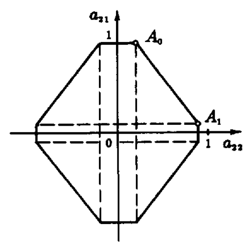

Fig. 1. Convex combination of stable interval matrices.

for some $d_i > 0$. By multiplying both sides of (13) by $(1 - a_{ii}^k)(1 - a_{ii}^0)d_j$, summing for $j \neq i$, and using (14), we get

$$1 - a_{ii}^k > d_i^{-1}\sum_{j\neq i}^{n} d_j a_{ij}^k, \qquad i = 1, 2, \cdots, n; \quad k = 0, 1, \cdots, N. \tag{15}$$

which implies that

$$1 - \left(\sum_{k=1}^{N}\alpha_k a_{ii}^k\right) > d_i^{-1}\sum_{j\neq i}^{n} d_j \left(\sum_{k=1}^{N}\alpha_k a_{ij}^k\right), \qquad i = 1, 2, \cdots, n. \tag{16}$$

Thus, $I - \sum_{k=0}^{N}\alpha_k A_k$ is PQDD, and the proof is complete.

Note that under the conditions of Theorem 3, the matrices$A_0, A_1, \cdots, A_N$ are simultaneously PQDD, as established by (15), and so are their convex combinations.

The following continuous-time version of Theorem 3 can be proved similarly.

Theorem 4: Let $A_0 = A_0^D + A_0^C$, with $A_0^C \geq 0$, be an $n \times n$Hurwitz-stable matrix, and $A_k = A_k^D + A_k^C$, with $A_k^C \geq 0$, be $n \times n$ matrices such that $-A_k$ proportionally dominates $-A_0$, $k = 1, 2, \cdots, N$, either all rowwise or all columnwise. Then, a convex combination $\sum_{k=0}^{N}\alpha_k A_k$, with $\alpha_k \geq 0$, $k = 0, 1, \cdots, N$, $\sum_{k=0}^{N}\alpha_k = 1$, is also Hurwitz-stable.

To illustrate the result of Theorem 3, we consider two positive matrices

$$A_0 = \begin{bmatrix} 0.4 & 0.4 \\ 1 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}.$$

both of which can easily be shown to be Schur-stable. Scaling the rows of $I - A_0$ and $I - A_1$ by the reciprocals of their diagonal elements we observe that $I - A_1$ proportionally dominates $I - A_0$. Then Theorem 3 implies that any convex combination

$$\alpha A_0 + (1-\alpha)A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.1 + 0.9\alpha & 0.9 - 0.7\alpha \end{bmatrix}, \qquad \alpha \in [0, 1]$$

is also stable. The significance of this result can be seen by considering the stability regions in the (a_{21}, a_{22}) plane: While Theorem 1 produces two thin rectangular regions associated with A_0 and A_1, Theorem 3 combines these into a much larger one as shown in Fig. 1.

Notice that in this example, although $I - A_0$ and $I - A_1$ aresimultaneously PQDD, there exist no diagonal-type Liapunov function that would prove simultaneous stability [5] of A_0 and A_1; this function being a natural one in the context of M-matrices. This fact points out the significance of connective stability, which is diagonal-type simultaneous stability that can be established *via* simultaneous PQDD conditions [7].

VI. Conclusions

We have obtained necessary and sufficient conditions for stability ofMorishima-type interval matrices using the results available in the context of connective stability. The conditions have been further broadened to include convex combination of interval matrices. In future research, attempts shall be made to apply the results to establish connective stability of convex polytopes of nonlinear time varying matrix systems.

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Unified Formulation of Variable Structure Control Schemes for Robot Manipulators

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***Abstract*— A general target model is proposed in the task space to represent motion trajectory, interaction force trajectory, and second-order function relating the motion errors and the interaction force errors. Using variable structure model reaching control (VSMRC) strategy, the model is achieved in the sliding mode with robust performance. Reaching transient can be eliminated or guaranteed with prescribed quality. By choosing a suitable model for the application, robust motion control, impedance control, hybrid position/force control, or constrained motion control are achieved, respectively.**

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I. Introduction

Tasks such as following a contour require robots interaction with the environment. In these cases, interaction force are present and needed to be accommodated rather than rejected [1]. To accomplish such tasks successfully, several approaches have been proposed such as impedance control [2]–[6], hybrid position/force control [7], [8], and constrained motion control [9]–[16].

With impedance control [2] the manipulator control system should be designed, not to track a motion or force trajectory alone, but rather to regulate the mechanical impedance of the manipulator. By proper choice of the impedance, dynamic interaction may be controlled to obtain proper force response. Kazerooni *et al.* [3] presented the specification of the desired impedance in the frequency domain to preserve the stability of whole system in the presence of bounded modeling uncertainties. This problem is further discussed in [4] where several sufficient stability conditions are derived via the Small Gain Theorem and Nyquist Criterion. Adaptive control is used in [5], [6] to deal with parametric uncertainties of the robot, however, transient of adaptive systems is not guaranteed.

In hybrid position/force control method proposed by Raibert and Craig [7], task space is divided into two orthogonal subspace. Position is commanded and controlled in one subspace, and force in the other one. Since modeling of the environment is not clearly given, there are some difficulties in formulating the mathematical model of the problem and force tracking control is not shown in [7], [8]. Assuming known environmental stiffness, Kelly and Carelli developed an adaptive hybrid position/force control for dealing with parametric uncertainties and the resulted controller required the measurement of force derivative.

Constrained motion control has been extensively studied in recent years. In the formulation, the robot is assumed to be in contact with a rigid frictionless surface, and as a result, kinematic constraint is imposed on its motion. The general theoretical framework is developed by McClamroch and Wang [9] using a controller based on a modification of the computed torque method. For considering parametric uncertainties, adaptive constrained motion control is discussed in [17], [14] where only bounded force tracking error is guaranteed unless persistent excitation condition is satisfied.

Variable structure control (VSC) method [18], which possesses several advantages such as simplicity, perfect adaption to both parametric uncertainties and external disturbances, has been applied for robot motion control by Young [19] and Slotine [20]. Recent, Yao, *et al.* [15], [16] developed VSC for constrained motion. However, motion control and force control are treated differently and separate switching functions are selected in order to achieve the desired response in the sliding mode. In this note, a systematic approach is presented to formulate VSC schemes by representing the various control objectives in a unified way. The main feature is to introduce a general target model which is specified in the task space to represent motion trajectory, interaction force trajectory, and a desired second-order function relating the motion errors and the interaction force errors. By choosing a suitable task space and target model, specific applications such as motion control, impedance control, hybrid position/force control, or constrained motion control can be achieved respectively.

With variable structure model reaching control (VSMRC) strategy [21], a dynamic compensator is incorporated in forming the sliding surface such that the resulted sliding mode can be identified with the target model. Using VSC, the system is maintained in the sliding mode and model reaching is realized. The proposed controller does not require measurement of force derivative and is simple in the structure. To illustrate applications of the approach, selection of the desired target model for various control schemes is discussed.

II. Formulation of General Robot Dynamics

General dynamic equations of a rigid link manipulator having n-DOF in contact with the environment are considered. It is assumed that the robot is nonredundant and contact with the environment is made by its end-effector and occurs at a point. Let $x \in R^n$ denote the position and Euler angles of the end-effector frame in Cartesian space. It is related to the joint displacement $q \in R^n$ by

$$x = x(q); \quad \dot{x} = J(q)\dot{q}; \quad J(q) = \frac{\partial x(q)}{\partial q} \tag{1}$$

where $J(q)$ is assumed to be nonsingular in the finite work space Ω. In the joint space, the dynamic equations can be written as [8]

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + J^T(q)F + \tilde{f}(t) = \tau \tag{2}$$

where τ is the applied joint torque, $M(q)$ is the inertia matrix, $C(q, \dot{q})\dot{q}$ is the Coriolis and centrifugal force vector, $G(q)$ is the gravitational load of the robot, $\tilde{f}(t)$ is the external disturbance. $F \in R^n$ is the vector of forces/moments on the environment exerted by the robot at the end-effector.

Depending on different contacting environments, different task space will be defined. Here, a general task space is assumed to be defined by

$$\begin{aligned} &r = r(x, t) = r(x(q), t), \qquad r \in R^n \\ &\dot{r} = J_r(x, t)\dot{x} + v_t = J_q(q)\dot{q} + v_t, \\ &J_r = \frac{\partial r(x, t)}{\partial x}, \quad v_t = \frac{\partial r(x, t)}{\partial t}, \quad J_q = J_r(x(q), t)J(q) \\ &\ddot{r} = J_q\ddot{q} + \dot{J}_q\dot{q} + \dot{v}_t \end{aligned} \tag{3}$$

where $J_q(q, t)$ is assumed to be nonsingular in the finite work space Ω.

Using the transformation (3) in (2) and multiplying both sides by J_q^{-T}, dynamic equation (2) can be expressed in terms of the task space variable r as

$$H(r, t)\ddot{r} + C(r, \dot{r}, t)\dot{r} + G(r, t) + F_t + \tilde{F}(r, t) = T_r - F_r \tag{4}$$

where

$$\begin{aligned} H(r, t) &= J_q^{-T}M(q)J_q^{-1}; \\ C(r, \dot{r}, t) &= J_q^{-T}C(q, \dot{q})J_q^{-1} - J_q^{-T}M(q)J_q^{-1}\dot{J}_qJ_q^{-1} \\ G(r, t) &= J_q^{-T}G(q); \quad \tilde{F}(r, t) = J_q^{-T}\tilde{f}(t) \\ F_t &= -H(r, t)\dot{v}_t - C(r, \dot{r}, t)v_t; \\ F_r &= J_r^{-T}F; \quad T_r = J_q^{-T}\tau. \end{aligned} \tag{5}$$

Due to parametric uncertainties of the robot and the environment, only estimated values $\hat{H}(r, t)$, $\hat{C}(r, \dot{r}, t)$, $\hat{G}(r)$, $\hat{f}_t(r, t)$ of $H(r, t)$, $C(r, \dot{r}, t)$, $G(r, t)$, $F_t(r, t)$ in (4) are available. The modeling errors and external disturbances are bounded by

$$\|\Delta H(r, t)\| \le \delta_H(r, t); \quad \|\Delta C(r, \dot{r}, t)\| \le \delta_C(r, \dot{r}, t);$$

$$\|\Delta G(r, t)\| \le \delta_G(r, t)$$

$$\|\Delta F_t(r, \dot{r}, t)\| \le \delta_{F_t}; \quad \|\tilde{F}(r, t)\| \le \delta_{\tilde{F}}(r, t) \tag{6}$$

where ΔA represents the modeling error of matrix (or vector) A defined by $\Delta A = A - \hat{A}$. $\|\cdot\|$ denotes a norm of $\cdot$ which is a vector or a matrix. Without loss of generality, in this paper, $\|\cdot\|_2$ is used, i.e., $\|A\| = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^TA)}$, where $\sigma(\cdot)$ denotes singular value of $\cdot$, $\lambda(\cdot)$ means eigenvalue of $\cdot$, and $\cdot_{\max}$ (or $\cdot_{\min}$) is the maximum (or minimum) value of $\cdot$. The positive scalar bounds δ_H, δ_C, δ_G, δ_{F_t}, and $\delta_{\tilde{F}}$ are assumed known.

Remark 1: Suppose that the estimated values $\hat{M}(q)$, $\hat{C}(q, \dot{q})$, $\hat{G}(q)$ of $M(q)$, $C(q, \dot{q})$, $G(q)$ are available. Let $\delta_M(q)$, $\delta_C(q, \dot{q})$, $\delta_G(q)$ and $\delta_{\bar{f}}(t)$ denote the known bounds of the modeling errors $\Delta M(q)$, $\Delta C(q, \dot{q})$, $\Delta G(q)$ and the external disturbance $\bar{f}(t)$ in the joint space, respectively. From (5), the estimated values in (4) can be calculated, and their corresponding bounds in (6) can be determined by

$$\begin{aligned} \delta_H(r, t) &= \delta_M(q)\|J_q^{-1}\|^2 \\ \delta_C(r, \dot{r}, t) &= \delta_C(q, \dot{q})\|J_q^{-1}\|^2 + \delta_M(q)\|J_q^{-1}\|\|J_q^{-1}\dot{J}_q J_q^{-1}\| \\ \delta_G(r, t) &= \|J_q^{-1}\|\delta_G(q) \\ \delta_{\bar{F}}(r, t) &= \|J_q^{-1}\|\delta_{\bar{f}}(t) \\ \delta_{F_t} &= \delta_H(r, t)\|\dot{v}_t\| + \delta_C(r, \dot{r}, t)\|v_t\|. \end{aligned} \tag{7}$$

Equation (4) has the following properties [22].

Property 1: For any finite work space $\Omega = \{q: \|q - q_0\| \leq q_{\max}\}$ in which J_q is nonsingular, $H(r, t)$ is a symmetric positive definite (s.p.d.) matrix with $k_r' I_{n\times n} \leq H(r, t) k_r'' I_{n\times n}$ where k_r', k_r'' are positive scalars, q_0 and $q_{\max}$ are some constants.

Property 2: The matrix $N(r, \dot{r}, t) = \dot{H}(r, t) - 2C(r, \dot{r}, t)$ is a skew-symmetric matrix.

III. Unified VSC Design

Suppose $r_d(t)$, $F_{cd}(t) \in R^n$ are given as the desired robot motion and interaction force trajectories in the task space. Let $e_r = r(t) - r_d(t)$, $e_{F_c} = F_c - F_{cd}(t)$ be the errors of motion and controlled interaction force, respectively. F_c is the controlled force which may not be equal to F_r in (4) since F_r is the total interaction force which may include friction force and its n components may not be independent. the unified formulation of VSC design for the robot can be stated as that of determining a control law so that the system errors achieve the following general target model in the task space under the modeling errors (6)

$$M_m \ddot{e}_r + B_m \dot{e}_r + K_m e_r = -K_f e_{F_c} \tag{8}$$

where M_m, B_m, K_m, $K_f \in R^{n\times n}$ are constant matrices which are usually chosen as diagonal to get decoupled responses. M_m is assumed to be nonsingular. The model (8) should be chosen to be stable when the robot interacts with the environment.

Using VSMRC strategy, a dynamic compensator is formulated

$$\dot{z} = Az + K_{pz}e_r + K_{vz}\dot{e}_r + K_{fz}e_{F_c}, \quad A, K_{pz}, K_{vz}, K_{fz} \in R^{n\times n} \tag{9}$$

where z is the n-dimensional state vector of the compensator, A is any diagonal negative semidefinite matrix, K_{pz}, K_{vz}, K_{fz} will be specified to shape the dynamic sliding mode so that the target model (8) is achieved. The compensator is employed in forming the switching function

$$s(e_r, \dot{e}_r, z) = \dot{e}_r + F_1 e_r + F_2 z; \qquad F_1, F_2 \in R^{n\times n} \tag{10}$$

where F_1 and F_2 are any diagonal constant matrices, F_2 is nonsingular. Differentiating (10)

$$\dot{s} = \ddot{e}_r + F_1 \dot{e}_r + F_2 \dot{z}. \tag{11}$$

Substituting (9) into (11) and using (10) to eliminate z

$$\begin{aligned} \ddot{e}_r + (F_1 - F_2AF_2^{-1} + F_2K_{vz})\dot{e}_r + (F_2K_{pz} - F_2AF_2^{-1}F_1)e_r \\ = -F_2K_{fz}e_{F_c} + \dot{s} - F_2AF_2^{-1}s. \end{aligned} \tag{12}$$

Since F_2 and A are diagonal matrices, $F_2AF_2^{-1} = A$. Choosing K_{pz}, K_{vz}, K_{fz}, as

$$\begin{aligned} K_{vz} &= F_2^{-1}(M_m^{-1}B_m - F_1 + A) \\ K_{pz} &= F_2^{-1}(M_m^{-1}K_m + AF_1) \\ K_{fz} &= F_2^{-1}M_m^{-1}K_f \end{aligned} \tag{13}$$

(12) is given by

$$M_m\ddot{e}_r + B_m\dot{e}_r + K_m e_r = -K_f e_{F_c} + M_m(\dot{s} - As). \tag{14}$$

In the sliding mode $\{s = 0, \dot{s} = 0\}$, (14) will be identical to the target model (8). That is to say, the target model is achieved in the sliding mode.

Theorem 1: For the robot manipulator described by (4), the system achieve the desired general target model (8) under the modeling errors (6) if the following control torque is applied:

$$T_r = \hat{H}(r, t)\ddot{r}_{\rm eq} + \hat{C}(r, \dot{r}, t)\dot{r}_{\rm eq} + \hat{G}(r, t) + \hat{F}_t + F_r - K_s s - d\frac{s}{\|s\|} \tag{15}$$

where

$$\dot{r}_{\rm eq} = \dot{r}_d(t) - F_1 e_r - F_2 z; \quad \ddot{r}_{\rm eq} = \ddot{r}_d(t) - F_1\dot{e}_r - F_2\dot{z}. \tag{16}$$

K_s is any s.p.d. matrix. d is a positive scalar satisfying

$$d \geq \delta_H(r, t)\|\ddot{r}_{\rm eq}\| + \delta_C(r, \dot{r}, t)\|\dot{r}_{\rm eq}\| + \delta_G(r, t) + \delta_{F_t} + \Delta_{\bar{F}(r, t)} + \epsilon \tag{17}$$

where ϵ is any positive scalar. Furthermore, the reaching time t_r when the system reaches the sliding mode satisfies

$$t_r \leq \frac{2}{c_1}\ln\left(1 + \frac{c_1}{c_2}\sqrt{V_0}\right) \tag{18}$$

where

$$\begin{aligned} c_1 &= \frac{2\lambda_{\min}(K_s)}{k_r''} \\ c_2 &= \epsilon\sqrt{\frac{2}{k_r''}} \\ V_0 &= \frac{1}{2}s^T(0)H(r(0), 0)s(0) \end{aligned} \tag{19}$$

and the reaching transient response is shaped by

$$\|s\| \le \sqrt{\frac{2}{k_r'}}\left[\left(\sqrt{V_0}+\frac{c_2}{c_1}\right)\exp^{-(c_1/2)t}-\frac{c_2}{c_1}\right]. \tag{20}$$

Proof: For the robot manipulator (4), we choose a positive definite function as $V=\frac{1}{2}s^T H(r,t)s$ From Property 1, we have

$$\frac{1}{2}k_r'\|s\|^2 \le V \le \frac{1}{2}k_r''\|s\|^2. \tag{21}$$

Differentiating V with respect to time yields

$$\begin{aligned}\dot V &= s^T H(r,t)\dot s+\frac{1}{2}s^T\dot H(r,t)s = s^T H(\ddot r-\ddot r_{\rm eq})+s^T C(r,\dot r,t)s\\ &= s^T[T_r - H(r,t)\ddot r_{\rm eq} - C(r,\dot r,t)\dot r_{\rm eq}\\ &\quad - G(r,t)-F_t-F_r-\tilde F(r,t)]\end{aligned} \tag{22}$$

where Property 2 has been used to eliminate the term $\frac{1}{2}s^T\dot H s$. Substituting control torque (15) into it and noticing (21), we have

$$\begin{aligned}\dot V &= -s^T K_s s - d\|s\| + s^T[-\Delta H(r,t)\ddot r_{\rm eq} - \Delta C(r,\dot r,t)\dot r_{\rm eq}\\ &\quad -\Delta G(r,t)-\Delta F_t - \tilde F(r,t)]\\ &\le -s^T K_s s - d\|s\| + \|s\|[\|\Delta H(r,t)\|\,\|\ddot r_{\rm eq}\|\\ &\quad + \|\Delta C(r,\dot r,t)\|\,\|\dot r_{\rm eq}\|\\ &\quad + \|\Delta G(r,t)\| + \|\Delta F_t\| + \|\tilde F(r,t)\|]\\ &\le -\lambda_{\min}(K_s)\|s\|^2 - \epsilon\|s\| \le -c_1 V - c_2\sqrt{V}.\end{aligned} \tag{23}$$

Thus

$$\sqrt{V} \le \left(\sqrt{V_0}+\frac{c_2}{c_1}\right)\exp^{-(c_1/2)t}-\frac{c_2}{c_1} \tag{24}$$

and this means in finite time $V=0$, i.e., $s=0$. Moreover, from (21), the reaching transient response is shaped by (20). The upper limit of the reaching time t_r is solved by setting right hand of (24) equal zero which is given by (18). Proof is completed. △

Remark 2: In the above theorem, s exponentially reaches zero, and the reaching time t_r inversely depends on $\lambda_{\min}(K_s)$ and ϵ. Therefore, by suitable choices of K_s and ϵ, the reaching transient can be guaranteed with prescribed quality. The reaching transient can also be eliminated by choosing initial value of the dynamic compensator z as $z(0)=-F_2^{-1}(\dot e_r(0)+F_1 e_r(0))$ such that $s(0)=0$. The system is maintained in the sliding mode all the time and reaching transient is eliminated. □

Remark 3: The estimated matrices $\hat H, \hat C, \hat G, \hat F_t$ may be chosen to reduce on-line computation load. For example, by choosing $\hat H = \hat H(r_d,t)$, $\hat C=\hat C(r_d,\dot r_d,t)$, $\hat G=\hat G(r_d,t)$ in which $r_d(t)$ is the nominal trajectory, then can be calculated off-line as long as $\hat H$, $\hat C$, $\hat G$ are within the bounds given by (6). In any finite workspace, the scalar bounds δ_H, δ_C, δ_G, δ_{F_t}, $\delta_{\tilde F}$ can also be determined off-line. F_r is directly obtained from force measurement. In this way, the control law (15) does not require much on-line computation time. □

Remark 4: The control torque (15) is implemented in the joint space as

$$\tau = \hat M(q)\ddot q_{\rm eq}+\hat C(q,\dot q)\dot q_{\rm eq}+\hat G(q)+J^T F - J_q^T\left[K_s s + d\frac{s}{\|s\|}\right] \tag{25}$$

where $\dot q_{\rm eq}=J_q^{-1}(\dot r_{\rm eq}-v_t)$, $\ddot q_{\rm eq}=J_q^{-1}(\ddot r_{\rm eq}-\dot J_q\dot q_{\rm eq}-\dot v_t)$. □

IV. Specific Applications

In this section, we show how to choose the desired general target model (8) to achieve robust motion control, impedance control, hybrid position/force control, or constrained motion control, respectively.

4.1 Motion Control

When the robot moves in a free space, it is required to follow a desired motion trajectory $q_d(t)$ or $x_d(t)$ [8], [19], [20]. Since there is no environmental constraints, $F=0$, $F_c=0$. The task space vector r can be chosen as either the joint space vector q with $J_q=I_{n\times n}$, $v_t=0$ or the Cartesian space vector x with $J_q=J(q)$, $v_t=0$, and the corresponding controller will be implemented in the joint space or in the Cartesian space. The general model (8) is specified as

$$M_m\ddot e_r + B_m\dot e_r + K_m e_r = 0 \tag{26}$$

where $e_r = r - r_d$ is tracking error, $r_d=q_d(t)$ or $x_d(t)$, and M_m, B_m, K_m are s.p.d. matrices so that (26) is asymptotically stable. The control law is given by (25) such that the model (26) is achieved in finite time under the modeling errors (6). From (26), $e_r\to 0$, and the system follows the desired trajectory $r_d(t)$.

4.2 Impedance Control

Specification of the impedance is usually given in terms of a desired motion trajectory and a desired dynamic relationship between motion errors and interaction force to guarantee stable contact with the environment. In Cartesian space, the desired impedance is specified as [5], [6], [3]

$$M_m\ddot e + B_m\dot e + K_m e = -F, \qquad e = x - x_d(t) \tag{27}$$

where $x_d(t)$ is the desired motion trajectory in Cartesian space.

Suppose that the environment can be described by the following general mass-damper-spring system [2], [3]

$$M_e\ddot x + B_e\dot x + K_e(x-x_e) = F \tag{28}$$

where x_e is the equilibrium position. M_e, B_e, K_e which represents the inertia, damping, and stiffness of the environment, respectively, are symmetric positive semidefinite (s.p.sd.) matrices [3]. Combining (27) with (28), the closed-loop system is described by

$$\begin{aligned}(M_m+M_e)\ddot x + (B_m+B_e)\dot x + (K_m+K_e)x\\ = M_m\ddot x_d + B_m\dot x_d + K_m x_d + K_e x_e\end{aligned} \tag{29}$$

which is asymptotically stable if M_m, B_m, K_m are chosen as s.p.d. matrices.

By setting parameter $K_f = I_{n\times n}$, $F_c = F$, $F_{cd}(t)=0$, and choosing the task space vector r as Cartesian space vector x, (correspondingly $r_d(t)=x_d(t)$), the target model (8) is identical to the desired impedance (27).

4.3 Hybrid Position and Force Control

Suppose the environment in undeformation is described by a set of m time-varying hypersurfaces

$$\Phi(x,t)=0;\quad \Phi(x,t)=[\phi_1(x,t),\cdots,\phi_m(x,t)]^T, \qquad m\le n \tag{30}$$

which are mutually independent for any t, and $\phi_i(x,t)$ is assumed to be twice differentiable with respect to x and t. Then, there exist a set of $(n-m)$ scalar functions $\{\psi_1(x, t), \cdots, \psi_{n-m}(x, t)\}$, which are twice differentiable with respect to x and t, such that $\{\phi_i(x, t), i = 1, \cdots m; \psi_j(x, t), j = 1, \cdots, n-m\}$ are mutually independent for any t [15]. In this case, the task space is defined as

$$\begin{aligned} r &= [r_f^T, r_p^T]^T \\ r_f &= [\phi_1(x, t), \cdots, \phi_m(x, t)]^T \\ r_p &= [\psi_1(x, t), \cdots, \psi_{n-m}(x, t)]^T. \end{aligned} \tag{31}$$

Directions of curvilinear coordinates r_f are aligned with the normal directions of the environment in undeformation. Therefore, subspace r_f represents the constrained subspace, in which force control is required, and r_p can be considered as the unconstrained subspace in which motion control is needed. In the normal directions of the contact surfaces, the environment is assumed to be an elastic model with s.p.d. stiffness matrix K_e (either from the force sensor or from the contact surfaces), i.e.,

$$f_n = K_e r_f; \quad f_n \in R^m; \quad K_e \in R^{m \times m} \tag{32}$$

where f_n are the normal contact force components. The problem is to design a controller so that the robot with the modeling errors (6) exerts the desired normal contact force $f_{nd}(t)$ on the environment while following the desired motion trajectory $r_{pd}(t)$.

The controlled force is $F_c = [f_n^T, 0]^T$. The desired motion and force trajectories are specified as $r_d = [r_{fd}(t)^T, r_{pd}(t)^T]^T$ and $F_{cd} = [f_{nd}(t)^T, 0]^T$ where $r_{fd}(t)$ is specified as

$$r_{fd}(t) = \hat{K}_e^{-1} f_{nd} \tag{33}$$

in which $\hat{K}_e$ represents the estimated stiffness matrix. The target model (8) is chosen as

$$\begin{bmatrix} M_{m1} & \\ & M_{m2} \end{bmatrix} \ddot{e}_r + \begin{bmatrix} B_{m1} & \\ & B_{m2} \end{bmatrix} \dot{e}_r + \begin{bmatrix} K_{m1} & \\ & K_{m2} \end{bmatrix} e_r = -\begin{bmatrix} I_{m \times m} & \\ & 0 \end{bmatrix} e_{F_c}$$

$$e_r = \begin{bmatrix} e_{r_f} \\ e_{r_p} \end{bmatrix} = \begin{bmatrix} r_f \\ r_p \end{bmatrix} - \begin{bmatrix} r_{fd} \\ r_{pd} \end{bmatrix}$$

$$e_{Fc} = \begin{bmatrix} e_{f_n} \\ 0 \end{bmatrix} = \begin{bmatrix} f_n \\ 0 \end{bmatrix} - \begin{bmatrix} f_{nd}(t) \\ 0 \end{bmatrix} \tag{34}$$

where $M_{m1}, B_{m1} \in R^{m \times m}$, $M_{m2}, B_{m2}, K_{m2} \in R^{(n-m)\times(n-m)}$ are any s.p.d. matrices. K_{m1} is any s.p.sd. matrix. Substituting (32), (33) into (34), the closed-loop equation is given by

$$\begin{aligned} M_{m1}\ddot{e}_{r_f} + B_{m1}\dot{e}_{r_f} + (K_{m1} + K_e)e_{r_f} &= (K_e - \hat{K}_e)r_{fd} \\ M_{m2}\ddot{e}_{r_p} + B_{m2}\dot{e}_{r_p} + K_{m2}e_{r_p} &= 0 \end{aligned} \tag{35}$$

which is asymptotically stable and $e_{r_p} \to 0$. In the case of known environmental stiffness, i.e., $K_e = \hat{K}_e$, from (35), we obtain

$$e_{r_f} = r_f - r_{fd} \to 0; \quad e_{fn} = f_n - f_{nd} = K_e e_{r_f} \to 0. \tag{36}$$

Hence in the constrained subspace r_f, the robot exerts the desired normal force $f_{nd}(t)$ on the environment while in the unconstrained subspace r_p, the robot follows the desired motion $r_{pd}(t)$.

In the case that the stiffness is unknown, for constant f_{nd}, the steady state motion and force tracking errors in the normal direction are

$$\begin{aligned} e_{r_f}(\infty) &= -(K_{m1} + K_e)^{-1}(K_e - \hat{K}_e)\hat{K}_e^{-1} f_{nd} \\ e_{f_n}(\infty) &= [I - K_e(K_{m1} + K_e)^{-1}](K_e - \hat{K}_e)\hat{K}_e^{-1} f_{nd} \end{aligned} \tag{37}$$

By choosing $K_{m1} = 0$, we have $e_{f_n}(\infty) = 0$. In spite of unknown stiffness, the robot can exert the constant desired normal contact force f_{nd} with controllable time constant of the system in (35).

Remark 5: In the hybrid position/force control [7], [8], the constrained subspace and the unconstrained subspace are chosen as orthogonal subspace. Here, the orthogonality of the constrained subspace r_f and the unconstrained subspace r_p is not needed. This is convenience for implementation as we can simply choose some joint angles q_i as the supplemental coordinates r_p. □

Remark 6: In [5], a hybrid impedance/force control technique is proposed as an extension of the hybrid position/force control which control impedance along the unconstrained directions while controlling force along the constrained directions. This can be done by modifying the model (34) as

$$\begin{bmatrix} M_{m1} & \\ & M_{m2} \end{bmatrix} \ddot{e}_r + \begin{bmatrix} B_{m1} & \\ & B_{m2} \end{bmatrix} \dot{e}_r + \begin{bmatrix} K_{m1} & \\ & K_{m2} \end{bmatrix} e_r = -\begin{bmatrix} I & \\ & I \end{bmatrix} e_{F_c}$$

$$e_r = \begin{bmatrix} r_f \\ r_p \end{bmatrix} - \begin{bmatrix} r_{fd} \\ r_{pd} \end{bmatrix}$$

$$e_{F_c} = \begin{bmatrix} f_n \\ f_{r_p} \end{bmatrix} - \begin{bmatrix} f_{nd}(t) \\ 0 \end{bmatrix} \tag{38}$$

where M_{m1}, B_{m1}, K_{m1} are any s.p.d. matrices, M_{m2}, B_{m2}, K_{m2} are s.p.d. matrices representing the desired inertia, damping and stiffness in the uncontrained directions, f_{r_p} is the interaction force in the unconstrained directions such as friction force. □

4.4 Constrained Motion Control

In constrained motion control [9]–[12], the environment is assumed to be rigid frictionless surface and described by (30) (in this formulation, the frictionless surface assumption is unnecessary). Motion of the robot end-effector is constrained on the surfaces. The generalized constrained forces in Cartesian space is given by [9]

$$F_n = \left(\frac{\partial \Phi(x, t)}{\partial x}\right)^T \lambda, \qquad \lambda \in R^m \tag{39}$$

where λ is a vector of generalized multipliers associated with the constraints which usually represents normal contact force. The task space is defined by (31) with the constraint surfaces (30) and the generalized constrained forces (39) given by

$$r_f = 0; \quad F_{rn} = J_r^{-T} F_n = [\lambda^T \quad 0]^T \tag{40}$$

in which motion of the robot on the constraint surfaces is completely determined by the variables r_p. The problem is to design a robust controller so that the robot exerts the desired constrained force $\lambda_d(t)$ on the environment while following the desired motion $r_{pd}(t)$ on the constraint surfaces.

The controlled force is $F_c = [\lambda, 0]^T$. The desired motion and force trajectories are specified as $r_d = [0, r_{pd}(t)^T]^T$ and $F_{cd} = [\lambda_d(t)^T, 0]^T$. The target model (8) is chosen as

$$\begin{bmatrix} M_{m1} & \\ & M_{m2} \end{bmatrix} \ddot{e}_r + \begin{bmatrix} B_{m1} & \\ & B_{m2} \end{bmatrix} \dot{e}_r + \begin{bmatrix} 0 & \\ & K_{m2} \end{bmatrix} e_r = -\begin{bmatrix} I_{m\times m} & \\ & 0 \end{bmatrix} e_{F_c}$$

$$e_r = \begin{bmatrix} r_f \\ r_p \end{bmatrix} - \begin{bmatrix} 0 \\ r_{pd} \end{bmatrix}$$

$$e_{F_c} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} - \begin{bmatrix} \lambda_d(t) \\ 0 \end{bmatrix} \tag{41}$$

where $M_{m1}, B_{m1} \in R^{m\times m}$, $M_{m2}, B_{m2}, K_{m2} \in R^{(n-x)\times(n-m)}$ are any s.p.d. matrices. On the constraint surface (30), i.e., $r_f = 0$, the model (41) is reduced to

$$\lambda - \lambda_d(t) = 0$$
$$M_{m2}\ddot{e}_{r_p} + B_{m2}\dot{e}_{r_p} + K_{m2}e_{r_p} = 0 \quad \text{or} \quad e_{r_p} = r_p P - r_{pd} \to 0. \tag{42}$$

That is, the robot exerts the desired generalized constrained force $\lambda_d(t)$ on the environment and follows the desired motion $r_{pd}(t)$ on the constraint surfaces.

V. Conclusion

The main feature of the proposed formulation is that the robot control system is designed not to track a motion or force trajectory alone, but to achieve a general target model which relates the motion errors and the interaction force errors. Using VSC, the target model is achieved in the sliding mode, and the reaching transient can be eliminated or guaranteed with prescribed quality. With specific target model suitably chosen according to different control objectives, the unified controller can be applied to different applications such as motion control, impedance control, hybrid position/force control, and constrained motion control. The approach may also be extended to a rather large class of control system which is under further investigation. The problem associated with VSC is the control law being discontinuous across the switching surface. In practice, direct implementation of the control law will lead to control chattering, which may affect the system performance. This problem should be given careful consideration in the application of VSC. Experimental results for trajectory control of a robot as well as simulation results of the robot elastically interacting with a time-varying surface and the robot moving on a semicircle constraint surface [22] show that control chattering is eliminated by smooth implementation of VSC law using the concept of boundary layer [20].

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