Lyapunov Stability

– Stability of Equilibrium Points

1. Stability of Equilibrium Points - Definitions

In this section we consider n-th order nonlinear time varying continuous time (CT) systems of the form

$$\dot{x} = f(t,\ x), \qquad x(t_o) = x_o \tag{L.1}$$

or nonlinear time varying discrete time (DT) systems of the form

$$x\,(k+1) = f\,(k, x\,(k)), \qquad x\,(k_o) = x_o\,, \tag{L.2}$$

where $x \in R^n$, $t \in R_+$, $k \in Z_+$ and x_o is the initial state at t_o (k_o respectively).

Definition [**Equilibrium State**]

An equilibrium state x_e is such that

- $f\,(t, x_e) = 0$, for all t, for CT systems.
- $f\,(k, x_e) = x_e$, for all k, for DT systems.

Without loss of generality, we will assume that 0 is an equilibrium state. Notice that non-linear systems (and some linear systems) may have more than one equilibrium state.

Definition [Ref.1] [**Stability and Uniform Stability in the sense of Lyapunov**]

The equilibrium state 0 of (1) is (locally) *stable in the sense of Lyapunov* if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_o) > 0$ such that, if $\|x(t_o)\| < \delta$ then $\|x(t)\| < \varepsilon$ for all $t > t_o$ (respectively k_o for DT).

In addition, if δ can be chosen independent of t_o, i.e., $\delta(\varepsilon)$, then, the origin is (locally) *uniformly stable*.

Definition [Ref.1] [**Asymptotic Stability and Uniform Asymptotic Stability**]

The equilibrium state 0 of (1) is (locally) *asymptotically stable* if

1. It is stable in the sense of Lyapunov and
2. There exists a $\delta'(t_o)$ such that, if $\|x(t_o)\| < \delta'$, then, $x(t) \to 0$ as $t \to \infty$.

The equilibrium state 0 of (1) is (locally) *uniformly asymptotically stable* if

1. It is uniformly stable in the sense of Lyapunov and
3. There exists a δ' , independent of t_o, such that, if $\|x(t_o)\| < \delta'$, then, $x(t) \to 0$ as $t \to \infty$, uniformly in t_o; that is, for each $\varepsilon > 0$, there exists T=T(ε)>0, independent of t_o, such that

$$\|x(t)\| < \varepsilon, \quad \forall t \ge t_0 + T(\varepsilon),\ \forall \|x(t_0)\| < \delta'$$

Note: The definition of stability in the sense of Lyapunov is closely related to that of continuity of solutions. An equilibrium is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

Definition [**Global asymptotic stability**]

The equilibrium state 0 of (1) is globally asymptotically stable, if it is asymptotically stable for any $\delta' > 0$.

Definition [**Exponential stability**]

The equilibrium state 0 of (1) is exponentially stable, if it is stable in the sense of Lyapunov and there exists a $\delta' > 0$ and constants $M < \infty$ and $\alpha > 0$ such that

$$\|x(t)\| \le e^{-\alpha(t-t_o)} M \|x_o\| \quad \text{(L.3)}$$

for all $\|x(t_o)\| < \delta'$. α is called the rate of exponential convergence.

2. Lyapunov Stability Theorems For Autonomous (or Time-Invariant) Systems

When f in (1) does not depend on time t explicitly, i.e.,

$$\dot{x} = f(x), \qquad x(t_o) = x_o \quad \text{(L.4)}$$

for continuous time or

$$x(k+1) = f(x(k)), \qquad x(k_o) = x_o, \quad \text{(L.5)}$$

for discrete time, then, the system becomes an autonomous system. The behavior of an autonomous system is invariant to shifts in the time origin. Thus, the solution $x(t)$ depends on x_0 and t-t_0 only, and is independent of t_0. This leads to the following fact:

For autonomous system, uniform (asymptotic) stability is the same as (asymptotic) stability.

Definition [**Positive Definite (Semi-Definite) Function (PDF)**]

A continuously differentiable function V: $R^n \to R$ is called *positive definite* in a region $U \subset R^n$ containing the origin if

a. $V(0)=0$

b. $V(x)>0$, $x \in U$ and $x \neq 0$

A function is called *positive semi-definite* if Condition b is replaced by $V(x) \ge 0$.

Theorem L.1 [Ref1] [**Lyapunov Theorem**]

For autonomous systems, let $D \subset R^n$ be a domain containing the equilibrium point of origin. If there exists a continuously differentiable positive definite function V: $D \to R$ such that

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} f(x) = -W(x) \quad \text{(L.6)}$$

is negative semi-definite in D, then, the equilibrium point 0 is stable. Moreover, if W(x) is positive definite, then, the equilibrium is asymptotically stable.

In addition, if $D=R^n$ and V is radially unbounded, i.e.,

$$\|x\| \to \infty \quad \Rightarrow \quad V(x) \to \infty \quad \text{(L.7)}$$

then, the origin is globally asymptotically stable. Δ

For autonomous systems, when W(x) in the above theorem is only positive semi-definite, asymptotic stability may still be obtained by applying the following simplified version of LaSalle's Theorem.

Theorem L.2 [Ref1] [**LaSalle's Invariance Principle Theorem**]

For autonomous systems, let $D \subset R^n$ be a domain containing the equilibrium point of origin. If there exists a continuously differentiable positive definite function $V: D \to R$ such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x) = -W(x) \le 0 \tag{L.8}$$

in D. Let

$$S = \left\{ x \in D \mid \dot{V}(x) = 0 \right\} \tag{L.9}$$

and suppose that no solution can stay identically in S, other than the origin. Then, the origin is asymptotically stable.

In addition, if $D=R^n$ and V is radially unbounded, the origin is globally asymptotically stable. Δ

2.2 Linear Time Invariant System

Theorem L.3 The following conditions are equivalent:

(a) *The equilibrium* 0 *of the nth order system*

$$\dot{x} = Ax \tag{L.10}$$

is globally asymptotically stable (*exponentially stable*).

(b) *All eigenvalues of A have negative real parts*.

(c) *For any positive definite symmetric matrix Q, there exists a unique positive definite symmetric matrix P which is the solution of the following Lyapunov equation*[1]

$$PA + A^T P = -Q. \tag{L.11}$$

Note: (c) indicates that the PDF function $V(x) = x^T P x$ is a Lyapunov function for the system.

Proof: We will demonstrate that (c) is a necessary and sufficient condition for (a) and (b).

Sufficiency: Assume that given a positive definite symmetric matrix Q there exists a positive definite symmetric matrix P which satisfies (L.11).

Define the PDF function $V(x) = x^T P x$. Taking the time derivative of V along the trajectories of (L.10), we obtain

$$\dot{V}(x) = x^T \left\{ A^T P + PA \right\} x = -x^T Q x \tag{L.12}$$

Thus, V and $-\dot{V}$ are both PDF, and the system is globally asymptotically stable. To prove exponential stability, we notice that

$$x^T Q x \ \ge\ \lambda_{\min}(Q) x^T x\,, \qquad x^T P x \ \le\ \lambda_{\max}(P) x^T x\,,$$

[1] The MATLAB command for solving Lyapunov equation is "lyap" in continuous time and "dlyap" in discrete time.

where $\lambda_{\min}(Q)$ and $\lambda_{\max}(P)$ are respectively the minimum eigenvalue of Q and the maximum eigenvalue of P, both of which are positive. Thus, defining $\alpha \triangleq \lambda_{\min}(Q)/\lambda_{\max}(P) > 0$, and using the notation $V(t) \triangleq V(x(t))$, we obtain

$$\frac{-\dot{V}(x)}{V(x)} \geq \frac{\lambda_{\min}(Q)x^T x}{\lambda_{\max}(P)x^T x} = \alpha$$

Thus,

$$\dot{V}(t) \leq -\alpha V(t). \tag{L.13}$$

Integrating (L.13), we obtain

$$V(t) \leq e^{-\alpha t}V(0) \tag{L.14}$$

which implies that $V \to 0$ exponentially. Since

$$V(x) \geq \lambda_{\min}(P)x^T x = \lambda_{\min}(P)\|x\|_2,$$

where $\lambda_{\min}(P) > 0$ is the minimum eigenvalue of P, $\|x\|$ must also converge to zero exponentially.

Necessity: We first define the unit ball:

$$B_1 \triangleq = \left\{ v \in R^n \mid v^T v = 1 \right\},$$

and the vector induce $\| \ \|_2$ norm of a matrix $M \in R^{n \times n}$:

$$\|M\|_2 = \max_{v \in B_1}\|Mv\|_2 = \max_{v \in B_1}\left\{\sqrt{v^T M^T M v}\right\} = \sqrt{\lambda_{\max}(M^T M)} = \sigma_{\max}(M) \ \cdot$$

Assume that the system given by (L.10) is asymptotically stable. Thus, all eigenvalues of A have negative real parts and, as a consequence, A is nonsingular and so is the solution matrix e^{At}, for $0 \leq t < \infty$. Since Q is positive definite, there exists a nonsingular matrix, $Q^{1/2}$, such that $Q = (Q^{1/2})^T(Q^{1/2})$. Thus, the matrix $e^{A^T t}Qe^{At}$ is positive definite for $0 \leq t < \infty$. Since all eigenvalues of A have negative real parts, the matrix

$$P = \int_0^\infty e^{A^T t}Qe^{At}dt \tag{L.15}$$

exists, is unique and $\|P\|_2 < \infty$. (Remember that $e^{\lambda t}, t^m e^{\lambda t} \in L_1 \cap L_\infty$ for any $\lambda \in C$ such that $\mathrm{Re}(\lambda) < 0$ and any $0 < m < \infty$). Thus, P, as defined by (L.15), is positive definite. We now prove that P, as defined by (L.15), satisfies the Lyapunov equation (L.11).

$$A^T P + PA = \int_0^\infty \left\{A^T e^{A^T t}Qe^{At} + e^{A^T t}Qe^{At}A\right\} dt$$

$$= \int_0^\infty d/dt\left\{e^{A^T t}Qe^{At}\right\} dt = \lim_{t \to \infty}\left\{e^{A^T t}Qe^{At}\right\} - Q = -Q,$$

since $\lim_{t \to \infty}\left\|e^{At}\right\|_2 = 0$. Q.E.D.

2.3 LTI Discrete Time Systems

Definition [Change of $V(k, x)$ relative to a state trajectory]

Consider the system (L.2). The change of $\Delta V(k+1, x)$ relative to (L.2) is given by

$$\Delta V(k+1,x) = V(k+1,x(k+1)) - V(k,x(k)) \qquad \text{(L.16)}$$
$$= V(k+1, f(k,x(k))) - V(k,x(k)).$$

Theorem L.4 The following conditions are equivalent:

(a) *The equilibrium 0 of the nth order system*

$$x(k+1) = Ax(k) \qquad \text{(L.17)}$$

is globally asymptotically stable (exponentially stable).

(b) *All eigenvalues of A have magnitudes less than 1.*

(c) *For any positive definite symmetric matrix Q, there exist a unique positive definite symmetric matrix P which is the solution of the following* Discrete Time Lyapunov equation

$$A^T PA - P = -Q. \qquad \text{(L.18)}$$

Proof: The proof is very similar to the continuous time case and it's left as an exercise. Q.E.D.

2.4 Lyapunov's Indirect Method

Theorem L.5 [Ref1] *Consider the autonomous system (L.4) with the origin as an equilibrium point. If $f: D \to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let*

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0} \qquad \text{(L.19)}$$

Then,

(a) The origin is locally asymptotically stable if A is asymptotically stable or all eigenvalues of A have negative real parts.

(b) The origin is unstable if one or more of the eigenvalues of A has positive real part. Δ

Note:

(1). Both the Lyapunov's indirect method (Theorem L.5) and direct method (Theorem L.1) can be used to judge the local stability of an equilibrium point when the linearized system matrix *A* is either asymptotically stable or unstable. However, the indirect method does not tell anything about the *region of attraction*[2] (or domain of attraction) while the direct method gives at least some conservative estimate of the domain of attraction. For example, if conditions for asymptotic stability in Theorem 1 are satisfied and $\Omega_c=\{x\in R^n \mid V(x) \le c\}$ is bounded and contained in D, then, every trajectory starting in Ω_c remains in Ω_c and approaches the origin as $t\to\infty$. Thus Ω_c is an estimate of the region of attraction.

(2). When some of the eigenvalues of A have zero real parts and all the rest eigenvalues have negative real parts, the local stability of the origin cannot be concluded from the above theorem. In such a case, the local stability of the origin depends on higher-order nonlinear terms also. Advanced stability theorems such as Center Manifold Theorem may be used to judge the local stability of the origin.

[2] The region in which all trajectories converge to the equilibrium point as t approaches ∞.

Lyapunov Stability Theorems For Non-autonomous (or Time-Varying) Systems

Consider the non-autonomous system (L.1) where f: $[0,\infty)\times D\to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,\infty)\times D$, and $D\subset R^n$ is a domain that contains the equilibrium point of origin x=0. Note that an equilibrium at origin of a non-autonomous system could be a translation of a non-zero time-varying solution of an autonomous system (e.g., trajectory tracking of an autonomous system). The solution of a non-autonomous system may depend on both $t-t_0$ and t_0, and the Lyapunov function V(x,t) in general depends on t also. To characterize the positive definiteness of a time function, we introduce the following addition definitions.

Definition [Ref.1] [**Class-*K* Function**]

A continuous function α: $[0,a)\to R_+$ is said to be a class-K function if,

(a) $\alpha(0) = 0$.

(b) α is strictly increasing.

It is said to belong to class K_∞ if $a=\infty$ and $\alpha(r)\to\infty$ as $r\to\infty$. ♦

Definition [Ref.1] [**Class-*KL* Function**]

A continuous function β: $[0,a)\times[0,\infty)\to R_+$ is said to belong to class-KL if, for each fixed t, the mapping $\beta(r,t)$ belongs to class K with respect to r and, for each fixed r, the mapping $\beta(r,t)$ is decreasing with respect to t and $\beta(r,t)\to 0$ as $t\to\infty$. ♦

Definition [**Locally Positive Definite Function (LPDF)**]

A continuous function V: $R_+\times D\to R_+$ is said to be a Locally Positive Definite Function (LPDF) if there exists a class K function α such that

(a) $V(t,x)\ge\alpha(\|x\|)$ for all $t\ge 0$ and for all $\|x\|\le r$, for some $r>0$.

(b) $V(t,0)=0$ ♦

Definition [**Positive Definite Function (PDF)**]

A continuous function V: $R_+\times R^n\to R_+$ is said to be a Positive Definite Function (PDF) if there exists a class K_∞ function α such that

(a) $V(t,x)\ge\alpha(\|x\|)$ for all $t\ge 0$ and for all $x\in R^n$.

(b) $V(t,0)=0$ ♦

Definition [**Decrescent Function**]

A continuous V: $R_+\times D\to R_+$ is said to be locally decrescent if there exists a class K function α such that

$$V(t,x)\le\alpha(\|x\|) \text{ for all } t\ge 0 \text{ and for all } \|x\|\le r, \text{ for some } r>0.$$

It is decrescent if α is a class K_∞ function and the above inequality is valid for all x in R^n. ♦

Examples

1) $V(x_1, x_2) = x_1^2 + x_2^2$ is a PDF and decrescent.
2) $V(t, x_1, x_2) = (t+1)(x_1^2 + x_2^2)$ is a PDF but not decrescent.
3) $V(t, x_1, x_2) = e^{-t}(x_1^2 + x_2^2)$ is not a PDF.
4) $V(x_1, x_2) = x_1^2 + \sin^2(x_2)$ is a LPDF and locally decrescent (but not PDF and decrescent).

3.1 Continuous Time Systems

Definition [Derivative of $V(t, x)$ relative to a state trajectory]

Consider the system (L.1). The Derivative of $V(t, x)$ relative to (L.1) is given by

$$\dot{V}(t,x) = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,x), \qquad \text{(L.20)}$$

where

$$\frac{\partial V(t,x)}{\partial x} \triangleq \left[\frac{\partial V(t,x)}{\partial x_1} \ \frac{\partial V(t,x)}{\partial x_2} \ \cdots \ \frac{\partial V(t,x)}{\partial x_n} \right]$$

Theorem L.6 [Ref2] *The equilibrium point 0 of* (L.1) *is locally stable in the sense of Lyapunov if there exists a LPDF V(t,x) such that*

$$\dot{V}(t,x) \le 0$$

for all $t \ge t_o$ *and all x such that* $\|x\| < r$ *for some* $r > 0$. Δ

Theorem L.7 [Ref2] *The equilibrium point* 0 *of* (L.1) *is locally uniformly stable in the sense of Lyapunov if there exists a locally decrescent LPDF* V(t,x) *such that*

$$\dot{V}(t,x) \le 0$$

for all $t \ge 0$ *and all x such that* $\|x\| < r$ *for some* $r > 0$. Δ

Theorem L.8 [Ref2] *The equilibrium point* 0 *of* (L.1) *is locally uniformly asymptotically stable if there exists a locally decrescent LPDF V(t,x) such that* $-\dot{V}(t,x)$ *is a LPDF.* Δ

Theorem L.9 [Ref2] *The equilibrium point 0 of* (L.1) *is globally uniformly asymptotically stable if there exists a decrescent PDF V(t,x) such that* $-\dot{V}(t,x)$ *is a PDF.* Δ

In adaptive control problems, it is often the case that $\dot{V}(t,x)$ is only negative semi-definite, i.e., $\dot{V}(t,x) \le 0$. If the system is autonomous, then, LaSalle's Invariance Principle Theorem L.2 may be applied to obtain asymptotic tracking. *For non-autonomous system, LaSalle's Invariance Theorem L.2 cannot be applied.* Instead, the following Barbalat's lemma should be used.

Lemma L.1 [Ref1] **Barbalat's Lemma**

Let $\phi(t)$ be a uniformly continuous real function of t defined for t≥0. Suppose that $\lim_{t\to\infty}\int_0^t \phi(\tau)d\tau$ *exists and is finite. Then,*

$$\phi(t) \to 0 \quad \text{as} \quad t \to \infty \qquad \Delta$$

Using Barbalat's lemma, the following Lyapunov-like lemma can be obtained.

Lemma L.2 [Ref2] **"Lyapunov-Like Lemma"**

If a scalar function $V(t,x)$ satisfies the following conditions

- *$V(t,x)$ is lower bounded*
- *$\dot{V}(t,x)$ is negative semi-definite.*
- *$\dot{V}(t,x)$ is uniformly continuous in time (A sufficient condition is that $\ddot{V}(t,x)$* is bounded)

then, $\dot{V}(t,x) \to 0 \quad \text{as} \quad t \to \infty$ Δ

3.2 Discrete Time Systems

Definition [Change of $V(k, x)$ relative to a state trajectory]

Consider the system (L.2). The change of $\Delta V(k+1,x)$ relative to (L.2) is given by

$$\begin{aligned}\Delta V(k+1,x) &= V(k+1,x(k+1)) - V(k,x(k)) \\ &= V(k+1, f(k,x(k))) - V(k,x(k)).\end{aligned} \qquad \text{(L.21)}$$

Theorem L.10 *The equilibrium point 0 of* (L.2) *is locally stable in the sense of Lyapunov if there exists a LPDF $V(k,x)$ such that*

$$\Delta V(k+1,x) \le 0$$

for all $k \ge k_o$ and all x such that $\|x\| < r$ for some $r > 0$. Δ

Theorem L.11 *The equilibrium point 0 of* (L.2) *is globally uniformly asymptotically if there exists a decrescent PDF $V(k,x)$ such that* $\Delta V(k+1,x)$ is negative definite.

3.3 Linear Time-Varying Systems

The solution of the linear time-varying system described by

$$\dot{x} = A(t)x \qquad \text{(L.22)}$$

is given by

$$x(t) = \Phi(t,t_0)x(t_0)$$

where $\Phi(t,t_0)$ is the state transition matrix.

Theorem L.12 [Ref1] *The equilibrium point 0 of (L.22) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality*

$$\|\Phi(t,t_0)\| \le k e^{-\gamma(t-t_0)}, \qquad \forall t \ge t_0 \ge 0 \tag{L.23}$$

for some positive constant k and γ.

Theorem L.12 shows that, for linear systems, uniform asymptotic stability of the origin is equivalent to exponential stability. Note that, *for linear time-varying system, in general, uniform asymptotic stability cannot be characterized by the location of the eigenvalues of the matrix A*

Theorem L.13 [Ref1] *Suppose that the equilibrium point 0 of (L.22) is uniformly asymptotically stable, and A(t) is continuous and bounded. Let Q(t) be a continuous, bounded, symmetric positive definite matrix. Then, there is a continuously differentiable, bounded, symmetric positive definite matrix P(t) such that*

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \tag{L.24}$$

Hence, $V(t,x)=x^T P(t)x$ is a Lyapunov function for the system that satisfies the conditions of Theorem L.9 . Δ

Proof: Let

$$P(t) = \int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)d\tau \tag{L.25}$$

It can be verified that P(t) given above satisfies (L.24). Details are omitted.

3.4 Linearization (Lyapunov's Indirect Method)

Theorem L.14 [Ref1] *Consider the non-autonomous system (L.1) with the origin as an equilibrium point. Suppose that $f: R_+ \times D \to R^n$ is continuously differentiable, and the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t. Let*

$$A(t) = \frac{\partial f}{\partial x}(t,x)\bigg|_{x=0} \tag{L.26}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system (L.1) if it is an exponentially stable equilibrium point for the linearized linear system (L.22). Δ

References

[Ref1] Khalil, H. K. (1996), *Nonlinear Systems*, Second edition, Prentice-Hall.

[Ref2] Slotine, J.J.E. and Li, Weiping (1991), *Applied Nonlinear Control*, Prentice-Hall.