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ADAPTIVE ROBUST CONTROL FOR A CLASS OF NON-MINIMUM PHASE NONLINEAR SYSTEM

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ABSTRACT

The paper presents the state feedback adaptive robust control approach to track the reference input for a class of nonminimum phase nonlinear systems. The key for this approach is to combine the adaptive robust control design techniques and the input-to-state property to deal with a class of non-minimum phase nonlinear systems with unknown parameter and unstructural uncertainties. The control design will guarantee that the tracking error dynamics is stabilized with bounded internal states and the closed-loop system is robust to the unstructural uncertainties.

1 INTRODUCTION

Control design for the nonlinear system has been the main stream in the control community during the past two decades. Recent results include the notion and properties of input-to-state stability [1], the approaches for nonlinear adaptive control with backstepping [2], the adaptive robust control for semi-strict feedback structure [3], and explicit integrator forwarding controller for semi-strict feedforward system [4]. The control design have been successfully dealt with the structural and unstructural uncertainties via adaptive control and robust control. A comprehensive survey paper [5] reviewed the constructive nonlinear control development from the historical point of view and predicted the trend of the constructive procedure in the nonlinear control design.

Most of the current control design has been focused on the minimum phase nonlinear system. For the minimum phase system, it has stable internal dynamics. In [6], the original nonlinear system is transformed into two subsystems via coordination transformation and the controller design is only focused on the linear subsystem via input-output linearization. Tracking control problem for nonlinear system is one of the most challenging problem in the control community due to the practical importance and the theoretical difficulties. An interesting dynamic inversion method [7] and modified version [8] have been proposed for tracking problems of nonminimum phase nonlinear systems. The main drawback is that it is non-causal, since the future information is needed to compute the desired trajectories. Another approach proposed in [9] [10] needed to solve the partial differential equation to find the center manifold of the composite system including the so-called exosystem. How to solve the partial differential equation is still a challenging problem in the computation field.

In [11], a robust output feedback stabilization controller is developed for a class of nonminimum phase nonlinear systems. Motivated by the fundamental limitation through the feedback control for the nonminimum phase system [12], the unstable internal dynamics is stabilized first when the output is viewed as the virtual control input. Then the following controller design via backstepping is to ensure the error between the output and the virtual control input and its derivatives converge to zeros asymptotically. With the input-to-state stability concept, the control law

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will guarantee that the control input and the internal signals are bounded and the system is stabilized even with the unstructural uncertainties.

In the paper, the state feedback adaptive robust tracking controller design is addressed for a class of nonminimum phase nonlinear system. The output feedback stabilization controller design [11] is extended for state feedback adaptive robust control to track the desired trajectories for a class of nonminimum phase nonlinear systems. The adaptive robust control proposed in [3] has been well developed for semi-strict feedback structure to deal with parameter and unstructural uncertainties in the nonlinear control community, which assumed that the system is minimum phase. In our tracking controller design, adaptive robust control techniques is combined with the input-to-state stability concept to design the control law for a class of nonminimum phase nonlinear systems.

The paper is organized as follows. Section 2 gives the motivation from the cheap control strategy for the nonminimum phase linear systems. Section 3 introduces the class of nonlinear system to be addressed. Section 4 proposes an adaptive robust tracking control design. The proposed control scheme are illustrated through the simulation of adaptive robust tracking control of inverted pendulum on a cart in Section 5.

2 STABILIZATION FOR SECOND ORDER NONMINIMUM PHASE LINEAR SYSTEM

In this section, stabilization of a simple second order nonminimum phase linear system is used to illustrate the motivation of the proposed adaptive robust control for a class of nonminimum phase system. Consider a second order linear system transformable into cascade form

$$\begin{aligned}\dot{z} &= a_0 z + b_0 y\\ \dot{y} &= a_1 z + b_1 y + bu\end{aligned} \tag{1}$$

where $z, y, u \in R$ and $a_0, a_1, b_0, b_1 \in R$. $b_0 \neq 0$ and $b_1 \neq 0$. $a_0 < 0$ is the zero of the transfer function from control input u to output y, which means that the system is nonminimum phase system.

The objective of the controller design is to find the state feedback stabilization controller for the dynamical system (1) which minimizes the functional

$$J = \frac{1}{2}\int_0^{\infty} [y^2 + \varepsilon^2 u^2] dt \tag{2}$$

where $0 < \varepsilon << 1$.

It is well known from the view of the cheap optimal control [12] that the minimum energy to stabilize the system (1) is equivalent to the minimum of the functional

$$J = \frac{1}{2}\int_0^{\infty} y^2 dt \tag{3}$$

while the output y is viewed as the control input to stabilize the zero dynamics

$$\dot{z} = a_0 z + b_0 y \tag{4}$$

In other words, the smallest achievable 2-norm of the output y is equal to the least amount control energy needed to stabilize the unstable internal dynamics [12]. Thus, the original stabilization problem is decomposed into two steps, one that stabilizes the unstable internal dynamics with the virtual control $\alpha(z)$ and the other that regulates $y - \alpha(z)$ to zero quickly.

3 PROBLEM FORMULATION

Consider a class of single input single output(SISO) nonminimum phase nonlinear systems transformable to the cascade form

$$\begin{aligned}\dot{\zeta} &= F(\zeta, x_1) + \Delta_0(\zeta, x, t)\\ \dot{x}_i &= x_{i+1} + \theta^T \varphi_i(\zeta, x_1, \cdots, x_i, t) + \Delta_i(\zeta, x, t), 1 \leq i \leq r-1\\ \dot{x}_r &= u + \theta^T \varphi_r(\zeta, x_1, \ldots, x_r) + \Delta_r(\zeta, x, t)\\ y &= x_1\end{aligned} \tag{5}$$

where $\zeta \in R^{n-r}$ and $x = [x_1, \cdots, x_r]^T \in R^r$ are the measured states. $y \in R$ and $u \in R$ are the measured output and the control input respectively. The known basis functions $\varphi(\zeta, x_1, \ldots, x_r)$ are assumed to be sufficiently smooth. The unknown nonlinearity functions $\Delta_i(\zeta, x, t), i = 0, \ldots, r$ represent the unstructured uncertainties.

Remark 1. *For nonminimum phase linear system with relative degree equal to r and transfer function give by*

$$G(s) = \frac{b_0 + b_1 s + \cdots + b_{n-r} s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-r} + s^n} = \frac{n(s)}{d(s)}$$

in which $b_{n-r} \neq 0$, and $n(s)$ and $d(s)$ are coprime, it is well known that the system can be transformed into the normal form *via coordination transformation*

$$
\begin{aligned}
\dot{\zeta}_i &= \zeta_{i+1}, 1 \le i \le n-r-1 \\
\dot{\zeta}_{n-r} &= \frac{1}{b_{n-r}}(-b_0\zeta_1 - \cdots - b_{n-r-1}\zeta_{n-r}) + \frac{1}{b_{n-r}}x_1 \\
\dot{x}_i &= x_{i+1}, 1 \le i \le r-1 \\
\dot{x}_r &= \sum_{i=1}^{n-r} c_i\zeta_i + \sum_{i=1}^{r} d_i x_i + b_{n-r}u \\
y &= x_1
\end{aligned}
$$

where c_i and d_i depend on a_i and b_i. It is noted that the above nonminimum phase linear system is a special case of the nonminimum phase nonlinear system in (5). ◇

Let $y_d(t)$ be the desired bounded output trajectory, which is assumed to be known with bounded derivatives up to r^{th} order. The control objective is to design a state feedback adaptive robust controller to track the desired output $y_d(t)$ as close as possible in spite of the structural and unstructural nonlinearities for the above possible non-minimum phase nonlinear system, while the controller will guarantee that the control input $u(t)$ and internal states ζ are bounded.

3.1 PRACTICAL ASSUMPTION

The following practical assumptions are normally made for the above nonlinear system.

Assumption 1. The unknown parameters θ lie in a known bounded region Ω_θ and the unknown nonlinearity functions $\Delta_0(\zeta,x,t)$ and $\Delta_i(\zeta,x,t), i = 1,\ldots,r$ are bounded by known functions, i.e.,

$$
\begin{aligned}
\theta \in \Omega_\theta &= \{\theta : \theta_{min} \le \theta \le \theta_{max}\} \\
\Delta_0 \in \Omega_{\Delta_0} &= \{\Delta_0 : |\Delta_0(\zeta,x,t)| \le \delta_0(\zeta,t)\} \\
\Delta_i \in \Omega_{\Delta_i} &= \{\Delta_i : |\Delta_i(\zeta,x,t)| \le \delta_i(\zeta,\bar{x}_i,t)\}, i = 1,\ldots,r
\end{aligned} \tag{6}
$$

where $\bar{x}_i = [x_1,\cdots,x_i]^T$, δ_0 and δ_i are bounded known functions.

Assumption 2. *There exists bounded trajectory $\zeta_d(t)$ such that*

$$
\dot{\zeta}_d = F(\zeta_d, y_d) \tag{7}
$$

Remark 2. *If the internal dynamics is a minimum phase system, it is well known that ζ_d exists. If it is a nonminimum phase system, there may exist analytic solution for some nontrivial solution for some desired trajectory $y_d(t)$. For example, a dimensionless dynamical model for the planar inverted pendulum is given as*

$$
\begin{aligned}
\ddot{\theta} + \varepsilon\cos\theta\ddot{x} - \beta\sin\theta &= 0 \\
\varepsilon\cos\theta\ddot{\theta} + \ddot{x} - \varepsilon\dot{\theta}^2\sin\theta &= u \\
y &= x
\end{aligned} \tag{8}
$$

where x is the position of the cart center of the mass, the angle θ is relative to the vertical axis, and u is the control input, ε, β are positive constants with $\varepsilon < 1$.

It is well known that the inverted pendulum is an underactuated and nonminimum phase nonlinear system. If the desired output trajectory is given as $y = \frac{1}{2}at^2$ where a is the constant acceleration, the desired trajectory for θ can be computed analytically from the first equation of (8).

$$
\theta_r = \arctan\frac{\varepsilon a}{\beta}
$$

On the other hand, if it is too difficult to find the analytic solution for the internal desired trajectory, some approximation method such as power series expansion may be used to find the bounded solution of the internal states [10, 13]. The desired trajectory for internal dynamics can also be computed numerically based on the nominal system dynamics via the nonlinear dynamic inversion approach with the assumption that $y_d(t) \to 0$ as $t \to \infty$ [7, 8, 14]. This approach is non-causal, since the computation of $\zeta_d(t)$ needs the preview information of $y_d(t)$. It is a drawback for this technique, but ζ_d can be computed off-line efficiently.

4 Discontinuous Projection Mapping

In the tracking controller design, the adaptive robust control techniques proposed in [3, 15] will be used to deal with the unknown parameters and unstructured uncertainties effectively to achieve high performance. In the adaptive robust control approach, the discontinuous projection mapping method is used to guarantee that the estimation parameter stay in the known region all the time. Let $\hat{\theta}$ and $\tilde{\theta}$ denote the estimates of the parameters θ and the estimates error, i.e., $\tilde{\theta} = \hat{\theta} - \theta$, respectively. The resulting adaptive control law can be defined as

$$
\dot{\hat{\theta}} = Proj_{\hat{\theta}}(\Gamma\tau_r) \tag{9}
$$

where the projection mapping $Proj_{\hat{\theta}}(\bullet)$ is defined with assumption that $\Gamma > 0$ is a diagonal matrix, and τ_r will be synthesized during the controller design.

$$Proj_{\hat{\theta}}(\bullet) = \begin{cases} 0 & \text{if} \begin{cases} \hat{\theta}_i = \hat{\theta}_{imax} & \text{and} \quad \bullet > 0 \\ \hat{\theta}_i = \hat{\theta}_{imin} & \text{and} \quad \bullet < 0 \end{cases} \\ \bullet & \text{Otherwise} \end{cases} \tag{10}$$

It can be shown [3, 15] that the projection mapping has the following properties

$$\begin{array}{ll} \text{P1} & \hat{\theta} \in \bar{\Omega}_\theta = \{\hat{\theta} : \theta_{min} \le \theta \le \theta_{max}\} \\ \text{P2} & \tilde{\theta}^T(\Gamma^{-1}Proj_{\hat{\theta}}(\Gamma\bullet) - \bullet) \le 0, \forall \bullet \end{array} \tag{11}$$

5 Adaptive Robust Controller for NMP nonlinear systems

In the following state feedback adaptive robust controller design, it first apply the input-to-state stability to track the internal states $\zeta_d(t)$ when the output x_1 is viewed as the virtual control input α_0 and then design the control input u to guarantee that the tracking error dynamics is stabilized.

5.1 Step 0

Define

$$\tilde{\zeta} = \zeta - \zeta_d \tag{12}$$

From (5), the internal dynamics is governed by

$$\dot{\zeta} = F(\zeta, x_1) + \Delta_0(\zeta, x, t) \tag{13}$$

Substituting the desired internal dynamics (7)

$$\dot{\zeta}_d(t) = F(\zeta_d, y_d) \tag{14}$$

into (13), the error dynamics of the internal states are

$$\begin{aligned} \dot{\tilde{\zeta}} =& F(\zeta, x_1) + \Delta_0(\zeta, x, t) - F(\zeta_d, x_d) \\ =& f(\tilde{\zeta}, x_1 - y_d, t) + \Delta_0(\zeta, x, t) \end{aligned} \tag{15}$$

Assumption 3. There exists smooth function $\alpha_0(\tilde{\zeta}, t) = \alpha_0(\zeta, \zeta_d, t)$ with $\alpha_0(0, t) = 0$ such that the error dynamics of the internal state is input-to-state practically stable(ISpS) [16, 17] with respect to z_1.

$$\dot{\tilde{\zeta}} = f(\tilde{\zeta}, z_1 + \alpha_0(\tilde{\zeta}, t), t) + \Delta_0(\zeta, x, t) \tag{16}$$

where

$$z_1 = x_1 - \alpha_0(\tilde{\zeta}, t) - y_d(t) \tag{17}$$

And there exists a exp-ISpS positive definite Lyapunov function $V_0(\tilde{\zeta})$ such that

$$\begin{aligned} \gamma_1(||\tilde{\zeta}||) \le V_0(\tilde{\zeta}) \le \gamma_2(||\tilde{\zeta}||) \\ \dot{V}_0(\tilde{\zeta}) \le -c_{\tilde{\zeta}} V_0(\zeta) + \gamma_3 |z_1|^2 + d_{\tilde{\zeta}} \end{aligned} \tag{18}$$

where γ_1, γ_2 and γ_3 belong to the class of κ_∞ functions, $c_\zeta > 0$ and $d_\zeta \ge 0$. $\diamondsuit$

The adaptive robust control design using standard backstepping techniques will start from the error z_1 rather than the error of the internal state $\tilde{\zeta}$. The second and the third term in (18) will be taken into account through the input-to-state practical stability assumption.

5.2 Step 1

From (5), the error dynamics $z_1 = x_1 - \alpha_0(\zeta, \zeta_d, t) - y_d(t)$ can be written as

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 - \dot{\alpha}_0(\zeta, \zeta_d) - \dot{y}_d(t) \\ &= x_2 + \theta^T \varphi_1(\zeta, x_1, t) + \Delta_1(\zeta, x_1, t) \\ &\quad - \frac{\partial \alpha_0}{\partial \zeta}(F(\zeta, x_1) + \Delta_0(\zeta, x, t)) - \frac{\partial \alpha_0}{\partial \zeta_d} F(\zeta_d, y_d) - \frac{\partial \alpha_0}{\partial t} - \dot{y}_d(t) \end{aligned} \tag{19}$$

The dynamics has both parameters uncertainties and unstructured nonlinearities. The principle of adaptive robust control techniques can be applied to design a virtual control law α_1 for x_2. The control law has to be robust to the unmodeled uncertainty $\Delta_0(\zeta, x, t)$ and $\Delta_1(\zeta, x, t)$. Following the adaptive robust controller design techniques [3, 16], define

$$\begin{aligned} \phi_1 =& \varphi_1 \\ \tau_1(\zeta, x_1, t) =& \phi_1(\zeta, x_1, t) z_1 \end{aligned} \tag{20}$$

The control law is designed as

$$\begin{aligned} &\alpha_1(\hat{\theta}, \zeta, x_1, \zeta_d, y_d, t) = \alpha_{1f} + \alpha_{1s} \\ &\alpha_{1f} = -\hat{\theta}^T \varphi(\zeta, x_1, t) + \frac{\partial \alpha_0}{\partial \zeta} F(\zeta, x_1) + \frac{\partial \alpha_0}{\partial \zeta_d} F(\zeta_d, y_d) + \frac{\partial \alpha_0}{\partial t} \\ &\alpha_{1s} = \alpha_{1s1} + \alpha_{1s2}, \alpha_{1s1} = -k_1 z_1 \end{aligned} \tag{21}$$

where $k_1 = \beta_1 + g_1 \|\Gamma\phi_1\|^2 + \gamma_3$, $\beta_1 > 0$, $g_1 > 0$ and $\gamma_3 > 0$ and the robust term α_{1s2} is any smooth function to satisfy the following condition

$$\begin{array}{ll} \text{i} & z_1(\alpha_{1s2} - \tilde{\theta}^T\phi_1(\zeta,x_1,t) + \tilde{\Delta}_1(\zeta,x_1,t)) \le \varepsilon_{12} \\ \text{ii} & z_1\alpha_{1s2} \le 0 \end{array} \tag{22}$$

where $\tilde{\Delta}_1(\zeta,x_1,\zeta_d,t) =: \Delta_1(\zeta,x,t) + \frac{\partial\alpha_0}{\partial\zeta}\Delta_0(\zeta,x,t)$, $\varepsilon_{12} > 0$ is a design parameters which can be arbitrarily small. One smooth function satisfying the above robust condition is

$$\alpha_{1s2} = -\frac{1}{4\varepsilon_{12}}(||\theta_{max} - \theta_{min}|| \times ||\phi_1|| + \delta_1 + |\frac{\partial\alpha_0}{\partial\zeta}|\delta_0)^2 z_1 \tag{23}$$

Define

$$z_2 = x_2 - \alpha_1(\hat{\theta},\zeta,x_1,y_d,t) - \dot{y}_d \tag{24}$$

Substituting the controller (21) into (19), the error dynamics of z_1 is

$$\dot{z}_1 = z_2 - k_1 z_1 + \alpha_{1s2} - \tilde{\theta}^T\phi_1 + \tilde{\Delta}_1 \tag{25}$$

Define a positive definite function $V_1(z) = \frac{1}{2}z_1^2$. From (25), its time derivative is

$$\begin{aligned} \dot{V}_1 &= z_1 z_2 - k_1 z_1^2 + z_1(\alpha_{1s2} + \tilde{\Delta}_1(\zeta,x_1,t)) - \tilde{\theta}^T\phi_1 z_1 \\ &= z_1 z_2 - k_1 z_1^2 + z_1(\alpha_{1s2} + \tilde{\Delta}_1(\zeta,x_1,t)) - \tilde{\theta}^T\tau_1 \end{aligned} \tag{26}$$

5.3 Step i

For the control law design in the intermediate step $2 \le i < r$, the mathematical induction procedure will be used. The above procedure and the design techniques will be employed to design the control through constructive procedure. Define $\bar{x}_i = [x_1,\cdots,x_i]$,$\bar{y}_{di} = [y_d,\cdots,y_d^{(i-1)}]$. For step i, define the error $z_i = x_i - \alpha_{i-1} - y_d^{(i-1)}(t)$, in which α_{i-1} is derived through the recursive backstepping design procedure.

$$\begin{aligned} &\alpha_i(\hat{\theta},\zeta,\bar{x}_i,\zeta_d,\bar{y}_{di},t) = \alpha_{if} + \alpha_{is} \\ &\alpha_{if} = -\hat{\theta}^T\phi_i + \frac{\partial\alpha_{i-1}}{\partial\zeta}F(\zeta,x_1) \\ &+ \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \frac{\partial\alpha_{i-1}}{\partial\zeta_d}F(\zeta_d,y_d) + \sum_{j=0}^{i-2}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)} + \frac{\partial\alpha_{i-1}}{\partial t} \\ &\alpha_{is} = \alpha_{is1} + \alpha_{is2}, \alpha_{is1} = -z_{i-1} - k_i z_i \end{aligned} \tag{27}$$

where $k_i = \beta_i + d_i\|\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\|^2 + g_j\|\Gamma\phi_j\|^2$.

For the robust term α_{is2} design, it should satisfy the same kind of condition as (22).

$$\begin{array}{ll} \text{i} & z_i(\alpha_{is2} - \tilde{\theta}^T\phi_i + \tilde{\delta}_i) \le \varepsilon_{i2} \\ \text{ii} & z_i\alpha_{is2} \le 0 \end{array} \tag{28}$$

where $\tilde{\delta}_i = \Delta_i(\zeta,x,t) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\Delta_i(\zeta,x,t) - \frac{\partial\alpha_{i-1}}{\partial\zeta}\Delta_0(\zeta,x,t)$, $\phi_i = \varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j$ and $\tau_i = \tau_{i-1} + \phi_j z_j$ Then the error dynamics of z_i at the *i-th* step is

$$\dot{z}_i = -z_{i-1} - k_i z_i + z_{i+1} + \alpha_{is2} - \tilde{\theta}^T\phi_i + \tilde{\delta}_i - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{29}$$

Consider the augmented positive definite function $V_i = V_{i-1} + \frac{1}{2}z_i^2$. From (26) (29), its time derivative is

$$\dot{V}_i = z_i z_{i+1} - \sum_{j=1}^{i}k_j z_j^2 + \sum_{j=1}^{i}z_j(\alpha_{js2} + \tilde{\delta}_j - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}) - \tilde{\theta}^T\tau_i \tag{30}$$

5.4 Step r

This is the final control design step, in which the control input u will be synthesized such that the estimated state x_r tracks the virtual control input $\alpha_{r-1}(\bar{x}_r,\bar{y}_{dr})$. The error dynamics of $z_r = x_r - \alpha_{r-1} - y_d^{(r-1)}$ can be obtained from (5)(33).

$$\begin{aligned} \dot{z}_r &= \dot{x}_r - \dot{\alpha}_{r-1}(\zeta,x_1,\cdots,x_{r-1},y_d,\cdots,y_d^{(r-2)},t) - y_d^{(r)} \\ &= \theta^T\varphi_r(\zeta,x) + u + \Delta_r(\zeta,x,t) \\ &- \dot{\alpha}_{r-1}(\hat{\theta},\zeta,x_1,\cdots,x_{r-1},y_d,\cdots,y_d^{(r-2)},t) - y_d^{(r)} \end{aligned} \tag{31}$$

Design the control law as

$$u = \alpha_r + y_d^{(r)}(t) \tag{32}$$

where

$$\begin{aligned} &\alpha_r = \alpha_{rf} + \alpha_{rs} \\ &\alpha_{rf} = -\hat{\theta}^T\phi_r(\zeta,x) + \frac{\partial\alpha_{r-1}}{\partial\zeta}F(\zeta,x_1) \\ &+ \sum_{j=1}^{r-1}\frac{\partial\alpha_{r-1}}{\partial x_j}x_{j+1} + \frac{\partial\alpha_{r-1}}{\partial\zeta_d}F(\zeta_d,y_d) + \sum_{j=0}^{r-2}\frac{\partial\alpha_{r-1}}{\partial y_d^{(j)}}y_d^{(j+1)} + \frac{\partial\alpha_{r-1}}{\partial t} \\ &\alpha_{rs} = \alpha_{rs1} + \alpha_{rs2}, \alpha_{rs1} = -z_{r-1} - k_r z_r \end{aligned} \tag{33}$$

in which $k_r > 0$.
For the robust term α_{rs2} design, it should satisfy the same kind of constraint as (28).

$$\begin{array}{ll} \text{i} & z_r(\alpha_{rs2} - \tilde{\theta}^T\phi_r + \tilde{\delta}_r) \le \varepsilon_{r2} \\ \text{ii} & z_r\alpha_{rs2} \le 0 \end{array} \tag{34}$$

Theorem 1. *Given the desired trajectory $y_d(t)$ with the bounded solution for the internal states $\zeta_d(t)$ satisfying the assumption (2), the following results hold if the adaptive law is applied.*

$$\dot{\hat{\theta}} = Proj_{\hat{\theta}}(\Gamma\tau_r) \tag{35}$$

A. *The control input and all internal states are bounded. In addition, the control law also guarantees that*

$$V(\tilde{\zeta},z) \le \exp(-k_v t)V(\tilde{\zeta}(0),z(0)) + \frac{\bar{\varepsilon}}{k_v}[1-\exp(-k_v t)] \tag{36}$$

where $V(\tilde{\zeta},z) = V_0(\tilde{\zeta}) + V_r(z)$, $k_v = \min(c_\zeta,\bar{k}_v), \bar{k}_v = \min(2\beta_1,\cdots,2\beta_r)$, *and* $\bar{\varepsilon} = \sum_{i=1}^r \varepsilon_{i2} + d_\zeta$.

B. *If the unstructured uncertainties of $\Delta_i, i = 0,\dots,r$ vanish after a finite time t_0, the error of $z_i, i = 1,\dots,r$ asymptotically converge to zero for any controller gain k_i and ε_{i2}. And the difference $e_1(t) = x_1(t) - y_d(t)$ converges to zero asymptotically.*

Proof. Noting that $z_{r+1} = 0$ and (30),

$$\begin{aligned} V_r(z) &\le \sum_{i=1}^r(-\beta_i z_i^2 + \varepsilon_{i2}) - \gamma_3 z_1^2 \\ &\le -\sum_{i=1}^r \beta_i z_i^2 - \gamma_3 z_1^2 + \sum_{i=1}^r \varepsilon_{i2} \end{aligned} \tag{37}$$

So the derivative of $V(\tilde{\zeta},z)$ can be computed as

$$\begin{aligned} \dot{V}(\tilde{\zeta},z) &\le -c_{\tilde{\zeta}}V_0(\tilde{\zeta}) + \gamma_3 z_1^2 + d_{\tilde{\zeta}} - \sum_{i=1}^r \beta_i z_i^2 - \gamma_3 z_1^2 + \sum_{i=1}^r \varepsilon_{i2} \\ &\le -c_{\tilde{\zeta}}V_0(\tilde{\zeta}) - \sum_{i=1}^r \beta_i z_i^2 + d_\zeta + \sum_{i=1}^r \varepsilon_{i2} \\ &\le -c_\zeta V_0(\tilde{\zeta}) - \bar{k}_v V_r(z) + \bar{\varepsilon} \\ &\le -k_v V(\tilde{\zeta},z) + \bar{\varepsilon} \end{aligned} \tag{38}$$

which proved the claim (36).

Now consider the situation that there has no unmodeled uncertainty after a finite time. Choose a positive definite function $V_{an}(z,\tilde{\theta})$ as

$$V_{an}(z,\tilde{\theta}) = V_r(z) + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{39}$$

From the robust condition (22) and (28), it is easy to say that $z_i\alpha_{is2} \le 0, i = 1,\cdots,r$. Noting the condition(28), we have

$$\begin{aligned} \dot{V}_{an} &\le \sum_{i=1}^r(-k_i z_i^2 + z_i\alpha_{is2} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}) - \tilde{\theta}^T\tau_r + \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ &\le -\sum_{i=1}^r \beta_i z_i^2 + \tilde{\theta}^T\Gamma^{-1}(\dot{\hat{\theta}} - \Gamma\tau_r) \\ &\le -\sum_{i=1}^r \beta_i z_i^2 + \tilde{\theta}^T(\Gamma^{-1}Proj_{\hat{\theta}}(\Gamma\tau_r) - \tau_r) \\ &\le -\sum_{i=1}^r \beta_i z_i^2 \end{aligned} \tag{40}$$

Thus, $\dot{V}_{an}$ is semi-negative positive. From the Lyapunov theory, $z = [z_1,\dots,z_r]$ are bounded. And it is easy to check that $\dot{z}$ is bounded. So z will converge to zero asymptotically as $t \to \infty$ by the Barbalat's lemma.

Finally, the tracking error $e_1(t) = x_1 - y_d(t)$ is computed from (17)

$$e_1(t) = z_1 + \alpha_0(\tilde{\zeta},t) \tag{41}$$

thus

$$\begin{aligned} |e_1(t)| = |x_1 - y_d| &= |z_1 + \alpha_0(\tilde{\zeta},t)| \\ &\le |z_1| + |\alpha_0(\tilde{\zeta},t)| \end{aligned} \tag{42}$$

With the condition in (18) and the input-to-state stability, the $\tilde{\zeta}$ converges to zero as $t \to \infty$. It is noted that the internal state ζ is bounded and converges to ζ_d, which may not equal to zero for most nonminimum phase nonlinear systems. Noting that $z_1 \to 0$ as $t \to \infty$, the tracking error asymptotically converges to zero as $t \to \infty$ from (42).

6 Simulation

In this section, the proposed control scheme is illustrated through tracking control design for the inverted pendulum system.

6.1 DYNAMICAL MODEL

The inverted pendulum scheme is shown in Figure 1. It consists of a point mass m attached to the massless pivot which is mounted on a cart M. It can only move along the horizontal direction. Let q denote the translational position of the center of the cart from its equilibrium point, and θ is the clockwise angular rotation of the pendulum. F is the horizontal force acting on the center of the cart. The friction force F_{fric} includes the viscous friction $b_1\dot{q}$ and Coulomb friction $F_f \text{sgn}(\dot{q})$, where b_1 and F_f are the viscous friction coefficient and Coulomb coefficient, respectively. $b_0\dot{\theta}$ is the friction torque for the pendulum.

The equations of the motion for the system are given by [18].

$$
\begin{aligned}
J\ddot{\theta} + mL\cos\theta\ddot{q} + b_0\dot{\theta} - mgL\sin\theta + \Delta_0 = 0 \\
mL\cos\theta\ddot{\theta} + (M+m)\ddot{q} - mL\dot{\theta}^2\sin\theta + b_1\dot{q} + F_f\text{sgn}(\dot{q}) + \Delta_1 = F \\
y = q
\end{aligned}
\tag{43}
$$

where $J = mL^2$.

Before the controller design is proceeded, the dynamics equation is transformed into the cascade form. Define the new variables for the nonlinear model (43) as

$$
\begin{aligned}
x &= \sqrt{\frac{M+m}{J}}q & \tau &= t \\
\varepsilon &= \frac{mL}{\sqrt{J(M+m)}} & \beta &= \frac{mgL}{J} \\
c_0 &= b_0\frac{1}{J} & c_1 &= b_1\frac{1}{M+m} \\
f_1 &= F_f\sqrt{\frac{1}{J(M+m)}} & u &= F\sqrt{\frac{1}{J(M+m)}}
\end{aligned}
\tag{44}
$$

Then the dimensionless equation of the motion is equivalent to

$$
\begin{aligned}
\ddot{\theta} + \varepsilon\cos\theta\ddot{x} - \beta\sin\theta + \bar{\Delta}_0 = 0 \\
\varepsilon\cos\theta\ddot{\theta} + \ddot{x} - \varepsilon\dot{\theta}^2\sin\theta + c_1\dot{x} + f_1\text{sgn}(\dot{x}) + \bar{\Delta}_1 = u
\end{aligned}
\tag{45}
$$

Remark 3. *In the above coordinate transformation, $0 < \varepsilon < 1$ is a kind of coupling coefficient, which is usually much less than 1.*

The damping term $c_0\dot{\theta}$ for the inverted pendulum is not included in order to facilitate the following coordinate transformation and simplify the adaptive robust controller design. If the damping term is included, the coordinate transformation exists but is too complicated to be used in the controller design. However, the damping term would help us stabilize and track the inverted pendulum in practice.

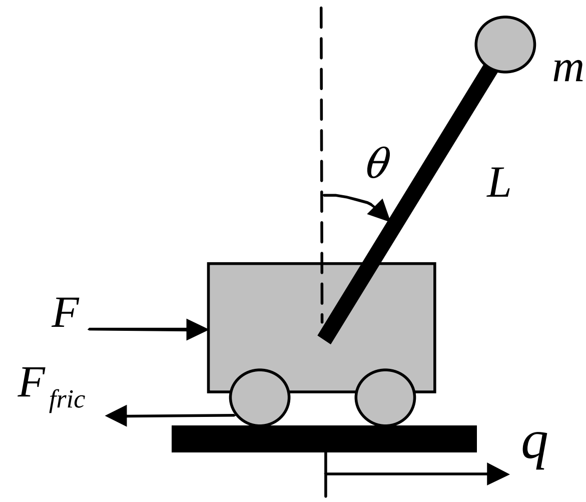

Figure 1. Inverted pendulum

Define the new coordinates for dimensionless nonlinear model of the inverted pendulum (45).

$$
\begin{aligned}
\zeta_1 &= \log(\tan\theta + \sqrt{1+\tan^2\theta}) + \varepsilon x & \zeta_2 &= \frac{\dot{\theta}}{\cos\theta} + \varepsilon\dot{x} \\
x_1 &= x & x_2 &= \dot{x}
\end{aligned}
\tag{46}
$$

The corresponding inverse coordinate transformation can be obtained from (46)

$$
\theta = \tan^{-1}(\sinh(\zeta_1 - \varepsilon x_1)) \tag{47}
$$

where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

The nonlinear dynamics of the inverted pendulum in Figure 1 is described in the new coordinates as

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= (\beta + \frac{\dot{\theta}^2}{\cos\theta})\sinh(\zeta_1 - \varepsilon x_1) + \bar{\Delta}_0 \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{\Delta_\theta}(-\varepsilon\beta\cos\theta\sin\theta + \varepsilon\dot{\theta}^2\sin\theta - c_1x_2 - f_1\text{sgn}(x_2) + u + \bar{\Delta}_1)
\end{aligned}
\tag{48}
$$

where $\Delta_\theta = 1 - \varepsilon^2\cos^2\theta > 0$.

The controller objective is to design the tracking controller such that the control effort and the internal states are bounded for the feasible desired trajectory $y_d(t) = x_1$.

6.2 CONTROLLER DESIGN

Based on the coordinate transformation, the desired values for ζ_{d1}, ζ_{d2} can be computed via

$$\begin{aligned}\zeta_{d1} =& \log(\tan\theta_{d1} + \sqrt{1+\tan^2\theta_{d1}}) + \varepsilon y_d \\ \zeta_{d2} =& \frac{\dot{\theta}_{d2}}{\cos\theta_{d1}} + \varepsilon \dot{y}_d\end{aligned} \tag{49}$$

Define the error variables of the internal states $\tilde{\zeta} = [\tilde{\zeta}_1 \quad \tilde{\zeta}_2]^T$

$$\begin{aligned}\tilde{\zeta}_1 =& \zeta_1 - \zeta_{d1} \\ \tilde{\zeta}_2 =& \zeta_2 - \zeta_{d2}\end{aligned} \tag{50}$$

From (47) and (50), the error dynamics of $\tilde{\zeta}$ is given as

$$\begin{aligned}\dot{\tilde{\zeta}}_1 =& \tilde{\zeta}_2 \\ \dot{\tilde{\zeta}}_2 =& (\beta + \frac{\dot{\theta}^2}{\cos\theta})\sinh(\zeta_1 - \varepsilon x_1) - (\beta + \frac{\dot{\theta}_{d1}^2}{\cos\theta_{d1}})\sinh(\zeta_{d1} - \varepsilon y_d) + \bar{\Delta}_0\end{aligned} \tag{51}$$

From the principles of the backstepping design approach, the virtual controller α_0 for $\sinh(\zeta_1 - \varepsilon x_1)$ can be chosen as

$$\alpha_0 = -\frac{1}{\beta}((k_{02} + \frac{\beta^2}{4\gamma_4})s_0 + \frac{s_0}{4\varepsilon_0}(g_f^2 + \bar{\delta}_0^2)) \tag{52}$$

where $|\bar{\Delta}_0| \le \bar{\delta}_0$, $g_f = (\beta + \frac{\dot{\theta}_{d1}^2}{\cos\theta_{d1}})\sinh(\zeta_{d1} - \varepsilon y_d)$, the sliding surface $s_0 = \tilde{\zeta}_2 + k_{01}\tilde{\zeta}_1$ and $k_{02} - k_{01} > 0$. It can be proved that the ISpS assumption is satisfied with respect to z_1, where $z_1 = \sinh(\zeta_1 - \varepsilon x_1) - \alpha_0(\zeta, \zeta_d, t)$ for $\theta \in (-\frac{\pi}{2} \quad \frac{\pi}{2})$.

The derivative of z_1 is computed

$$\dot{z}_1 = \cosh(\zeta_1 - \varepsilon x_1)(\zeta_2 - \varepsilon x_2) - \dot{\alpha}_0(\zeta, \zeta_d, t) \tag{53}$$

Decompose $\dot{\alpha}_0$ into two parts: computable part $\dot{\alpha}_{0c}$ and uncomputable part $\dot{\alpha}_{0u}$. Then the virtual control α_1 was proposed for $\cosh(\zeta_1 - \varepsilon x_1)(\zeta_2 - \varepsilon x_2)$ rather than x_2.

$$\alpha_1 = \alpha_{1f} + \alpha_{1s1} + \alpha_{1s2} \tag{54}$$

where $\alpha_{1f} = \dot{\alpha}_{0c}$ and $\alpha_{1s1} = -k_1 z_1$

The robust term α_{1s2} is any function satisfying the following condition:

$$\begin{cases}\text{i} & z_1(\alpha_{1s2} - \dot{\alpha}_{0u}) \le \varepsilon_1 \\ \text{ii} & z_1\alpha_{1s2} \le 0\end{cases} \tag{55}$$

Then the dynamics of z_1 is computed

$$\dot{z}_1 = -k_1 z_1 + z_2 + \alpha_{1s2} - \dot{\alpha}_{0u} \tag{56}$$

where $z_2 = \cosh(\zeta_1 - \varepsilon x_1)(\zeta_2 - \varepsilon x_2) - \alpha_1$

Compute the derivative of z_2 as

$$\begin{aligned}\dot{z}_2 =& \sinh(\zeta_1 - \varepsilon x_1)(\zeta_2 - \varepsilon x_2)^2 + \cosh(\zeta_1 - \varepsilon x_1)(\dot{\zeta}_2 - \varepsilon \dot{x}_2) \\ & - \dot{\alpha}_{1c} - \dot{\alpha}_{1u} \\ =& b_k(\vartheta^T \varphi_2 + u) + \alpha_{2f} + \Delta_2\end{aligned} \tag{57}$$

where α_{2f} and Δ_2 include all of the computable and uncomputable part, respectively. $b_k(\theta, \dot{\theta})$ is the gain. $\vartheta = [c_1 \quad f_1]^T$ is the unknown parameters, and $\varphi_2 = [-x_2 \quad -\mathrm{sgn}(x_2)]^T$ is the regressor vector.

Define the control law u

$$u = \frac{1}{b_k}(\alpha_{2f} + \alpha_{2s1} + \alpha_{2s2}) + \hat{\vartheta}^T \phi_2 \tag{58}$$

where $\alpha_{2s1} = -z_1 - k_2 z_2$

The robust term α_{2s2} is any function satisfying the following condition:

$$\begin{cases}\text{i} & z_2(\alpha_{2s2} - b_k\tilde{\vartheta}^T\varphi_2 + \Delta_2) \le \varepsilon_2 \\ \text{ii} & z_2\alpha_{2s2} \le 0\end{cases} \tag{59}$$

Then the dynamics of z_2 is

$$\dot{z}_2 = -z_1 - k_2 z_2 + (\alpha_{2s2} - b_k\tilde{\vartheta}^T\varphi_2 + \Delta_2) \tag{60}$$

In the final step, the discontinuous projection adaptive control law is chosen as

$$\dot{\hat{\vartheta}} = \mathrm{Proj}(\Gamma\tau_2) \tag{61}$$

where $\tau_2 = \phi_2 z_2 = b_k \varphi_2 z_2$.

6.3 SIMULATION RESULTS

The system parameters are given in Table 1. The desired trajectory $y_d = \frac{1}{2}at^2$ where $a = 0.35g$ with $g = 9.81m/s^2$. The corresponding desired pendulum angle is set as $\theta_d = \arctan(\frac{\varepsilon a}{\beta}) = 0.33\text{rad} = 19.3°$. The initial condition for simulation is $[\theta, \quad \dot{\theta}, \quad q, \quad \dot{q}] = [-20°, \quad 0, \quad 0, \quad 0]$. There are

Table 1. System Parameters of the Inverted Pendulum

Properties	Symbol	Value
Mass of the cart	M	0.44Kg
Tip mass on the pendulum	m	0.14 Kg
Length of the pendulum	L	0.215 m
Inertia	$J = mL^2$	$0.00647 Kg.m^2$
Viscous friction coefficient	b_1	1.5 N/m/s
Coulomb friction coefficient	F_f	0.5 N

some external disturbances $\Delta_0 = 0.05 \sin 2t$ and $\Delta_1 = 0.1$. For the parameter adaptation, the coefficients are set in the region $1 \le b_1 \le 2$ and $0.2 \le F_f \le 0.8$. The results are plotted in Figure 2, Figure 3, Figure 4, Figure 5, and Figure 6. In Figure 5, the control input increases with time, since the viscous friction force between the cart wheel and the supporting surface is proportional to the velocity. The tracking error is bounded in Figure 3.

7 CONCLUSION

This paper presents a state feedback adaptive robust controller to track the desired output for a class of nonminimum phase nonlinear systems. The basic assumption for the approach is that there exist bounded trajectories for the unstable internal dynamics when the desired trajectory is bounded. If it is too difficult to find the analytic solution for the internal desired trajectory, some approximation method such as dynamic inversion-based output tracking and power series expansion may be used to find the numerical solution.

The well developed adaptive robust controller, which is based on standard backstepping techniques and the widely used discontinuous projection method, is combined with input-to-state stability concept to construct tracking controller for a class of nonminimum phase nonlinear systems with parameter uncertainties and unstructed uncertainties. Simulation results for tracking problems of the inverted pendulum are presented to illustrate the proposed adaptive robust feedback controller scheme.

REFERENCES

[1] Sontag, E., and Wang, Y., 1996, "New characteristics of input-to-state stability," *IEEE Trans. Automat. Contr.*, vol. 41, no. 9, pp. 1283–1294.

[2] Krstic, K., Kanellakopoulos, I., and Kokotovic,P., 1995, *Nonlinear and Adaptive Control Design*. New York, NY: John Wesley and Sons Inc.

[3] Yao, B., 1997, "High performance adaptive robust control of nonlinear systems: a general framework and new

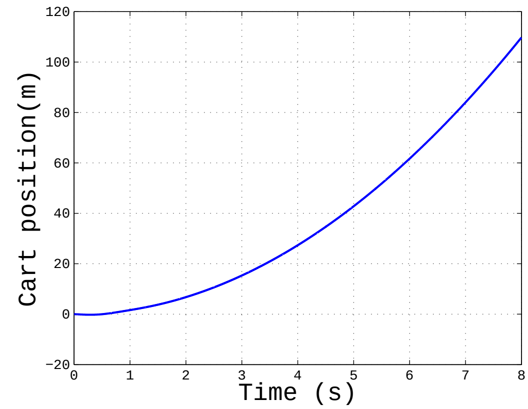

Figure 2. Cart position for adaptive robust tracking problem

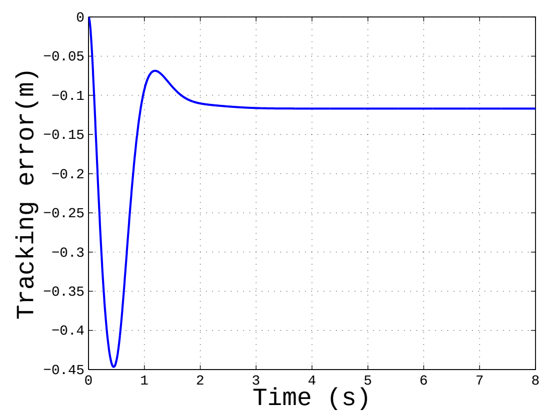

Figure 3. Tracking error for adaptive robust tracking problem

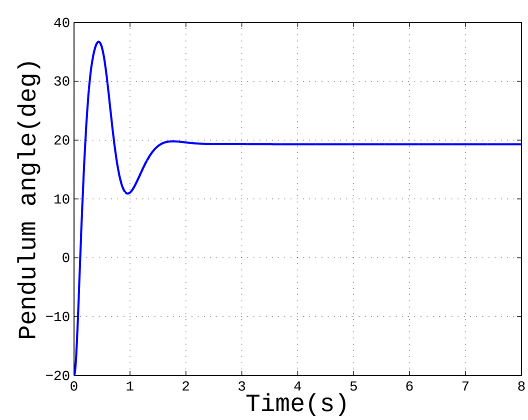

Figure 4. Pendulum angle for adaptive robust tracking problem

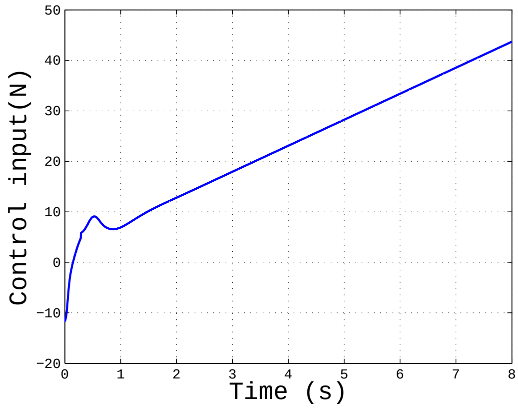

Figure 5. Control input for adaptive robust tracking problem

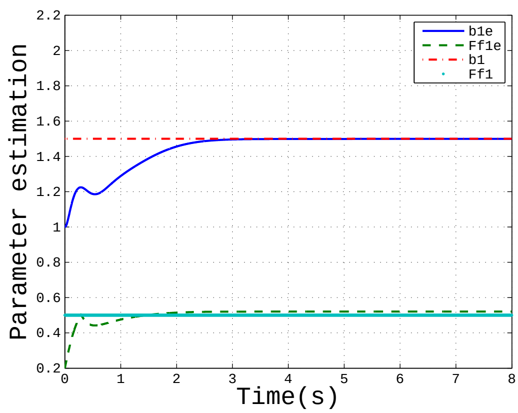

Figure 6. Parameter estimation for adaptive robust tracking problem

schemes," in *Proc.IEEE Conf. Decision Control*, San Diego, CA, pp. 2489–2494.

[4] Krstic, M., 2004, "Feedback linearizability and explicit integrator forwarding controllers for a class of feedforward systems," *IEEE Trans. Automat. Contr.*, vol. 49, no. 10, pp. 1668–1682.

[5] Kokotovic, P., and Arcak, M., 2001, "Constructive nonlinear control: a historical perspective," *Automatica*, vol. 37, pp. 637–662.

[6] Jankovic, M., and Fontaine,D., 1996, "Tora example: Cascade and passivity-based control design," *IEEE Trans. Automat. Contr.*, vol. 4, no. 3, pp. 292–297.

[7] Devansia, S., Chen, D., and Paden, B., 1996, "Nonlinear inversion based output tracking," *IEEE Trans. Automat. Contr.*, vol. 41, no. 7, pp. 930–942.

[8] Tomlin, C., and Sastry, S., 1997, "Bounded tracking for nonminimum phase nonlinear systems with fast zero dynamics," *International Journal of Control*, vol. 68, no. 4, pp. 819–847.

[9] Isidori, A., and Byrnes, C., 1990, "Output regulation of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 35, no. 2, pp. 131–140.

[10] Huang, J., 2000, "Asymptotic tracking of a nonminimum phase nonlinear system with nonhperbolic zero dynamics," *IEEE Trans. Automat. Contr.*, vol. 45, no. 3, pp. 542–546.

[11] Xie, B., and Yao,B., 2006, "Robust output feedback stabilization of a class of nonminimum phase nonlinear systems," in *Proc. American Control Conf.*, Minneapolis, MN, pp. 4981–4986.

[12] Seron, M., Braslavsky,J., and Kokotovic,P., 1999, "Feedback limitation in nonlinear systems: from bode integrals to cheap control," *IEEE Trans. Automat. Contr.*, vol. 44, no. 4, pp. 829–833.

[13] Gurumoorthy, R., and Sanders,S., 1993, "Controlling nonminimum phase nonlinear systems-the inverted pendulum on a cart example," in *Proc. American Control Conf.*, San Francisco, CA, pp. 680–685.

[14] Hunt, L., Meyer, G., and Su, R.,1996, "Noncausal inverse for linear systems," *IEEE Trans. Automat. Contr.*, vol. 41, no. 4, pp. 608–611.

[15] Yao, B., and Tomizuka, M., 1997, "Adaptive robust control of siso nonlinear systems in a semi-strict feedback form," *Automatica*, vol. 33, no. 5, pp. 893–900.

[16] Yao, B., and Xu, L., 2001, "Observer based adaptive robust control of a class of nonlinear systems with dynamic uncertainties," *Int. J. Robust Nonlinear Contrrol*, vol. 11, pp. 335–356.

[17] Jiang, Z., Teel, A., and Praly, L., 1994, "Nonlinear observer-controller design for a class of nonlinear systems," *Mathematics of control Signal Systems*, vol. 7, pp. 95–120.

[18] Goodwin,G., Graebe,S., and Salgado, M.,2001, *Control System Design*. Eaglewood Cliffs, NJ: Prentice- Hall Inc.