Proceedings of IMECE2005
2005 ASME International Mechanical Engineering Congress and Exposition
November 5-11, 2005, Orlando, Florida USA

IMECE2005-81313

MULTI-OBJECTIVE OPTIMIZATION OF TIP TRACKING CONTROL USING LMI

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ABSTRACT

This paper presented multi-objective optimization of tip tracking control for non-collocated flexible beam. The desired trajectory is specified at the tip displacement of the flexible structure, which undergoes translation base motion actuated by a linear motor. The system model is first formulated from modal truncation approach for the flexible structure representing a single Cartesian robot manipulator. The linear system model of the flexible structure always has structural uncertainties. Robust stability and robust performance on tip tracking can be expressed as H_2/H_∞ norm constraints, which are converted into the Linear Matrix Inequality (LMI). The multi-objective controller design is solved by the convex minimization. In order to reduce the conservatism generated when the same Lyapunov matrix is selected, the Lyapunov matrix is scaled for different norm constraints. Simulation results have demonstrated favorable tip tracking of the proposed robust controller.

1 INTRODUCTION

Robot manipulator is widely used to help the monotonous, tedious jobs in the industrial practice and space applications. The conventional rigid manipulator are built to be rigid with high stiffness in order to be easily controlled. Those robot manipulators are shown to be inefficient in terms of high power consumption, low motion speed, high actuator capacity, and low payload ratio [3, 13, 14]. In order to improve the efficiency by reducing the weight and increasing the speed of the robot manipulator, it is widely accepted that the lightweight structure may provide a solution especially in the space robotic systems. However, reducing the weight or increasing the speed will lead to the onset of the low frequency oscillation that can limit their time response and accuracy [14]. These flexible modes need to be taken into account in the control design in order to preserve the quality and the accuracy of the flexible structure's operation. The lightweight structure or flexible structure will lead to the high degree of elastic motion especially in the high speed operation of the structure. In addition, the dynamical models for multi-link flexible structure are coupled and nonlinear, which will make it much more difficult to model, identify, and control the multi-link flexible structure.

It is much difficult, if not impossible, to derive the high accuracy dynamical model for the flexible manipulator for the control purpose. The control objective for a high performance flexible structure is to track the desired smooth trajectory of motion at the tip or the joint. The trajectory at the joint is assumed that the manipulator is rigid enough and the effect of the flexible modes is negligible. It is more reasonable to design the tip trajectory for the tip tracking problems provided that the link deflection is within the limited deflection.

Advances in the control of the flexible structure is necessary to reduce the power consumption and increase the speed in order to satisfy the increasingly stringent demand in the industrial practice. Motion control of the flexible structure has been investigated by many researchers. A variety of control schemes

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have been successfully proposed to deal with the motion control problems. Among these control approaches, one finds linear control algorithms such as linear quadratic regulator (LQR), command shaping, adaptive learning, and pole placement. Those algorithms generally address the problems of collocated flexible systems in that the sensor and the actuator are placed at the same location. However, in order to address the exact tip tracking control, a non-collocated system usually arises, whose dynamics are normally of non-minimum phase systems. When coupled with the loop transfer recovery (LTR), the linear quadratic Gaussian (LQG) could improve the performance with the non-minimum phase systems [19].

The nonlinear control methods such as the computed torque have also been proposed, but mainly for rigid manipulators. In order to deal with the unmodelled uncertainty, some robust control approaches such as H_∞ and variable structure control (VSC) have received much attention in the last decades [1, 3].

In [1, 3, 19], only one H_∞ or H_2 objective is specified. In practice, however, different specifications may impose on the closed-loop system of the flexible structure. In this scenario, the control problem is a multi-objective control problem, in which controllers have to be designed to satisfy several different practical requirements. It is hard to solve the general multi-objective control problem with the available computing tool today. In this paper, we will confine ourselves on H_2/H_∞ problems, which can be formulated using linear matrix inequality (LMI) and solved using the interior point methods.

Linear matrix inequality (LMI) has gained wide acceptance in control and optimization communities due to its elegant formulation and the availability of the efficient solvers [5, 8, 11, 16]. The robust stability like H_∞ or performance requirement like H_2 can be formulated as the linear matrix inequalities. The LMI can be solved efficiently by the available software packages [8]. In addition, the multi-objective control problem can be formulated as combination of LMI. In the dynamical output feedback control, these formulated LMIs are nonlinear with the controller parameters. It is much difficult to solve this kind of non-convex optimization problem right now. One possible way to recover the convexity is to require that the Lyapunov matrices should be the same [10, 11, 16]. The solution is indeed a suboptimal solution. In order to reduce the conservatism, one may scale the Lyapunov matrices to reduce the effect of the above artificial constraint.

The rest of the paper details the application of the multi-objective optimization to the tip tracking control of the flexible structure. We first derived the dynamical model from modal truncation approach. Then the control problem for tip tracking is formulated through the linear matrix inequalities. The controller is solved as a multi-objective optimization problem. Finally, the simulation results are given to demonstrate the effectiveness and advantages of the proposed approaches and the results are summarized.

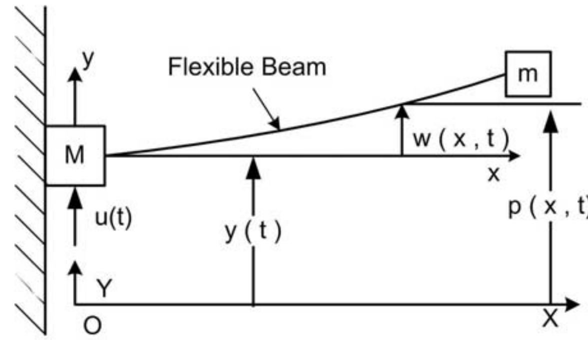

Figure 1. SCHEMATIC DIAGRAM OF THE FLEXIBLE MANIPULATOR.

2 DYNAMICAL MODEL

The virtual experimentation is designed to simulate the flexible structure with base translational motion. The schematic of the flexible structure is shown in Figure1,which represents a single Cartesian robot manipulator. The structure includes one actuator and one position sensor at the base and the displacement sensor at the tip. In addition, the payload is attached on the tip to simulate the industrial application. The flexible beam is lightly damped.

The structure with uniform flexible cantilever beam undergoes translation base motion actuated by a linear motor. In literature, the flexible manipulator with rotational joint has been investigated in modeling, identification, and control. This configuration is different in that it is actuated by the translational force rather than the angular torque. For simplicity, the gravity is ignored when deriving the dynamical model for the system. It is also assumed that the deflection of the cantilever beam is small.

There have several approaches to obtain the nominal model of the above flexible structure like the Hamiltonian and energy equation method, the modal analysis method, and the finite impulse response method. Of course, different approaches will have different fidelity to apply in different situations. In this paper, the truncated mode method with clamped-mass boundary condition is adopted to derive the nominal model.

In Figure 1, the coordinates OXY represents the inertial frame, and the coordinates oxy is the local reference frame fixed with the base actuator. The base actuator position is denoted by $y(t)$, and the control force $u(t)$ is applied at the base. The other system parameters used throughout the section are: M, the mass of base; L, the length of the beam; m, the tip mass payload; EI, the uniform flexural rigidity; $w(x,t)$, the elastic deflection of beam measured from the undeformed position at x; $p(x,t)$, the position of the beam at x in the Y direction.

Based on the small deflection assumption, the position of beam at x in Figure 1 can be approximated by $p(x,t) = y(t) + w(x,t)$.

Following the procedure in [21], the total kinetic energy and the total potential energy are

$$E_k = \frac{1}{2}M\dot{y}^2(t) + \frac{1}{2}\rho\int_0^L \dot{p}^2(x,t)dx + \frac{1}{2}m\dot{p}^2(L,t)$$
$$E_p = \frac{1}{2}EI\int_0^L [p''(x,t)]^2 dx$$

where dot and prime represent the derivative with respect to the time and space x respectively, and ρ is the linear mass density of the flexible beam. The corresponding governing equation of the flexible beam can then be derived by the extended Hamilton's principle:

$$\rho\ddot{w}(x,t) + EIw''''(x,t) = -\rho\ddot{y}(t) \tag{1}$$

The boundary conditions for solving (1) are obtained as follows. At the base, noting that the beam is clamped onto the base M, we have

$$\begin{aligned} w(0,t) &= 0 \\ w'(0,t) &= 0 \\ u(t) - M\ddot{y}(t) &= EIw'''(0,t) \end{aligned} \tag{2}$$

At the tip, the mass m is free to move. Thus

$$\begin{aligned} w''(L,t) &= 0 \\ EIw'''(L,t) &= m(\ddot{w}(L,t) + \ddot{y}(t)) \end{aligned} \tag{3}$$

The dynamical model of flexible structure is described by the partial ordinary equation in (1). This will increase the difficulty to obtain the solution since the solution is affected by the boundary conditions in addition to the commonly encountered initial condition in the ordinary differential equation. In the above case, the boundary conditions are non-homogeneous, which makes it much difficult to obtain the solution. In the following, the truncated mode approach will be used to obtain the set of ordinary differential equations that well approximate the above partial differential equation to simplify the analysis and control designs. For this purpose, let us first find the eigenfunctions of the elastic motion $w(x,t)$ of the beam when the base is assumed to be clamped, i.e., $y(t) = 0$. With a clamped base, the corresponding partial differential equation in (1) is

$$\rho\ddot{w}(x,t) + EIw''''(x,t) = 0 \tag{4}$$

with the boundary conditions at $x = 0$ simplified to

$$\begin{aligned} w(0,t) &= 0 \\ w'(0,t) &= 0 \end{aligned} \tag{5}$$

and $x = L$

$$\begin{aligned} w''(L,t) &= 0 \\ EIw'''(L,t) &= m\ddot{w}(L,t) \end{aligned} \tag{6}$$

The equation (4) with the boundary condition in (5) and (6) are of the homogeneous form. The separation of variables method can be used to find the infinite number of the eigenfunctions (or mode function) of the flexible beam, $\varphi_j(x), j = 1,2,\ldots,n,\ldots$. With the normalization, the eigenfunctions satisfy the following orthogonal properties.

$$\begin{aligned} \textstyle\int_0^L \rho\varphi_i(x)\varphi_j(x)dx + m\varphi_i(L)\varphi_j(L) &= \delta_{i,j} \\ EI\textstyle\int_0^L \varphi_i''(x)\varphi_j''(x)dx &= \omega_i^2\delta_{i,j} \end{aligned} \tag{7}$$

where $\delta_{i,j}$ is the Kronecker Delta, $i,j = 1,2,\ldots,n,\ldots$, and $\omega_i = (\frac{EI}{\rho L^4})^{1/2}\beta_i^2$ is the natural frequency, in which β_i is the ith roots of the following characteristic equation

$$1 + cos(\beta)cosh(\beta) + \frac{m}{\rho L}(cos(\beta)sinh(\beta) - sin(\beta)cosh(\beta)) = 0$$

In addition, it is noted from the boundary condition at $x = L$ that

$$EI\int_0^L \varphi_i(x)\varphi_j''''(x)dx = \omega_i^2\delta_{i,j} + EI\varphi_i(L)\varphi_j'''(L) \tag{8}$$

Assuming that only the first n modes have to be considered and all higher modes can be neglected, the elastic deflection of the beam with the non-homogeneous PDE in (1) and the boundary conditions (2) and (3) can be approximated by

$$w(x,t) = \sum_{j=1}^{n} \varphi_j(x)q_j(t) \tag{9}$$

Substituting (9) into the boundary condition in (3):

$$EI\sum_{j=1}^{n} \varphi_j'''(L)q_j(t) = m\sum_{j=1}^{n} \varphi_j(L)\ddot{q}_j(t) + m\ddot{y}(t)$$

And substituting (9) into (1):

$$\rho\ddot{y}(t) + \rho\sum_{j=1}^{n} \varphi_j(x)\ddot{q}_j(t) + EI\sum_{j=1}^{n} \varphi_j''''(x)q_j(t) = 0 \tag{10}$$

Multiply both sides by $\varphi_i(x), i = 1,2,\ldots,n$ and integrate from 0 to L

$$\rho \int_0^L \varphi_i(x)dx\ddot{y}(t) + \rho \sum_{j=1}^n \int_0^L \varphi_i(x)\varphi_j(x)dx\ddot{q}_j(t) + EI\sum_{j=1}^n \int_0^L \varphi_i(x)\varphi_j''''(x)dxq_j(t) = 0 \quad (11)$$

Noticing (8), the last term in above equation becomes

$$\begin{aligned} & EI\sum_{j=1}^n \int_0^L \varphi_i(x)\varphi_j''''(x)dxq_j(t) \\ &= \sum_{j=1}^n \omega_i^2\delta_{i,j}q_j(t) + \sum_{j=1}^n \varphi_i(L)EI\varphi_j'''(L)q_j(t) \\ &= \omega_i^2 q_i(t) + m\sum_{j=1}^n \varphi_i(L)\varphi_j(L)\ddot{q}_j(t)) \\ &+ m\varphi_i(L)\ddot{y}(t) \end{aligned} \quad (12)$$

With the orthogonal properties in (7), substituting (12) into (11):

$$(\rho \int_0^L \varphi_i(x)dx + m\varphi_i(L))\ddot{y}(t) + \ddot{q}_i(t) + \omega_i^2 q_i(t) = 0 \quad (13)$$

And integrate the governing equation (1) from 0 to L with the boundary condition (3), the boundary condition (2) will be

$$u(t) - M\ddot{y}(t) = \rho \int_0^L (\ddot{y}(t) + \ddot{w}(x,t))dx + m(\ddot{y}(t) + \ddot{w}(L,t)) \quad (14)$$

And substituting (9) into the (14):

$$(M + m + \rho L)\ddot{y}(t) + \sum_{j=1}^n (\rho \int_0^L \varphi_j(x)dx + m\varphi_j(L))\ddot{q}_j(t) = u(t) \quad (15)$$

The nominal model of the flexible structure can be obtained by rearranging (13), (15) in the matrix form, which includes the rigid mode and the first n flexible modes

$$\begin{pmatrix} M_{rr} & M_{rf} \\ M_{rf}^T & M_{ff} \end{pmatrix} \begin{pmatrix} \ddot{q}_r \\ \ddot{q}_f \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K_{ff} \end{pmatrix} \begin{pmatrix} q_r \\ q_f \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} \quad (16)$$

where

$$\begin{aligned} q_r(t) &= q_0(t) = y(t) \\ q_f(t) &= [q_1(t)\ldots q_n(t)]^T \\ M_{rr} &= M + m + \rho L \\ [M_{rf}]_j &= \rho \int_0^L \varphi_j(x)dx + m\varphi_j(L) \\ [M_{ff}]_{ij} &= \delta_{i,j} \\ [K_{ff}]_{ij} &= \omega_i^2\delta_{i,j} \end{aligned}$$

When the damping is included in the physical system, the system becomes

$$M\ddot{q} + C\dot{q} + Kq = \begin{pmatrix} u \\ 0 \end{pmatrix} \quad (17)$$

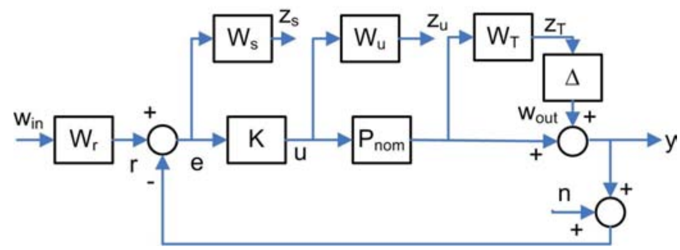

Figure 2. CONTROL SYNTHESIS BLOCK DIAGRAM.

where $\begin{pmatrix} M_{rr} & M_{rf} \\ M_{rf}^T & M_{ff} \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & C_{ff} \end{pmatrix}$, $K = \begin{pmatrix} 0 & 0 \\ 0 & K_{ff} \end{pmatrix}$ are the mass matrix, damping matrix, and stiffness matrix respectively, and $C_{ff} = diag([2\xi_1\omega_1,\ldots,2\xi_n\omega_n])$, where $\xi_i \ll 1$ is the damping ratio.

3 CONTROL PROBLEM FORMULATION

The closed-loop configuration in the tip tracking control is shown in Figure 2. The transfer function P_{nom} represents the nominal linear time-invariant model of the flexible structure. K is the linear time-invariant controller. W_s, W_u, W_T are frequency weighting function, and W_r is the shaping filter that transforms the disturbance input w_{in} to the desired trajectory r. The e, u, y, n are the error signal, control input, measurable output, and sensor noise, respectively.

It is well known that the H_∞ performance is convenient for the robustness with parameter uncertainty and unmodelled dynamics, and meeting the frequency domain requirements such as bandwidth, low frequency gain, or roll off [10]. The control design problem in the flexible structure may be well casted into the H_∞ framework of the control optimization problem, as follows.

The nominal model from the modal analysis approach have numerous lightly damped flexible modes. As the high frequency modes are truncated, the nominal dynamical model always has some unmodelled uncertainties. It is desirable to make the closed-loop system robust stable with the unmodelled uncertainties. We consider the closed-loop configuration in the tip tracking control shown in Figure 2.

When the robust stability is concerned, the modeling error is represented by the frequency weighting W_T multiplying Δ, where $\|\Delta\|_\infty \le 1$. The frequency weighting $W_T(s)$ can be obtained from experiment [16] or physical requirement [15] to account for the unmodelled dynamics . It is well known [4] that closed-loop system is robust stable iff

$$\|W_T \frac{P_{nom}K}{1 + P_{nom}K}\|_\infty \le 1 \quad (18)$$

Besides the robust stability requirement, it is always desirable to maximize the disturbance rejection in the control bandwidth while keeping the control effort within certain limits. This can be done conveniently in the H_∞ framework by minimizing the H_∞ norm from input disturbance w_{in} to error signal z_s.

$$min(\|W_r W_s \frac{1}{1+P_{nom}K}\|_\infty) \quad (19)$$

while keeping the H_∞ norm from the input disturbance w_{in} to the weighted control effort z_u within certain limits. For example,

$$\|W_u W_r \frac{K}{1+P_{nom}K}\|_\infty \leq 1 \quad (20)$$

In addition, the effect of the sensor noise n on the measurement output needs to be minimized, since the sensor is never absolutely accurate or noise-free. This requirement can be handled by the H_2 design, which is useful to handle stochastic aspects including the measurement noise. In the H_2 framework, the sensor noise n is a realization of a unit variance white noise process. And the H_2 norm minimization problem is to minimize H_2 norm from the sensor noise n to the output z_s.

$$min(\|W_s \frac{1}{1+P_{nom}K}\|_2) \quad (21)$$

If there exist some linear controller K, it will guarantee that the average RMS power of z_s is minimized.

Different practical requirements will need different control designs. One approach such as H_2 control design can only satisfy H_2 norm requirement. There is no guarantee that H_∞ requirement can be met. If different norm constraints need to be satisfied at the same time, it is a multi-objective optimization problem. In the multi-objective control design, the controller will need to satisfy the performance index including the H_2 norm and H_∞ requirement. It means that the controller K is designed to minimize H_∞ norm in (18), (19) and H_2 norm in(21) . Before we solve the problem, the problem is transformed into state space form by linear fractional transformation (LFT) .

Linear fractional transformation forms the basis of the multi-objective control design. Figure 3 shows the standard multi-objective synthesis framework. The structural perturbation or uncertainties Δ is not included, as the robust stability problem will be implicitly solved via enforcing the robust stability performance requirement 18 in the following controller design. It is claimed [3] that any linear interconnections of inputs, outputs, commands, perturbations, and controller can be rearranged to match the diagrams in Figure 3. It is easy to transform the system into the state space form with software package [8]. Consider the closed-loop system in Figure 3, in which the generalized plant P

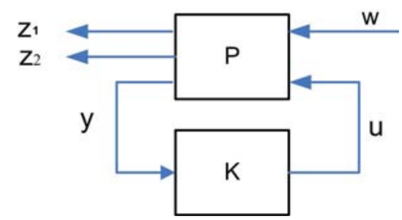

Figure 3. MULTIOBJECTIVE H_2/H_∞ SYNTHESIS .

is causal and linearly connected, K is the linear controller. In the multi-objective H_2/H_∞ synthesis configuration, the plant P is a state space system:

$$P: \begin{cases} \dot{x} = Ax + B_1 w + Bu \\ z_1 = C_1 x + D_1 w + E_1 u \\ z_2 = C_2 x + D_2 w + E_2 u \\ y = Cx + Fw \end{cases} \quad (22)$$

where $x \in \Re^{n_p}$ is the plant state including the state from nominal plant P_{nom} and weighting function W_r, W_s, W_T, W_u. And $z_1 \in \Re^{n_{z_1}}$, $z_2 \in \Re^{n_{z_2}}$ are the objective signals related to the robust stability and performance requirements of the control system, $y \in \Re^{n_y}$ is measurement output, $w \in \Re^{n_w}$ is the exogenous input including the input disturbance w_{in}, output disturbance w_{out} and the sensor noise n, and $u \in \Re^{n_u}$ is the control input. It is assumed that the dimensions are compatible in all of the above matrices. It should be noted that the frequency dependent weighting W_T has been included into the above formulation. In the above specific configuration, the performance output is defined as

$$z_1 = \begin{pmatrix} z_s \\ z_u \\ z_T \end{pmatrix}, \qquad z_2 = \begin{pmatrix} z_s \\ z_u \end{pmatrix} \quad (23)$$

In the control design, the objective is to design the output feedback controller **K** to satisfy the robust stability and performance requirement, subject to the constraint that the closed-loop system is internally stable.

$$K: \begin{cases} \dot{\xi} = A_k \xi + B_k y \\ u = C_k \xi + D_k y \end{cases} \quad (24)$$

where $\xi \in \Re^{n_c}$ is the controller state, $u \in \Re^{n_u}$, is the control input, that satisfies the closed-loop system specification. Then the closed-loop system satisfies our intent of the H_∞ robust stability from w to z_1 and the H_2 performance from w to z_2.

4 MULTI-OBJECTIVE OPTIMIZATION PROBLEM

To begin with, we review the H_2 and H_∞ design problem in the state space form. In the H_2 design, the optimization problem is

$$\min(\gamma_2) \tag{25}$$

where $\gamma_2 = \|T_{w\to z_2}\|_2$

Similarly, the H_∞ design problem is

$$\min(\gamma_1) \tag{26}$$

where $\gamma_1 = \|T_{w\to z_1}\|_\infty$

The so-called multi-objective H_2/H_∞ control problems is concerned in this paper. The goal is to design a controller K such that the H_∞ norm from $w \to z_1$ and H_2 norm from $w \to z_2$ satisfy the following constraints

$$\begin{cases} \|T_{w\to z_1}\|_\infty \le \gamma_1 \\ \|T_{w\to z_2}\|_2 \le \gamma_2 \end{cases} \tag{27}$$

As a typical control application of flexible structure, one may optimize the H_2 performance requirement while the controller satisfy the robust stability constraint. that is

$$\min(\gamma_2) \tag{28}$$

where $\gamma_2 = \|T_{w\to z_2}\|_2$, with the constraint that

$$\|T_{w\to z_1}\|_\infty \le 1$$

If there exists solution for the controller in the above formulation, the H_∞ norm may be close to 1. If the H_∞ norm is too close to 1, the system may be unstable when implementing the controller. In order to determine the trade-off between H_2 and H_∞ norm requirements, we reformulate the above functional as

$$\min(\alpha_1\gamma_1^2 + \alpha_2\gamma_2^2) \tag{29}$$

such that

$$\|T_{w\to z_2}\|_2 \le \gamma_2 \text{ and } \|T_{w\to z_1}\|_\infty \le \gamma_1$$

where $\alpha_1 \ge 0, \alpha_2 \ge 0$ are given and satisfy $\alpha_1 + \alpha_2 = 1$. The problem becomes the well known multi-objective optimization problem. The constraint in α_1 and α_2 is for the normalization purpose. For different α_1 and α_2, one can determine α_1 and α_2 through the Pareto optimal controllers [2, 9]. For simplicity, in this paper it is assumed that α_1 and α_2 are given.

5 OUTPUT FEEDBACK USING LMI

With the nominal model of the structure and the specification of the closed-loop system, it is possible to follow the LMI synthesis and analysis technique to design dynamical output feedback controller to satisfy the robust stability and performance requirement under the multi-objective optimization framework.

Linear matrix inequality (LMI) has been accepted as the powerful computation tool for the control and optimization problem recently [5, 10]. Most known control problems with analytic solution such as ARE can be transferred into LMI [5]. The well formulated LMI is a convex problem and can be solved with polynomial-time worst case complexity [5, 11]. In the control community, it is regarded as a practical solution by transferring the control design problems into LMI if analytical solutions do not exist or are too difficult to find. In the scenario that analytical solutions exist, the LMI solution can be used to verify the analytical solution by numerical computation.

In the LMI analysis problem with the output feedback controller K , the closed-loop system in state space form is

$$\begin{cases} \dot{x}_c = A_c x_c + B_c w \\ z_1 = C_{1c} x_c + D_{1c} w \\ z_2 = C_{2c} x_c + D_{2c} w \end{cases} \tag{30}$$

where $x_c \in \Re^{n_p+n_c}$ is the closed-loop state.

In the following, the feasibility condition for the H_2 and H_∞ are given in [5, 7, 8, 11, 16]. The results can also be easily derived by computing $\frac{dV}{dt} < 0$ where $V(x) = x^T P^{-1} x$ and $P > 0$ for $\forall x, w$.

For example, the system $\dot{x} = Ax$ is stable iff there exists $P > 0$ such that $A^T P + PA < 0$ is valid for $\forall x$. In the LMI framework, it means that we want to find the feasibility solution of $A^T P + PA < 0$ with $P = P^T > 0$.

To begin with our derivation on the H_2 and H_∞ synthesis condition, two lemmas are reviewed [5] [8].

Lemma 1. *The closed loop norm H_2 from $w \to z_2$ is less than γ_2 if there exists positive symmetric P_2 and Q such that the following LMI are feasible.*

$$\begin{cases} \begin{pmatrix} A_c P_2 + P_2 A_c^T & B_c \\ B_c^T & -I \end{pmatrix} < 0 \\ \begin{pmatrix} Q & C_{2c} P_2 \\ P_2 C_{2c}^T & P_2 \end{pmatrix} > 0 \\ Trace(Q) < \gamma_2^2 \\ D_{2c} = 0 \end{cases} \tag{31}$$

Lemma 2. *The closed loop norm H_∞ from $w \to z_1$ is less than γ_1 if there exists positive symmetric P_∞ such that the following*

LMI are feasible.

$$\begin{pmatrix} A_c P_\infty + P_\infty A_c^T & B_c & P_\infty C_{1c}^T \\ B_c^T & -I & D_{1c}^T \\ C_{1c} P_\infty & D_{1c} & -\gamma_1^2 I \end{pmatrix} < 0 \quad (32)$$

If trying to solve the general multi-objective problems in (29) and to derive the synthesis inequalities, there would have some trouble since the controller parameters nonlinearly depends on the closed loop Lyapunov matrices [10, 11, 16]. In (31) and (32) , two matrices P_2 ,P_∞ in synthesis conditions are involved. The feasibility problem for the multi-objective optimization is nonlinear and non-convex in general. Therefore, it would be difficult, if not impossible, to solve numerically by convex optimization approach like LMI techniques. In order to recover the convexity to make the computation tractable, one possible approach is to add additional artificial constraint that these two Lyapunov matrices are equal, which means that a single Lyapunov function $V(x) = x^T P^{-1} x$ where $P \geq 0$ is chosen. i.e.,

$$P = P_2 = P_\infty \geq 0 \quad (33)$$

The approach in (33) to solve the multi-objective optimization problem has been widely used in the state feedback and the output feedback. The problem is easy to solve by the available software package like LMI toolbox [8].

Remark 1. *From the optimization point of view, the controller would indeed be a suboptimal one since the artificial constraint in (33) is added. Hence, this approach would certainly generate some conservatism. How to justify the conservatism is an open issue up to date [10,11].*

Remark 2. *It should be noted that the optimal values of γ_1 and γ_2 are the upper bound of the H_∞ and H_2 norm, respectively. After implementing the controller from the above approach, the norms of H_2 and H_∞ should be analyzed and directly computed with different Lyapunov matrix without any conservatism [10,11].*

In summary, the multi-objective optimization problem with artificial constraint $P_2 = P_\infty$ in (33) is

$$\min(\alpha_1 \gamma_1^2 + \alpha_2 \gamma_2^2) \quad (34)$$

with

$$\begin{gathered} \|T_w \to z_1\|_\infty \leq \gamma_1 \\ \|T_w \to z_2\|_2 \leq \gamma_2 \\ \alpha_1 \geq 0 \text{ and } \alpha_2 \geq 0 \\ \alpha_1 + \alpha_2 = 1 \end{gathered}$$

In order to solve the multi-objective optimization problem, the H_2 and H_∞ norm constraint (32,31) should be transformed such that the new LMI will depend on the controller parameters explicitly and linearly. These ideas has been found recently by applying a series of congruence transformation and nonlinear transformation [8, 10, 11].

Following the same idea [8, 11], one can prove that through these following transformations, the original H_2 and H_∞ in (31) and (32) , which depend non-linearly on the Lyapunov matrix P and $\begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ are transformed into new LMIs that are affine to the new variables of R, S, A_k, B_k, C_k, D_k only.

$$\left(P, \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}\right) \to v = \left(R,\ S,\ \begin{pmatrix} K & L \\ M & N \end{pmatrix}\right) \quad (35)$$

$$P \to \mathbf{P}(v) := \begin{pmatrix} R & I \\ I & S \end{pmatrix} \quad (36)$$

$$\begin{pmatrix} A_c P & B_c(v) \\ C_{1c} P & D_1 c(v) \end{pmatrix} \to \begin{pmatrix} \mathbf{A}(v) & \mathbf{B}(v) \\ \mathbf{C}_1(v) & \mathbf{D}_1(v) \end{pmatrix} \quad (37)$$

$$\begin{pmatrix} A_c P & B_c(v) \\ C_{2c} P & D_2 c(v) \end{pmatrix} \to \begin{pmatrix} \mathbf{A}(v) & \mathbf{B}(v) \\ \mathbf{C}_2(v) & \mathbf{D}_2(v) \end{pmatrix} \quad (38)$$

where

$$\begin{pmatrix} \mathbf{A}(v) & \mathbf{B}(v) \\ \mathbf{C}_1(v) & \mathbf{D}_1(v) \end{pmatrix} = \begin{pmatrix} AR + BM & A + BNC & B_1 + BNF \\ K & SA + LC & SB_1 + LF \\ C_1 R + E_1 M & C_1 + E_1 NC & D_1 + E_1 NF \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}(v) & \mathbf{B}(v) \\ \mathbf{C}_2(v) & \mathbf{D}_2(v) \end{pmatrix} = \begin{pmatrix} AR + BM & A + BNC & B_1 + BNF \\ K & SA + LC & SB_1 + LF \\ C_2 R + E_2 M & C_2 + E_2 NC & D_2 + E_2 NF \end{pmatrix}$$

It is easy to derive the corresponding H_2 and H_∞ synthesis condition by substituting the synthesis condition in (31) and (32) dependent on controller K and Lyapunov matrix P with the new variables R, S, K, L, M, N dependent on the new variable v with the above transformation [5, 8, 11]. After the new variable v is known, one can find the controller parameters through inverse transformation [11]

$$\begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} = \begin{pmatrix} U & SB \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K - SAR & L \\ M & N \end{pmatrix} \begin{pmatrix} V^T & 0 \\ CR & N \end{pmatrix} \quad (39)$$

where U, V are nonsingular matrix with $SR + UV^T = I$

In the above approach, the Lyapunov matrices of P_2 and P_∞ are artificially set to be equal in (33) in order to make the problems solvable, which certainly leads to some conservatism. In order to reduce the conservatism, one possible approach is to

Table 1. System specification of the flexible structure

Properties	Symbol	Value
Young Modulus	E	$71.7 \times 10^9 N/m^2$
Thickness	h	2.3mm
Width	b	39.8mm
Length	L	300mm
Linear density	ρ	0.2563Kg/m
Base Mass	M	2.7Kg
Payload at the tip	m	0 till 0.5kg

scale the Lyapunov matrix in the multi-objective problem such that $P = P_2 = \alpha P_\infty > 0$ where $\alpha > 0$ [11, 16].

In the H_∞ synthesis condition , one can multiply both sides by α. Then the H_∞ constraint (32) becomes

$$\begin{pmatrix} A_c(\alpha P_\infty) + (\alpha P_\infty)A_c^T & \alpha B_c & (\alpha P_\infty)C_{1c}^T \\ \alpha B_c^T & -\alpha I & \alpha D_{1c}^T \\ C_{1c}(\alpha P_\infty) & \alpha D_{1c} & -\alpha\gamma_1^2 I \end{pmatrix} < 0 \quad (40)$$

Apply a congruence transformation through multiplying the above linear matrix inequality by $\frac{1}{\alpha}$ on second row and second column, the H_∞ synthesis condition becomes by noting that $P_2 = \alpha P_\infty$

$$\begin{pmatrix} A_c P_2 + P_2 A_c^T & B_c & P_2 C_{1c}^T \\ B_c^T & -\frac{1}{\alpha} I & D_{1c}^T \\ C_{1c} P_2 & D_{1c} & -\alpha\gamma_1^2 I \end{pmatrix} < 0 \quad (41)$$

Then the optimal value γ_1 and γ_2 can be computed for different α by following the same congruence transformation as before. For any given α, these values γ_1 and γ_2 are the upper bound on the actual optimal value [11]. Since the optimization problem is a single degree freedom of the multi-objective optimization problem, it is possible to perform linear search to find the best bound while the convexity of the problem is preserved [11, 16].

6 SIMULATION

The multi-objective optimization approach is applied to the flexible structure in Figure 1. The geometric and material properties given in Table 1 are used to do the simulation. The nominal model is derived from mode-truncation approach with one rigid mode and two flexible modes. And the desired tip trajectory is a set-point value of $p(L,t) = 1m$.

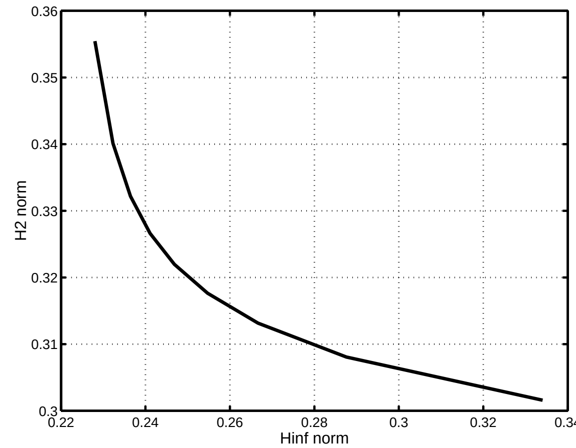

Figure 4. MULTI-OBJECTIVE TRADEOFF CURVE.

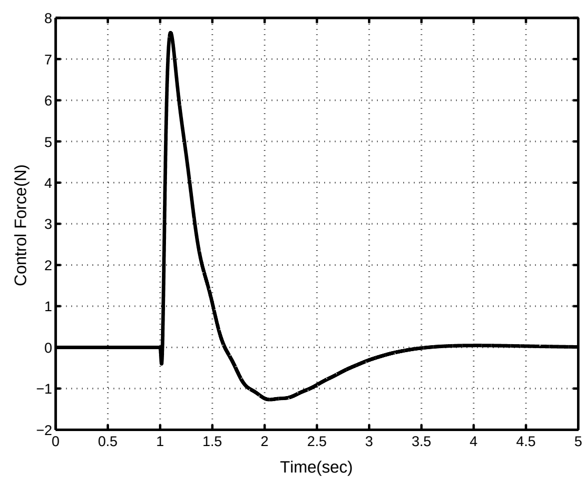

Figure 5. CONTROL FORCE $u(t)$.

In the H_2 and H_∞ control design problem, the weighting selection is very important in the control schematic (2) . For the non-minimum phase system, the time delay from the actuator at the base to the tip is taken into account by the all pass filter ($W_f = \frac{-0.01s+1}{0.01s+1}$). In the tip tracking, the reference model or the ideal complementary transfer function is specified as $T_{id}(s) = \frac{25}{s^2+10s+25}$, which satisfies the settling time and overshoot requirement for the step response. So $W_r(s) = W_f(s)T_{id}(s)$. In the H_2 or H_∞ framework, the interesting output of z_1 and z_2 should include the weighted control input z_u to make the optimization problem well posed. Otherwise, it is like the cheap control strategy with unbounded control input. For simplicity, the weighting function for control input is chosen as $W_u(s) = 0.001$. Since the ideal sensitivity function $S_{id}(s) = 1 - T_{id}(s) = \frac{s(s+10)}{s^2+10s+25}$, we choose the weighting function $W_s(s) = S_{id}(s)^{-1}$,i.e., $W_s(s) = \frac{s^2+10s+25}{s(+10)}$.

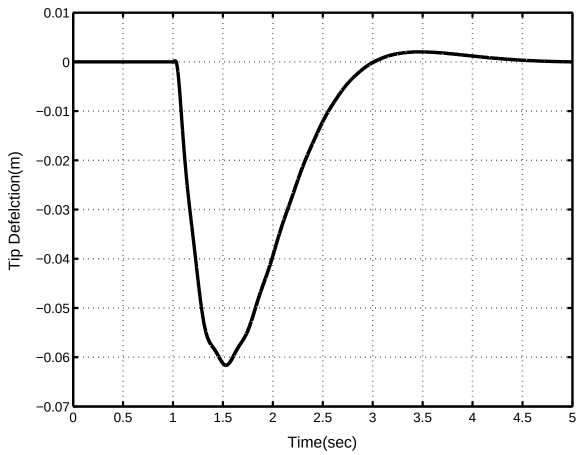

Figure 6. FLEXIBLE BEAM TIP DEFLECTION $y(L,t)$.

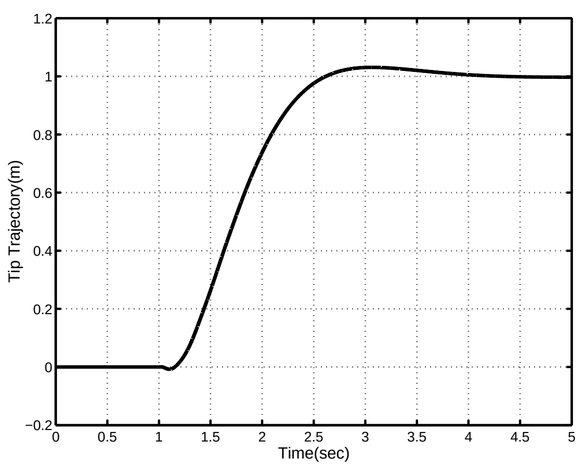

Figure 7. FLEXIBLE BEAM TIP TRAJECTORY $p(L,t)$.

In the control problem setup, $W_s(s)$ need to be strictly stable. So we choose $W_s(s) = 0.09\frac{s^2+10s+25}{(s+0.01)(s+10)}$ with gain adjustment. The weighting function $W_T(s)$ is used to account for the high frequency unmodelled dynamics beyond the first two flexible modes. In our case, the weighting is chosen as $W_T(s) = 0.5\frac{s^2+60s+864}{s^2+1200s+600^2}$. For more information on how to choose $W_T(s), W_u(s), W_s(s)$ and some robust feedback problem setup, see [4, 17].

After the control problem is formulated, the H_2 and H_∞ performance are computed. The optimal H_2 norm alone is $\gamma_2 = 0.286$ and the optimal H_∞ norm alone is $\gamma_1 = 0.219$. In the multi-objective optimization problem (34) , the weight factors $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$ are determined approximately from the Pareto-like multiobjective tradeoff curve in Figure 4. The curve is derived by computing the γ_1 and γ_2 in the multi-objective optimization problem for each combination of α_1 and α_2, provided that $P_2 = P_\infty$. After the weighing factor α_1 and α_2 are given, the output feedback control law is designed in the multi-objective optimization framework. The corresponding optimal γ_1 and γ_2 are 0.241 and 0.327, respectively. In order to reduce the conservatism, the best scaling factor α is linearly searched from 0.1 to 10 with step equal to 0.1. Of course, some initial guess from the logarithmical search may help focus on the most possible minimum value for α. With $P_2 = \alpha P_\infty$ as additional artificial constraint in the multi-objective optimization problem (34) , the optimal value is achieved at $\alpha = 0.7$ with $\gamma_1 = 0.247$ and $\gamma_2 = 0.305$. The index value is reduce from 7.76×10^{-2} to 7.37×10^{-2}.

For the illustration purpose, the flexible tip deflection $y(L,t)$ and tip trajectory $p(L,t)$ are shown in Figure 6 and Figure 7. It can be seen in Figure 7 that the control law achieves fast and oscillation free tip tracking performance. In addition, the control force in Figure 5 are bounded within reasonable range.

7 CONCLUSION

In this paper, the multi-objective optimization has been proposed for the tip tracking control on the flexible structure. The dynamical model of the structure is obtained through the modal analysis. The discrepancies between the dynamical model and the "real structure" is treated as output disturbance in the H_∞ framework. The robust stability and tracking error minimization from the sensor noise are transferred into the H_∞ and H_2 norm constraint respectively. Then the control problem is formulated as multi-objective optimization problem and solved using LMI techniques. These LMI are transformed to be affine with new controller variables by a series of nonlinear transformation and congruence transformation. In order to reduce the conservatism generated by forcing all of the Lyapunov matrices to be equal, additional freedom is sought through scaling the Lyapunov matrix. The design results has demonstrated favorable tip tracking control from multi-objective optimization approach using LMI.

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