CE371 Structural Analysis I

Curvature formulation

Let $v = v(x)$, i.e. the deflection v is a function of position, x, along the element.

The curvature φ of the element at x is defined by

$$\varphi = \frac{d\theta}{ds} = \frac{\dfrac{d\theta}{dx}}{\dfrac{ds}{dx}} = \frac{\dfrac{d\theta}{dx}}{\sqrt{\left(\dfrac{dx}{dx}\right)^2 + \left(\dfrac{dv}{dx}\right)^2}} = \frac{\dfrac{d\theta}{dx}}{\sqrt{1+v'^2}}$$

where θ is the tangential angle and s is the arc length. Note that curvature has units of inverse of distance. (Aside: the inverse of curvature is known as the radius of curvature.)

The $\dfrac{d\theta}{dx}$ derivative in the above equation can be found using the identity $\tan\theta = \dfrac{dv}{dx} = v'$

from which we could find

$$\frac{d}{dx}\left(\tan\theta\right) = \sec^2\theta \frac{d\theta}{dx} = v''$$

and

$$\frac{d\theta}{dx} = \frac{1}{\sec^2\theta}\frac{d}{dx}\left(\tan\theta\right) = \frac{1}{1+\tan^2\theta}v'' = \frac{1}{1+v'^2}v'' = \frac{v''}{1+v'^2}$$

Combining the above equations gives

$$\varphi = \frac{v''}{\left(1+v'^2\right)^{3/2}} = \frac{\dfrac{d^2v}{dx^2}}{\left[1+\left(\dfrac{dv}{dx}\right)^2\right]^{3/2}}$$

In our problems $\dfrac{dv}{dx}$ is very small. Therefore, $\left(\dfrac{dv}{dx}\right)^2$ is negligible and thus we can approximate the curvature as

$$\varphi = \frac{d^2v}{dx^2}$$