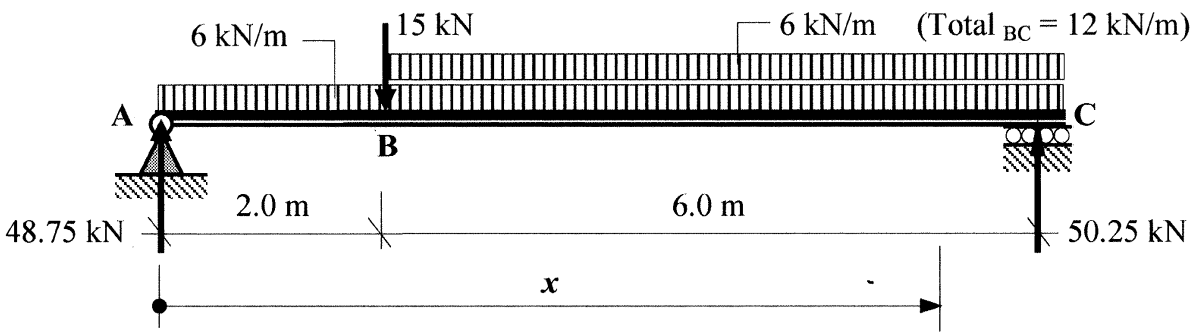
**Important Note**

In the following solutions, square brackets, i.e. [ ], are used to enclose the lever arms for the forces in calculating the bending moments. These brackets are integrated as a unit. During the calculation for slope and deflection, they are ignored if the contents are negative, i.e. when the position $x$ of the section being considered is to the left of the load associated with the bracket. Note that the bending moment expressions are derived using free-bodies that start from the left end of the beams.

# Solution

**Topic: Statically Determinate Beams – Deflection**

**Problem Number: 4.11** 




(See Problem 4.2 for the support reactions)

The equation for the **bending moment** at $x$ is:

$$EI\frac{d^2y}{dx^2} = M_x = +\,48.75x - (6x^2)/2 - 15.0[x-2] - 6.0[x-2]^2/2 \qquad \text{Equation (1)}$$

The equation for the **slope** at $x$ is:

$$EI\frac{dy}{dx} = EI\theta = +\,24.38x^2 - x^3 - 7.5[x-2]^2 - [x-2]^3 + A \qquad \text{Equation (2)}$$

The equation for the **deflection** at $x$ is:

$$EIy = EI\delta = +\,8.13x^3 - 0.25x^4 - 2.5[x-2]^3 - 0.25[x-2]^4 + Ax + B \qquad \text{Equation (3)}$$

where $A$ and $B$ are constants of integration related to the **boundary conditions**.

**when $x = 0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,8.13(0)^3 - 0.25(0)^4 - \underbrace{2.5[-2]^3}_{\textit{ignore}} - \underbrace{0.25[-2]^4}_{\textit{ignore}} + A(0) + B$$

$\therefore\ B = 0$

**when $x = 8.0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,8.13(8.0)^3 - 0.25(8.0)^4 - 2.5[6.0]^3 - 0.25[6.0]^4 + A(8.0)$$

$\therefore\ A = -\,284.32$

The general equations for the slope and deflection at any point along the length of the beam are given by substituting for A and B in equations (2) and (3)

The equation for the **slope** at $x$:

$$EI\theta = +\,24.38x^2 - x^3 - 7.5[x-2]^2 - [x-2]^3 - 284.32 \qquad \text{Equation (4)}$$

The equation for the **deflection** at $x$:

$$EI\delta = +\,8.13x^3 - 0.25x^4 - 2.5[x-2]^3 - 0.25[x-2]^4 - 284.32x \qquad \text{Equation (5)}$$

The position of the maximum deflection at the point of zero slope can be determined from equation (4) as follows:
Assume that zero slope occurs when $2.0 \leq x \leq 8.0$ and neglect [ ] when negative

$$EI\theta = 0 = +24.38x^2 - x^3 - 7.5[x-2]^2 - [x-2]^3 - 284.32$$

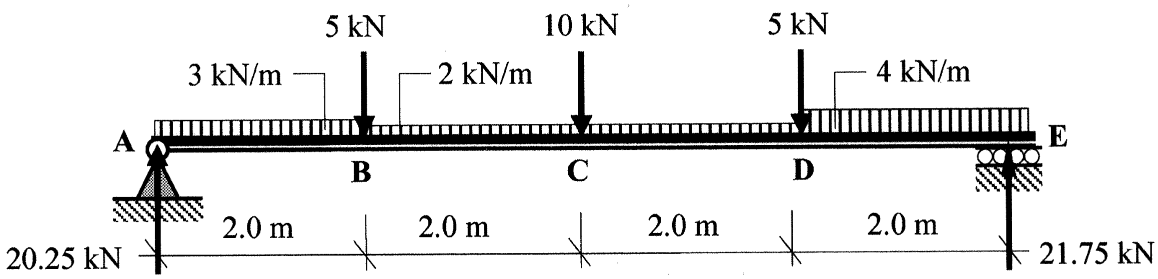
Solve the resulting cubic equation by trial and error.
Guess $x = 3.9$ m (i.e. slightly to the left of the mid-span)
$+24.38(3.9)^2 - 3.9^3 - 7.5(1.9)^2 - (1.9)^3 - 284.32 = -6.75$ Increase $x$
try $x = 3.95$
$+24.38(3.95)^2 - 3.95^3 - 7.5(1.95)^2 - (1.95)^3 - 284.32 = -1.49$ Increase $x$
try $x = 3.96$
$+24.38(3.96)^2 - 3.97^3 - 7.5(1.97)^2 - (1.97)^3 - 284.32 = -0.44$
Accept $x = 3.96$ m

The maximum deflection is given by:
$\delta_{max.} = \{+8.13(3.96)^3 - 0.25(3.96)^4 - 2.5(1.96)^3 - 0.25(1.96)^4 - 284.32(3.96)\}/EI$
$\boldsymbol{\delta_{max.} = -705.03/EI}$

**Equivalent Uniformly Distributed Load Method:**
$\delta_{max.} \approx -(0.104M_{maximum}L^2)/EI$

The maximum bending moment = 105.2 kNm (see Problem 4.2)

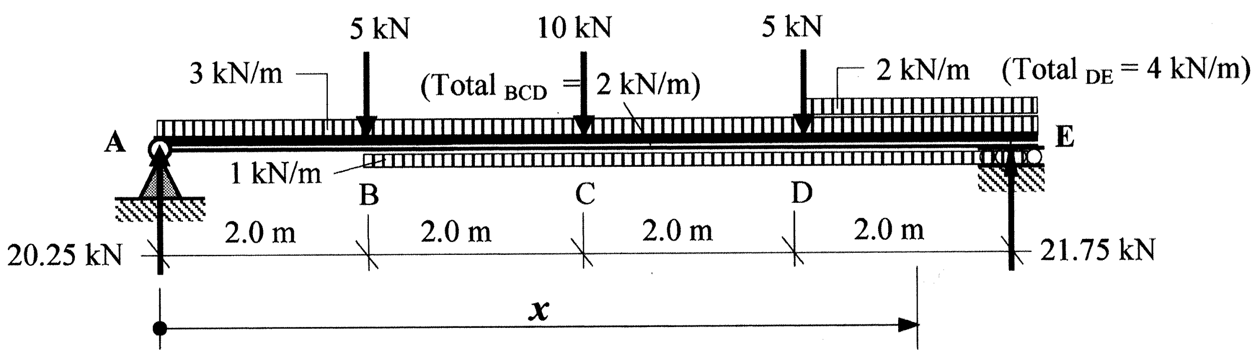
$\delta_{max.} \approx -(0.104 \times 105.2 \times 8.0^2)/EI = \mathbf{-700.2/EI}$




(See Problem 4.3 for the support reactions)

The distributed loads must continue to the end of the beam from the point where they begin. An equivalent load system is therefore required to ensure that the applied loads are represented in the equations.

**Equivalent Load System:**




The equation for the **bending moment** at $x$ is:

$$EI\frac{d^2y}{dx^2} = M_x = +\,20.25x - (3x^2)/2 - 5.0[x-2] + 1.0[x-2]^2/2 - 10.0[x-4]$$
$$-\,5.0[x-6] - 2.0[x-6]^2/2 \qquad \text{Equation (1)}$$

The equation for the **slope** at $x$ is:

$$EI\frac{dy}{dx} = EI\theta = +\,10.13x^2 - 0.5x^3 - 2.5[x-2]^2 + 0.17[x-2]^3 - 5.0[x-4]^2$$
$$-\,2.5[x-6]^2 - 0.33[x-6]^3 + A \qquad \text{Equation (2)}$$

The equation for the **deflection** at $x$ is:

$$EIy = EI\delta = +\,3.38x^3 - 0.125x^4 - 0.83[x-2]^3 + 0.04[x-2]^4 - 1.67[x-4]^3$$
$$-\,0.83[x-6]^3 - 0.08[x-6]^4 + Ax + B \qquad \text{Equation (3)}$$

where $A$ and $B$ are constants of integration related to the **boundary conditions**.

**when $x = 0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,3.38(0)^3 - 0.125(0)^4 - \cancel{0.83[-2.0]^3} + \cancel{0.04[-2.0]^4} - \cancel{1.67[-4.0]^3} - \cancel{0.83[-6.0]^3}$$
$$-\,\cancel{0.08[-6.0]^4} + A(0) + B$$

*ignore* *ignore* *ignore* *ignore* *ignore*

$\therefore\ B = 0$

**when $x = 8.0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,3.38(8.0)^3 - 0.125(8.0)^4 - 0.83[6.0]^3 + 0.04[6.0]^4 - 1.67[4.0]^3$$
$$-\,0.83[2.0]^3 - 0.08[2.0]^4 + A(8.0)$$

$\therefore\ A = -\,122.04$

The general equations for the slope and deflection at any point along the length of the beam are given by substituting for A and B in equations (2) and (3)

The equation for the **slope** at $x$:

$$EI\theta = +\,10.13x^2 - 0.5x^3 - 2.5[x-2]^2 + 0.17[x-2]^3 - 5.0[x-4]^2 - 2.5[x-6]^2$$
$$-\,0.33[x-6]^3 - 122.04$$

Equation (4)

The equation for the **deflection** at $x$:

$$EI\delta = +\,3.38x^3 - 0.125x^4 - 0.83[x-2]^3 + 0.04[x-2]^4 - 1.67[x-4]^3 - 0.83[x-6]^3$$
$$-\,0.08[x-6]^4 - 122.04x$$

Equation (5)

The position of the maximum deflection at the point of zero slope can be determined from equation (4) as follows:
Assume that zero slope occurs when $4.0 \le x \le 6.0$ and neglect [ ] when negative

$$EI\theta = 0 = +\,10.13x^2 - 0.5x^3 - 2.5[x-2]^2 + 0.17[x-2]^3 - 5.0[x-4]^2 - \cancel{2.5[x-6]^2}$$
$$-\,\cancel{0.33[x-6]^3} - 122.04$$

*ignore* *ignore*

Solve the resulting cubic equation by trial and error.

Guess $x = 4.1$ m $\quad EI\theta = +\,4.33 > 0 \quad \therefore$ reduce $x$

try $x = 4.05 \quad EI\theta = +\,1.86 > 0 \quad$ try $x = 4.02 \quad EI\theta = +\,0.38$

Accept $x = 4.02$ m

The maximum deflection is given by:

$$\delta_{\text{max.}} = \{+\,3.38(4.02)^3 - 0.125(4.02)^4 - 0.83(2.02)^3 + 0.04(2.02)^4 - 1.67(0.02)^3$$
$$-\,(122.04 \times 4.02)\}/EI$$

$$\boldsymbol{\delta_{\text{max.}} = -\,309.84/EI}$$

**Equivalent Uniformly Distributed Load Method:**

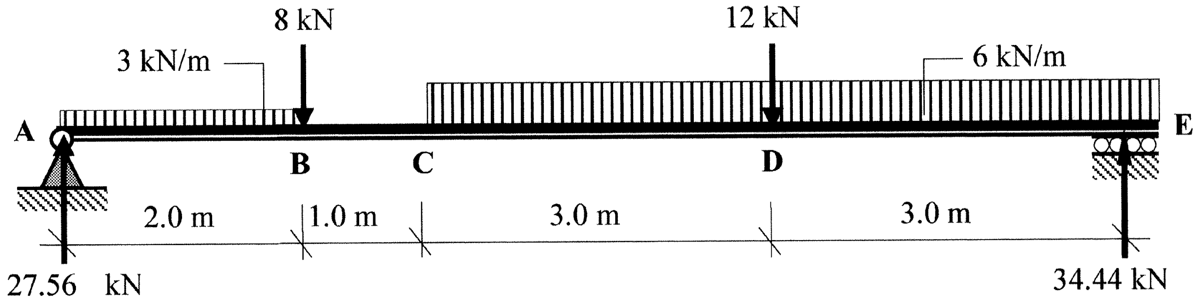
$\delta_{\text{max.}} \approx -\,(0.104M_{\text{maximum}}L^2)/EI$

The maximum bending moment = 49.0 kNm (see Problem 5.3)

$\delta_{\text{max.}} \approx -\,(0.104 \times 49.0 \times 8.0^2)/EI = \mathbf{-\,326.14/}\boldsymbol{EI}$

# Solution

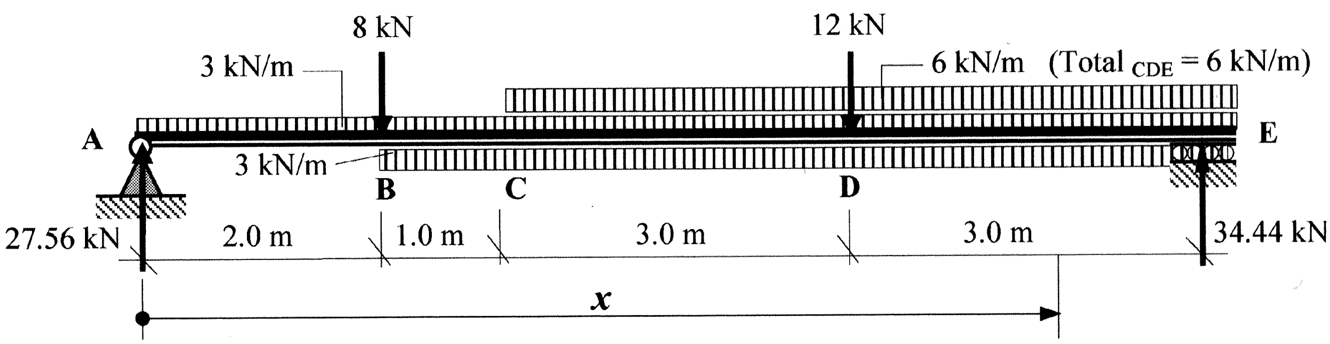
**Topic: Statically Determinate Beams - Deflection**
**Problem Number: 4.13** **Page No. 1**




(See Problem 4.4 for the support reactions)

The distributed loads must continue to the end of the beam from the point where they begin. An equivalent load system is therefore required to ensure that the applied loads are represented in the equations.

**Equivalent Load System:**




The equation for the **bending moment** at $x$ is:

$$EI\frac{d^2y}{dx^2} = M_x = +27.56x - (3x^2)/2 - 8.0[x-2] + 3.0[x-2]^2/2 - 6.0[x-3]^2/2$$
$$- 12.0[x-6] \qquad \text{Equation (1)}$$

The equation for the **slope** at $x$ is:

$$EI\frac{dy}{dx} = EI\theta = +13.78x^2 - 0.5x^3 - 4.0[x-2]^2 + 0.5[x-2]^3 - [x-3]^3 - 6.0[x-6]^2 + A$$
Equation (2)

The equation for the **deflection** at $x$ is:

$$EI\,y = EI\delta = +4.59x^3 - 0.125x^4 - 1.33[x-2]^3 + 0.125[x-2]^4 - 0.25[x-3]^4$$
$$- 2.0[x-6]^3 + Ax + B \qquad \text{Equation (3)}$$

where $A$ and $B$ are constants of integration related to the **boundary conditions**.

**when $x = 0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,4.59(0)^3 - 0.125(0)^4 - 1.33\cancel{[-2.0]}^3 + 0.125\cancel{[-2.0]}^4 - 0.25\cancel{[-3.0]}^4$$
$$- 2.0\cancel{[-6.0]}^3 - A(0) + B$$

ignore ignore ignore ignore

$\therefore B = 0$

**when $x = 9.0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,4.59(9.0)^3 - 0.125(9.0)^4 - 1.33[7.0]^3 + 0.125[7.0]^4 - 0.25[6.0]^4 - 2.0[3.0]^3$$
$$- A(9.0)$$

$\therefore A = -\,221.32$

The general equations for the slope and deflection at any point along the length of the beam are given by substituting for A and B in equations (2) and (3)

The equation for the **slope** at $x$:

$$EI\theta = +\,13.78x^2 - 0.5x^3 - 4.0[x-2]^2 + 0.5[x-2]^3 - [x-3]^3 - 6.0[x-6]^2 - 221.32$$

Equation (4)

The equation for the **deflection** at $x$:

$$EI\delta = +\,4.59x^3 - 0.125x^4 - 1.33[x-2]^3 + 0.125[x-2]^4 - 0.25[x-3]^4 - 2.0[x-6]^3$$
$$- 221.32x$$

Equation (5)

The position of the maximum deflection at the point of zero slope can be determined from equation (4) as follows:

Assume that zero slope occurs when $3.0 \le x \le 6.0$ and neglect [ ] when negative

$$EI\theta = 0 = +\,13.78x^2 - 0.5x^3 - 4.0[x-2]^2 + 0.5[x-2]^3 - [x-3]^3 - 221.32$$

Solve the resulting equation by trial and error.

Guess $x = 4.6$ m $EI\theta = -\,0.75 > 0$ $\therefore$ reduce $x$

try $x = 4.61$ m $EI\theta = +\,0.02 > 0$ Accept $x = 4.61$ m

The maximum deflection is given by:

$$\delta_{\text{max.}} = \{+\,4.59(4.61)^3 - 0.125(4.61)^4 - 1.33(2.61)^3 + 0.125(2.61)^4 - 0.25(2.61)^4$$
$$- (221.32 \times 4.61)\}/EI$$

$$\boldsymbol{\delta_{\text{max.}} = -\,656.5/EI}$$

**Equivalent Uniformly Distributed Load Method:**

$\delta_{\text{max.}} \approx -\,(0.104 M_{\text{maximum}} L^2)/EI$

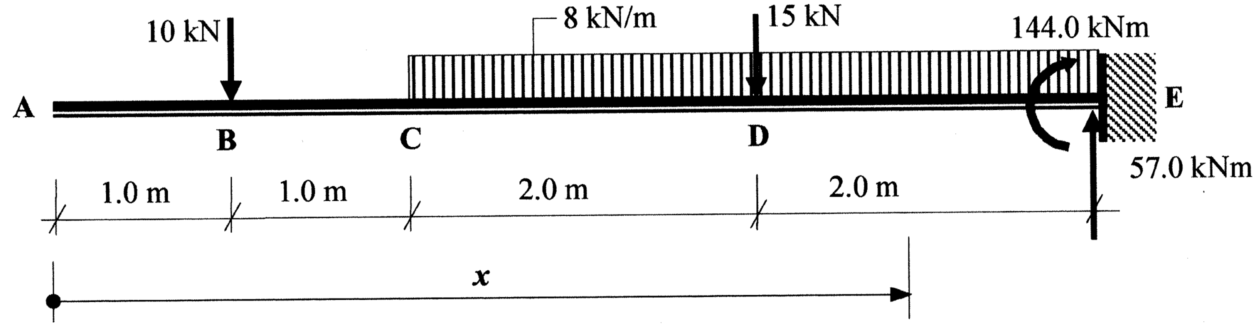
The maximum bending moment = 78.0 kNm (see Problem 5.4)

$\delta_{\text{max.}} \approx -\,(0.104 \times 78.0 \times 9.0^2)/EI = \boldsymbol{-\,657.4/EI}$

# Solution

**Topic: Statically Determinate Beams - Deflection**
**Problem Number: 4.14**






(See Problem 4.10 for the support reactions)

The equation for the **bending moment** at $x$ is:

$$EI\frac{d^2y}{dx^2} = M_x = -10.0[x-1] - 8.0[x-2]^2/2 - 15.0[x-4] \qquad \text{Equation (1)}$$

The equation for the **slope** at $x$ is:

$$EI\frac{dy}{dx} = EI\theta = -5.0[x-1]^2 - 1.33[x-2]^3 - 7.5[x-4]^2 + A \qquad \text{Equation (2)}$$

The equation for the **deflection** at $x$ is:

$$EI\,y = EI\delta = -1.67[x-1]^3 - 0.33[x-2]^4 - 2.5[x-4]^3 + Ax + B \qquad \text{Equation (3)}$$

where $A$ and $B$ are constants of integration related to the **boundary conditions.**

**when $x = 6.0$, $dy/dx = 0$** and substituting for $x$ and $y$ in equation (2)

$$EI\,(0) = -5.0(5.0)^2 - 1.33(4.0)^3 - 7.5(2.0)^2 + A$$

$$\therefore A = +240.12$$

**when $x = 6.0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = -1.67(5.0)^3 - 0.33(4.0)^4 - 2.5(2.0)^3 + (240.12 \times 6.0) + B$$

$$\therefore B = -1127.49$$

The general equations for the slope and deflection at any point along the length of the cantilever are given by substituting for A and B in equations (2) and (3).

The equation for the **slope** at $x$:

$$EI\theta = -5.0[x-1]^2 - 1.33[x-2]^3 - 7.5[x-4]^2 + 240.12 \qquad \text{Equation (4)}$$

The equation for the **deflection** at $x$:

$$EI\delta = -1.67[x-1]^3 - 0.33[x-2]^4 - 2.5[x-4]^3 + 240.12x - 1127.49$$

Equation (5)

# Solution

**Topic: Statically Determinate Beams - Deflection**
**Problem Number: 4.14** **Page No. 2**

The maximum deflection occurs at the free end of the cantilever i.e. when $x = 0$ neglecting all [ ] which are negative.

$\delta_{max.} = \mathbf{-1127.49\ /\ \mathit{EI}}$

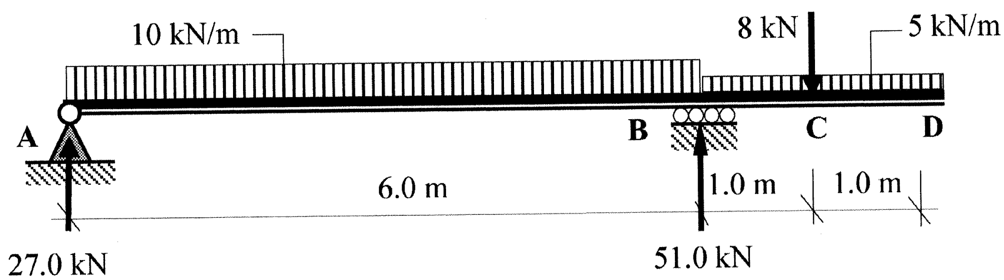
The deflection at any other location can be found by substituting the appropriate value of $x$, e.g.

At B: $x = 1.0$
$\delta_B = \{+ (240.12 \times 1.0) - 1127.49\}/EI$ $\quad$ $\delta_B = \mathbf{-887.4/\mathit{EI}}$

At C: $x = 2.0$
$\delta_C = \{- 1.67(1)^3 + (240.12 \times 2.0) - 1127.49\}/EI$ $\quad$ $\delta_C = \mathbf{-648.9/\mathit{EI}}$

At D: $x = 4.0$
$\delta_D = \{- 1.67(3.0)^3 - 0.33(2.0)^4 + (240.12 \times 4.0) - 1127.49\}/EI$ $\quad$ $\delta_D = \mathbf{-217.4/\mathit{EI}}$

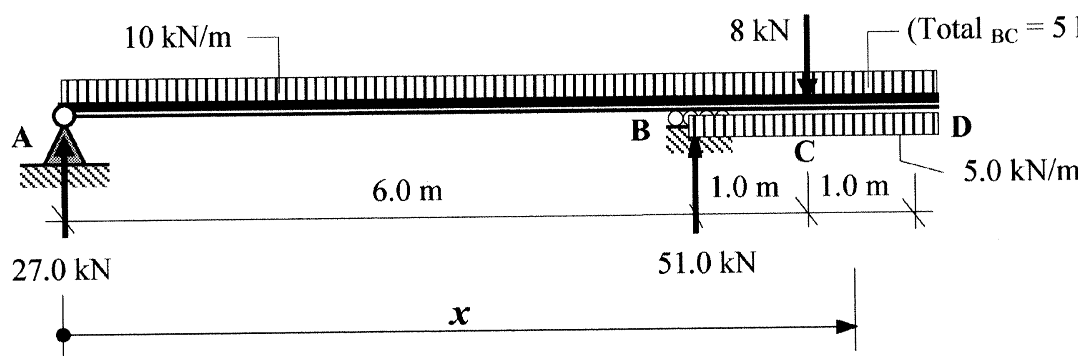
**Note: The Equivalent Uniformly Distributed Load Method only applies to single-span beams.**




(See Problem 4.6 for the support reactions)

The distributed loads must continue to the end of the beam from the point where they begin. An equivalent load system is therefore required to ensure that the applied loads are represented in the equations.

**Equivalent Load System:**




The equation for the **bending moment** at $x$ is:

$$EI\frac{d^2y}{dx^2} = M_x = +27.0x - (10x^2)/2 + 51.0[x-6] + 5.0[x-6]^2/2 - 8.0[x-7]$$

Equation (1)

The equation for the **slope** at $x$ is:

$$EI\frac{dy}{dx} = EI\theta = +13.5x^2 - 1.67x^3 + 25.5[x-6]^2 + 0.83[x-6]^3 - 4.0[x-7]^2 + A$$

Equation (2)

The equation for the **deflection** at $x$ is:

$$EI\,y = EI\delta = +4.5x^3 - 0.42x^4 + 8.5[x-6]^3 + 0.21[x-6]^4 - 1.33[x-7]^3 + Ax + B$$

Equation (3)

where $A$ and $B$ are constants of integration related to the **boundary conditions**.

# Solution

**Topic: Statically Determinate Beams - Deflection**
**Problem Number: 4.15**



**when $x = 0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,4.5(0)^3 - 0.42(0)^4 + 8.5\underbrace{\cancel{[-6.0]^3}}_{ignore} + 0.21\underbrace{\cancel{[-6.0]^4}}_{ignore} - 1.33\underbrace{\cancel{[-7.0]^3}}_{ignore} + A(0) + B$$

$\therefore\ B = 0$

**when $x = 6.0$, $y = 0$** and substituting for $x$ and $y$ in equation (3)

$$EI\,(0) = +\,4.5(6.0)^3 - 0.42(6.0)^4 + A(6.0)$$

$\therefore\ A = -\,71.28$

The general equations for the slope and deflection at any point along the length of the beam are given by substituting for A and B in equations (2) and (3)

The equation for the **slope** at $x$:

$$EI\theta = +\,13.5x^2 - 1.67x^3 + 25.5[x-6]^2 + 0.83[x-6]^3 - 4.0[x-7]^2 - 71.28$$

Equation (4)

The equation for the **deflection** at $x$:

$$EI\delta = +\,4.5x^3 - 0.42x^4 + 8.5[x-6]^3 + 0.21[x-6]^4 - 1.33[x-7]^3 - 71.28x$$

Equation (5)

The position of the maximum deflection between A and B at the point of zero slope can be determined from equation (4) as follows:

Assume that zero slope occurs when $3.0 \le x \le 6.0$ and neglect [ ] when negative

$$EI\theta = 0 = +\,13.5x^2 - 1.67x^3 - 71.28$$

Solve the resulting equation by trial and error.

| | | | |
|---|---|---|---|
| Guess | $x = 2.9$ m | $EI\theta = +\,1.53 > 0$ | $\therefore$ reduce $x$ |
| try | $x = 2.85$ m | $EI\theta = -\,0.29 < 0$ | Accept $x = 2.85$ m |

The maximum deflection is given by:

$$\delta_{AB\ max.} = \{+\,4.5(2.85)^3 - 0.42(2.85)^4 - (71.28 \times 2.85)\}/EI$$

$\boldsymbol{\delta_{AB\ max.} = -\,126.69/EI}$

The maximum deflection of the cantilever occurs when $x = 8.0$ m

$$\delta_D = \{+\,4.5(8.0)^3 - 0.42(8.0)^4 + 8.5(2.0)^3 + 0.21(2.0)^4 - 1.33(1.0)^3 - (71.28 \times 8.0)\}/EI$$

$\boldsymbol{\delta_{D\ max.} = +\,83.47/EI}$

**Equivalent Uniformly Distributed Load Method:**

This can be used to give a conservative estimate of $\delta_{AB}$ assuming AB to be a simply supported 6.0 m span without the cantilever

$$\delta_{max.} \approx -\,(0.104 M_{maximum} L^2)/EI$$

The maximum bending moment in span AB = 36.5 kNm (see Problem 5.6)

$$\delta_{max.} \approx -\,(0.104 \times 36.5 \times 6.0^2)/EI = \boldsymbol{-\,136.7/EI}$$