5.2 DOUBLE-INTEGRATION METHOD

The double-integration method of determining the equation of the elastic curve is a basic classical method using the differential equation. It is given the name "double integration" because one usually starts with the bending moment M, which relates to the curvature, d^2y/dx^2. Integration twice gives the equation for deflection y. Study of Fig. 5.1.2 will show that one may start from load intensity, shear, moment, or slope and integrate the appropriate number of times to get the deflection equation.

Use of the differential equation as a basic method is occasionally useful when the complete elastic curve equations are required. This method is, however, cumbersome when discontinuities exist in the function being integrated, creating a large number of integration constants. When only the deflections or slopes at certain *chosen points* on the beam are required, the virtual work (unit load) or the moment area methods treated later in this chapter are far more convenient.

Example 5.2.1

For the beam of Fig. 5.2.1, determine the equation of the elastic curve and also the maximum deflection using the method of double integration.

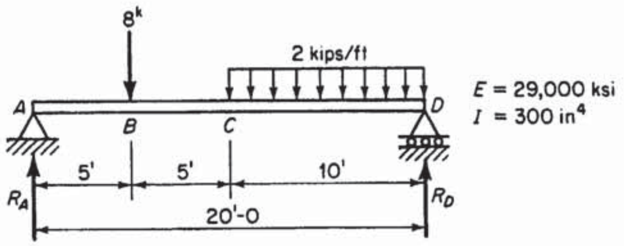


Figure 5.2.1 Beam of Example 5.2.1.

Solution

(a) *Determine the reactions.* Taking moments about end A,

$$R_D = \frac{2(10)15 + 8(5)}{20} = 17 \text{ kips}$$

Taking moments about end D,

$$R_A = \frac{2(10)5 + 8(15)}{20} = 11 \text{ kips}$$

Check that $R_A + R_D$ equals the total downward load of $8 + 2(10) = 28$ kips.

(b) *Establish the equations for bending moment.* Starting from end A, for $x \leq 5$,

$$M_x = R_A x = 11x$$

For $5 \leq x \leq 10$,

$$M_x = R_A x - 8(x - 5) = 11x - 8x + 40 = 3x + 40$$

For $10 \leq x \leq 20$,

$$M_x = R_A x - 8(x - 5) - 2(x - 10)\left(\frac{x - 10}{2}\right)$$

$$= 11x - 8x + 40 - x^2 + 20x - 100$$

$$= -x^2 + 23x - 60$$

(c) *Obtain the slope equations.* Using

$$\frac{d^2y}{dx^2} = -\frac{M_x}{EI}$$

and integrating once:

For $x \leq 5$,

$$\frac{dy}{dx} = \frac{-1}{EI}\left(11\frac{x^2}{2} + C_1\right)$$

For $5 \leq x \leq 10$,

$$\frac{dy}{dx} = \frac{-1}{EI}\left(3\frac{x^2}{2} + 40x + C_2\right)$$

For $10 \leq x \leq 20$,

$$\frac{dy}{dx} = \frac{-1}{EI}\left(-\frac{x^3}{3} + 23\frac{x^2}{2} - 60x + C_3\right)$$

(d) *Obtain the deflection equations.* Integration once of the slope equations gives the following:

For $x \leq 5$,

$$y = \frac{-1}{EI}\left(11\frac{x^3}{6} + C_1 x + C_4\right)$$

For $5 \leq x \leq 10$,

$$y = \frac{-1}{EI}\left(3\frac{x^3}{6} + 40\frac{x^2}{2} + C_2 x + C_5\right)$$

For $10 \leq x \leq 20$,

$$y = \frac{-1}{EI}\left(-\frac{x^4}{12} + 23\frac{x^3}{6} - 60\frac{x^2}{2} + C_3 x + C_6\right)$$

Since the deflection equations contain six constants of integration, six boundary conditions of geometry are required. These conditions are:

1. At $x = 0$, $y = 0$
2. At $x = 5$, $y_{\text{segment } AB} = y_{\text{segment } BC}$

3. At $x = 5$, $\left(\frac{dy}{dx}\right)_{\text{segment } AB} = \left(\frac{dy}{dx}\right)_{\text{segment } BC}$
4. At $x = 10$, $y_{\text{segment } BC} = y_{\text{segment } CD}$
5. At $x = 10$, $\left(\frac{dy}{dx}\right)_{\text{segment } BC} = \left(\frac{dy}{dx}\right)_{\text{segment } CD}$
6. At $x = 20$, $y = 0$

From condition 1,

$$0 = C_4 \tag{a}$$

From condition 2,

$$11\frac{(5)^3}{6} + C_1(5) = 3\frac{(5)^3}{6} + 40\frac{(5)^2}{2} + C_2(5) + C_5$$

$$5C_1 - 5C_2 - C_5 = 333.33 \tag{b}$$

From condition 3,

$$11\frac{(5)^2}{2} + C_1 = 3\frac{(5)^2}{2} + 40(5) + C_2$$

$$C_1 - C_2 = 100 \tag{c}$$

From condition 4,

$$3\frac{(10)^3}{6} + 40\frac{(10)^2}{2} + C_2(10) + C_5$$

$$= -\frac{(10)^4}{12} + 23\frac{(10)^3}{6} - 60\frac{(10)^2}{2} + C_3(10) + C_6$$

$$10C_2 - 10C_3 + C_5 - C_6 = -2500 \tag{d}$$

From condition 5,

$$3\frac{(10)^2}{2} + 40(10) + C_2 = -\frac{(10)^3}{3} + 23\frac{(10)^2}{2} - 60(10) + C_3$$

$$C_2 - C_3 = -333.33 \tag{e}$$

From condition 6,

$$0 = -\frac{(20)^4}{12} + 23\frac{(20)^3}{6} - 60\frac{(20)^2}{2} + C_3(20) + C_6$$

$$-20C_3 - C_6 = 5333.33 \tag{f}$$

Solving Eqs. (a) to (f) gives the following:

$$C_1 = -466.67; \qquad C_4 = 0$$
$$C_2 = -566.67; \qquad C_5 = +166.67$$
$$C_3 = -233.33; \qquad C_6 = -666.67$$

The deflection equations are, therefore, the following:
For $x \leq 5$,

$$y = \frac{-1}{EI}(1.833x^3 - 466.67x)$$

For $5 \leq x \leq 10$,

$$y = \frac{-1}{EI}(0.5x^3 + 20x^2 - 566.67x + 166.67)$$

For $10 \leq x \leq 20$,

$$y = \frac{-1}{EI}(-0.0833x^4 + 3.833x^3 - 30x^2 - 233.33x - 666.67)$$

(e) *Determine maximum deflection.* Since most of the load is on segment CD, it is likely that maximum deflection occurs in that segment. For maximum y, set dy/dx equal to zero. For $10 \leq x \leq 20$,

$$\frac{dy}{dx} = 0 = \frac{-1}{EI}\left(-\frac{x^3}{3} + 23\frac{x^2}{2} - 60x - 233.33\right)$$

$$x^3 - 34.5x^2 + 180x + 700 = 0$$

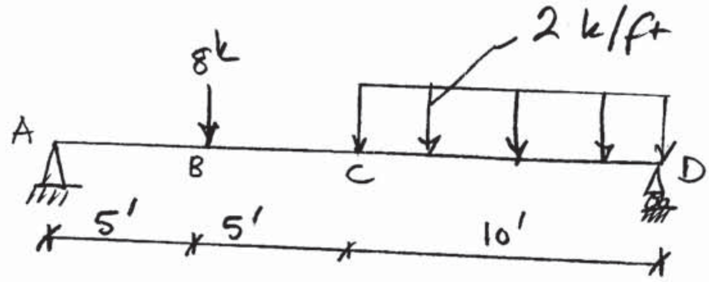
$$x = 10.24 \text{ ft}$$

$$\text{Max. } y = \frac{-1}{EI}[-0.0833(10.24)^4 + 3.833(10.24)^3 - 30(10.24)^2 - 233.33(10.24) - 666.67]$$

$$\text{Max. } y = \frac{3000}{EI} = \frac{3000(1728)}{29{,}000(300)} = 0.596 \text{ in.}$$

In this problem, the maximum deflection is very close to the center of span, where the deflection is exactly 3000 kip-ft^3/EI.

Let's solve the problem using, for lack of a better word, the "[ ]" ("square brackets") method.



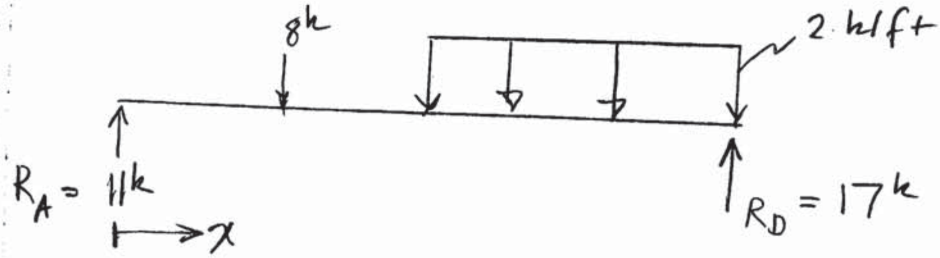
$E = 29{,}000$ ksi

$I = 300$ in^4

Taking moments about D,

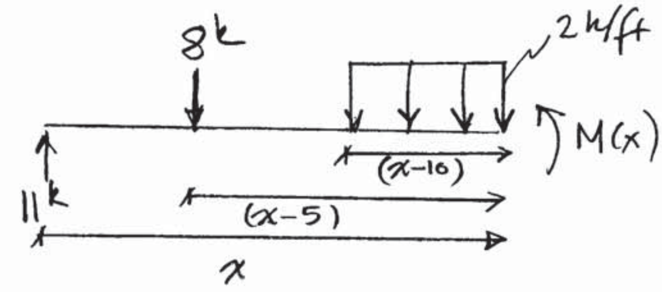
$$\Sigma M = 0 \; : \quad R_A \times 20 - 8 \times 15 - 2 \times 10 \times 5 = 0$$

$$R_A = 11 \text{ k} \uparrow$$



instead of expressing the bending moment over three segments, i.e. AB, BC, and CD, let's use the "[ ]" method to write all at once — of carse we will ignore the terms w/ [ ] when the argument inside the [ ] is negative.

To generate the most general expression, consider x to fall between 10' & 20', i.e. the bending moment expression for segment CD (it is the segment where R_A, 8^k point load, and the distributed load are all "active" in the bending moment expression):



$$\Sigma M = 0: \quad M(x) - 11x + 8[x-5] + 2[x-10]\frac{[x-10]}{2} = 0$$

$$\Rightarrow M(x) = 11x - 8[x-5] - [x-10]^2$$

again, remember that when the argument in a "[ ]" is less than zero, we will ignore that term

$$\varphi(x) = \frac{M}{EI} = \frac{1}{EI}\left(11x - 8[x-5] - [x-10]^2\right)$$

slope: $$\theta(x) = \frac{1}{EI}\left(\frac{11}{2}x^2 - \frac{8[x-5]^2}{2} - \frac{[x-10]^3}{3}\right) + A$$

deflection: $$v(x) = \frac{1}{EI}\left(\frac{11}{6}x^3 - \frac{8[x-5]^3}{6} - \frac{[x-10]^4}{12}\right) + Ax + B \qquad (*)$$

how to find the two constants A & B? Using the conditions that there is no vertical displacement at supports A & D, i.e. $v(0)=0$ and $v(20)=0$

$v(0)=0$: from (*) $\quad 0 = B$

$v(20)=0$: from (*) $\quad 0 = \frac{1}{EI}\left(\frac{11}{6}20^3 - \frac{8[15]^2}{6} - \frac{[10]^4}{12}\right) + A\cdot 20$

$$\Rightarrow A = -\frac{466.67}{EI}$$

hence, the eq'ns for slope & deflection become

$$\theta(x) = \frac{1}{EI}\left(\frac{11}{2}x^2 - 4[x-5]^2 - \frac{[x-10]^3}{3} - 466.67\right] \qquad (**)$$

$$v(x) = \frac{1}{EI}\left(\frac{11}{6}x^3 - \frac{4}{3}[x-5]^3 - \frac{[x-10]^4}{12} - 466.67x\right) \qquad (***)$$

where does the maximum deflection occur? It occurs where the slope is zero. Note that if slope = 0 @ more than one location, you need to compare the displacements at these locations to find "global" maximum deflection

$\theta(x) = 0$, $x = ?$ $\Rightarrow$ $x = 10.24\ ft$ (from (**))

subst. $x = 10.24'$ into (***)

$$v_{max} = v(10.24) = \frac{1}{EI}\left(\frac{11}{6}(10.24)^3 - \frac{4}{3}[5.24]^3 - \frac{[0.24]^4}{12} - 466.67 \times 10.24\right)$$

$$= -\frac{3000}{EI}$$

$$= -\frac{3000}{(29{,}000 \times 12^2)(300 \times \frac{1}{12^4})} \quad \leftarrow \text{consistent units in k \& ft, ft}^2\text{, ft}^4$$

$$= 0.497 \times 10^{-2}\ ft$$

$$v_{max} = 0.596\ in$$