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Output Feedback Adaptive Robust Control of Uncertain Linear Systems with Large Disturbances *

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Abstract

In this paper, the discontinuous projection based adaptive robust control (ARC) is extended to a class of uncertain linear systems with large disturbances. An observer is first designed to provide exponentially convergent estimates of the unmeasured states. This observer has an extended filter structure so that on-line parameter adaptation can be utilized to reduce the effect of the possible large nominal disturbance that has a known shape but unknown amplitude. Estimation errors are dealt with via robust feedback at each step of the design procedure. Compared to other existing robust adaptive schemes, the proposed method explicitly takes into account the effect of disturbances and uses both parameter adaptation and robust feedback to attenuate its effect for an improved performance. Furthermore, the upper bound on the absolute value of the tracking error over entire time-history is given and related to certain controller design parameters in a known form, which is more transparent than that in RAC design.

1 Introduction

During the last several years, a great deal of progress has been made in the control of uncertain nonlinear systems [1, 2, 3, 4, 5, 6, 7] and some of the results have been extended to the output feedback control. Kanellakopoulos *et al.* introduce backstepping procedure for a class of nonlinear systems, whose nonlinearities depend only on the measured signals [8]. In [9], Krstic *et al.* propose an adaptive controller for linear systems with parametric uncertainties by using nonlinear methods, such as tuning functions and nonlinear damping. The resulting controller possesses much better transient and steady state performance, as compared with the traditional one. Parameter convergence properties of this controller are also analyzed by Zhang *et al.* [10]. Recently, Ikhouane and Krstic showed that by using a switching σ-modification [11] or parameter projection [12] in the parameter adaptation law, the robustness of this scheme can be improved with respect to both unmodeled dynamics and bounded disturbances.

In this paper, we combine the approach developed in [9] with the adaptive robust control (ARC) design procedure [7] to construct controllers for a class of linear systems having both parametric uncertainties and bounded disturbances. Since only output signal is available for measurement, an observer is first designed to provide exponentially convergent estimates of the unmeasured states. This observer has an extended filter structure so that parameter adaptation can be used to reduce the effect of the possible large nominal disturbance, which is very important from the view point of application. The destabilizing effect of estimation errors is dealt with using robust feedback at each step of the design procedure. Compared with the RAC approaches [11, 12], the proposed scheme explicitly takes into account the effect of disturbance and puts more emphasis on the robust control law design. In fact, the parameter adaptation law in ARC can be switched off at any time without affecting global stability or sacrificing the guaranteed transient performance result since the resulting controller becomes a deterministic robust controller. Furthermore, the proposed controller achieves a guaranteed transient performance and a prescribed final tracking accuracy, i.e., the upper bound on the absolute value of the tracking error over entire time-history is given and related to certain controller design parameters in a *known* form, which is more transparent than that in RAC design.

2 Problem Statement

Consider the following single-input single-output plant described by

$$y(t) = \frac{B(s)}{A(s)}u(t) + \frac{D(s)}{A(s)}\Delta(y,t) + d_y(t) \qquad (1)$$

in which $A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, $B(s) = b_m s^m + \cdots + b_1 s + b_0$ and $D(s) = d_l s^l + \cdots + d_1 s + d_0$, where

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$l, m < n$. The plant parameters a_i and b_i are unknown constants. For simplicity, d_i are assumed to be known here; the results can be easily extended to the case where d_i are unknown constants. $d_y(t)$ is the output disturbance and $\Delta(y,t)$ represents any disturbance coming from the intermediate channels of the plant. In the following, Δ will be handled as follows: we first use the prior information about the nature of the disturbance to construct a nominal disturbance model $\Delta_n = q^T(y,t)c$, in which $q(y,t) = [q_p(y,t), \cdots, q_1(y,t)]^T \in \mathcal{R}^p$ represents the known basis shape functions, $c = [c_p, \cdots, c_1]^T$ represents the unknown magnitudes; this nominal model will be explicitly used in the controller design to improve achievable performance. The disturbance modeling error $\tilde{\Delta} = \Delta - \Delta_n$ will be dealt with via robust feedback to achieve a robust performance. With this disturbance modeling, a state space realization of the plant (1) is given by (without loss of generality, assume $m < l$):

$$\begin{aligned}
\dot{x}_1 &= x_2 - a_{n-1}x_1 \\
\dot{x}_2 &= x_3 - a_{n-2}x_1 \\
&\vdots \\
\dot{x}_{n-l-1} &= x_{n-l} - a_{l+1}x_1 \\
\dot{x}_{n-l} &= x_{n-l+1} - a_l x_1 + d_l q^T c + d_l \tilde{\Delta} \\
&\vdots \\
\dot{x}_{\rho-1} &= x_\rho - a_{m+1}x_1 + d_{m+1} q^T c + d_{m+1}\tilde{\Delta} \\
\dot{x}_\rho &= x_{\rho+1} - a_m x_1 + d_m q^T c + d_m \tilde{\Delta} + b_m u \\
&\vdots \\
\dot{x}_n &= -a_0 x_1 + d_0 q^T c + d_0 \tilde{\Delta} + b_0 u \\
y &= x_1 + d_y
\end{aligned} \tag{2}$$

where $\rho = n - m$ is the relative degree of the system. The unknown parameter vector is defined as $\theta \triangleq [-a_{n-1}, \cdots, -a_0, b_m, \cdots, b_0, c_p, \cdots, c_1]^T \in \mathcal{R}^{m+n+p+1}$. The following notations are used: $\bullet_i$ for the i-th component of the vector $\bullet$, $\bullet_{min}$ for the minimum value of $\bullet$, and $\bullet_{max}$ for the maximum value of $\bullet$. The operation $<$ for two vectors is performed in terms of the corresponding elements of the vectors. The following standard assumptions are made:

A1. The plant is minimum phase. i.e., the polynomial $B(s)$ is Hurwitz. The plant order (n), relative degree ($\rho = n-m$) and the sign of the high frequency gain ($sgn(b_m)$) are known. ◇

A2. The extent of parametric uncertainties θ, uncertain nonlinearities $\tilde{\Delta}$, d_y, and $\dot{d}_y$ are known, i.e.,

$$\begin{aligned}
\theta &\in \Omega_\theta \triangleq \{\theta : \ \theta_{min} < \theta < \theta_{max}\} \\
\tilde{\Delta} &\in \Omega_\Delta \triangleq \{\tilde{\Delta} : |\tilde{\Delta}(y,t)| \le \delta(t)\} \\
d_y &\in \Omega_d \triangleq \{d_y : |d_y(t)| \le \delta_d(t)\} \\
\dot{d}_y &\in \Omega_f \triangleq \{\dot{d}_y : |\dot{d}_y(t)| \le \delta_f(t)\}
\end{aligned} \tag{3}$$

where θ_{min}, θ_{max}, $\delta(t)$, $\delta_d(t)$ and $\delta_f(t)$ are known. ◇

Given the reference trajectory $y_r(t)$, the objective is to synthesize a control input u such that the output y tracks $y_r(t)$ as closely as possible in spite of various model uncertainties. The reference signal y_r and its first ρ derivatives are assumed to be known, bounded, and, in addition, $y_r^{(\rho)}$ is piecewise continuous.

3 State Estimation

Since only the output y is available for measurement, we first design an observer [1] to provide exponentially convergent estimates of the unmeasured states. Rewriting (2) in the form

$$\begin{aligned}
\dot{x} &= A_0 x + (\bar{k} - a)x_1 + dq^T c + bu + d\tilde{\Delta} \\
y &= x_1 + d_y
\end{aligned} \tag{4}$$

where

$$A_0 = \begin{bmatrix} -k_1 & \\ \vdots & I_{n-1} \\ -k_n & 0 \cdots 0 \end{bmatrix}, \ \bar{k} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$$
$$a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}, \ b = \begin{bmatrix} 0_{(\rho-1)\times 1} \\ \bar{b} \end{bmatrix}, \ \bar{b} = \begin{bmatrix} b_m \\ \vdots \\ b_0 \end{bmatrix} \tag{5}$$
$$d = \begin{bmatrix} 0_{(n-l-1)\times 1} \\ \bar{d} \end{bmatrix}, \ \bar{d} = \begin{bmatrix} d_l \\ \vdots \\ d_0 \end{bmatrix}$$

By suitably choosing $\bar{k}$, the observer matrix A_0 will be stable. Thus, there exists a symmetric positive definite (s.p.d.) matrix P such that

$$PA_0 + A_0^T P = -I, \qquad P = P^T > 0 \tag{6}$$

Following the design procedure of [1], we define filters

$$\begin{aligned}
\dot{\xi}_n &= A_0 \xi_n + \bar{k} y, \\
\dot{\xi}_i &= A_0 \xi_i + e_{n-i} y, \quad 0 \le i \le n-1 \\
\dot{v}_i &= A_0 v_i + e_{n-i} u, \quad 0 \le i \le m \\
\dot{\psi}_i &= A_0 \psi_i + d q_i, \quad 0 \le i \le p
\end{aligned} \tag{7}$$

where e_i denotes the i-th standard basis vector in $\mathcal{R}^n$. Note that, departing from [1], the last equation of (7) is introduced so that parameter adaptation can be used to reduce the effect of unknown nominal disturbance represented by unknown parameters c_i. The state estimates can thus be represented by

$$\hat{x} = \xi_n - \sum_{i=0}^{n-1} a_i \xi_i + \sum_{i=0}^{m} b_i v_i + \sum_{i=1}^{p} c_i \psi_i \tag{8}$$

Let $\varepsilon_x = x - \hat{x}$ be the estimation error, from (4), (7) and (8), it can be verified that the observer error dynamics is given by

$$\dot{\varepsilon}_x = A_0 \varepsilon_x + (a - \bar{k})d_y + d\tilde{\Delta} \tag{9}$$

The solution of this equation can be written as $\varepsilon_x = \varepsilon + \varepsilon_u$, where ε is the zero input response satisfying $\dot{\varepsilon} = A_0 \varepsilon$ and ε_u is the zero state response. Noting Assumption A2 and the fact that matrix A_0 is stable, we have

$$\varepsilon_u \in \Omega_\varepsilon \triangleq \{\varepsilon_u : |\varepsilon_u(t)| \le \delta_\varepsilon(t)\} \tag{10}$$

where $\delta_\varepsilon(t)$ is known. In the following controller design, ε and ε_u will be treated as disturbances and accounted for using different robust control functions at each step to achieve a guaranteed final tracking accuracy.

Remark 1 *The filter states ξ_i and v_i can be obtained from the algebraic expressions*

$$\begin{aligned} \xi_n &= -A_0^n \eta, & \\ \xi_i &= A_0^i \eta, & 0 \le i \le n-1 \\ v_i &= A_0^i \lambda & 0 \le i \le m \end{aligned} \tag{11}$$

where η and λ are the states of the following filters

$$\begin{aligned} \dot{\eta} &= A_0 \eta + e_n y \\ \dot{\lambda} &= A_0 \lambda + e_n u \end{aligned} \tag{12}$$

$\Diamond$

4 ARC Backstepping Design

4.1 Parameter Projection

Let $\hat{\theta}$ denote the estimate of θ and $\tilde{\theta}$ the estimation error (i.e., $\tilde{\theta} = \hat{\theta} - \theta$). Under Assumption A2, the discontinuous projection based ARC design [7] can be used to solve the robust tracking control problem for (1). Specifically, the parameter estimate $\hat{\theta}$ is updated through a parameter adaptation law having the form given by

$$\dot{\hat{\theta}} = Proj_{\hat{\theta}}(\Gamma \tau) \tag{13}$$

where the projection mapping $Proj_{\hat{\theta}}(\bullet)$ is defined by [13] (for simplicity, assume that Γ is a diagonal matrix in the following)

$$Proj_{\hat{\theta}}(\bullet) = \begin{cases} 0 & \text{if } \hat{\theta}_i = \theta_{imax} \text{ and } \bullet > 0 \\ 0 & \text{if } \hat{\theta}_i = \theta_{imin} \text{ and } \bullet < 0 \\ \bullet & \text{otherwise} \end{cases} \tag{14}$$

It can be shown [5] that the projection mapping has the following nice properties

$$\begin{array}{ll} \textbf{P1} & \hat{\theta} \in \bar{\Omega}_\theta = \{\hat{\theta}: \ \theta_{min} \le \hat{\theta} \le \theta_{max}\} \\ \textbf{P2} & \tilde{\theta}^T(\Gamma^{-1} Proj_{\hat{\theta}}(\Gamma \bullet) - \bullet) \le 0, \quad \forall \bullet \end{array} \tag{15}$$

4.2 Step 1

The design combines the backstepping design in [9] with the ARC design procedure in [7]. From (2), the derivative of the output tracking error $z_1 = y - y_r$ is

$$\dot{z}_1 = x_2 - a_{n-1} y + a_{n-1} d_y + \dot{d}_y - \dot{y}_r \tag{16}$$

Since x_2 is not measurable, we replace it by its expression from (8)

$$x_2 = \xi_{n,2} - \xi_{(2)} a + v_{(2)} \bar{b} + \psi_{(2)} c + \varepsilon_{x2} \tag{17}$$

where $\varepsilon_{x2} = \varepsilon_2 + \varepsilon_{u2}$ is the estimation error of x_2, and

$$\begin{aligned} &\xi_{(2)} \triangleq [\xi_{n-1,2}, \cdots, \xi_{0,2}], \quad v_{(2)} \triangleq [v_{m,2}, \cdots, v_{0,2}], \\ &\psi_{(2)} \triangleq [\psi_{p,2}, \cdots, \psi_{1,2}] \end{aligned} \tag{18}$$

in which $\bullet_{i,j}$ represents the jth element of $\bullet_i$. Substituting (17) into (16), gives

$$\dot{z}_1 = b_m v_{m,2} + \xi_{n,2} + \theta^T \bar{\omega} - \dot{y}_r + \tilde{\Delta}_1 + \varepsilon_2 \tag{19}$$

where $\omega^T \triangleq [\xi_{(2)}, \ v_{(2)}, \psi_{(2)}] + e_1^{*T} y, \bar{\omega} \triangleq \omega - e_{n+1}^* v_{m,2}$, $\tilde{\Delta}_1 \triangleq a_{n-1} d_y + \dot{d}_y + \varepsilon_{u2}$, and e_i^* is the i-th standard basis vector in $\mathcal{R}^{n+m+p+1}$. If we treat $v_{m,2}$ as the input, we can synthesize a virtual ARC control law α_1 for $v_{m,2}$ such that z_1 is as small as possible:

$$\begin{aligned} \alpha_1(y, \eta, \bar{\lambda}_{m+1}, \psi_{(2)}, \hat{\theta}, t) &= \alpha_{1a} + \alpha_{1s} \\ \alpha_{1a} &= -\tfrac{1}{\hat{b}_m}\{\xi_{n,2} + \hat{\theta}^T \bar{\omega} - \dot{y}_r\} \end{aligned} \tag{20}$$

where $\bar{\lambda}_i = [\lambda_1, \cdots, \lambda_i]^T$, α_{1a} functions as an adaptive control law used to achieve an improved model compensation, and α_{1s} is a robust control law to be synthesized later. Noting Assumption A1 that $sgn(b_m)$ is known, without loss of generality, we assume $b_m > 0$. Then, from P1 of (15), $\hat{b}_m \ge b_{min} > 0$, where b_{min} is the lower bound in A2 for b_m. Thus the control function (20) is well defined. Let $z_2 = v_{m,2} - \alpha_1$ denote the input discrepancy. Substituting (20) into (19) leads to

$$\dot{z}_1 = b_m(z_2 + \alpha_{1s}) - \tilde{\theta}^T \phi_1 + \tilde{\Delta}_1 + \varepsilon_2 \tag{21}$$

where $\phi_1 \triangleq \bar{\omega} + e_{n+1}^* \alpha_{1a}$.

In [9], it needs to incorporate the *tuning functions* in the construction of control functions. Here, due to the use of discontinuous projection (14), the adaptation law (13) is discontinuous and thus *cannot* be used in the control law design at each step since backstepping design requires that the control function synthesized at each step be sufficiently smooth in order to obtain its partial derivatives. In the following, it will be shown that this design difficulty can be overcome by strengthening the robust control law design. The robust control function α_{1s} consists of three terms given by

$$\alpha_{1s} = -\tfrac{1}{b_{min}} k_{1s} z_1 + \alpha_{1s1} + \alpha_{1s2} \tag{22}$$

where α_{1s1} and α_{1s2} are robust control functions designed in the following and k_{1s} is any nonlinear feedback gain satisfying

$$k_{1s} \ge \|C_{\phi 1} \Gamma \phi_1\|^2 + g_1, \quad g_1 > 0 \tag{23}$$

in which $C_{\phi 1}$ is a positive definite constant diagonal matrix to be specified later. Substituting (22) into (21),

$$\begin{aligned} \dot{z}_1 = b_m z_2 - \tfrac{b_m}{b_{min}} k_{1s} z_1 + b_m(\alpha_{1s1} + \alpha_{1s2}) \\ -\tilde{\theta}^T \phi_1 + \tilde{\Delta}_1 + \varepsilon_2 \end{aligned} \tag{24}$$

Define a positive semi-definite (p.s.d.) function $V_1 = \frac{1}{2} w_1 z_1^2$, where $w_1 > 0$ is a weighting factor. From(24), its time derivative satisfies

$$\begin{aligned} \dot{V}_1 \le b_m w_1 z_1 z_2 - w_1 k_{1s} {z_1}^2 + w_1 z_1 (b_m \alpha_{1s1} \\ -\tilde{\theta}^T \phi_1 + \tilde{\Delta}_1) + w_1 z_1 (b_m \alpha_{1s2} + \varepsilon_2) \end{aligned} \tag{25}$$

where

$$|\tilde{\Delta}_1| \le \tilde{\delta}_1(t) \triangleq max\{|a_{n-1}|\} \delta_d(t) + \delta_f(t) + \delta_{\varepsilon 2} \tag{26}$$

Since $\tilde{\delta}_1$ is known, there exists a robust control function α_{1s1} satisfying the following conditions [7]

$$\begin{array}{ll} \text{condition i} & z_1\{b_m \alpha_{1s1} - \tilde{\theta}^T \phi_1 + \tilde{\Delta}_1\} \le \epsilon_{11} \\ \text{condition ii} & z_1 \alpha_{1s1} \le 0 \end{array} \tag{27}$$

where ϵ_{11} is a design parameter which can be arbitrarily small. Essentially, condition i of (27) shows that

α_{1s1} is synthesized to dominate the model uncertainties coming from both $\tilde{\theta}$ and $\tilde{\Delta}_1$ with the level of control accuracy being measured by the design parameter ϵ_{11}, and condition *ii* is to make sure that α_{1s1} is dissipating in nature so that it does not interfere with the functionality of the adaptive control part α_{1a}.

In principle, the same strategy can be used to design a robust control function α_{1s2} to handle the effect of ε_2. However, since the bound of ε_2 is unknown, we cannot pre-specify the level of control accuracy. So we relax the condition *i* to

$$z_1(b_m\alpha_{1s2}+\varepsilon_2)\le \epsilon_{12}\ \varepsilon_2^2 \tag{28}$$

with ϵ_{12} being a design parameter [14].

4.3 Step 2

From (20), (11), (12), and the rearrangements from (16) to (19)

$$\begin{aligned}
\dot{\alpha}_1 &= \dot{\alpha}_{1c}+\dot{\alpha}_{1u}\\
\dot{\alpha}_{1c} &= \tfrac{\partial\alpha_1}{\partial y}(\xi_{n,2}+\hat{\theta}^T\omega)+\tfrac{\partial\alpha_1}{\partial\eta}\dot{\eta}+\textstyle\sum_{j=1}^{m+1}\tfrac{\partial\alpha_1}{\partial\lambda_j}\dot{\lambda}_j\\
&\quad+\textstyle\sum_{j=1}^{p}\tfrac{\partial\alpha_1}{\partial\psi_{j,2}}\dot{\psi}_{j,2}+\tfrac{\partial\alpha_1}{\partial t}\\
\dot{\alpha}_{1u} &= \tfrac{\partial\alpha_1}{\partial y}(-\tilde{\theta}^T\omega+\tilde{\Delta}_1+\varepsilon_2)+\tfrac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}
\end{aligned} \tag{29}$$

In (29), noting that $\dot{\eta}$, $\dot{\lambda}_j$ and $\dot{\psi}_j$ can be calculated from (12) and (7) respectively, $\dot{\alpha}_{1c}$ is calculable and can be used in the design of control functions, but $\dot{\alpha}_{1u}$ cannot due to various uncertainties. Therefore, $\dot{\alpha}_{1u}$ has to be dealt with via robust feedback in this step design. From (7) and (29), we express the derivative of $z_2=v_{m,2}-\alpha_1$ as

$$\dot{z}_2=v_{m,3}-k_2v_{m,1}-\dot{\alpha}_{1c}-\dot{\alpha}_{1u} \tag{30}$$

Consider the augmented p.s.d. function $V_2=V_1+\frac{1}{2}w_2z_2^2$, where $w_2>0$. Noting (25) and (30), its derivative is

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1|_{\alpha_1}+w_2z_2\{\tfrac{w_1}{w_2}b_mz_1+v_{m,3}-k_2v_{m,1}\\
&\quad-\dot{\alpha}_{1c}-\dot{\alpha}_{1u}\}
\end{aligned} \tag{31}$$

where $\dot{V}_1|_{\alpha_1}$ denotes $\dot{V}_1$ under the condition that $v_{m,2}=\alpha_1$ (or $z_2=0$). Similar to (20), the virtue control input $\alpha_2=v_{m,3}-z_3$ consists of two parts given by

$$\begin{aligned}
\alpha_2(y,\eta,\bar{\lambda}_{m+2},\psi,\hat{\theta},t) &= \alpha_{2a}+\alpha_{2s}\\
\alpha_{2a} &= -\tfrac{w_1}{w_2}\hat{b}_mz_1+k_2v_{m,1}+\dot{\alpha}_{1c}\\
\alpha_{2s} &= -k_{2s}z_2+\alpha_{2s1}+\alpha_{2s2},\\
k_{2s} &\ge g_2+\|\tfrac{\partial\alpha_1}{\partial\hat{\theta}}C_{\theta 2}\|^2+\|C_{\phi 2}\Gamma\phi_2\|^2
\end{aligned} \tag{32}$$

where $g_2>0$ is a constant, $C_{\theta 2}$ and $C_{\phi 2}$ are positive definite constant diagonal matrixes , α_{2s1} and α_{2s2} are robust control functions to be chosen later. Substituting (32) and (29) into (31) leads to

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1|_{\alpha_1}+w_2z_2z_3-w_2k_{2s}z_2^2+w_2z_2(\alpha_{2s1}-\tilde{\theta}^T\phi_2\\
&\quad+\tilde{\Delta}_2)+w_2z_2(\alpha_{2s2}-\tfrac{\partial\alpha_1}{\partial y}\varepsilon_2)-w_2z_2\tfrac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}
\end{aligned} \tag{33}$$

in which $\phi_2=e_{n+1}^*\frac{w_1}{w_2}z_1-\frac{\partial\alpha_1}{\partial y}\omega$ and $\tilde{\Delta}_2=-\frac{\partial\alpha_1}{\partial y}\tilde{\Delta}_1$. Since $|\tilde{\Delta}_2|\le\tilde{\delta}_2(t)\triangleq|\frac{\partial\alpha_1}{\partial y}|\tilde{\delta}_1$, similar to (27) and (28), the robust control functions α_{2s1} and α_{2s2} are chosen to satisfy

$$\begin{array}{ll}
\text{condition i} & z_2(\alpha_{2s1}-\tilde{\theta}^T\phi_2+\tilde{\Delta}_2)\le\epsilon_{21}\\
\text{condition ii} & z_2\alpha_{2s1}\le 0\\
\text{condition iii} & z_2(\alpha_{2s2}-\tfrac{\partial\alpha_1}{\partial y}\varepsilon_2)\le\epsilon_{22}\ \varepsilon_2^2
\end{array} \tag{34}$$

where ϵ_{21} and ϵ_{22} are design parameters. Noting (25) and (33), the derivative of V_2 satisfies

$$\begin{aligned}
\dot{V}_2 &\le w_2z_2z_3-\textstyle\sum_{j=1}^2 w_jk_{js}{z_j}^2+w_1z_1\{b_m\alpha_{1s2}-\tilde{\theta}^T\phi_1\\
&\quad+\tilde{\Delta}_1\}+w_2z_2\{\alpha_{2s2}-\tilde{\theta}^T\phi_2+\tilde{\Delta}_2\}+w_1z_1(b_m\alpha_{1s2}\\
&\quad+\varepsilon_2)+w_2z_2(\alpha_{2s2}-\tfrac{\partial\alpha_1}{\partial y}\varepsilon_2)-w_2z_2\tfrac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}
\end{aligned} \tag{35}$$

4.4 Step i $(3\le i\le\rho-1)$

Mathematical induction will be used to prove the general results for all intermediate design steps. At step i, the same ARC design as in the above two steps will be employed to construct a control function α_i for $v_{m,i+1}$. For step j, $\forall\ 3\le j\le i-1$, let $z_j=v_{m,j}-\alpha_{j-1}$ and recursively define the following functions

$$\phi_j=-\tfrac{\partial\alpha_{j-1}}{\partial y}\omega,\qquad \tilde{\Delta}_j=-\tfrac{\partial\alpha_{j-1}}{\partial y}\tilde{\Delta}_1 \tag{36}$$

Lemma 1 *At step i, choose the control function α_i as*

$$\begin{aligned}
\alpha_i(y,\eta,\bar{\lambda}_{m+i},\psi,\hat{\theta},t) &= \alpha_{ia}+\alpha_{is}\\
\alpha_{ia} &= -\tfrac{w_{i-1}}{w_i}z_{i-1}+k_iv_{m,1}+\dot{\alpha}_{(i-1)c}\\
\alpha_{is} &= -k_{is}z_i+\alpha_{is1}+\alpha_{is2},\\
k_{is} &\ge g_i+\|\tfrac{\partial\alpha_{i-1}}{\partial\hat{\theta}}C_{\theta i}\|^2+\|C_{\phi i}\Gamma\phi_i\|^2
\end{aligned} \tag{37}$$

where $\dot{\alpha}_{(i-1)c}=\frac{\partial\alpha_{i-1}}{\partial y}(\xi_{n,2}+\hat{\theta}^T\omega)+\frac{\partial\alpha_{i-1}}{\partial\eta}\dot{\eta}+\sum_{j=1}^{m+i-1}\frac{\partial\alpha_{i-1}}{\partial\lambda_j}\dot{\lambda}_j+\sum_{j=1}^{p}\frac{\partial\alpha_{i-1}}{\partial\psi_j}\dot{\psi}_j+\frac{\partial\alpha_{i-1}}{\partial t}$, *$g_i$ is a positive constant, $C_{\theta i}$ and $C_{\phi i}$ are positive definite constant diagonal matrixes, w_i is a weighting factor, α_{is1} and α_{is2} satisfy*

$$\begin{array}{ll}
i & z_i(\alpha_{is1}-\tilde{\theta}^T\phi_i+\tilde{\Delta}_i)\le\epsilon_{i1}\\
ii & z_i\alpha_{is1}\le 0\\
iii & z_i(\alpha_{is2}-\tfrac{\partial\alpha_{i-1}}{\partial y}\varepsilon_2)\le\epsilon_{i2}\ \varepsilon_2^2
\end{array} \tag{38}$$

Then the i-th error subsystem is

$$\begin{aligned}
\dot{z}_i &= z_{i+1}-\tfrac{w_{i-1}}{w_i}z_{i-1}-k_{is}z_i+(\alpha_{is1}-\tilde{\theta}^T\phi_i+\tilde{\Delta}_i)\\
&\quad+(\alpha_{is2}-\tfrac{\partial\alpha_{i-1}}{\partial y}\varepsilon_2)-\tfrac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}
\end{aligned} \tag{39}$$

and the derivative of the augmented p.s.d. function $V_i=V_{i-1}+\frac{1}{2}w_iz_i^2$ satisfies

$$\begin{aligned}
\dot{V}_i &\le w_iz_iz_{i+1}-\textstyle\sum_{j=1}^i w_jk_{js}{z_j}^2+w_1z_1(b_m\alpha_{1s1}\\
&\quad-\tilde{\theta}^T\phi_1+\tilde{\Delta}_1)+\textstyle\sum_{j=2}^i w_jz_j(\alpha_{js1}-\tilde{\theta}^T\phi_j+\tilde{\Delta}_j)\\
&\quad+w_1z_1(b_m\alpha_{1s2}+\varepsilon_2)+\textstyle\sum_{j=2}^i w_jz_j(\alpha_{js2}\\
&\quad-\tfrac{\partial\alpha_{j-1}}{\partial y}\varepsilon_2)-\textstyle\sum_{j=2}^i w_jz_j\tfrac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}
\end{aligned} \tag{40}$$

The Lemma can be proved in the same way as in [7].

4.5 Step ρ

This is the final design step, in which the actual control u will be synthesized such that $v_{m,\rho}$ tracks the desired control function $\alpha_{\rho-1}$. We express the derivative of z_ρ as

$$\begin{aligned}
\dot{z}_\rho &= v_{m,\rho+1}-k_\rho v_{m,1}+u-\dot{\alpha}_{(\rho-1)c}-\tfrac{\partial\alpha_{\rho-1}}{\partial y}(-\tilde{\theta}^T\omega\\
&\quad+\tilde{\Delta}_1+\varepsilon_2)-\tfrac{\partial\alpha_{\rho-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}
\end{aligned} \tag{41}$$

if we treat the $v_{m,\rho+1}+u$ as the input, (41) has the same form as the intermediate step i. Therefore, the general form (36)~(40) applies to Step ρ. Since u is the actual control input, we can choose it as $u=\alpha_\rho-v_{m,\rho+1}$, where α_ρ is given by (37). Then, $z_{\rho+1}=u+v_{m,\rho+1}-\alpha_\rho=0$.

Theorem 1 *Let the parameter estimates be updated by the adaptation law (13) in which* $\tau = \sum_{j=1}^{\rho} w_j \phi_j z_j$. *If controller parameters* $C_{\theta j}$ *and* $C_{\phi k}$ *are chosen such that* $c_{\phi kr}^2 \geq \frac{\rho w_k}{4} \sum_{j=2}^{\rho} \frac{w_j}{c_{\theta jr}^2}$, *where* $c_{\theta jr}$ *and* $c_{\phi kr}$ *are the r-th elements of* $C_{\theta j}$ *and* $C_{\phi k}$ *respectively. Then, the control law guarantees that*

A. *In general, the control input and all internal signals are bounded. Furthermore,* V_ρ *is bounded above by*

$$V_\rho(t) \leq exp(-\lambda_\rho t)V_\rho(0) + \frac{\bar{\epsilon}_{\rho 1} + \bar{\epsilon}_{\rho 2}\|\varepsilon_2\|_\infty^2}{\lambda_\rho}[1 - exp(-\lambda_\rho t)] \quad (42)$$

where $\lambda_\rho = 2min\{g_1, \cdots, g_\rho\}$, $\bar{\epsilon}_{\rho 1} = \sum_{j=1}^{\rho} w_j \epsilon_{j1}$, $\bar{\epsilon}_{\rho 2} = \sum_{j=1}^{\rho} w_j \epsilon_{j2}$, *and* $\|\cdot\|_\infty$ *stands for the infinity norm. Noting that* $\varepsilon_2(t)$ *exponentially converges to zero,* $V_\rho(t)$ *is ultimately bounded by* $V_\rho(\infty) \leq \frac{\bar{\epsilon}_{\rho 1}}{\lambda_\rho}$.

B. *If after a finite time* t_0, $\Delta = 0$ *and* $d_y = 0$, *i.e., in the presence of parametric uncertainties only, then, in addition to results in A, asymptotic output tracking (or zero final tracking error) is also achieved.* $\triangle$

The Theorem can be proved in the same way as in [7].

Remark 2 *Results in A of Theorem 1 indicate that the proposed controller has an exponentially converging transient performance with the exponentially converging rate* λ_ρ *and the final tracking error being able to be adjusted via certain controller parameters* (g_i *and* ϵ_{i1}) *freely in a* known *form. Theoretically, this result is what a well-designed robust controller can achieve. In fact, when the parameter adaptation law (13) is switched off, the proposed ARC law becomes a deterministic robust control law and Results A of the Theorem remain valid [5, 6].*

B of Theorem 1 implies that without using high gain, the controller may have a very small tracking error due to the reduced parametric uncertainties. Theoretically, Result B is what a well-designed adaptive controller can achieve. $\diamond$

Remark 3 *It is seen from (42) that transient tracking error is affected by the initial value* $V_\rho(0)$. *To further reduce transient tracking error, the idea of filter initialization [9] can be used to render* $V_\rho(0) = 0$. $\diamond$

5 Conclusions

In this paper, an output feedback ARC scheme based on discontinuous projection method is presented for a class of linear systems having model uncertainties. In contrast to other existing robust adaptive control schemes, the proposed controller uses on-line parameter adaptation to compensate for the effect of disturbances that can be modeled for an improved performance. The uncompensated disturbances and the estimation errors are effectively handled via robust feedback to achieve a robust performance. Furthermore, the resulting controller achieves a guaranteed transient performance and a prescribed final tracking accuracy in the presence of both parametric uncertainties and bounded disturbance. In the presence of parametric uncertainties only, asymptotic tracking is achieved without using an infinite fast switching control law or an infinite-gain feedback.

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