VSC Motion and Force Control of Robot Manipulators in the Presence of Environmental Constraint Uncertainties

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Due to task kinematic modelling inaccuracy, constraint functions imposed on robot manipulators may not be known exactly. In this article, a variable structure control (VSC) method is developed for robust motion and constrained force control of robot manipulators in the presence of parametric uncertainties, external disturbances, and constraint function uncertainties. The method is based on a particular structure of the constrained robot, in which motion control and force control are treated together. The proposed VSC controller provides the sliding mode and reaching transient response with prescribed qualities. A sufficient condition to guarantee the robot does not lose contact with the constraint surface is given. Detailed simulation results illustrate the proposed method. © *1994 John Wiley & Sons, Inc.*

タスクの力学モデルが持つ不正確さのために、ロボット・マニピュレータに含まれる束縛関数を正確に知ることは、困難である。この研究では、パラメトリックな不確実性、外部の障害物、そして束縛関数に不確実性を持つロボット・マニピュレータの高速な動作と強制力を制御するために、可変構造制御法 (VSC) を開発した。この方法は、強制ロボットの特定の構造を対照にしており、動作と力の制御を同時におこなう。提案した VSC コントローラは、スライディング・モードと到達過渡応答を、指定の精度で制御する。十分条件は、ロボットが強制されている面との接触を失わないことを保証することで与えられる。制御方法の詳細については、シミュレーションを使い説明する。

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1. INTRODUCTION

Many tasks of practical interest such as assembly and manufacturing require mechanical interaction of a robot with its environment. In these applications, interaction forces are generated that need to be controlled rather than rejected.[1] To accomplish such tasks successfully, several approaches have been proposed such as hybrid position/force control,[2–4] constrained motion control,[5–18] and impedance control.[19,20]

Constrained motion control has been extensively studied in recent years. In constrained motion control, the robot end-effector is assumed to be in contact with rigid frictionless surfaces.[4,9] As a result, kinematic constraints are imposed on the manipulator motion, which correspond to some algebraic constraints among the manipulator state variables. It is necessary to control both the motion on the constraint surfaces and the generalized constrained force.

A general theoretical framework of the constrained motion control is rigorously developed by McClamroch and Wang.[7] The proposed controller is based on a modification of the computed torque method. Lyapunov's direct method is utilized in Wang and McClamroch[8] to develop a class of decentralized position and force controllers. Mill and Goldenberg[9] applied descriptor theory to constrained motion control. The controller is derived based on a linearized dynamic model of the manipulator. Recently, adaptive constrained motion control methods have been proposed in refs. 13–15 to deal with parametric uncertainties of the robot. A variable structure control (VSC) method is developed in refs. 17 and 18 to overcome the problem of parametric uncertainties as well as external disturbances.

The above methods are all based on the assumption that the constraint functions are fully known. This assumption is the key to obtaining the needed nonlinear coordinate transformation so that the robot dynamic equation can be transformed into a reduced form that enables position control and force control to be designed separately.[7,9,13–15]

Due to task kinematic modelling inaccuracy, it is not possible to know the constraint functions exactly. In this article, the robust motion and constrained force control problem of robot manipulators in the presence of constraint function uncertainties as well as parametric uncertainties and external disturbances will be considered. By exploiting a particular structure of the constrained robot, motion control and constrained force control are treated together and designed via a variable structure control (VSC) method.[21–25] The proposed VSC controller can guarantee both the sliding mode and reaching transient response with prescribed qualities. Contact problem is also considered, and a sufficient condition to keep the robot in contact with the constraint surfaces is obtained. Detailed simulation results of the robot moving on a surface with unknown slope are given to illustrate the proposed method.

This article is organized as follows. Dynamic equations of the robot and problem formulation are given in Section 2. Section 3 presents the proposed VSC controller. The contact problem is discussed in Section 4. Simulation results are presented in Section 5, and a conclusion is given in Section 6.

2. DYNAMIC MODEL OF ROBOT MANIPULATORS

The dynamic equation of a general rigid link manipulator having n degrees of freedom (dof) can be written as:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + J^T(q)F + \tilde{f}(t) = \tau \quad (1)$$

where $q \in R^n$ is the joint displacement vector; $\tau \in R^n$ is the applied joint torque, $M(q) \in R^{n\times n}$ is the inertia matrix, $C(q, \dot{q})\dot{q} \in R^n$ is the vector function characterizing the Coriolis and centrifugal force, $g(q) \in R^n$ is the gravitational load of the robot, $\tilde{f}(t) \in R^n$ is the external disturbance, $J(q) = \partial x(q)/\partial q \in R^{n_0\times n}$ is the Jacobian matrix, which is assumed to be of full rank in a finite workspace Ω_q; $x \in R^{n_0}$ is the position and Euler angles of the end-effector in Cartesian space; and $F \in R^{n_0}$ is the vector of forces/moments on the environment exerted by the robot at the end-effector (corresponding to Euler angles, moments are decomposed along non-orthogonal rotation axes defining Euler angles). The following properties exist for equation (1):

Property 1.[26] $M(q)$ is a symmetric positive definite (s.p.d.) matrix, and there exists scalar $k' > 0$ such that $k'I_n \le M(q)\ \forall q \in \Omega_q$, in which the ordering is in the sense of positive definite matrices, i.e., $M(q) - k'I_n$ is an s.p.d. matrix. Moreover, for the robot with all revolute joints, there exists scalar $k'' > 0$ such that $M(q) \le k''I_n\ \forall q \in \Omega_q$. For a general robot, because $M(q)$ is analytic about q, $M(q) < k''I_n$ is valid for any finite workspace $\Omega_q = \{q : \|q - q_0\| \le q_{max}\}$ where q_0 and q_{max} are some constants.

Property 2.[26] The matrix $N(q, \dot{q}) = \dot{M}(q) - 2C(q, \dot{q})$ is a skew-symmetric matrix.

It is assumed that measurements of position, velocity, and the controlled interaction force are available. Only the case of the robot end-effector in contact with rigid constraint surfaces will be considered. If initially the robot end-effector is not in contact with the surfaces, any motion control scheme or impedance control scheme[19] can be used to bring the end-effector into contact with the surfaces. Care must be taken to prevent large impact force during the transition when the robot comes into contact with the surfaces such as using a zero or low approach velocity, large end-effector velocity damping along the force control direction,[3] etc. In the following, it is assumed that the end-effector is already in contact with the constraint surfaces, and the control exercised over the constrained force is such that the force will always hold the end-effector on the constraint surfaces. Hence, the problems due to loss of contact will not be considered.[9,7] In Section 4, a sufficient condition to guarantee this assumption is given.

Suppose that the environment is described by a set of m rigid hypersurfaces[7,9]:

$$\Phi(x) = 0 \quad \Phi(x) = [\phi_1(x), \ldots, \phi_m(x)]^T \quad m \le n_0 \tag{2}$$

which are mutually independent, and $\phi_i(x)$ is assumed to be twice differentiable with respect to x. The interaction force F can be written as:

$$F = D^T(x)\lambda + A_t f_t(\mu, v_{end}, \lambda) \triangleq L(x, \dot{x}, \mu)\lambda$$
$$D(x) = \frac{\partial \Phi(x)}{\partial x} \quad L \in R^{n_0 \times m} \tag{3}$$

where $\lambda \in R^m$ is a vector of Lagrange multipliers associated with the constraints that usually represent normal contact force components[7]; $D^T(x)\lambda$ represents the constraint force, i.e., normal contact force in the Cartesian space; $A_t \in R^{n_0 \times m}$ represents unit tangent directions of the surfaces; and $f_t \in R^m$ is the vector of tangential friction force, the magnitude of which depends on unknown friction coefficients μ and λ and direction depends on end-effector velocity v_{end}. In the assumption of frictionless contact surface ($f_t = 0$), Eq. (3) reduces to the form given by refs. 7 and 9.

When the motion of the robot is constrained on the surfaces (Eq. (2)), only $(n - m)$ coordinates of the position vector can be specified independently. In them, $(n_0 - m)$ coordinates are used to characterize the motion of the end-effector on the constraint surfaces. The others are used to characterize the joint self movement of the robot due to its redundancy. As pointed out in refs. 7 and 9, control of all the position coordinates of the robot is unnecessary, and only $(n - m)$ generalized position coordinates need to be controlled in the constrained motion of the robot. Therefore, $(n - m)$ mutually independent generalized curvilinear coordinates $\Psi(q) = [\psi_1(q), \ldots, \psi_{n-m}(q)]^T$ will be controlled (independence means that $\frac{\partial \Psi(q)}{\partial q}$ has full row rank). $\Psi(q)$ are assumed to be twice continuously differentiable and independent of $\Phi(x(q))$ in the workspace Ω_q, i.e., $\left[\left(\frac{\partial \Psi(q)}{\partial q}\right)^T, \left(\frac{\partial \Phi(x(q))}{\partial q}\right)^T\right]^T$ is nonsingular. Thus, once $\Psi(q)$ is regulated to the desired value $\Psi_d(t)$, combining with the constraints in Eq. (2), the position of the robot is uniquely determined. Notice that $\Psi(q)$ can be selected as some joint angle q_i. For example, because $D(x(q))J(q)$ is of full rank m, without loss of generality, we can assume the first m columns of $D(x(q))J(q)$ are independent. In this case, we can choose $\Psi(q) = [q_{m+1}, \ldots, q_n]^T$.

Similarly, only m independent components f of the interaction force F need to be controlled, which is supposed to be given by:

$$f = T(x)F \qquad f \in R^m \qquad T \in R^{m \times n_0} \tag{4}$$

From Eq. (3), we have

$$\lambda = (TL)^{-1} f \qquad F = L(TL)^{-1} f \tag{5}$$

where the invertibility of matrix TL is guaranteed by the assumption of independence of the components of f. Define a set of curvilinear coordinates as:

$$r = [r_f^T, r_p^T]^T \quad r_f = [\phi_1(x(q)), \ldots, \phi_m(x(q))]^T$$
$$r_p = [\psi_1(q), \ldots, \psi_{n-m}(q)]^T \tag{6}$$

with inverse transformation $q = q(r)$ for the finite workspace $q \in \Omega_q$, $r \in \Omega_r$. In general, for an r, there can exist several joint positions q for which $r = r(q)$. By restricting q to some finite workspace Ω_q, this inverse transformation is unique. Thereafter, all the results are developed for these finite workspaces.

Differentiate Eq. (6)

$$\dot{r} = J_q \dot{q} \tag{7}$$

where:

$$J_q = \frac{\partial r(q)}{\partial q} \qquad J_q = [(D(x(q))J(q))^T J_{qp}^T]^T$$
$$J_{qp} = \frac{\partial \Psi(q)}{\partial q} \in R^{(n-m)\times n} \qquad J_q \in R^{n\times n} \tag{8}$$

Using the transformations from Eqs. (6) and (7) in Eq. (1), then multiplying both sides by J_q^{-T}, dynamic equation (1) with the constraints from Eq. (2) and interaction force of Eq. (3) can be expressed in terms of variable r as:

$$M(r)\ddot{r} + C(r, \dot{r})\dot{r} + g(r) + \bar{F}(r, t) + B(r)f = J_q^{-T}\tau$$
$$r = \begin{bmatrix} 0 \\ r_p \end{bmatrix} \tag{9}$$

where:

$$M(r) = J_q^{-T}(q(r))M(q(r))J_q^{-1}(q(r))$$
$$C(r, \dot{r}) = J_q^{-T}(q(r))C(q(r), \dot{q}(r))J_q^{-1}(q(r)) - J_q^{-T}(q(r))M(q(r))J_q^{-1}(q(r))\dot{J}_q(q(r))J_q^{-1}(q(r))$$
$$g(r) = J_q^{-T}g(q(r))$$
$$\bar{F}(r, t) = J_q^{-T}(q(r))\bar{f}(t)$$
$$B(r) = J_q^{-T}J^T(q(r))L(x(r))[T(x(r))L(x(r))]^{-1} \tag{10}$$

in which the constraints are simply described by $r_f = 0$. Motion of the robot on the constraint surfaces is uniquely determined by the coordinates r_p. This particular structure will be utilized later in problem formulation and controller design. From Properties 1 and 2, the following properties can be obtained for Eq. (9).[17]

Property 3. For any finite work space Ω in which J_q is nonsingular, $M(r)$ is a symmetric positive definite matrix with:

$$k_r' I_{n\times n} \le M(r) \le k_r'' I_{n\times n} \quad \forall q \in \Omega_q \tag{11}$$

where $k_r' = \frac{k'}{c_1^2}$, $k_r'' = \frac{k''}{c_2^2}$, $c_1 = \sup_{q\in\Omega_q}[\sigma_{max}(J_q(q))]$, $c_2 = \inf_{q\in\Omega_q}[\sigma_{min}(J_q(q))]$.

Property 4. The matrix $N(r, \dot{r}) = \dot{M}(r) - 2C(r, \dot{r})$ is a skew-symmetric matrix.

We are now in a position to formulate the robust motion and constrained force control problem of robot manipulators in the presence of constraint function uncertainties. Due to task kinematic modelling inaccuracy or for computation efficiency, only estimates of the constraint functions from Eq. (2) are assumed to be known:

$$\hat{\Phi}(x) = 0 \tag{12}$$

Because the only requirement in choosing the generalized coordinates $\Psi(x)$ is to guarantee the independency of $\Psi(x)$ and $\Phi(x)$, i.e., J_q being nonsingular, it is possible to do so if Eq. (12) is a reasonably good estimate of $\Phi(x)$. The extent of allowable estimation error will be made clear in Section 3. Correspondingly, estimated values of the transformation Jacobian matrices from Eq. (8) are given by:

$$\hat{J}_q(q) = [(\hat{D}(x(q))J(q))^T, J_{qp}^T(q)]^T \tag{13}$$

The terms in Eq. (9) cannot be known exactly, which may be due to parametric uncertainties of the robot and the contact surfaces. It is assumed that only the estimated values $\hat{M}(r)$, $\hat{C}(r, \dot{r})$, $\hat{g}(r)$, $\hat{B}(r)$, $\hat{J}_q(r)$ of $M(r)$, $C(r, \dot{r})$, $g(r)$, $B(r)$, $J_q(r)$ in Eq. (9) are available. Their modelling errors and the external disturbances are bounded by:

$$\|\Delta M(r)\| \le \delta_M(r) \qquad \Delta M(r) = M(r) - \hat{M}(r)$$
$$\|\Delta C(r, \dot{r})\| \le \delta_C(r, \dot{r}) \qquad \Delta C(r, \dot{r}) = C(r, \dot{r}) - \hat{C}(r, \dot{r})$$
$$\|\Delta g(r)\| \le \delta_g(r) \qquad \Delta g(r) = g(r) - \hat{g}(r)$$
$$\|\Delta B(r)\| \le \delta_B(r) \qquad \Delta B(r) = B(r) - \hat{B}(r)$$
$$\|\bar{F}(r, t)\| \le \delta_{\bar{F}}(r, t) \tag{14}$$

where $\|\cdot\|$ denotes a norm of $\cdot$, which is a vector or a matrix. Without loss of generality, in this article $\|\cdot\|_2$ is used for vector $\cdot$ and $\|\cdot\|$ of a matrix $\cdot$ is the correspondingly induced norm, i.e., $\|A\| = \sigma_{max}(A) = \lambda_{max}^{1/2}(A^TA)$ where $\sigma(\cdot)$ denotes the singular value of $\cdot$, $\lambda(\cdot)$ means the eigenvalue of $\cdot$, and $\cdot_{max}$(or $\cdot_{min}$) is the maximum (or minimum) value of $\cdot$. The positive scalar bounds $\delta_M(r)$, $\delta_c(r, \dot{r})$, $\delta_g(r)$, $\delta_B(r)$ and $\delta_{\bar{F}}(r, t)$ are assumed known.

Remark 1. A way to determine boundary values is to use matrix property[27,pp367–370]:

$$\|A\| \le \|E\| \quad \text{if} \quad |a_{ij}| \le e_{ij} \quad \forall ij \tag{15}$$

(Strictly, the above property is valid for a matrix norm induced by an absolute vector norm. Generally used vector norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_p$, $\|\cdot\|_\infty$ are all absolute vector norms). For example, we can easily determine:

$$m_{ij_{min}} \le m_{ij}(r) \le m_{ij_{max}} \quad \forall ij \tag{16}$$

where the components $\cdot_{max}$ (or $\cdot_{min}$) are the maximum (or minimum) value of the corresponding components of $\cdot$. Let $\hat{m}_{ij}(r) = \frac{1}{2}[m_{ij_{max}} + m_{ij_{min}}]$, then $|\Delta M_{ij}(r)| \le \frac{1}{2}(m_{ij_{max}} - m_{ij_{min}})$, and $\delta_M(r) = \frac{1}{2}\|(M_{max} - M_{min}\|$ where $M_{max} = (m_{ij_{max}})$, $M_{min} = (m_{ij_{min}})$.

Remark 2. Suppose that the estimated values $\hat{M}(q)$, $\hat{C}(q, \dot{q})$, $\hat{g}(q)$ of $M(q)$, $C(q, \dot{q})$, $g(q)$ in the joint space equation (1), and $\hat{J}_q$ of J_q in (10) are available. Let $\delta_M(q)$, $\delta_C(q, \dot{q})$, $\delta_g(q)$, $\delta_{J_q^{-1}}(q)$, $\delta_{\dot{J}q}(q)$, and $\delta_{\tilde{f}}(t)$ denote the bounds of the modelling errors $\Delta M(q)$, $\Delta C(q, \dot{q})$, $\Delta g(q)$, $\Delta J_q^{-1}(q)$, $\Delta \dot{J}_q(q)$ and the external disturbance $\tilde{f}(t)$ in the joint space, respectively, which are defined in the same way as in Eq. (14) and are assumed known. From Eq. (10), the estimated values in Eq. (9) can be calculated by:

$$\begin{aligned} \hat{M}(r) &= \hat{J}_q^{-T}(q)\hat{M}(q)\hat{J}_q^{-1}(q) \\ \hat{C}(r, \dot{r}) &= \hat{J}_q^{-T}(q)\hat{C}(q, \dot{q})\hat{J}_q^{-1}(q) - \hat{M}(r)\hat{\dot{J}}_q(q)\hat{J}_q^{-1}(q) \\ \hat{g}(r) &= \hat{J}_q^{-T}\hat{g}(q) \end{aligned} \tag{17}$$

Noticing matrix properties $|\|A\| - \|E\|| \le \|A - E\| \le \|A\| + \|E\|$ and $\|AE\| \le \|A\|\|E\|$, and $\|\Delta M(r)\| = \|J_q^{-T}(q)\Delta M(q)J_q^{-1}(q) + \Delta J_q^{-T}(q)\hat{M}(q)J_q^{-1}(q) + \hat{J}_q^{-T}(q)\hat{M}(q)\Delta J_q^{-1}(q)\|$, etc, the bounds in Eq. (14) can be determined by:

$$\begin{aligned} \delta_M(r) &= \delta_M(q)(\delta_{J_q^{-1}} + \|\hat{J}_q^{-1}\|)^2 \\ &\quad + \|\hat{M}(q)\|\delta_{J_q^{-1}}(\delta_{J_q^{-1}} + 2\|\hat{J}_q^{-1}\|) \\ \delta_C(r, \dot{r}) &= \delta_C(q, \dot{q})(\delta_{J_q^{-1}} + \|\hat{J}_q^{-1}\|)^2 \\ &\quad + \|\hat{C}(q, \dot{q}\|\delta_{J_q^{-1}}(\delta_{J_q^{-1}} + 2\|\hat{J}_q^{-1}\|) \\ &\quad + \delta_M(r)(\delta_{\dot{J}q} + \|\hat{\dot{J}}_q\|)(\delta_{J_q^{-1}} + \|\hat{J}_q^{-1}\|) \\ &\quad + \|\hat{M}(r)\|[\delta_{\dot{J}q}(\delta_{J_q^{-1}} + \|\hat{J}_q^{-1}\|) + \delta_{J_q^{-1}}\|\hat{\dot{J}}_q\|] \\ \delta_g(r) &= (\delta_{J_q^{-1}} + \|\hat{J}_q^{-1}\|)\delta_g(q) + \delta_{J_q^{-1}}\|\hat{g}(q)\| \\ \delta_{\tilde{F}}(r, t) &= (\delta_{J_q^{-1}} + \|\hat{J}_q^{-1}\|)\delta_{\tilde{f}}(t) \end{aligned} \tag{18}$$

Suppose $r_{pd}(t) = \Psi_d(t) \in R^{n-m}$ is given as the desired robot motion trajectory, and $f_d(t) \in R^m$ is the desired force trajectory. Let:

$$e_p = r_p(t) - r_{pd}(t) \quad e_f = f(t) - f_d(t) \tag{19}$$

be the tracking errors of motion and interaction force. Consider the robot manipulator described by Eq. (9), the end-effector of which is in contact with the environment from Eq. (2) with interaction force given by Eq. (5). The robot is under the modelling errors of Eq. (14), which account for parametric uncertainties, external disturbances, and constraints uncertainties. The robust motion and force controller design problem can be stated as that of designing a control law so that $e_p \to 0$, $e_f \to 0$ as $t \to \infty$.

Remark 3. In the presence of constraint function uncertainties, the desired motion and constrained force cannot be specified in terms of all components, for example, $x_d(t)$ and $F_d(t) \in R^n$, for consistency with all physical constraints that are not known in advance. But a subset of the independent components of motion and constrained force, for example, $r_p(t) \in R^{n-m}$ and $f(t) \in R^m$, can be specified independently of the uncertainties of the constraints actually imposed on the robot, and can be controlled to track the desired values. The other components are determined by the actual imposed constraints so that the whole robot motion complies to the constraints. Therefore, the actual position and constrained force of the controlled robot will vary with the imposed constraints, which cannot be determined exactly beforehand. By using the nonlinear transformation from Eq. (6), the problem of constraint function uncertainties is reduced to the problem of transformation Jacobian matrix uncertainty in the constrained motion equation (9), which is suitable for problem formulation and controller design.

3. VSC MOTION AND FORCE CONTROL OF ROBOT MANIPULATORS

In this section, based on the particular structure of the constrained motion equation (9), a VSC method is developed for solving the above robust control problem. We make the assumption that the uncertainty of the transformation Jacobian matrix satisfies:

$$\|\hat{J}_q(q)\Delta J_q^{-1}(q)\| \le \gamma(q) \quad \text{and} \quad \gamma < 1 \tag{20}$$

A sufficient condition to guarantee Eq. (20) is that $\delta_{J_q^{-1}} < \dfrac{1}{\|\hat{J}_q\|}$.

For VSC, the switching function is selected as:

$$s = [s_f^T, s_p^T]^T \quad s_f = K_f \int_0^t e_f(v)dv \quad s_p = \dot{e}_p + K_p e_p \quad (21)$$

where $K_p \in R^{(n-m)\times(n-m)}$ is any positive definite matrix, $K_f \in R^{m\times m}$, is a weighting matrix. The resulted sliding mode equation $\{s = 0, \dot{s} = 0\}$ is described by:

$$\int_0^t e_f(v)dv = 0 \quad \text{and} \quad f(t) - f_d(t) = 0$$
$$\dot{e}_p + K_p e_p = 0 \qquad e_p \to 0 \quad (22)$$

in which position and force responses are decoupled, and the robot asymptotically follows the desired motion trajectory $r_{pd}(t)$ while exerting the desired constrained force components $f_d(t)$ on the environment. Moreover, by suitable choice of K_p, prescribed quality can be guaranteed in the sliding mode.

The control torque can be determined so that the system reaches the sliding mode in finite time and has prescribed reaching transient response against the modelling errors.

Theorem 1. *For the robot manipulator described by Eq. (9) with the modelling errors of Eq. (14), the system follows the desired motion trajectory $r_{pd}(t)$ while exerting the desired force trajectory $f_d(t)$ if the following control torque is applied:*

$$\tau = \hat{J}_q^T(r)[T_1 + T_2]$$
$$T_1 = \hat{M}(r)\ddot{r}_{eq} + \hat{C}(r, \dot{r})\dot{r}_{eq} + \hat{g}(r) + \hat{B}(r)f - K_s s$$
$$T_2 = -\varepsilon\frac{s}{\|s\|} \quad (23)$$

where:

$$\dot{r}_{eq} = \begin{bmatrix} -s_f \\ \dot{r}_{pd}(t) - K_p e_p \end{bmatrix} \quad \ddot{r}_{eq} = \begin{bmatrix} -K_f e_f \\ \ddot{r}_{pd}(t) - K_p \dot{e}_p \end{bmatrix} \quad (24)$$

K_s is any positive definite matrix. ε is any positive scalar satisfying:

$$\varepsilon \geq \frac{1}{1-\gamma}[\delta_M(r)\|\ddot{r}_{eq}\| + \delta_C(r, \dot{r})\|\dot{r}_{eq}\| + \delta_g(r) + \delta_{\bar{F}}(r, t) + \delta_B\|f\| + \gamma\|T_1\| + \varepsilon_1] \quad (25)$$

where ε_1 is any positive scalar. Furthermore, the reaching time t_r when the system reaches the sliding mode satisfies:

$$t_r \leq \frac{1}{c_3}\ln\left(1 + \frac{1}{c_4}\left(\frac{2V_0}{k_r'}\right)^{1/2}\right) \quad (26)$$

where:

$$c_3 = \frac{\lambda_{min}(K_s)}{k_r''} \quad c_4 = \frac{\varepsilon_1}{\lambda_{min}(K_s)}\left(\frac{k_r''}{k_r'}\right)^{1/2}$$
$$V_0 = \frac{1}{2}s^T(0)M(r(0))s(0) \quad (27)$$

and the reaching transient response is shaped by:

$$\|s\| \leq \left(\left(\frac{2V_0}{k_r'}\right)^{1/2} + c_4\right)\exp^{-c_3 t} - c_4 \quad (28)$$

Proof: For the robot manipulator of Eq. (9), we choose a positive definite function $V = \frac{1}{2}s^T M(r)s$. From Property 3, we have:

$$\frac{1}{2}k_r'\|s\|^2 \leq V \leq \frac{1}{2}k_r''\|s\|^2 \quad (29)$$

Differentiating V with respect to time yields:

$$\dot{V} = s^T M(r)\dot{s} + \frac{1}{2}s^T\dot{M}(r)s$$
$$= s^T\{M(r)\begin{bmatrix} 0 \\ \ddot{r}_p \end{bmatrix} - M(r)\ddot{r}_{eq}\} + s^T C(r, \dot{r})s \quad (30)$$

where Property 4 has been used to eliminate the term $\frac{1}{2}s^T\dot{M}(r)s$, i.e., $s^T\dot{M}(r)s = s^T C(r, \dot{r})s$. Noticing from Eq. (9):

$$\dot{V} = s^T[J_q^{-T}\tau - M(r)\ddot{r}_{eq} - C(r, \dot{r})\dot{r}_{eq} - g(r) - \bar{F}(r, t) - B(r)f]$$
$$= s^T[I + \Delta J_q^{-T}\hat{J}_q^T)\hat{J}_q^{-T}\tau - M(r)\ddot{r}_{eq} - C(r, \dot{r})\dot{r}_{eq} - g(r) - \bar{F}(r, t) - B(r)f] \quad (31)$$

Substituting control torque from Eq. (23) into it, we have:

$$\dot{V} = s^T\left[-\Delta M(r)\ddot{r}_{eq} - \Delta C(r, \dot{r})\dot{r}_{eq} - \Delta g(r) - \bar{F}(r, t) - \Delta B(r)f - K_s s - \varepsilon\frac{s}{\|s\|}\right] + s^T(\Delta J_q^{-T}\hat{J}_q^T)(T_1 + T_2)$$
$$\leq -s^T K_s s - \varepsilon\|s\| + \|s\|\,\|[\Delta M(r)\ddot{r}_{eq} + \Delta C(r, \dot{r})\dot{r}_{eq} + \Delta g(r) + \bar{F}(r, t) + \Delta B(r)f]\| + \|s\|\,\|\Delta J_q^{-T}\hat{J}_q^T\|\,(\|T_1\| + \|T_2\|)$$

$$\leq -s^T K_s s - \varepsilon\|s\| + \|s\|[\|\Delta M(r)\|\|\ddot{r}_{eq}\| + \|\Delta C(r, \dot{r})\|\|\dot{r}_{eq}\| + \|\Delta g(r)\| + \|\bar{F}(r, t)\| + \|\Delta B\|\|f\|] + \|s\|\gamma[\|T_1\| + \varepsilon]$$
$$\leq -\lambda_{min}(K_s)\|s\|^2 - \varepsilon_1\|s\| - \|s\|[(1-\gamma)\varepsilon - \delta_M(r)\|\ddot{r}_{eq}\| - \delta_C(r, \dot{r})\|\dot{r}_{eq}\| - \delta_g(r) - \delta_{\bar{F}} - \delta_B\|f\| - \gamma\|\hat{J}_q^{-T}\tau_1\| - \varepsilon_1]$$
$$\leq -\lambda_{min}(K_s)\|s\|^2 - \varepsilon_1\|s\| \tag{32}$$

From Eq. (29):

$$\dot{V} \leq -2c_3 V - (2k_r')^{1/2} c_4 c_3 V^{1/2} \tag{33}$$

Thus:

$$V^{1/2} \leq \left(V_0^{1/2} + \left(\frac{k_r'}{2}\right)^{1/2} c_4\right) \exp^{-c_3 t} - \left(\frac{k_r'}{2}\right)^{1/2} c_4 \tag{34}$$

and this means in finite time $V = 0$, i.e., $s = 0$. Moreover, from Eq. (29), the reaching transient response is shaped by Eq. (28). The upper limit of the reaching time t_r is solved by setting the right hand of Eq. (34) equal to zero, which is given by Eq. (26). ■

Remark 4. Because there are no requirements on the estimated matrices $\hat{M}(r)$, $\hat{C}(r, \dot{r})$, $\hat{g}(r)$, they can be chosen in a simple structure that will facilitate on-line computation. For example, they can be calculated off-line by choosing $\hat{M}(r) = \hat{M}(r_d)$, $\hat{C}(r, \dot{r}) = \hat{C}(r_d, \dot{r}_d)$, $\hat{g}(r) = \hat{g}(r_d)$ where $r_d(t)$ is the nominal trajectory. Similarly, $\hat{J}_q$ can be calculated off-line by choosing $\hat{J}_q(r) = \hat{J}_q(r_d)$ as long as its modelling error satisfies Eq. (20). Therefore, the control law given in Eq. (23) does not require much on-line computation time because the scalar bounds $\delta_M(r)$, $\delta_C(r, \dot{r})$, $\delta_g(r)$, $\delta_{\bar{F}}$, δ_B, γ can be determined off-line. However, better estimates of $\hat{M}(r)$, $\hat{C}(r, \dot{r})$, $\hat{g}(r)$, etc, will reduce the bound in Eq. (25) imposed on ε, which represents the magnitude of the discontinuous control T_2 in Eq. (23).

Remark 5. From Eq. (28), s exponentially reaches zero at a rate determined by the controller parameter K_s. From Eq. (26), the reaching time t_r is inversely dependent on $\lambda_{min}(K_s)$ and ε_1. By suitable choices of K_s and ε_1, the reaching transient can be guaranteed with prescribed quality.

Remark 6. The control law from Eq. (23) is discontinuous across the sliding surface. Possibly due to neglected dynamics such as sampling time, the control law leads to control of chattering in practice. In constrained motion, control torque chattering can cause force response chattering that will degrade the system performance. To overcome this phenomenon, smooth implementation of VSC law will be used. For example, with the concept of the boundary layer,[22] in the small neighbourhood $\{\|s\| \leq \Delta\}$ of the sliding surface $\{s = 0\}$, the discontinuous term $\frac{s}{\|s\|}$ is replaced by a smooth function $\frac{s}{\Delta}$ where Δ is the boundary layer thickness. Generally, a smaller Δ gives a more accurate approximation but is less robust to the neglected dynamics. The control law from Eq. (23) is the limited case when $\Delta \rightarrow 0$. Therefore, a trade-off exists between the accuracy that can be achieved and robustness of the neglected dynamics.

4. CONTACT CONDITION

Theorem 1 is based on the assumption that during constrained motion the robot does not lose contact with its environment. If the environment is a bilateral constraint surface, this assumption is satisfied automatically. However, in practice, the environment may be a unilateral constraint surface. The robot can only push the environment, which is supposed to correspond to $f_i \geq 0$, $i = 1, \ldots, m$, but cannot pull the environment, which corresponds to $f_i < 0$, $i = 1, \ldots, m$. In this case, the assumption is satisfied if the constrained force f is kept positive all the time by applying the control torque from Eq. (23). We now examine the system transient performance. For simplicity, in the control law from Eq. (23), let $K_f = k_f I_{m\times m}$, $K_p = k_p I_{(n-m)\times(n-m)}$, $K_s = k_s I_{n\times n}$, and let ε be equal to the right-hand side of Eq. (25).

In the Appendix, it will be shown that:

$$\|e_f\| \leq c_5 = \frac{1}{c_6}[c_7 + c_8\|e_p\| + c_9\|s\|] \quad \text{if} \quad c_6 > 0 \tag{35}$$

where:

$$c_6 = \left[k_r' - \frac{2}{1-\gamma}(\delta_M + \gamma\|\hat{M}\|)\right]k_f - \frac{2}{1-\gamma}(\delta_B + \gamma\|\hat{B}\|)$$

$$c_7 = \frac{2}{1-\gamma}[(\delta_M + \gamma\|\hat{M}\|)\|\ddot{r}_{pd}\| + (\delta_C + \gamma\|\hat{C}\|)\dot{r}_{pd}\| + (\delta_g + \gamma\|\hat{g}\|) + \delta_{\bar{F}} + (\delta_B + \gamma\|\hat{B}\|)\|f_d\| + \varepsilon_1] - \varepsilon_1$$

$$c_8 = \frac{2}{1-\gamma}[(\delta_C + \gamma\|\hat{C}\|)k_p + (\delta_M + \gamma\|\hat{M}\|)k_p^2]$$

$$c_9 = \frac{2}{1-\gamma}[(\delta_C + \gamma\|\hat{C}\|) + \gamma k_s + (\delta_M + \gamma\|\hat{M}\|)k_p] + \|C\| + k_s \tag{36}$$

$\|s\|$ satisfies:

$$\|s\| \begin{cases} \leq k_r''' \|s(0)\| \exp^{-c_3 t} - c_4[1 - \exp^{-c_3 t}] & \text{if} \quad t < t_r \\ = 0 & \phantom{\text{if}} \quad t \geq t_r \end{cases} \tag{37}$$

where $k_r''' = \frac{k_r''}{k_r'}$, and $\|e_p\|$ is (suppose $k_p \neq c_3$):

$$\|e_p\| \leq \begin{cases} \exp^{-k_p t}\|e_p(0)\| + \frac{k_r'''\|s(0)\| + c_4}{k_p - c_3}(\exp^{-c_3 t} - \exp^{-k_p t}) - \frac{c_4}{k_p}(1 - \exp^{-k_p t}) & \text{if } t < t_r \\ \exp^{-k_p t}\|e_p(0)\| + \exp^{-k_p t}\left[\frac{k_r'''\|s(0)\| + c_4}{k_p - c_3}(\exp^{(k_p - c_3)t_r} - 1) - \frac{c_4}{k_p}(\exp^{-k_p t_r} - 1)\right] & \text{if } t \geq t_r \end{cases} \tag{38}$$

Theorem 2. *If* $c_6 > 0$ *and the desired force trajectory* $f_d(t)$ *satisfy:*

$$f_{di} \geq c_5 \quad \forall\, t \geq 0 \; i = 1, \ldots, m \tag{39}$$

where c_5 *is defined in Eq. (35), in which the system tracking error* $\|s\|$ *and* $\|e_p\|$ *satisfy Eqs. (37) and (38), respectively, the system will not lose contact with its environment when the control law from Eq. (23) is applied, i.e.,* $f_i \geq 0$.

Proof:

$$f_i = e_{fi} + f_{di} \geq -\|e_f\| + f_{di} \geq 0 \tag{40}$$

in which Eqs. (35) and (39) have been used. ■

Remark 7. In Theorem 2, larger initial tracking errors $e_p(0)$, $s(0)$ and the modelling errors from Eq. (14), mean that γ will require a larger c_5, i.e, a stronger restriction on the system given by Eq. (39). Therefore, they should be chosen as small as possible. For example, $e_p(0)$ and $s(0)$ can always be made equal to zero by suitably choosing the desired motion trajectories. In the ideal case of known dynamics without modelling errors and initial tracking errors, $c_6 = k_f k_r' > 0$, $c_5 = \frac{1}{k_f k_r'}\varepsilon_1$. As the controller parameter ε_1 can be chosen as small as possible, Theorem 2 is satisfied for any practical desired force trajectories, i.e., $f_{di} > 0$. Generally, c_5 and c_6 are continuous functions of the modelling errors and initial tracking errors. Theorem 2 will be satisfied, at least for small modelling errors and initial tracking errors. Notice that Theorem 2 is only a sufficient condition; in practice less restriction on $f_d(t)$ is expected.

Remark 8. If $k_r' > \frac{2}{1-\gamma}(\delta_M + \gamma\|\hat{M}\|)$, Theorem 2 always can be satisfied for any practical desired force trajectory by choosing a large enough controller parameter k_f because c_5 is inversely proportional to k_f. A large k_f represents a "high gain" force feedback. Thus, "high gain" in the force feedback loop results in improved system performance, which agrees with the general discussion in ref. 7. Practically, due to the noise in force measurement and time delay, high gain in the force feedback loop can cause unstable phenomenon. Therefore, a compromise should be made in choosing k_f.

5. SIMULATION

For a two-link Cartesian space type robot (UMS-2 Robot), as shown in Figure 1, the matrices in the dynamic equation (1) are given by:

$$M(q) = \begin{bmatrix} \beta_2 & \\ & \beta_1 \end{bmatrix} \quad C(q, \dot{q}) = 0$$

$$g(q) = \begin{bmatrix} \beta_2 g \\ 0 \end{bmatrix} \quad J(q) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \tag{41}$$

where $\beta = [\beta_1, \beta_2]^T = [m_2, m_1 + m_2]^T$.

External disturbance is assumed as $\tilde{f}(t) = [10, 10]^T \sin(\pi t)$. Actual parameter values of the robot are $m_1 = 10$ kg. $m_2 = 10$ kg, $g = 9.8$ m/s^2, $d = 0.5$ m. The exact values of m_2 are assumed to be unknown with its estimated values $\hat{m}_2 = 8$ kg. Then, $\beta = [10, 20]^T$ with estimated values $\hat{\beta} = [8, 18]^T$ and bound $\delta_\beta = [\delta_{\beta_1}, \delta_{\beta_2}]^T = [2, 2]^T$.

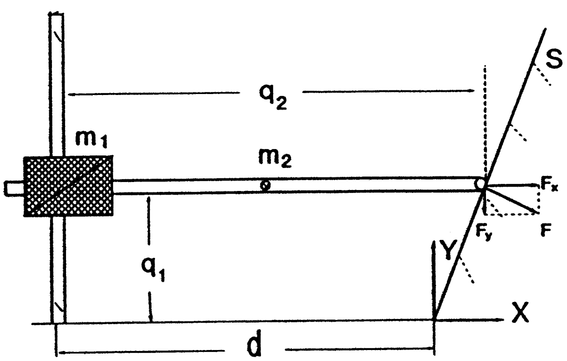

Figure 1. Configuration of the robot.

The robot is assumed to be in contact with a surface S shown in Figure 1, which is described by:

$$x - \alpha y = 0 \quad \alpha \le \delta_\alpha = 0.3 \tag{42}$$

where the actual α used is 0.3. The surface S has an unknown slope with an unknown coefficient of dry friction $\mu = 0.2$ (estimates $\hat{\mu} = 0$). The constrained force is:

$$f_n = D^T(x)\lambda \quad D(x) = [1 \quad -\alpha] \tag{43}$$

The interaction force is given by Eq. (3):

$$F = \begin{bmatrix} 1 \\ -\alpha \end{bmatrix} \lambda + \begin{bmatrix} \dfrac{\alpha}{1+\alpha^2} \\ \dfrac{1}{1+\alpha^2} \end{bmatrix} \mu \|f_n\| \mathrm{sgn}(\dot{y}) = L\lambda$$

$$L = \begin{bmatrix} 1 + \alpha\mu\,\mathrm{sgn}(\dot{y}) \\ -\alpha + \mu\,\mathrm{sgn}(\dot{y}) \end{bmatrix} \tag{44}$$

Suppose that the interaction force component $F_x \in R^m = R$ and position coordinate $y \in R^{n-m} = R$ should be controlled as long as α is finite. (F_x and y can be directly measured no matter what the actual constraints will be imposed). Then, $f = F_x = T(x)F$, $T(x) = [1 \quad 0]$ in Eq. (4) and $\Psi(q) = y(q)$. The transformation from Eq. (6) and its Jacobian matrices from Eq. (8) are given by:

$$r = \begin{bmatrix} r_f \\ r_p \end{bmatrix} = \begin{bmatrix} x - \alpha y \\ y \end{bmatrix} \quad J_q = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} J(q) \tag{45}$$

and the matrices in constrained dynamics from Eq. (9) are given by:

$$M(r) = \begin{bmatrix} \beta_1 & \alpha\beta_1 \\ \alpha\beta_1 & \beta_2 + \alpha^2\beta_1 \end{bmatrix} \quad C(r, \dot{r}) = 0 \quad g(r) = \begin{bmatrix} 0 \\ \beta_2 g \end{bmatrix}$$

$$B(r) = \begin{bmatrix} 1 \\ \dfrac{(1+\alpha^2)\mu\,\mathrm{sgn}(\dot{y})}{1 + \alpha\mu\,\mathrm{sgn}(\dot{y})} \end{bmatrix} \quad \bar{F} = \begin{bmatrix} 10 \\ 10(1+\alpha) \end{bmatrix} \sin(\pi t) \tag{46}$$

Since the actual slope of surface S is unknown, constraint function is approximated by:

$$\hat{\Phi}(x) = 0 \quad \hat{\Phi}(x) = x - \hat{\alpha} y \quad \hat{\alpha} = 0 \tag{47}$$

Therefore, estimates of $M(r)$, $C(r, \dot{r})$, $g(r)$, $B(r)$, $J_q(r)$ in Eq. (9) are:

$$\hat{M}(r) = \begin{bmatrix} \hat{\beta}_1 & 0 \\ 0 & \hat{\beta}_2 \end{bmatrix} \quad \hat{C} = 0 \quad \hat{g}(r) = \begin{bmatrix} 0 \\ \hat{\beta}_2 g \end{bmatrix}$$

$$\hat{B}(r) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{J}_q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} J(q) \tag{48}$$

The bounds of the modelling errors from Eq. (18) can be chosen as:

$$\delta_M(r) = \left\| \begin{bmatrix} \delta_{\beta_1} & \delta_\alpha \delta_{\beta_1} \\ \delta_\alpha \delta_{\beta_1} & \delta_{\beta_2} + \delta_\alpha^2 \delta_{\beta_1} \end{bmatrix} \right\| = 5.49 \quad \delta_C = 0$$

$$\delta_g(r) = \delta_{\beta_2} g = 19.6$$

$$\delta_B = \frac{(1+\delta_\alpha^2)\mu}{1 - \delta_\alpha \mu} = 0.232 \quad \delta_{\bar{F}}(r, t) = 19.6 \tag{49}$$

and:

$$\|\hat{J}_q(q) \Delta J_q^{-1}(q)\| \le \left\| \begin{bmatrix} 0 & \delta_\alpha \\ 0 & 0 \end{bmatrix} \right\| = 0.3 \tag{50}$$

where Remark 1 has been used. By choosing $\gamma = 0.3 < 1$, the assumption from Eq. (20) is satisfied. The switching function is given by Eq. (21) where $K_f = 0.1$, $K_p = 20$. The control torque is calculated from Eq. (23) where $\frac{s}{\|s\|}$ is replaced by $\frac{s}{\Delta}$ when $\|s\| \le \Delta = 0.1$ to eliminate chattering problem. Parameter values of the controller used are $K_s = \mathrm{diag}\{10, 10\}$, $\varepsilon_1 = 1$, and ε is calculated by Eq. (25). The desired motion and force trajectories are $r_{pd} = 0.3(1 - \cos$

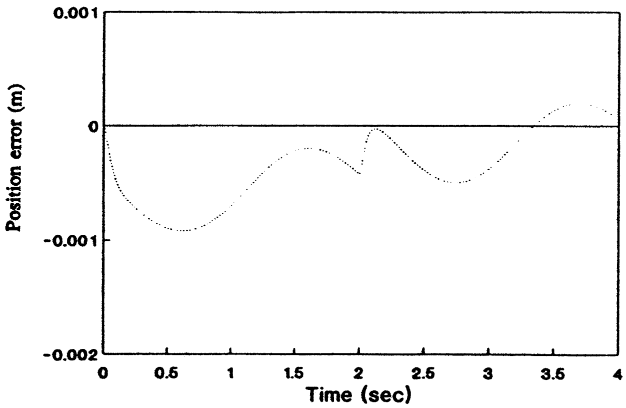

Figure 2. Tracking error of position coordinate y.

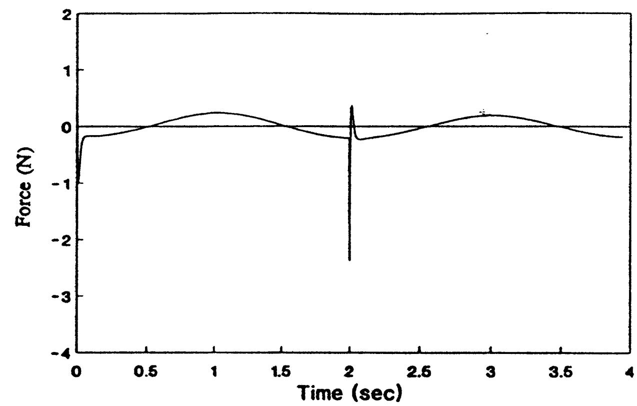

Figure 4. Tracking error of force component F_x.

$(0.5\pi t))$ and $f_d = 30 + 10 \cos(\pi t)$. Sampling time is 0.00125 s.

In the first simulation, zero initial tracking errors are used. The time response of the position tracking error is shown in Figure 2, from which we can see that the suggested VSC controller has good position tracking ability. Figures 3 and 4 present the time response of the contact force component F_x and its tracking error, respectively. The system also exhibits good force-tracking ability. The time response of the switching function is shown in Figure 5, and the applied joint torque is presented in Figure 6. The sudden changes occurring at about $t = 2$ s in these figures are caused by a change of surface friction force because of the direction change of the robot end-effector velocity. From these figures, we also see that by using the boundary layer technique, the robot does not exhibit chattering phenomenon.

In the second simulation, non-zero initial tracking errors are used to test the reaching transient

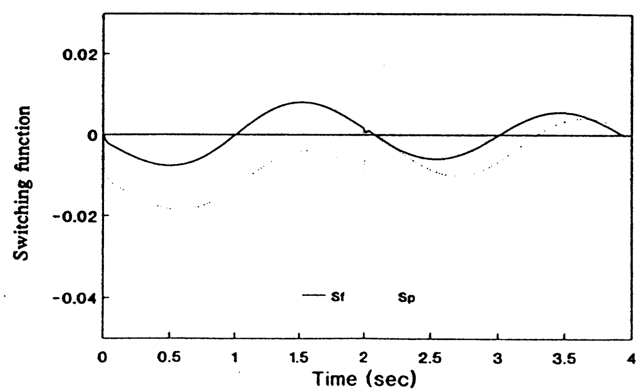

Figure 5. Time response of switching function.

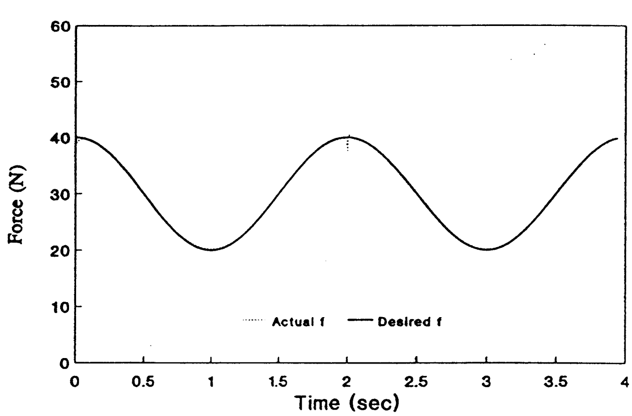

Figure 3. Time response of force components F_x.

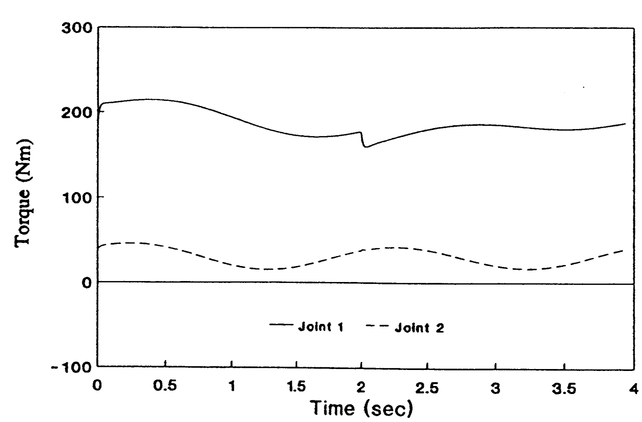

Figure 6. Joint torque of the robot.

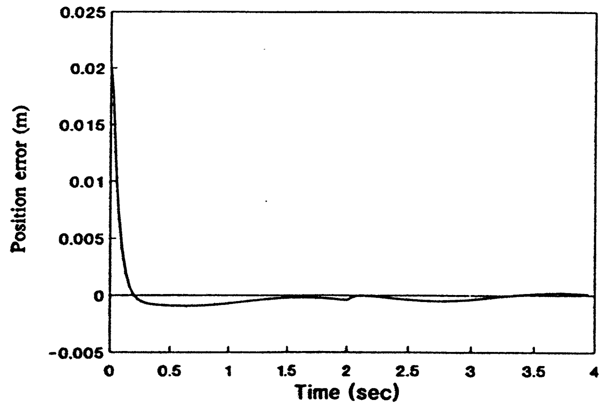

Figure 7. Tracking error of position coordinate y.

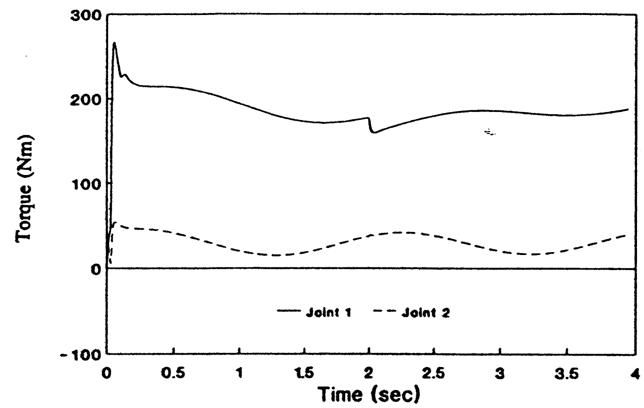

Figure 10. Joint torque of the robot.

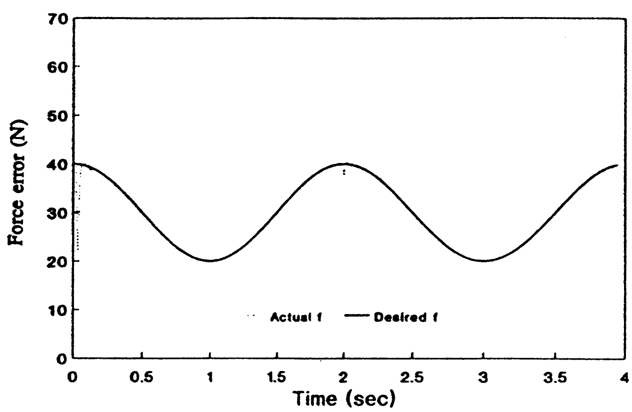

Figure 8. Time response of force component F_x.

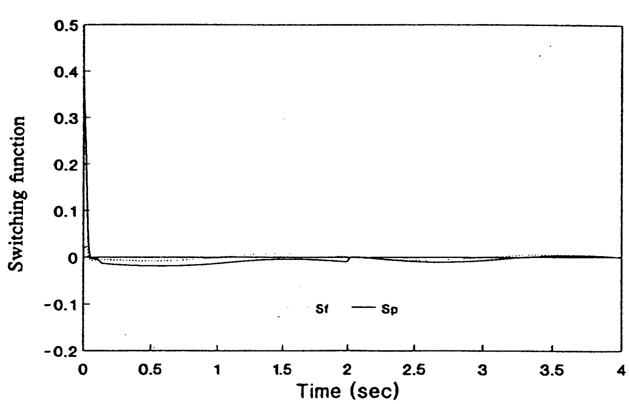

Figure 9. Time response of switching function.

of the system. The time response of position tracking error is shown in Figure 7. Figure 8 presents the time response of contact force component F_x. The time response of the switching function is shown in Figure 9, from which we can see that the suggested VSC controller has the prescribed reaching transient. The applied joint torque is presented in Figure 10. Again, the system exhibits good position and force-tracking ability except in the reaching transient. Note that in the reaching transient the system has a large force-tracking error, which could cause the system to lose contact with its environment. Therefore, zero initial tracking error is recommended, which can be achieved by suitably choosing the desired motion and force trajectory.

6. CONCLUSION

The design of a robust motion and force-tracking controller in the presence of parametric uncertainties, external disturbances, and constraint function uncertainties is presented in this article. In the presence of constraint uncertainties, some components of motion and constrained force can be specified freely and controlled to track the desired values. Actual position and constrained force of the controlled robot will vary with the physically imposed constraints so that the robot complies to the imposed constraints. By exploiting a particular structure of constrained motion dynamics, motion and force control are designed together with prescribed qualities guaranteed both in the sliding mode and in the reaching transient. The contact problem is considered and a sufficient condition to ensure contact is given. The simulation results verify the performance of the proposed method.

APPENDIX

From Eq. (23), we have:

$$
\begin{aligned}
\|T_1\| \le\ & \|\hat{M}(r)\|\|\ddot{r}_{eq}\| + \|\hat{C}(r,\dot{r})\|\|\dot{r}_{eq}\| \\
& + \|\hat{g}(r)\| + \|\hat{B}\|\|f\| + k_s\|s\| \\
\varepsilon =\ & \frac{1}{1-\gamma}[\delta_M(r)\|\ddot{r}_{eq}\| + \delta_C(r,\dot{r})\|\dot{r}_{eq}\| + \delta_g(r) \\
& + \delta_{\tilde{F}}(r,t) + \delta_B\|f\| + \gamma\|T_1\| + \varepsilon_1] \\
\le\ & \frac{1}{1-\gamma}[\delta_M + \gamma[\|\hat{M}\|)\|\ddot{r}_{eq}\| \\
& + (\delta_C + \gamma\|\hat{C}\|)\|\dot{r}_{eq}\| + \delta_g(r) + \gamma\|\tilde{g}\| \\
& + \delta_{\tilde{F}}(r,t) + (\delta_B + \gamma\|\hat{B}\|)\|f\| + \gamma k_s\|s\| + \varepsilon_1]
\end{aligned} \tag{51}
$$

Substituting the control torque of Eq. (23) into Eq. (9), and noticing $\dot{s} = \ddot{r} - \ddot{r}_{eq}$, we have:

$$
\begin{aligned}
\|M\dot{s}\| =\ & \| -\Delta M\ddot{r}_{eq} - \Delta C\dot{r}_{eq} - Cs - \Delta g(r) \\
& -\tilde{F} - \Delta B(r)f - k_s s - \varepsilon\frac{s}{\|s\|} \\
& + \Delta J_q^{-T}\hat{J}_q^T\left[T_1 - \varepsilon\frac{s}{\|s\|}\right]\| \\
\le\ & \delta_M\|\ddot{r}_{eq}\| + \delta_C\|\dot{r}_{eq}\| + \|C\|\|s\| + \delta_g \\
& + \delta_{\tilde{F}} + \delta_B\|f\| + k_s\|s\| + \varepsilon + \gamma[\|T_1\| + \varepsilon] \\
=\ & 2\varepsilon - \varepsilon_1 + (\|C\| + k_s)\|s\|
\end{aligned} \tag{52}
$$

From Property 3:

$$
\|M\dot{s}\| \ge k_r'\|\dot{s}\| \ge k_r'\|\dot{s}_f\| = k_r'k_f\|e_f\| \tag{53}
$$

Because:

$$
\begin{aligned}
\|f\| \le\ & \|f_d\| + \|e_f\| \\
\|\dot{r}_{eq}\| \le\ & \|s_f\| + \|\dot{r}_{pd}\| + k_p\|e_p\| \le \|s\| + \|\dot{r}_{pd}\| + k_p\|e_p\| \\
\|\ddot{r}_{eq}\| \le\ & k_f\|e_f\| + \|\ddot{r}_{pd}\| + k_p\|\dot{e}\| \\
\le\ & k_f\|e_f\| + \|\ddot{r}_{pd}\| + k_p\|s_p\| + k_p^2\|e_p\|
\end{aligned} \tag{54}
$$

from Eqs. (51)–(53) one obtains Eq. (35). Eq. (37) follows from Eqs. (28) and (29). Because $\dot{e}_p + k_p e_p = s_p$ and $\|s_p\| \le \|s\|$, we have:

$$
\|e_p\| \le exp^{-k_p t}\|e_p(0)\| + \int_0^t \exp^{-k_p(t-v)}\|s_p(v)\|dv \tag{55}
$$

which leads to Eq. (38).

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