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Neural Network-based Adaptive Robust Control of a Class of Nonlinear Systems in Normal Form *

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Abstract

In this paper, Neural networks (NNs) and adaptive robust control (ARC) design philosophy are integrated to design performance oriented control laws for a class of n-th order nonlinear systems in a normal form in the presence of both repeatable and non-repeatable uncertain nonlinearities. Unknown nonlinearities can exist in the input channel also. All unknown but repeatable nonlinearities are approximated by outputs of multi-layer NNs. Discontinuous projection method with fictitious bounds is used to tune NN weights on-line with no prior information for a controlled learning process. Robust terms are constructed to attenuate model uncertainties effectively for a guaranteed output tracking transient performance and a guaranteed final tracking accuracy. If the unknown nonlinear functions are in the functional ranges of NNs and the ideal weights fall within the prescribed range, asymptotic output tracking is also achieved. Furthermore, by choosing the prescribed range appropriately, the controller may have a well-designed built-in anti-integration windup mechanism.

1 Introduction

It is generally difficult to treat various nonlinearities under a unified framework. Sometimes, due to the limited knowledge about certain physical phenomena, it is even impossible to describe those phenomena by explicit nonlinear functions precisely. These factors make the design of high performance controllers for nonlinear systems difficult.

Due to their universal approximation capabilities [1], NNs have been effectively used in modeling complex nonlinear phenomena and system identification. In the research field of neural networks itself, focus is on the investigation of various NN characteristics, such as network structure, stability [2], convergence, and uniqueness of weights [3], *etc.* However, in all these papers, in order to guarantee the stability of neural networks and/or the uniqueness of the weights, the NN weights have to satisfy some restrictive conditions, which may limit the approximation capability of neural networks in practice since weights can only be tuned in a relatively small region. As such, researchers are still keeping on looking for NN structures with less restrictive conditions for the convergence of NN weights. Fortunately, when neural networks are used for control design purposes, the main focus is on the performance of the closed-loop system in terms of output tracking as long as all signals are bounded. Whether or not the NN weights converge to their ideal values may not be the key issue. As such, the NN weights can be tuned in a relatively large region. Consequently, the approximation range of a neural network becomes large, and a better approximation capability can be expected, which is helpful in the control of nonlinear systems when little is known about the nonlinearities in the system. Thus, in this paper, not much attention will be paid to the convergence of weights of neural networks, and only the boundednesses of all the signals in neural networks are sought.

Neural networks have been applied to the control field recently [4] and various results have been achieved [5, 6, 7]. Two main issues have to be dealt with in the use of neural networks for nonlinear control design. Firstly, the ideal synaptic weights of a neural network for approximating an unknown nonlinear function are usually unknown. Certain algorithms have to be derived to tune these unknown NN weights on-line if NN is used to deal with various unknown nonlinear functions. In terms of control terminology, adaptation laws are needed. Secondly, the ideal NN weights for the neutral network to reconstruct an unknown nonlinear function exactly may not exist, i.e., the unknown nonlinear function to be approximated may not be in the functional range of the neural network. The approximation error between the ideal output of a neural network and the true nonlinear function cannot be assumed to be zero although it may be very small within a compact set. Thus, the issue of robustness to the approximation errors also need to be considered when certain on-line tuning rules are derived for the NN weights. In [5], based on the assumption that the both the input-hidden weights and the bounds of the hidden-output weights are known, backpropagation neural networks were used to design a robust adaptive controller (RAC) for multi-

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link rigid robots. In [6], with the σ-modification type weight-tuning law, adaptive neural network control schemes were proposed for nonlinear systems with uncertainties not satisfying matching conditions, where the input-hidden weights are also assumed to be known. Backstepping method was used in [8] to design a neural network controller to guarantee the semi-global stability of the closed system. RBF networks were used in [7] to adaptively compensate for the plant nonlinearities, and the resulting adaptive controller achieves global stability and the final tracking accuracy. All these works are based on the assumption that the input-hidden weights of neural networks are known. It may be beneficial if this assumption can be relaxed so that one can fully explore the generality and flexibility of neural networks. Furthermore, since the σ-modification type weight tuning method is used [5, 6], asymptotic output tracking cannot be achieved even when the unknown nonlinear function is in the functional range of the neural network. In other words, ideal perfect learning capability of neural networks is lost. In addition, transient tracking performance is in general not known. Transient period may be long and large transient tracking errors may exhibit.

Recently, the adaptive robust control (ARC) approach has been proposed in [9] for nonlinear systems in the presence of both parametric uncertainties and non-repeatable uncertain nonlinearities. The resulting ARC controllers achieve a guaranteed output tracking transient performance and final tracking accuracy in general. In addition, in the presence of parametric uncertainties only, asymptotic output tracking is achieved. The strong performance results achieved by ARC controllers motivate us to investigate whether the essential idea of ARC approach can be extended to the NN based controller designs to further improve the achievable performance of NN based controllers. At the same time, since only a special class of unknown nonlinear functions–a linear combination of known basis functions with unknown weights–has been considered in [9], such an extension is also of significant theoretical values since a more general class of unknown functions can be dealt with via neural networks.

In this paper, NNs and ARC design philosophy will be integrated to design performance oriented control laws for uncertain nonlinear systems. All unknown but repeatable nonlinear functions will be approximated by the outputs of multi-layer NNs. Discontinuous projection method with fictitious bounds [10] will be used to tune NN weights on-line to achieve a controlled learning. Robust control is constructed to attenuate various model uncertainties effectively for a guaranteed output tracking transient performance and a guaranteed final tracking accuracy in general-a transient tracking performance that existing NN based robust adaptive controllers (RACs) [5, 6] cannot achieve. In addition, if the unknown nonlinear function is in the functional range of the NN and the ideal weights fall within the prescribed range, asymptotic output tracking is also achieved to retain the perfect learning capability of NNs-a performance that existing NN based RACs [5, 6] cannot have.

2 Problem Formulation

The system to be considered is as follows [7]

$$x^{(n)} = f(\mathbf{x}) + b(\mathbf{x})u(t) + \Delta(\mathbf{x},t) \quad (1)$$

where x is the output, $\mathbf{x} = [x, x^{(1)}, \cdots, x^{(n-1)}]^T$ is the state vector with $x^{(i)}$ being the i-th time derivative of x, $u(t)$ is the input, $f(\mathbf{x})$ is the lumped unknown nonlinearity, $b(\mathbf{x})$ is the input gain, and $\Delta(\mathbf{x},t)$ is the lumped *non-repeatable* nonlinearities such as disturbances.

Assumption 1 *$f(\mathbf{x})$ can be approximated by the output of a three-layer NN [1] with the approximation error bounded by*

$$|f(\mathbf{x}) - \mathbf{w}_f^T \mathbf{g}_f(\mathbf{V}_f \mathbf{x}_a)| \le d_f(x), \forall \mathbf{x} \in \mathcal{R}^n \quad (2)$$

where $\mathbf{x}_a = [\mathbf{x}^T, -1]^T$ is the augmented input vector to the NN (-1 term denotes the input bias), $d_f(x) \ge 0$ is the bound of the approximation error, $\mathbf{w}_f = [w_{f1}, \cdots, w_{fr_f}]^T$ is the hidden-output weight vector, $\mathbf{V}_f = [\mathbf{v}_{f1}, \cdots, \mathbf{v}_{fr_f}]^T \in \mathcal{R}^{r_f \times (n+1)}$ is the input-hidden weight matrix with $\mathbf{v}_{fi} \in \mathcal{R}^{(n+1)\times 1}$, r_f is the number of hidden-layer neurons, and $\mathbf{g}_f(\mathbf{V}_f \mathbf{x}_a) = [g_{f1}(\mathbf{v}_{f1}^T \mathbf{x}_a), \cdots, g_{fr_f}(\mathbf{v}_{fr_f}^T \mathbf{x}_a)]^T$ is the activation function vector.

Assumption 2 *$b(\mathbf{x})$ is nonzero with known sign. Without loss of generality, assume $b(\mathbf{x}) \ge b_l > 0$, $\forall \mathbf{x} \in \mathcal{R}^n$ with b_l being a known positive constant.*

Assumption 3 *$|\Delta| \le \delta(\mathbf{x},t) d(t)$ holds with δ being a known function, and $d(t)$ being an unknown but bounded positive time-varying function.*

Let the desired output be $x_d(t)$, and its state vector be $\mathbf{x}_d(t) = [x_d, x_d^{(1)}, \cdots, x_d^{(n-1)}]^T$. *The objective is to design a control law for u such that $\mathbf{x}$ tracks $\mathbf{x}_d$ as closely as possible.* Define $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_d(t)$. Let $\hat{\star}$ represent the estimate of $\star$, and $\tilde{\star} = \hat{\star} - \star$ be the estimation error of $\star$.

3 NNARC with Known Input-hidden Weights

In order to explain the proposed NNARC clearly and compare it with other existing methodologies, we will solve the problem step by step. For these purposes, it is firstly assumed that the input-hidden weights of the network are known as done in [5, 6] and the input gain is a unity [5], i.e., $b \equiv 1$. In the subsequent sections, these assumptions will be removed, and on-line estimates of the input-hidden weights and approximation of $b(\mathbf{x})$ will be considered.

3.1 Discontinuous Projection Mapping

The general discontinuous projetion mapping is as follows

$$\mathbf{Proj}_{\hat{\star}}(\bullet) = \{Proj_{\hat{\star}}(\bullet_{ij})\} \quad (3)$$

with its ij-th entry defined as [11]

$$Proj_{\hat{\star}}(\bullet_{ij}) = \begin{cases} 0 & if \begin{cases} \hat{\star}_{ij} = \hat{\rho}_{u,\star_{ij}} \text{ and } \bullet_{ij} > 0 \\ \hat{\star}_{ij} = \hat{\rho}_{l,\star_{ij}} \text{ and } \bullet_{ij} < 0 \end{cases} \\ \bullet_{ij} & otherwise \end{cases} \quad (4)$$

where $\hat{\rho}_{l,\star_{ij}}$ and $\hat{\rho}_{u,\star_{ij}}$ are the fictitious lower and upper bound of $\star_{ij}$. Define $\hat{\rho}_{\star_{ij}} = max\{|\hat{\rho}_{l,\star_{ij}}|, |\hat{\rho}_{u,\star_{ij}}|\}$. Denote $\hat{\rho}_\star = \{\hat{\rho}_{\star_{ij}}\}$, and $\rho_\star = \{\rho_{\star_{ij}}\}$.

3.2 NNARC Design

A concise tracking error metric is defined by [7]:

$$s(t) = (\frac{d}{dt} + \lambda)^{n-1}\tilde{x}(t) \quad \text{with} \quad \lambda > 0 \tag{5}$$

(5) can be rewritten as $s(t) = \lambda^T\tilde{\mathbf{x}}(t)$ with the i-th entry of vector λ being $C_{n-1}^{i-1}\lambda^{n-i} = \frac{(n-1)!}{(n-i)!(i-1)!}\lambda^{n-i}$.

Consider the following control law and adaptation law

$$u = u_a + u_s \tag{6}$$

$$u_a = -a_r(t) - \hat{\mathbf{w}}_f^T\mathbf{g}_f,\ u_s = u_{s1} + u_{s2},\ u_{s1} = -ks \tag{7}$$

$$\dot{\hat{\mathbf{w}}}_f = \mathbf{Proj}_{\hat{\mathbf{w}}_f}(\Gamma_{wf}s\mathbf{g}_f) \tag{8}$$

where $k > 0$, $a_r(t) = \lambda_v^T\tilde{\mathbf{x}} - x_d^{(n)}$ with $\lambda_v^T = [0, \lambda^{n-1}, \cdots, C_{n-1}^{i-2}\lambda^{n-i+1}, \cdots, (n-1)\lambda]$, Γ_{w_f} is symmetric positive definite (s.p.d.), and u_{s2} satisfies [10]

$$s\left(f(\mathbf{x}) - \hat{\mathbf{w}}_f^T\mathbf{g}_f(\mathbf{V}_f\mathbf{x}_a) + u_{s2} + \Delta\right) \le \varepsilon_s, \tag{9}$$

$$su_{s2} \le 0 \tag{10}$$

where $\varepsilon_s = \left|1 + \frac{|(||\rho_{wf}||_2 - ||\hat{\rho}_{wf}||_2)|}{||\hat{\rho}_{wf}||_2}\right|^2 \varepsilon_1 + \varepsilon_2 + ||d||_\infty^2\varepsilon_3$ with $||d||_\infty$ being L_∞ norm of $d(t)$, ε_1, ε_2, and ε_3 being positive design constants, and $||\bullet||_2$ denoting the 2-norm of a vector $\bullet$. Control law (6) is referred to as NNARC-I.

Remark 1 *The robust term u_{s2} in (7) may be chosen as [10]*

$$u_{s2} = -k_{s2}s \tag{11}$$

where $k_{s2} \ge \frac{h_1^2}{4\varepsilon_1} + \frac{h_2^2}{4\varepsilon_2} + \frac{h_3^2}{4\varepsilon_3}$ with $h_1 \ge ||\mathbf{g}_f||(||\hat{\mathbf{w}}_f||_2 + ||\hat{\rho}_{wf}||_2)$, $h_2 \ge d_f$, $h_3 \ge \delta(x,t)$, and $||\mathbf{g}_f|| = \sqrt{\sum_{i=1}^{i=r_f}||g_{fi}||_\infty^2}$. Usually, g_{fi} is a sigmoid function and $|g_{fi}| \le 1$. Thus, one can choose $h_1 = 2\sqrt{r_f}||\hat{\rho}_{wf}||_2$.

Theorem 1 *With NNARC-I and the adaptation law (6)-(8), the following results hold:*

A. *In general, all signals are bounded. $s(t)$ exponentially converges to a small value and is bounded above by*

$$s^2(t) \le exp(-2kt)s^2(0) + \frac{\varepsilon_s}{k} \tag{12}$$

B. *By setting $\mathbf{x}_d(0) = \mathbf{x}(0)$, the actual tracking error is asymptotically bounded by*

$$||\tilde{x}^{(i)}||_\infty \le 2^i\lambda^{i-n+1}\sqrt{\frac{\varepsilon_s}{k}} \tag{13}$$

C. *If $f(\mathbf{x}) = \mathbf{w}_f^T\mathbf{g}_f(\mathbf{V}_f\mathbf{x}_a)$, then, in addition to A and B, asymptotic output tracking is achieved provided that $\Delta = 0$, and the ideal weight $\mathbf{w}_f$ lies within the fictitious bound (i.e., $\hat{\rho}_{l,w_{fi}} \le w_{fi} \le \hat{\rho}_{u,w_{fi}}, \forall i = 1, \cdots, r_f$).* ◊

Proof. Using the similar approach as in [10], Result **A** and **C** can be proved through the use of two positive definite functions $V_1 = \frac{1}{2}s^2(t)$ and $V_{a1} = V_1 + \frac{1}{2}\tilde{\mathbf{w}}_f^T\Gamma_{wf}^{-1}\tilde{\mathbf{w}}_f$, respectively. **B** can be derived from **A** [12]. □

Remark 2 *Results A and B show that a guaranteed transient performance and final tracking accuracy is achieved in general; the decaying rate $2k$ and the bound of the final tracking error ($||\tilde{x}||_\infty \le \lambda^{-n+1}\sqrt{\varepsilon_s}$) can be adjusted by tuning k, ε_1, ε_2, ε_3, and λ. These results are much stronger than those in [5, 6], where the transient performance is unknown. Result C shows that NNARC-I can accomplish its learning goal, which cannot be attained in the previous research [5, 6]. The saturation problem may be alleviated by choosing the fictitious bounds of $\mathbf{w}_f$ appropriately [10].*

4 NNARC with Unknown Input-hidden Weights

It is now assumed that $\mathbf{V}_f$ is unknown and $b(\mathbf{x}) \equiv 1$. For simplicity, the sigmoid function will be used. Other type of activation functions can be worked out similarly as long as the activation functions and their derivatives are bounded. It is still assumed that $b(\mathbf{x}) \equiv 1$ in this section. This assumption will be removed in the next section.

Theorem 2 *$\mathbf{w}_f^T\mathbf{g}_f(\mathbf{V}_f\mathbf{x}_a)$ can be approximated by its estimate $\hat{\mathbf{w}}_f^T\mathbf{g}_f(\hat{\mathbf{V}}_f\mathbf{x}_a)$ in the following form [13]*

$$\mathbf{w}_f^T\mathbf{g}_f = \hat{\mathbf{w}}_f^T\hat{\mathbf{g}}_f - \tilde{\mathbf{w}}_f^T(\hat{\mathbf{g}}_f - \hat{\mathbf{g}}_f'\hat{\mathbf{V}}_f\mathbf{x}_a) - \hat{\mathbf{w}}_f^T\hat{\mathbf{g}}_f'\tilde{\mathbf{V}}_f\mathbf{x}_a + d_{fNN} \tag{14}$$

where $\hat{\mathbf{g}}_f = \mathbf{g}_f(\hat{\mathbf{V}}_f\mathbf{x}_a)$, $\hat{\mathbf{g}}_f' = diag\{\hat{g}_{f1}', \cdots, \hat{g}_{fr_f}'\}$ with $\hat{g}_{fi}' = g_{fi}'(\hat{\mathbf{v}}_{fi}^T\mathbf{x}_a) = \frac{dg_{fi}(z)}{dz}|_{z=\hat{\mathbf{v}}_{fi}^T\mathbf{x}_a}$, $i = 1, \cdots, r_f$, and residual term $d_{fNN} = -\tilde{\mathbf{w}}_f^T\hat{\mathbf{g}}_f'\mathbf{V}_f\mathbf{x}_a + \mathbf{w}_fO(\tilde{\mathbf{V}}_f\mathbf{x}_a)$ with $O(\tilde{\mathbf{V}}_f\mathbf{x}_a)$ being the sum of the higher order terms. $|d_{fNN}| \le \alpha_f^T\mathbf{Y}_f$ holds with α_f being an unknown vector constituting of positive entries, and $\mathbf{Y}_f = [1, ||\mathbf{x}_a||_2, ||\hat{\mathbf{w}}_f||_2||\mathbf{x}_a||_2, ||\hat{\mathbf{V}}_f||_F||\mathbf{x}_a||_2]^T$ ($||\bullet||_F$ denotes the Frobenius norm of a matrix $\bullet$, which is defined as $||\bullet||_F^2 = \text{Trace}\{\bullet^T\bullet\}$). ◊

The control law and adaptation law are given as follows

$$u = u_a + u_s \tag{15}$$

$$u_a = -a_r(t) - \hat{\mathbf{w}}_f^T\hat{\mathbf{g}}_f \tag{16}$$

$$u_s = u_{s1} + u_{s2},\ u_{s1} = -ks - \hat{\alpha}_f^T\mathbf{Y}_fsgn(s) \tag{17}$$

$$\dot{\hat{\mathbf{w}}}_f = \mathbf{Proj}_{\hat{\mathbf{w}}_f}(\Gamma_{wf}s(\hat{\mathbf{g}}_f - \hat{\mathbf{g}}_f'\hat{\mathbf{V}}_f\mathbf{x}_a)) \tag{18}$$

$$\dot{\hat{\mathbf{V}}}_f = \mathbf{Proj}_{\hat{\mathbf{V}}_f}[\Gamma_{vf}^T\mathbf{x}_as\hat{\mathbf{w}}_f^T\hat{\mathbf{g}}_f'] \tag{19}$$

$$\dot{\hat{\alpha}}_f = \mathbf{Proj}_{\hat{\alpha}_f}(\Gamma_{\alpha f}|s|\mathbf{Y}_f) \tag{20}$$

where $\Gamma_{vf} \in \mathcal{R}^{(n+1)\times(n+1)}$ and $\Gamma_{\alpha f} \in \mathcal{R}^{4\times4}$ are s.p.d. adaptation rate matrices. The lower fictitous bound for $\hat{\alpha}_{fi}$ is chosen to be zero since $\alpha_{fi} > 0$. u_{s2} satisfies

$$s\left(f(\mathbf{x}) - \hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f + u_{s2} + \Delta\right) \le \varepsilon_s, \quad (21)$$

$$s u_{s2} \le 0 \quad (22)$$

Control law (15) will be referred to as NNARC-II.

Substituting control law (15) to the system equation (1) yields the following dynamic equation of $s(t)$

$$\dot{s} = -ks + f(\mathbf{x}) - \hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f - \hat{\alpha}_f^T \mathbf{Y}_f sgn(s) + \Delta + u_{s2} \quad (23)$$

Theorem 3 *With NNARC-II and the adaptation laws (15)-(20), the following results hold:*

A. *In general, all signals are bounded. $s(t)$ exponentially decays to a small value and is bounded above by*

$$s^2(t) \le exp(-2kt)s^2(0) + \frac{\varepsilon_s}{k} \quad (24)$$

B. *By setting $\mathbf{x}_d(0) = \mathbf{x}(0)$, the actual tracking error is asymptotically bounded by*

$$||\tilde{x}^{(i)}||_\infty \le 2^i \lambda^{i-n+1} \sqrt{\frac{\varepsilon_s}{k}} \quad (25)$$

C. *If $f(\mathbf{x}) = \mathbf{w}_f^T \mathbf{g}_f(\mathbf{x})$, then, in addition to A and B, asymptotic output tracking is achieved provided that $\Delta = 0$, and all of the ideal weights actually lie within the corresponding fictitious bounds.*

D. *When the discontinuous sign function $sgn(s)$ in (17) is replaced by the continuous saturation function $sat(\frac{s}{\psi})$ (ψ is the thickness of the boundary layer), Results A and B still remain valid.* ◇

Proof. A. Consider the positive definite function $V = \frac{1}{2}s^2(t)$. Noting the dynamic equation (23) and conditions (21), as well as the fact that $\hat{\alpha}_{fi} \ge 0$, we have

$$\begin{aligned}
\dot{V} &= s\dot{s} \\
&= -ks^2 + s\left(f(\mathbf{x}) - \hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f - \hat{\alpha}_f^T \mathbf{Y}_f sgn(s) + \Delta + u_{s2}\right) \\
&= -ks^2 - \hat{\alpha}_f^T \mathbf{Y}_f |s| + s\left[f(\mathbf{x}) - \hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f + \Delta + u_{s2}\right] \\
&\le -ks^2 + \varepsilon_s
\end{aligned} \quad (26)$$

which leads to the inequality (24).

B. Using the same method as in the proof B of Theorem 1, inequality (25) can be resulted [12].

C. Consider the positive definite function as follows

$$V = \frac{1}{2}\left[s^2(t) + \tilde{\mathbf{w}}_f^T \Gamma_{wf}^{-1} \tilde{\mathbf{w}}_f + \text{Trace}\{\tilde{\mathbf{V}}_f \Gamma_{vf}^{-1} \tilde{\mathbf{V}}_f^T\} + \tilde{\alpha}_f^T \Gamma_{\alpha f}^{-1} \tilde{\alpha}_f\right] \quad (27)$$

When $f(\mathbf{x}) = \mathbf{w}_f^T \mathbf{g}_f(\mathbf{V}_f \mathbf{x}_a)$ and $\Delta = 0$, using adaptation laws (18), (19), (20) and condition (22), we have

$$\begin{aligned}
\dot{V} &= s\dot{s} + \tilde{\mathbf{w}}_f^T \Gamma_{wf}^{-1} \dot{\tilde{\mathbf{w}}}_f + \text{Trace}\{\tilde{\mathbf{V}}_f \Gamma_{vf}^{-1} \dot{\tilde{\mathbf{V}}}_f^T\} + \tilde{\alpha}_f^T \Gamma_{\alpha f}^{-1} \dot{\tilde{\alpha}}_f \\
&= -ks^2 + su_{s2} - s\left[\tilde{\mathbf{w}}_f^T(\hat{\mathbf{g}}_f - \hat{\mathbf{g}}_f' \hat{\mathbf{V}}_f \mathbf{x}_a)\right] \\
&\quad - s\left(\hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f' \tilde{\mathbf{V}}_f \mathbf{x}_a\right) + s d_{fNN} - |s| \hat{\alpha}_f^T \mathbf{Y}_f \\
&\quad + \tilde{\mathbf{w}}_f^T \Gamma_{wf}^{-1} \mathbf{Proj}_{\hat{\mathbf{w}}_f}[\Gamma_{wf} s(\hat{\mathbf{g}}_f - \hat{\mathbf{g}}_f' \hat{\mathbf{V}}_f \mathbf{x}_a)] \\
&\quad + \text{Trace}\{\tilde{\mathbf{V}}_f \Gamma_{vf}^{-1} \mathbf{Proj}_{\hat{\mathbf{V}}_f}(\Gamma_{vf} \mathbf{x}_a s \hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f')\} \\
&\quad + \tilde{\alpha}_f^T \Gamma_{\alpha f} \mathbf{Proj}_{\hat{\alpha}_f}(\Gamma_{\alpha f} |s| \mathbf{Y}_f) \\
&\le -ks^2 + |s| \alpha_f^T \mathbf{Y}_f - |s| \hat{\alpha}_f^T \mathbf{Y}_f \\
&\quad + \tilde{\alpha}_f^T \Gamma_{\alpha f} \mathbf{Proj}_{\hat{\alpha}_f}(\Gamma_{\alpha f} |s| \mathbf{Y}_f) \\
&= -ks^2 - |s| \tilde{\alpha}_f^T \mathbf{Y}_f + \tilde{\alpha}_f^T \Gamma_{\alpha f} \mathbf{Proj}_{\hat{\alpha}_f}(\Gamma_{\alpha f} |s| \mathbf{Y}_f) \\
&\le -ks^2 \le 0
\end{aligned} \quad (28)$$

in which the second equality uses equation (23), Theorem 2, and adaptation laws, the first inequality uses Theorem 2, condition (22), and properties of projection mapping [10]. Using Barbalat's lemma, it can be proved that asymptotic output tracking is achieved.

D. Using the same Lyapunov function candidate as in A, it can be easily verified that $\dot{V} \le -ks^2 + \varepsilon_s$ since $-s\hat{\alpha}_f^T \mathbf{Y}_f sat(\frac{s}{\psi}) \le 0$ holds. Hence, Result A remains valid, so does Result B. □

Remark 3 *The discontinuous term $\hat{\alpha}_f^T \mathbf{Y}_f sgn(s)$ can be dropped from control law (17) if only A and B are needed.*

Remark 4 *Although conditions (21) are different from (9), u_{s2} in (21) can assume form (11) with the same h_1, h_2 and h_3 in Remark 1 since $||\hat{\mathbf{g}}_f|| = ||\mathbf{g}_f||$ for sigmoid functions.*

5 NNARC with Non-unity Input Gain

If $b(\mathbf{x})$ is known, the control law (15) changes to

$$u = \frac{1}{b(\mathbf{x})}[-ks - a_r(t) - \hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f - \hat{\alpha}_f^T \mathbf{Y}_f sgn(s) + u_{s2}] \quad (29)$$

With the control law (29), adaptation laws (18)-(20), all the results in Theorem 3 remain valid.

If $b(\mathbf{x})$ is unknown, an NN will be used to estimate it

Assumption 4 *$b(\mathbf{x})$ can be approximated as follows [1]*

$$|b(\mathbf{x}) - \mathbf{w}_b^T \mathbf{g}_b(\mathbf{V}_b \mathbf{x}_a)| \le d_b(\mathbf{x}), \ \forall \mathbf{x} \in \mathcal{R}^n \quad (30)$$

where notations are defined in a similar way as in Assumption 1.

Theorem 4 *$\mathbf{w}_b^T \mathbf{g}_b(\mathbf{V}_b \mathbf{x}_a)$ can be approximated by its estimate $\hat{\mathbf{w}}_b^T \mathbf{g}_b(\hat{\mathbf{V}}_b \mathbf{x}_a)$ in the following way*

$$\mathbf{w}_b^T \mathbf{g}_b = \hat{\mathbf{w}}_b^T \hat{\mathbf{g}}_b - \tilde{\mathbf{w}}_b^T(\hat{\mathbf{g}}_b - \hat{\mathbf{g}}_b' \hat{\mathbf{V}}_b \mathbf{x}_a) - \hat{\mathbf{w}}_b^T \hat{\mathbf{g}}_b' \tilde{\mathbf{V}}_b \mathbf{x}_a + d_{bNN} \quad (31)$$

where notations are defined in a similar way as in Theorem 2. $|d_{bNN}| \le \alpha_b^T \mathbf{Y}_b$ holds with α_b being an unknown vector constituting of positive entries, and $\mathbf{Y}_b = [1, ||\mathbf{x}_a||_2, ||\hat{\mathbf{w}}_b||_2 ||\mathbf{x}_a||_2, ||\hat{\mathbf{V}}_b||_F ||\mathbf{x}_a||_2]^T$. ◇

While the adaptation laws for $\hat{\mathbf{w}}_f$, $\hat{\mathbf{V}}_f$, and $\hat{\alpha}_f$ are given by (18)-(20), the following control law and more adaptation laws are proposed

$$u = u_a + u_s \tag{32}$$

$$u_a = -\frac{1}{\hat{\mathbf{w}}_b^T \hat{\mathbf{g}}_b}[a_r(t) + \hat{\mathbf{w}}_f^T \hat{\mathbf{g}}_f], \; u_s = u_{s1} + u_{s2}, \tag{33}$$

$$u_{s1} = -\frac{1}{b_l}\{ks + [\hat{\alpha}_f^T \mathbf{Y}_f + \hat{\alpha}_b^T \mathbf{Y}_b |u_a|] sgn(s)\} \tag{34}$$

$$\dot{\hat{\mathbf{w}}}_b = \mathbf{Proj}_{\hat{\mathbf{w}}_b}\{\Gamma_{wb}[u_a s(\hat{\mathbf{g}}_b - \hat{\mathbf{g}}_b' \hat{\mathbf{V}}_b \mathbf{x}_a)]\} \tag{35}$$

$$\dot{\hat{\mathbf{V}}}_b^T = \mathbf{Proj}_{\hat{\mathbf{V}}_b}\{\Gamma_{Vb}[u_a \mathbf{x}_a s \hat{\mathbf{w}}_b^T \hat{\mathbf{g}}_b']\} \tag{36}$$

$$\dot{\hat{\alpha}}_b = \mathbf{Proj}_{\hat{\alpha}_b}\{\Gamma_{\alpha_b}[|s|\mathbf{Y}_b|u_a|]\} \tag{37}$$

where $\hat{\rho}_{l,w_{bi}}$ and $\hat{\rho}_{u,w_{bi}}$ used in $\mathbf{Proj}_{\hat{\mathbf{w}}_b}$, and $\hat{\rho}_{b,v_{bij}}$ and $\hat{\rho}_{u,v_{bij}}$ in $\mathbf{Proj}_{\hat{\mathbf{V}}_b}$ are chosen such that $\hat{\mathbf{w}}_b^T \hat{\mathbf{g}}_b > 0$, $\forall \hat{w}_{bi} \in [\hat{\rho}_{l,w_{bi}}, \hat{\rho}_{u,w_{bi}}]$, $\forall \hat{v}_{bij} \in [\hat{\rho}_{l,v_{bij}}, \hat{\rho}_{u,v_{bij}}]$ since $b(\mathbf{x}) > 0$. u_{s2} satisfies the follow conditions

$$s\{[f(\mathbf{x}) + a_r(t)] + b(\mathbf{x})u_a + \Delta + b(\mathbf{x})u_{s2}\} \leq \varepsilon_{s2}, \tag{38}$$

$$su_{s2} \leq 0 \tag{39}$$

where $\varepsilon_{s2} = \varepsilon_s + \left|1 + \frac{|(||\rho_{wb}||_2 - ||\hat{\rho}_{wb}||_2)|}{||\hat{\rho}_{wb}||_2}\right|^2 \varepsilon_4$ with ε_s being the same as before, and ε_4 being a positive design constants. Control law (32) is referred to as NNARC-III.

Theorem 5 *With NNARC-III (32), the adaptation laws (18)-(20), and (35)-(37), the following results hold:*

A. *In general, all signals are bounded. $s(t)$ exponentially decays to a small value and is bounded above by*

$$s^2(t) \leq exp(-2kt)s^2(0) + \frac{\varepsilon_{s2}}{k} \tag{40}$$

B. *By setting $\mathbf{x}_d(0) = \mathbf{x}(0)$, the actual tracking error is asymptotically bounded by*

$$||\tilde{x}^{(i)}||_\infty \leq 2^i \lambda^{i-n+1}\sqrt{\frac{\varepsilon_{s2}}{k}} \tag{41}$$

C. *If $f(\mathbf{x}) = \mathbf{w}_f^T \mathbf{g}_f(\mathbf{V}_f \mathbf{x}_a)$, and $b(\mathbf{x}) = \mathbf{w}_b^T \mathbf{g}_b(\mathbf{V}_b \mathbf{x}_a)$, then, in addition to A and B, asymptotic output tracking is achieved provided that $\Delta = 0$, and all ideal weights lie within the fictitious bounds.*

D. *When the discontinuous sign function $sgn(s)$ in (33) is replaced by the continuous saturation function $sat(\frac{s}{\psi})$, Results A and B still remain valid.* ◊

Proof. Using the same technique in the proof of Theorem 3, **A** can be proved by noting that $\frac{b(\mathbf{x})}{b_l} > 1$ alwyas holds. **B** can be worked out by the same arguments as in the proof of **B** of Theorem 1. **C** can be proved by considering $V_{a3} = V_{a2} + \frac{1}{2}[\tilde{\mathbf{w}}_b^T \Gamma_{wb}^{-1} \tilde{\mathbf{w}}_b + \text{Trace}\{\tilde{\mathbf{V}} \Gamma_{vb}^{-1} \tilde{\mathbf{V}}_b^T\} + \tilde{\alpha}_b^T \Gamma_{\alpha_b}^{-1} \tilde{\alpha}_b]$. Result **D** can be verified in the same way as in the proof of **D** of Theorem 3. □

Remark 5 *The robust term u_{s2} in (33) can be selected in a similar form as (11) with $h_1 \geq \frac{1}{\sqrt{b_l}}||\hat{\mathbf{g}}_f||(||\hat{\mathbf{w}}_f|| + ||\hat{\rho}_{wf}||)$, $h_2 \geq \frac{1}{\sqrt{b_l}}[d_f + |u_a|d_b]$, $h_3 \geq \frac{1}{\sqrt{b_l}}\delta(x,t)$, and $h_4 \geq \frac{1}{\sqrt{b_l}}||\hat{\mathbf{g}}_b||(||\hat{\mathbf{w}}_b|| + ||\hat{\rho}_{wb}||)|u_a|$.*

6 Conclusion

In this paper, performance oriented NNARC control laws have been constructed for a class of n-th order uncertain nonlinear systems in a normal form. Discontinuous projection mappings with fictitious bounds are used to achieve a controlled learning even in the presence of neural network approximation error and non-repeatable nonlinearities such as external disturbances. Certain robust feedback is constructed to attenuate various model uncertainties effectively for a guaranteed output tracking transient performance and a guaranteed final tracking accuracy in general-a transient tracking performance that existing NN based robust adaptive controllers cannot achieve. The resulting NNARC has the nice feature that if the unknown nonlinear functions are in the functional ranges of the neural networks and the ideal weights fall within the prescribed range, asymptotic output tracking is also achieved–a performance that existing NN based robust adaptive controllers cannot have.

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