with AB removed and S" due to stress in AB. U is the stress due to unit pull along the line AB.

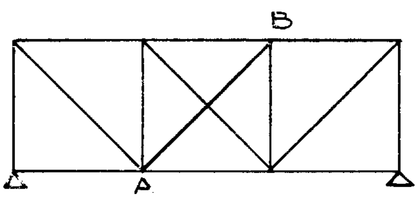


Fig. 168

For bar AB, S is the total stress (= x)
U is unity.
Then S' + S" = S' + ux

$$\frac{S'ul}{AE} + X \sum \frac{u^2 l}{AE} = X \frac{L_{AB}}{A_{ab}E}$$

$$X = \frac{\sum \frac{S'ul}{A}}{\sum \frac{u^2 l}{A} + \frac{l}{A}} = \frac{\sum \frac{S'ul}{A} \text{ excluding AB}}{\sum \frac{u^2 l}{A} \text{ including AB}}$$

The whole method and the resulting expression is not at all different from that already deduced for the stress in the tie of a two-hinged tied arch.

If the redundancy is multiple, simultaneous equations should be written, but they are simple in form.

In cases of multiple redundancy a satisfactory study of the situation may often be made by assuming reasonable values for redundant stresses and then drawing a Williot diagram. Discrepancies in the diagram indicate that the assumed values were in error and it is sometimes possible thus to estimate quickly the importance of this error. Evidently the first estimate must have been reasonably accurate if the method is to be useful.

Multiple Intersection Trusses. Multiple intersection trusses were formerly common in this country. Notable were the two-system and three-system Warren or lattice trusses, the two-system Warren with verticals, and the Whipple truss -- the last an especially popular and serviceable type -- and numerous others. The Bollman and Fink types, though usually containing indeterminate elements in the counter bracing, are not essentially indeterminate trusses in spite of their complicated appearance. Much interesting information on these types is to be found in "History of Bridge Engineering," Tyrrell.

For a presentation of approximate analyses, see "Theory of Structures," Spofford, Chapter 6, and for an unusually scholarly treatment of the whole subject, see "Modern Framed Structures," Johnson Bryan and Turneaure, Volume II.

The usual method of analysis is to break up the web into two systems and analyze as two trusses each carrying its own loads, finally adding the chord stresses. If the web systems are not tied together by verticals, this method will apparently give stresses in the web theoretically correct to within a few per cent if the systems are symmetrical. An odd number of panels is to be preferred because of this symmetry. Here again we may say loosely, that the stress paths of the two web systems are about equally stiff and will be stressed and deflect together.

If the web systems are tied together by verticals, the shears in the diagonals in any panel are equalized and such trusses are best analyzed by assuming the stress equal in the two diagonals. If, as is almost invariable, the diagonals in any panel are alike, the shear is equally distributed between them.

This truss, the double intersection Warren, with verticals, is the only one which has survived. Though not common, it has several points in its favor for railway bridges, especially simple details and, apparently, low secondaries. The slight indetermination in these trusses is not a serious objection to their use.

On the other hand, the argument has appeared at times that such structures have an advantage in safety because the failure of one member does not necessarily bring on collapse. The argument is of interest because it is apparently reappearing in connection with other more modern forms of indetermination.