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Observer-based adaptive robust control of a class of nonlinear systems with dynamic uncertainties[‡]

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SUMMARY

In this paper, a discontinuous projection-based adaptive robust control (ARC) scheme is constructed for a class of nonlinear systems in an extended semi-strict feedback form by incorporating a nonlinear observer and a dynamic normalization signal. The form allows for parametric uncertainties, uncertain nonlinearities, and dynamic uncertainties. The unmeasured states associated with the dynamic uncertainties are assumed to enter the system equations in an affine fashion. A novel nonlinear observer is first constructed to estimate the unmeasured states for a less conservative design. Estimation errors of dynamic uncertainties, as well as other model uncertainties, are dealt with effectively via certain robust feedback control terms for a guaranteed robust performance. In contrast with existing conservative robust adaptive control schemes, the proposed ARC method makes full use of the available structural information on the unmeasured state dynamics and the prior knowledge on the bounds of parameter variations for high performance. The resulting ARC controller achieves a prescribed output tracking transient performance and final tracking accuracy in the sense that the upper bound on the absolute value of the output tracking error over entire time-history is given and related to certain controller design parameters in a *known* form. Furthermore, in the absence of uncertain nonlinearities, asymptotic output tracking is also achieved. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: adaptive control; robust control; observer; nonlinear systems

1. INTRODUCTION

Almost every physical system is subject to certain degrees of model uncertainties, which makes the design of high-performance control algorithms a very challenging job. Normally, the causes of model uncertainties can be classified into two categories: (i) repeatable or constant unknown quantities such as the unknown physical parameters (e.g., the inertia load of any industrial drive systems) and (ii) non-repeatable unknown quantities such as external disturbances and imprecise

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modelling of certain physical terms. To account for these uncertainties, two nonlinear control methods, the deterministic robust control (DRC) [1–4] and the adaptive control (AC) [5–7] (or its robustified version, the robust adaptive control [8–11]), may apply. In general, the deterministic robust controllers can achieve a guaranteed transient and final tracking accuracy in the presence of both parametric uncertainties and uncertain nonlinearities. However, since no attempt is made to learn from past behaviour to reduce the effect of constant unknown quantities, the design is conservative and involves either switching [1] or infinite-gain [4] feedback for asymptotic tracking; both means are impractical and unattainable. In contrast, the adaptive controllers [5–7] are able to achieve asymptotic tracking in the presence of parametric uncertainties without resorting to infinite gain feedback. Recently, as in the robust adaptive control (RAC) of linear systems, much of the effort in nonlinear adaptive control area has been devoted to robustifying the adaptive backstepping designs [5] with respect to disturbances and significant progress has been made [8–11].

In References [12–15], an adaptive robust control (ARC) approach has been proposed for the design of a new class of high-performance robust controllers. By exploiting practically reasonable prior information on a physical system such as the bounds of parameter variations as much as possible, the approach effectively combines the design methods of DRC and AC. The resulting ARC controllers achieve the results of both DRC and AC while naturally overcoming their practical limitations. The approach has been applied to various applications and comparative experimental results [16–18] have verified the effectiveness and the high-performance nature of the proposed ARC strategy; for different applications [16, 17], tracking errors have been consistently reduced almost down to measurement resolution level during most of the execution period.

In addition to parametric uncertainties and uncertain nonlinearities, some systems are further subjected to *dynamic uncertainties* that depend on the *unmeasured* states of exogenous dynamic systems [19, 20]. Practical examples include the dynamic friction model in Reference [21] and the control of eccentric rotor in References [19, 22]. This class of systems have received a lot of attention in recent years since there are few results available on the general problem of robust control of nonlinear systems with partial state feedback. In Reference [19], Freeman and Kokotović constructed an adaptive controller for a class of extended strict feedback nonlinear systems in which the unmeasured states enter the systems in a linear affine fashion. As pointed out in the paper, it is unclear how the approach can be made robust to modelling errors such as uncertain nonlinearities. In Reference [23], Jiang and Praly generalized the idea of using a dynamic signal to dominate dynamic disturbances to the robust adaptive control of nonlinear systems subjected to dynamic uncertainties. Using a small-gain type of argument, they also presented in [20] a modified robust adaptive control (RAC) procedure [9] for a class of uncertain nonlinear systems subject to dynamic uncertainties satisfying an input-to-state stability property.

In this paper, by incorporating a nonlinear observer design and the dynamic signal introduced in Reference [20], the discontinuous projection-based ARC approach [12] will be extended to a class of nonlinear systems subjected to parametric uncertainties, uncertain nonlinearities, and dynamic uncertainties. The motivation for this research is that, in most situations, the structural information on the unmeasured state dynamics and on the way it interacts with the rest of the system dynamics is known. If these prior structural information can be utilized effectively, a less conservative controller can be synthesized and a better performance can be achieved. With this fact in mind, departing from Reference [20], in the absence of uncertain nonlinearities, the nominal structure of the unmeasured state dynamics are assumed to be known and the unmeasured states are assumed to enter the system equations in a linear affine fashion as in Reference [19]. With a mild assumption which is equivalent to the detectability assumption in linear systems, a nonlinear observer is first constructed to recover the unmeasured states of dynamic uncertainties. By doing so, the estimates of the unmeasured states can be used in the controller design to eliminate the effect of dynamic uncertainties for an improved achievable nominal performance—asymptotic tracking is obtained in the presence of parametric uncertainties and dynamic uncertainties. In addition, in the presence of uncertain nonlinearities, the observer error dynamics is made to be input-to-state stable. Estimation errors, as well as model uncertainties, are dealt with effectively via certain robust feedback as in the ARC design in Reference [12] to achieve a guaranteed robust performance. Consequently, the contributions of the paper are as follows. Firstly, compared to the adaptive control design [19] where uncertain nonlinearities are not considered, the proposed scheme is observer based and is robust to uncertain nonlinearities; in fact, a guaranteed robust performance is obtained even in the presence of uncertain nonlinearities. Secondly, compared to the RAC approach [20], the proposed scheme explicitly utilizes the structural information of the system and a better nominal performance, asymptotic tracking, is obtained in the absence of uncertain nonlinearities. In addition, the approach puts more emphasis on the robust control law design; in fact, the parameter adaptation law in the proposed ARC design can be switched off at any time without affecting global stability or sacrificing the prescribed transient performance result, since the resulting 'non-adaptive' controller becomes a deterministic robust controller. Because of this design philosophy, the proposed controller achieves a prescribed transient performance and final tracking accuracy, i.e. the upper bound on the absolute value of the tracking error over the entire time-history is given and related to certain controller design parameters in a *known* form, which is much more transparent than that in RAC design [20]. Lastly, the nonlinear observer design in this paper is motivated by the recent excellent research done in Reference [24]. However, in Reference [24], the effect of parametric uncertainties and uncertain nonlinearities are not considered. Thus, the paper also extends the nonlinear observer design.

The paper is organized as follows: Problem statement is presented in Section 2. A nonlinear state observer design is presented in Section 3. The proposed observer-based ARC controller is shown in Section 4. A design example and comparative simulation results are presented in Section 5, and conclusions are drawn in Section 6.

2. PROBLEM STATEMENT

The following nonlinear system will be considered in this paper:

$$\begin{aligned}
\dot{\eta} &= F_\eta(\bar{x}_l)\theta + G_\eta(\bar{x}_l)\eta + \bar{\Delta}(x,\eta,u,t) \\
\dot{x}_i &= x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i}(\bar{x}_i) + \Delta_i(x,\eta,u,t), \quad 1 \leqslant i \leqslant l-1 \\
\dot{x}_i &= x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i}(\bar{x}_i) + \varphi_{\eta i}^{\mathrm{T}}(\bar{x}_i)\eta + \Delta_i(x,\eta,u,t), \quad l \leqslant i \leqslant n-1 \\
\dot{x}_n &= u + \theta^{\mathrm{T}}\varphi_{\theta n}(x) + \varphi_{\eta n}^{\mathrm{T}}(x)\eta + \Delta_n(x,\eta,u,t) \\
y &= x_1
\end{aligned} \tag{1}$$

where $\bar{x}_l = [x_1, \ldots, x_l]^{\mathrm{T}} \in \mathbb{R}^l$, $\bar{x}_i = [x_1, \ldots, x_i]^{\mathrm{T}} \in \mathbb{R}^i$, and $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$. $y \in \mathbb{R}$ and $u \in \mathbb{R}$ are the control input and the output, respectively. $\eta \in \mathbb{R}^m$ represents the unmeasured states, and

$\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters. $F_\eta \in \mathbb{R}^{m\times p}$, $G_\eta \in \mathbb{R}^{m\times m}$, $\varphi_{\theta i} \in \mathbb{R}^p$ and $\varphi_{\eta i} \in \mathbb{R}^m$ are matrices or vectors of known smooth functions, which are used to describe the nominal model of the system. $\bar{\Delta}$ and Δ_i represent the lumped unknown nonlinear functions such as disturbances and modelling errors.

Throughout the paper, the following notations will be used. In general, $\bullet_i$ represents the ith component of the vector $\bullet$ and the operation $<$ for two vectors is performed in terms of the corresponding elements of the vectors. The following practical assumptions are made:

Assumption 1

The extent of parametric uncertainties and uncertain nonlinearities are known. In other words, parametric uncertainties θ and uncertain nonlinearities $\bar{\Delta}$ and Δ_i satisfy

$$\begin{aligned} \theta \in \Omega_\theta &\triangleq \{\theta\colon \theta_{\min} < \theta < \theta_{\max}\} \\ \bar{\Delta} \in \bar{\Omega}_\Delta &\triangleq \{\bar{\Delta}\colon |\bar{\Delta}(x,\eta,u,t)| \leqslant \bar{\delta}(\bar{x}_l)\} \\ \Delta_i \in \Omega_\Delta &\triangleq \{\Delta_i\colon |\Delta_i(x,\eta,u,t)| \leqslant \delta_i(\bar{x}_i)\} \end{aligned} \tag{2}$$

where $\theta_{\min}$, $\theta_{\max}$, $\bar{\delta}(\bar{x}_l)$ and $\delta_i(\bar{x}_i)$ are known.[§] ($|\cdot|$ denotes the usual Euclidean norm.)

Assumption 2

The η-subsystem, $\dot{\eta} = F_\eta(\bar{x}_l)\theta + G_\eta(\bar{x}_l)\eta + \bar{\Delta}(x,\eta,u,t)$, with η as the state and $\bar{x}_l(t)$ as the input, is bounded-input-bounded-state stable in the sense that for every $\eta_0 \in \mathbb{R}^m$ and every $\bar{x}_l(t) \in L_\infty^l[0,\infty)$, the solution $\eta(t)$ starting from the initial condition η_0 is bounded, i.e. $\eta(t) \in L_\infty^m[0,\infty)$.

Let $y_d(t)$ be the desired output trajectory, which is assumed to be known, bounded with bounded derivatives up to nth order. The objective is to synthesize a control input u such that the output y tracks $y_d(t)$ as closely as possible in spite of various model uncertainties.

Before leaving this section, we would like to make the following remark concerning the connection and the difference between system (1) and the systems extensively studied in the literature.

Remark 1

(i) In the absence of dynamic uncertainties, i.e. if there are no unmeasured state η in (1), system (1) with Assumption 1 reduces to the semi-strict feedback form studied in References [12, 15], where a discontinuous projection based ARC scheme has been presented for a guaranteed output tracking transient performance and final tracking accuracy [12]. (ii) If there are no uncertain nonlinearities, i.e., $\bar{\Delta} = 0$, and $\Delta_i = 0$, $\forall i$, the system (1) with Assumption 2 reduces to the extended strict feedback form studied by Freeman and Kokotović [19], where an adaptive controller was constructed. (iii) In Reference [20], Jiang and Praly considered a similar system which allows uncertain nonlinearities to depend on nonlinearly appearing parametric uncertainties also. However, the η-dynamics is assumed to be input-to-state practically stable, which is more stringent that Assumption 2. Also, the nominal structure of the η-dynamics is ignored.

§The design can be extended to the case where $\bar{\delta}$ and δ_i depend on t explicitly.

3. STATE ESTIMATION

Since η states are not measurable, a nonlinear observer need to be constructed to provide their estimates. Motivated by Reference [24], we first introduce a transformation of co-ordinate. Define a vector

$$\xi = \eta - \omega(\bar{x}_l) \tag{3}$$

where $\omega(\bar{x}_l) = [\omega_1(\bar{x}_l), \ldots, \omega_m(\bar{x}_l)]^{\mathrm{T}}$ is the vector of design functions yet to be determined. Its derivative is computed as

$$\begin{aligned}
\dot{\xi} &= \dot{\eta} - \dot{\omega}(\bar{x}_l) \\
&= F_\eta(\bar{x}_l)\theta + G_\eta(\bar{x}_l)\eta + \bar{\Delta} - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}(x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i} + \Delta_i) - \frac{\partial \omega}{\partial x_l}\varphi_{\eta l}^{\mathrm{T}}\eta \qquad (4)\\
&= F_\eta(\bar{x}_l)\theta + \left(G_\eta(\bar{x}_l) - \frac{\partial \omega}{\partial x_l}\varphi_{\eta l}^{\mathrm{T}}\right)\eta - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}(x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i}) + \bar{\Delta} - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}\Delta_i
\end{aligned}$$

For simplicity, let

$$A(\bar{x}_l) = G_\eta(\bar{x}_l) - \frac{\partial \omega}{\partial x_l}\varphi_{\eta l}^{\mathrm{T}} \tag{5}$$

Substituting (3) and (5) into (4), we have

$$\dot{\xi} = A(\bar{x}_l)(\xi + \omega(\bar{x}_l)) + F_\eta(\bar{x}_l)\theta - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}(x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i}) + \bar{\Delta} - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}\Delta_i \tag{6}$$

If θ were known, we would design a nonlinear observer

$$\dot{\hat{\xi}} = A(\bar{x}_l)(\hat{\xi} + \omega(\bar{x}_l)) + F_\eta(\bar{x}_l)\theta - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}(x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i}) \tag{7}$$

Then, the state estimation error $\varepsilon = \hat{\xi} - \xi$ would be governed by the following dynamic system

$$\dot{\varepsilon} = A(\bar{x}_l)\varepsilon + \Delta_\varepsilon, \quad \Delta_\varepsilon = -\bar{\Delta} + \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}\Delta_i \tag{8}$$

Since θ is not known, observer (7) is not implementable but it provides motivation for the design of following nonlinear filters:

$$\begin{aligned}
\dot{\zeta}_0 &= A(\bar{x}_l)\zeta_0 + A(\bar{x}_l)\omega(\bar{x}_l) - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}x_{i+1} \\
\dot{\zeta}_j &= A(\bar{x}_l)\zeta_j + F_{\eta j}(\bar{x}_l) - \sum_{i=1}^{l} \frac{\partial \omega}{\partial x_i}\varphi_{\theta i,j}, \quad 1 \leqslant j \leqslant p
\end{aligned} \tag{9}$$

where $\zeta_j \in \mathbb{R}^m$, F_{nj} represents the jth column of F_η, and $\varphi_{\theta i,j}$ is the jth element of the vector $\varphi_{\theta i}$. The state estimate can, thus, be represented by

$$\hat{\xi} = \zeta_0 + \sum_{j=1}^{p} \zeta_j \theta_j = \zeta_0 + \zeta\theta \tag{10}$$

where $\zeta = [\zeta_1, \ldots, \zeta_p] \in \mathbb{R}^{m \times p}$. From (9) and (10), it can be verified that the observer error dynamics is still described by (8). Therefore, the equivalent expression for the unmeasurable state η is

$$\eta = \zeta_0 + \zeta\theta + \omega(\bar{x}_l) - \varepsilon \tag{11}$$

In viewing the observer error dynamics (8), the following additional assumption is made.

Assumption 3

(A) There exists an exp-ISpS Lyapunov function¶ V_ε for the observer error dynamics (8), i.e. there exists a Lyapunov function, V_ε such that

$$\begin{aligned} \gamma_1(|\varepsilon|) \leqslant V_\varepsilon(\varepsilon) \leqslant \gamma_2(|\varepsilon|), \quad \forall \varepsilon \in \mathbb{R}^m \\ \dot{V}_\varepsilon \leqslant -c_\varepsilon V_\varepsilon(\varepsilon) + \gamma_\varepsilon(|\bar{x}_l|) + d_\varepsilon \end{aligned} \tag{12}$$

where γ_1, γ_2 and γ_ε are class $\mathscr{K}_\infty$ functions, $c_\varepsilon > 0$ and $d_\varepsilon \geqslant 0$ are two constants.

(B) The unperturbed system of (8) is assumed to be exponentially stable, i.e., when $\Delta_\varepsilon = 0$, the observation error ε converges to zero exponentially.

Remark 2

In Reference [25], it is shown that Assumption 3(A) is equivalent to the assumption that the observer error dynamics (8) with ε as the state and $\bar{x}_l$ as the input is input-to-state practically stable (ISpS). A control system $\dot{x} = f(x, u)$ is ISpS if there exist a class $\mathscr{KL}$ function β, a class $\mathscr{K}$ function γ, and a non-negative constant d such that, for any initial condition $x(0)$ and each input $u \in \mathscr{L}_\infty[0, t)$, the corresponding solution $x(t)$ satisfies, for all $t \geqslant 0$,

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|u_t\|) + d \tag{13}$$

where u_t is the truncated function of u at t and $\|\cdot\|$ stands for the $\mathscr{L}_\infty$ supremum norm. In Reference [20], a similar assumption as Assumption 3(A) is made with respect to the dynamic uncertainties.

Remark 3

It should be noted that Assumption 3 is a quite mild assumption and is in fact a nonlinear generalization of the detectability of the unmeasured η-dynamics. To see this, let us assume that $\varphi_{\eta l}^{\mathrm{T}}$ is a constant output vector and G_η is constant matrix. It is proved in the following that *the detectability of the pair* $(\varphi_{\eta l}^{\mathrm{T}}, G_\eta)$ *implies Assumption 3.*

Since the pair $(\varphi_{\eta l}^{\mathrm{T}}, G_\eta)$ is assumed to be detectable, there exists a constant vector $L \in R^m$ such that

$$A_\eta = G_\eta - L\varphi_{\eta l}^{\mathrm{T}} \tag{14}$$

¶The ISpS notion was first introduced in Reference [25].

is Hurwitz. By choosing $\omega(\bar{x}_l) = Lx_l$, from (5), it follows that $A(\bar{x}_l) = A_\eta$. Hence, there exist two positive-definite matrices $P > 0$ and $Q > 0$ such that

$$PA_\eta + A_\eta^{\mathrm{T}} P = -Q \tag{15}$$

Let

$$V_\varepsilon(\varepsilon) = \varepsilon^{\mathrm{T}} P \varepsilon \tag{16}$$

Then, we have

$$\gamma_1(|\varepsilon|) \leqslant V_\varepsilon(\varepsilon) \leqslant \gamma_2(|\varepsilon|) \tag{17}$$

where $\gamma_1(|\varepsilon|) = \lambda_{\min}(P)|\varepsilon|^2$ and $\gamma_2(|\varepsilon|) = \lambda_{\max}(P)|\varepsilon|^2$. Noting Assumption 1, from (8), the derivative of $V_\varepsilon(\varepsilon)$ satisfies

$$\begin{aligned}\dot{V}_\varepsilon &\leqslant -\lambda_{\min}(Q)|\varepsilon|^2 + 2|\varepsilon|\lambda_{\max}(P)(|\bar{\Delta}| + |L||\Delta_l|) \\ &\leqslant -\lambda_{\min}(Q)|\varepsilon|^2 + 2|\varepsilon|\lambda_{\max}(P)(\bar{\delta} + |L|\delta_l) \\ &\leqslant -c_\varepsilon\lambda_{\max}(P)|\varepsilon|^2 + \frac{\lambda_{\max}^2(P)}{\lambda_{\min}(Q) - c_\varepsilon\lambda_{\max}(P)}(\bar{\delta} + |L|\delta_l)^2\end{aligned} \tag{18}$$

where $c_\varepsilon > 0$ is any positive scalar satisfying $c_\varepsilon < \lambda_{\min}(Q)/\lambda_{\max}(P)$. Noting that $\bar{\delta} + |L||\delta_l$ is a function of $\bar{x}_l$ and t only and is bounded with respect (w.r.t) to t, there exist a class $\mathscr{K}_\infty$ function $\gamma_\varepsilon(|\bar{x}_l|)$ and a positive constant d_ε such that $\gamma_\varepsilon(|\bar{x}_l|) + d_\varepsilon \geqslant \lambda_{\max}^2(P)(\bar{\delta} + |L|\delta_l)^2/\lambda_{\min}(Q) - c_\varepsilon\lambda_{\max}(P)$. Thus, (18) becomes

$$\dot{V}_\varepsilon \leqslant -c_\varepsilon V_\varepsilon(\varepsilon) + \gamma_\varepsilon(|\bar{x}_l|) + d_\varepsilon \tag{19}$$

which implies that Assumption 3(A) is satisfied. Since A_η is Hurwitz, B of Assumption (3) is also satisfied.

Remark 4

If the unmeasured state η is of dimension 1, i.e. $m = 1$, then, as long as $\varphi_{\eta l}(\bar{x}_l)$ is non-zero, a nonlinear $\omega(\bar{x}_l)$ can be explicitly constructed as follows. Let k_η be a positive design constant and choose $\omega(\bar{x}_l)$ to satisfy

$$-k_\eta = g_\eta(\bar{x}_l) - \frac{\partial\omega}{\partial x_l}\varphi_{\eta l}(\bar{x}_l) \tag{20}$$

The differential equation (20) has the following explicit solution:

$$\omega(\bar{x}_l) = \int_0^{x_l} \frac{g(\bar{x}_l) + k_\eta}{\varphi_{\eta l}(\bar{x}_l)}\,\mathrm{d}x_l \tag{21}$$

In such a case, $A(\bar{x}_l) = -k_\eta$. It is thus obvious that Assumption 3 is satisfied. Furthermore, the exponential convergence rate for the unperturbed observer error dynamics (8) can be arbitrarily placed.

The following lemma, which is proved in Reference [20], will be used in the subsequent ARC controller design.

Lemma 1[‖]

If (12) holds, then, for any constants $\bar{c} \in (0, c_\varepsilon)$, any initial condition $\varepsilon^0 = \varepsilon(0)$ and $r^0 > 0$, for any function $\bar{\gamma}$ such that $\bar{\gamma}(\bar{x}_l) \geqslant \gamma_\varepsilon(|\bar{x}_l|)$, there exists a finite $T^0 = T^0(\bar{c}, r^0, \varepsilon^0) \geqslant 0$, a nonnegative function $D(t)$ defined for all $t \geqslant 0$ and a dynamic signal described by

$$\dot{r} = -\bar{c}r + \bar{\gamma}(\bar{x}_l) + d_\varepsilon, \quad r(0) = r^0 \tag{22}$$

such that $D(t) = 0$ for all $t \geqslant T^0$ and

$$V_\varepsilon(\varepsilon) \leqslant r(t) + D(t) \tag{23}$$

for all $t \geqslant 0$ where the solutions are defined.

From (12) and (23), it follows that [20]

$$|\varepsilon(t)| \leqslant \gamma_1^{-1}(r(t) + D(t)) \leqslant \gamma_1^{-1}(2r(t)) + \gamma_1^{-1}(2D(t)) \tag{24}$$

4. DISCONTINUOUS PROJECTION-BASED ARC BACKSTEPPING DESIGN

4.1. Parameter projection

Let $\hat{\theta}$ denote the estimate of θ and $\tilde{\theta}$ the estimation error (i.e. $\tilde{\theta} = \hat{\theta} - \theta$). Under Assumption 1, a discontinuous projection-based ARC design [12] will be constructed to solve the robust tracking control problem for (1). Specifically, the parameter estimate $\hat{\theta}$ is updated through a parameter adaptation law having the form given by

$$\dot{\hat{\theta}} = \mathrm{Proj}_{\hat{\theta}}(\Gamma\tau) \tag{25}$$

where Γ is a symmetric positive-definite (s.p.d) diagonal adaptation rate matrix, τ is an adaptation function to be synthesized later, and the projection mapping $\mathrm{Proj}_{\hat{\theta}}(\cdot) = [\mathrm{Proj}_{\hat{\theta}_1}(\cdot_1), \ldots, \mathrm{Proj}_{\hat{\theta}_p}(\cdot_p)]^T$ is defined by Sastry and Bodson [26] and Goodwin and Mayne [27]

$$\mathrm{Proj}_{\hat{\theta}_i}(\cdot_i) = \begin{cases} 0 & \text{if} \quad \hat{\theta}_i = \theta_{i\max} \text{ and } \cdot_i > 0 \\ 0 & \text{if} \quad \hat{\theta}_i = \theta_{i\min} \text{ and } \cdot_i < 0 \\ \cdot_i & \text{otherwise} \end{cases} \tag{26}$$

It can be shown [13] that for any adaptation function τ, the projection mapping (26) guarantees

$$\begin{aligned} &\text{P1} \quad \hat{\theta} \in \bar{\Omega}_\theta = \{\hat{\theta} : \theta_{\min} \leqslant \hat{\theta} \leqslant \theta_{\max}\} \\ &\text{P2} \quad \tilde{\theta}^T(\Gamma^{-1}\mathrm{Proj}_{\hat{\theta}}(\Gamma\tau) - \tau) \leqslant 0, \quad \forall\tau \end{aligned} \tag{27}$$

4.2. ARC controller design

In this subsection, the ARC backstepping design [12] will be employed and extended to the present case with dynamic uncertainties. In the first $l - 1$ steps, the design procedure is the same

[‖]See Reference [20], Lemma 3.1.

as that in Reference [12], since the first $l-1$ steps do not involve the unmeasured dynamic uncertainties η. From step l, we will deal with the effect of the unmeasured state η by replacing it with its estimate for high performance. The effect of state estimation error will be dealt with via certain robust feedback terms through the incorporation of a dynamic normalization signal for a guaranteed robust performance.

4.2.1. Step $1 \leqslant i \leqslant l-1$. At step i, $\forall 1 \leqslant i \leqslant l-1$, we construct a control function α_i for the virtual input x_{i+1} such that x_i tracks its desired ARC control law α_{i-1} synthesized at step $i-1$ (for simplicity, denotes $\alpha_0(t) = y_d(t)$). Let $z_i = x_i - \alpha_{i-1}$, $\tilde{\Delta}_1(x, t) \doteq \Delta_1(x, t)$, $\phi_1(x_1) = \varphi_{\theta 1}(x_1)$, and recursively define the following functions:

$$\phi_i(\bar{x}_i, \hat{\theta}) = -\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_{\theta j} + \varphi_{\theta i}(\bar{x}_i),$$
$$\tilde{\Delta}_i = -\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \Delta_j + \Delta_i(x, t). \tag{28}$$

Then we have the following lemma, which is proved in Reference [12].

Lemma 2

At step i, $\forall 1 \leqslant i \leqslant l-1$, choose the desired control function α_i as

$$\alpha_i(\bar{x}_i, \hat{\theta}, t) = \alpha_{ia} + \alpha_{is}$$
$$\alpha_{ia} = -z_{i-1} - \hat{\theta}^{\mathrm{T}} \phi_i + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial t} \tag{29}$$
$$\alpha_{is} = \alpha_{is1} + \alpha_{is2}, \quad \alpha_{is1} = -k_{is} z_i, \quad k_{is} \geqslant g_i + \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} C_{\theta i} \right|^2 + |C_{\phi i} \Gamma \phi_i|^2$$

where $g_i > 0$ is a constant, $C_{\theta i}$ and $C_{\phi i}$ are positive-definite constant diagonal matrices, and α_{is2} is a robust control term satisfying the following two conditions

$$\text{(i)} \quad z_i(\alpha_{is2} - \tilde{\theta}^{\mathrm{T}} \phi_i + \tilde{\Delta}_i) \leqslant \varepsilon_i$$
$$\text{(ii)} \quad z_i \alpha_{is2} \leqslant 0 \tag{30}$$

in which ε_i is a positive design parameter. Then the ith error subsystem is

$$\dot{z}_i = z_{i+1} - z_{i-1} - k_{is} z_i + (\alpha_{is2} - \tilde{\theta}^{\mathrm{T}} \phi_i + \tilde{\Delta}_i) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}, \tag{31}$$

and the derivative of the augmented p.s.d. function

$$V_i = V_{i-1} + \tfrac{1}{2} z_i^2, \quad V_0 = 0 \tag{32}$$

satisfies

$$\dot{V}_i = z_i z_{i+1} + \sum_{j=1}^{i} \left\{ -k_{js} z_j^2 + z_j(\alpha_{js2} - \tilde{\theta}^{\mathrm{T}} \phi_j + \tilde{\Delta}_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} z_j \right\} \tag{33}$$

(The fact that $\partial \alpha_0 / \partial \hat{\theta} = 0$ is used in the above descriptions.)

4.2.2. Step l. Noting (1) and (11), the derivative of $z_l = x_l - \alpha_{l-1}$ is

$$\dot{z}_l = x_{l+1} + \theta^T \varphi_{\theta l} + \varphi_{\eta l}^T(\zeta_0 + \zeta\theta + \omega(\bar{x}_l) - \varepsilon) + \Delta_l - \dot{\alpha}_{l-1} \tag{34}$$

where $\dot{\alpha}_{l-1} = \sum_{j=1}^{l-1} (\partial\alpha_{l-1}/\partial x_j)(x_{j+1} + \theta^T\varphi_{\theta j} + \Delta_j) + \partial\alpha_{l-1}/\partial\hat{\theta}\dot{\hat{\theta}} + \partial\alpha_{l-1}/\partial t$. If we treat x_{l+1} as the input, we can synthesize a virtual control law α_l for x_{l+1} such that z_1 is as small as possible. Since (34) contains unknown parameters θ, uncertain nonlinearity Δ_l, and the estimation error ε, the ARC approach proposed in Reference [12] will be generalized to accomplish the objective. The control function α_l consists of two parts given by

$$\begin{aligned} \alpha_l(\bar{x}_l, r, \zeta_0, \zeta, \hat{\theta}, t) &= \alpha_{la} + \alpha_{ls} \\ \alpha_{la} &= -z_{l-1} - \hat{\theta}^T\varphi_{\theta l} - \varphi_{\eta l}^T(\zeta_0 + \zeta\hat{\theta} + \omega(\bar{x}_l)) \\ &\quad + \sum_{j=1}^{l-1} \frac{\partial\alpha_{l-1}}{\partial x_j}(x_{j+1} + \hat{\theta}^T\varphi_{\theta j}) + \frac{\partial\alpha_{l-1}}{\partial t} \\ \alpha_{ls} &= \alpha_{ls1} + \alpha_{ls2}, \quad \alpha_{ls1} = -k_{ls}z_l, \quad k_{ls} \geqslant g_l \\ &\quad + \left|\frac{\partial\alpha_{l-1}}{\partial\hat{\theta}} C_{\theta l}\right|^2 + |C_{\phi l}\Gamma\phi_l|^2 + c_\theta|\psi_l|^2 \end{aligned} \tag{35}$$

where g_l and c_θ are positive constants, $\psi_l \triangleq \varphi_{\eta l}$, $C_{\theta l}$ and $C_{\phi l}$ are positive-definite constant diagonal matrices to be specified later. Let $z_{l+1} = x_{l+1} - \alpha_l$ denote the input discrepancy. Substituting (35) into (34) leads to

$$\dot{z}_l + k_{ls}z_l = z_{l+1} - z_{l-1} + \alpha_{ls2} - \tilde{\theta}^T\phi_l - \psi_l^T\varepsilon + \tilde{\Delta}_l - \frac{\partial\alpha_{l-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{36}$$

where $\phi_l \triangleq -\sum_{j=1}^{l-1}(\partial\alpha_{l-1}/\partial x_j)\varphi_{\theta j} + \varphi_{\theta l} + \zeta^T\varphi_{\eta l}$ and $\tilde{\Delta}_l \triangleq -\sum_{j=1}^{l-1}(\partial\alpha_{l-1}/\partial x_j)\Delta_j + \Delta_l$. Choose $V_l = V_{l-1} + \frac{1}{2}z_l^2$. From (33) and (36), its time derivative is

$$\dot{V}_l = z_l z_{l+1} + \sum_{j=1}^{l}\left\{-k_{js}z_j^2 + z_j(\alpha_{js2} - \tilde{\theta}^T\phi_j - \psi_j^T\varepsilon + \tilde{\Delta}_j) - \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}z_j\right\} \tag{37}$$

where $\psi_j = 0, \forall j < l$. Noting that $|\tilde{\Delta}_l| \leqslant \tilde{\delta}_l(\bar{x}_l, t) \triangleq \sum_{j=1}^{l-1}|\partial\alpha_{l-1}/\partial x_j|\,\delta_j + \delta_l$, the ARC design [14] can be applied to synthesize a robust control function α_{ls2} satisfying the following two conditions

$$\begin{aligned} &\text{(i)} \quad z_l(\alpha_{ls2} - \tilde{\theta}^T\phi_l - \psi_l^T\varepsilon + \tilde{\Delta}_l) \leqslant \epsilon_l(1 + \rho^2) \\ &\text{(ii)} \quad z_l\alpha_{ls2} \leqslant 0 \end{aligned} \tag{38}$$

where ϵ_l is a positive design parameter which represents the level of attenuation, and $\rho(t) \triangleq \gamma_1^{-1}(2D(t))$.

Remark 5

One smooth example of α_{ls2} satisfying (38) can be found in the following way. Let h_l be any smooth function satisfying

$$h_l \geqslant |\theta_M||\phi_l| + |\psi_l|\gamma_1^{-1}(2r) + \tilde{\delta}_l \tag{39}$$

where $\theta_M = \theta_{\max} - \theta_{\min}$. Then, α_{ls2} can be chosen as

$$\alpha_{ls2} = -\frac{1}{4\epsilon_l}(h_l^2 + |\psi_l|^2)z_l \tag{40}$$

It is obvious that Condition (38) (ii) is verified. From (24), it follows that

$$\begin{aligned} z_l(\alpha_{ls2} - \tilde{\theta}^{\mathrm{T}}\phi_l - \psi_l^{\mathrm{T}}\varepsilon + \tilde{\Delta}_l) &\leqslant z_l\alpha_{ls2} + |z_l||\theta_M||\phi_l| + |z_1||\psi_l|\gamma_1^{-1}(2r) + |z_l|\psi_l|\rho + |z_l||\tilde{\delta}_l \\ &= -\left(\frac{1}{2\sqrt{\epsilon_l}}h_l|z_l| - \sqrt{\epsilon_l}\right)^2 - \left(\frac{1}{2\sqrt{\epsilon_l}}|\psi_l||z_l| - \sqrt{\epsilon_l}\rho\right)^2 + \epsilon_l + \epsilon_l\rho^2 \\ &\leqslant \epsilon_l(1+\rho^2) \end{aligned} \tag{41}$$

Condition (i) of (38) is thus satisfied. Other smooth or continuous examples can be worked out in the same way as in References [12, 13, 15].

4.2.3. Step i ($l + 1 \leqslant i < n$). From (35) and (1),

$$\begin{aligned} \dot{\alpha}_{i-1} &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(x_{j+1} + \theta^{\mathrm{T}}\varphi_{\theta j} + \varphi_{\eta j}^{\mathrm{T}}\eta + \Delta_j) + \frac{\partial\alpha_{i-1}}{\partial r}\dot{r} \\ &\quad + \frac{\partial\alpha_{i-1}}{\partial\zeta_0}\dot{\zeta}_0 + \sum_{j=1}^{p}\frac{\partial\alpha_{i-1}}{\partial\zeta_j}\dot{\zeta}_j + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + \frac{\partial\alpha_{i-1}}{\partial t} \\ &= \dot{\alpha}_{(i-1)c} + \dot{\alpha}_{(i-1)u} \end{aligned} \tag{42}$$

where

$$\begin{aligned} \dot{\alpha}_{(i-1)c} &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\{x_{j+1} + \hat{\theta}^{\mathrm{T}}\varphi_{\theta j} + \varphi_{\eta j}^{\mathrm{T}}(\zeta_0 + \zeta\hat{\theta} + \omega(\bar{x}_l))\} + \frac{\partial\alpha_{i-1}}{\partial r}\dot{r} \\ &\quad + \frac{\partial\alpha_{i-1}}{\partial\zeta_0}\dot{\zeta}_0 + \sum_{j=1}^{p}\frac{\partial\alpha_{i-1}}{\partial\zeta_j}\dot{\zeta}_j + \frac{\partial\alpha_{i-1}}{\partial t} \\ \dot{\alpha}_{(i-1)u} &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\{-\tilde{\theta}^{\mathrm{T}}\varphi_{\theta j} - \varphi_{\eta j}^{\mathrm{T}}(\zeta\tilde{\theta} + \varepsilon) + \Delta_j\} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \end{aligned} \tag{43}$$

In (42) and (43), noting (9) and (22), $\dot{\alpha}_{(i-1)c}$ is calculable and can be used in the design of control functions, but $\dot{\alpha}_{(i-1)u}$ cannot due to various uncertainties. Therefore, it has to be dealt with via robust feedback in this step design. The details are given below.

To make the development general, mathematical induction will be used to prove the general results for all intermediate step designs. At step i, the ARC design similar to that used in step l will be employed to construct a control function α_i for x_{i+1}. Let $z_i = x_i - \alpha_{i-1}$ and recursively define the following functions

$$\phi_i = -\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_{\theta j} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \zeta^{\mathrm{T}} \varphi_{\eta j} + \zeta^{\mathrm{T}} \varphi_{\eta i} + \varphi_{\theta i}$$

$$\tilde{\Delta}_i = -\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \Delta_j + \Delta_i \tag{44}$$

$$\psi_i = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_{\eta j} + \varphi_{\eta i}$$

Lemma 3

At step i, $\forall l + 1 \leqslant i < n$, choose the desired control function α_i as

$$\alpha_i(\bar{x}_i, r, \zeta_0, \zeta, \hat{\theta}, t) = \alpha_{ia} + \alpha_{is}$$

$$\alpha_{ia} = -z_{i-1} - \hat{\theta}^{\mathrm{T}} \varphi_{\theta i} - \varphi_{\eta i}^{\mathrm{T}} \{\zeta_0 + \zeta\hat{\theta} + \omega(\bar{x}_l)\} + \dot{\alpha}_{(i-1)c} \tag{45}$$

$$\alpha_{is} = \alpha_{is1} + \alpha_{is2}, \quad \alpha_{is1} = -k_{is} z_i, \quad k_{is} \geqslant g_i + \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} C_{\theta i} \right|^2 + |C_{\phi i} \Gamma \phi_i|^2 + c_\theta |\psi_i|^2$$

where $g_i > 0$ are constants, $C_{\theta i}$ and $C_{\phi i}$ are positive-definite constant diagonal matrices, and α_{is2} satisfies

$$\begin{aligned} &\text{(i)} \quad z_i(\alpha_{is2} - \tilde{\theta}^{\mathrm{T}} \phi_i - \psi_i^{\mathrm{T}} \varepsilon + \tilde{\Delta}_i) \leqslant \epsilon_i (1 + \rho^2) \\ &\text{(ii)} \quad z_i \alpha_{is2} \leqslant 0 \end{aligned} \tag{46}$$

Then the ith error subsystem is

$$\dot{z}_i = z_{i+1} - z_{i-1} - k_{is} z_i + (\alpha_{is2} - \tilde{\theta}^{\mathrm{T}} \phi_i - \psi_i^{\mathrm{T}} \varepsilon + \tilde{\Delta}_i) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \tag{47}$$

and the derivative of the augmented p.s.d. function

$$V_i = V_{i-1} + \tfrac{1}{2} z_i^2 \tag{48}$$

satisfies

$$\dot{V}_i = z_i z_{i+1} + \sum_{j=1}^{i} \left\{ -k_{js} z_j^2 + z_j(\alpha_{js2} - \tilde{\theta}^{\mathrm{T}} \phi_j - \psi_j^{\mathrm{T}} \varepsilon + \tilde{\Delta}_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} z_j \right\} \tag{49}$$

Proof. It is easy to check that the steps $l+1$ and $l+2$ satisfy the lemma. So let us assume that the lemma is valid for step j, $\forall j \leqslant i-1$, and show that it is also true for step i to complete the induction process. From (44), we have

$$|\tilde{\Delta}_i| \leqslant \tilde{\delta}_i \triangleq \sum_{j=1}^{i-1} \left| \frac{\partial \alpha_{i-1}}{\partial x_j} \right| \delta_j + \delta_i \tag{50}$$

Since $\tilde{\delta}_i$ is known, there exist $\alpha_{is2}(\bar{x}_i, r, \zeta_0, \zeta, \hat{\theta}, t)$ satisfying (46) as in the step l design. The control law (45) can then be formed. We express the derivative of z_i as

$$\begin{aligned}\dot{z}_i &= \dot{x}_i - \dot{\alpha}_{i-1} \\ &= x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i} + \varphi_{\eta i}^{\mathrm{T}}(\zeta_0 + \zeta\theta + \omega(\bar{x}_l) - \varepsilon) + \Delta_i - \dot{\alpha}_{(i-1)c} \\ &\quad - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\{-\tilde{\theta}^{\mathrm{T}}\varphi_{\theta j} - \varphi_{\eta j}^{\mathrm{T}}(\zeta\tilde{\theta} + \varepsilon) + \Delta_j\} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}\end{aligned} \tag{51}$$

Substituting $x_{i+1} = z_{i+1} + \alpha_i$, (45) and (44) into (51), it is straightforward to verify that (47) and (49) are satisfied for i. This completes the induction process. □

4.2.4. Step n. This is the final design step. By letting $u = x_{n+1}$, the last equation of (1) has the same form as the intermediate equations $l + 1 \leqslant i \leqslant n - 1$. Therefore, the general form (44)–(49) applies to Step n. Since u is the actual control input, we can choose it as

$$u = \alpha_n(x, r, \zeta_0, \zeta, \hat{\theta}, t) \tag{52}$$

where α_n is given by (45).

Theorem 1

Let the parameter estimates be updated by the adaptation law (25) in which τ is chosen as

$$\tau = \sum_{j=1}^{n} \phi_j z_j \tag{53}$$

Let $c_{\theta ji}$ and $c_{\phi ki}$ be the ith diagonal elements of the diagonal matrices $C_{\theta j}$ and $C_{\phi k}$, respectively. If the controller parameters $C_{\theta j}$ and $C_{\phi k}$ are chosen such that $c_{\phi ki}^2 \geqslant (n/4)\sum_{j=2}^{n} 1/c_{\theta ji}^2$, $\forall k, i$, then, the control law (45) guarantees that

(A) In general, the control input and all internal signals are bounded. Furthermore, V_n is bounded above by

$$V_n(t) \leqslant \exp(-\lambda_n t)V_n(0) + \frac{\epsilon}{\lambda_n}[1 - \exp(-\lambda_n t)] + \frac{\epsilon}{\lambda_n}\int_0^t \exp[-\lambda_n(t - v)]\rho^2(v)\,\mathrm{d}v \tag{54}$$

where $\lambda_n = 2\min\{g_1, \ldots, g_n\}$ and $\epsilon = \sum_{j=1}^{n} \epsilon_j$. Noting that $\rho(\mathrm{t}) = 0$ for all $t \geqslant T^0$, $V_n(t)$ is ultimately bounded by

$$V_n(\infty) \leqslant \frac{\epsilon}{\lambda_n} \tag{55}$$

(B) If after a finite time t_f, $\bar{\Delta} = 0$ and $\Delta_i = 0$, i.e. in the presence of parametric uncertainties and dynamic uncertainties only, then, in addition to results in (A), asymptotic output tracking (or zero final tracking error) is also achieved.

Proof of the theorem is given in the appendix.

Remark 6

Results in A of Theorem 1 indicate that the proposed controller has an exponentially converging transient performance with the exponentially converging rate λ_n and the final tracking error being able to be adjusted via certain controller parameters (g_j and ϵ_j) freely in a *known* form. Theoretically, this result is what a well-designed robust controller can achieve. In fact, when the parameter adaptation law (25) is switched off, the proposed ARC law becomes a deterministic robust control law and Results A of the Theorem remain valid as in References [13, 15].

Results B of Theorem 1 implies that the proposed controller is able to make full use of the nominal structure of the system and eliminates the effect of parametric uncertainties and dynamic uncertainties through certain parameter adaptation laws. As a result, asymptotic output tracking is obtained without using infinite-gain feedback (i.e. none of the nonlinear gains used in the control law approaches infinity as time goes infinity). Theoretically, Result B is what a well-designed adaptive controller can achieve.

Remark 7

It is seen from (54) that the transient tracking error is affected by the initial value $V_n(0)$ also. To further reduce transient tracking error, the idea of filter initialization [5, 15] can be used to render $V_n(0) = 0$.

5. DESIGN EXAMPLE AND COMPARATIVE SIMULATION RESULTS

In order to illustrate the above ARC algorithm and compare it with the previous RAC design algorithms, simulation results are obtained for the following simple example which is used in Reference [20]:

$$\dot{\eta} = -\eta + x_1^2 + \bar{\Delta} \tag{56}$$

$$\dot{x}_1 = x_2 + \theta x_1^2 + \Delta_1 + 2\eta \tag{57}$$

$$\dot{x}_2 = u \tag{58}$$

$$y = x_1 \tag{59}$$

where θ is an unknown constant parameter, $\bar{\Delta}$ and $\Delta_1(t)$ are two unknown bounded disturbances and η is the unmeasured state. For comparison purpose, we take exactly the same simulation values as in Reference [20] for θ, $\bar{\Delta}$ and $\Delta_1(t)$, i.e. $\theta = 0.1$, $\bar{\Delta} = 0.5$ and $\Delta_1(t) = 0.6\sin(2t)$. The bounds describing the uncertain ranges in (2) are $\Omega_\theta = (-1, 3)$, $\bar{\delta} = 2$ and $\delta_1 = 2$. The control objective is to track a desired trajectory $x_d(t)$.

Let $\xi = \eta - \omega(x_1)$, with $\omega = -((1 + A)/2)x_1$ and A is a negative design parameter. Then, the observer design in Section 3 can be applied. Specifically, the filters are implemented as

$$\dot{\zeta}_0 = A\zeta_0 + A\omega(x_1) + x_1^2 - \frac{\partial \omega}{\partial x_1} x_2 \tag{60}$$

$$\dot{\zeta}_1 = A\zeta_1 - \frac{\partial \omega}{\partial x_1} x_1^2 \tag{61}$$

The equivalent expression for the unmeasured state η is

$$\eta = \zeta_0 + \zeta_1\theta + \omega(x_1) - \varepsilon \tag{62}$$

where ε is the state estimation error which is governed by the following dynamic system:

$$\dot{\varepsilon} = A\varepsilon - \bar{\Delta} - \frac{1+A}{2}\Delta_1(t) \tag{63}$$

Taking $V_\varepsilon = \frac{1}{2}\varepsilon^2$, we see that (12) is satisfied with $\gamma_\varepsilon \equiv 0$. Thus, Assumption 3 is satisfied, and the general design procedure in Section 4 can be applied to obtain an ARC controller. Furthermore, since ε is bounded above by an unknown constant as seen from (63), the dynamic normalization signal r is not needed and the resulting simplified ARC controller is given below

Step 1:

$$z_1 = x_1 - x_d \tag{64}$$

$$\phi_1 = x_1^2 + 2\zeta_1 \tag{65}$$

$$\alpha_{1a} = -\hat{\theta}x_1^2 - 2(\zeta_0 + \zeta_1\hat{\theta} + \omega(x_1)) + \dot{x}_d \tag{66}$$

$$\alpha_{1s1} = -[g_1 + (C_{\phi 1}\Gamma\phi_1)^2]z_1 \tag{67}$$

$$\alpha_{1s2} = -[(\theta_M\phi_1)^2 + \delta_1^2 + 1]z_1/(4\epsilon_1) \tag{68}$$

$$\alpha_1 = \alpha_{1a} + \alpha_{1s1} + \alpha_{1s2} \tag{69}$$

Step 2:

$$z_2 = x_2 - \alpha_1 \tag{70}$$

$$\phi_2 = -\frac{\partial\alpha_1}{\partial x_1}(x_1^2 + 2\zeta_1) \tag{71}$$

$$\alpha_{2a} = -z_1 + \frac{\partial\alpha_1}{\partial x_1}(x_2 + \hat{\theta}x_1^2 + 2\zeta_0 + 2\zeta_1\hat{\theta} + 2\omega) + \frac{\partial\alpha_1}{\partial\zeta_0}\dot{\zeta}_0 + \frac{\partial\alpha_1}{\partial\zeta_1}\dot{\zeta}_1 + \frac{\partial\alpha_1}{\partial t} \tag{72}$$

$$\alpha_{2s1} = -\left[g_2 + \left(\frac{\partial\alpha_1}{\partial\hat{\theta}}C_{\theta 2}\right)^2 + (C_{\phi 2}\Gamma\phi_2)^2\right]z_2 \tag{73}$$

$$\alpha_{2s2} = -\left[(\theta_M\phi_2)^2 + \left(\frac{\partial\alpha_1}{\partial\alpha_1}\right)^2(\delta_1^2 + 1)\right]z_2\Big/(4\epsilon_2) \tag{74}$$

$$u = \alpha_{2a} + \alpha_{2s1} + \alpha_{2s2} \tag{75}$$

The parameter adaptation law is given by

$$\dot{\hat{\theta}} = \mathrm{Proj}_{\hat{\theta}}(\Gamma\tau), \quad \tau = \phi_1 z_1 + \phi_2 z_2 \tag{76}$$

The following two cases are considered:

Case 1: Set-point regulation. By setting $x_d = 0$, trajectory tracking comes to the set-point regulation problem considered in Reference [20]. For the sake of easy comparison, the ARC scheme is adjusted to employ about the same degree of control effort as that of the RAC scheme

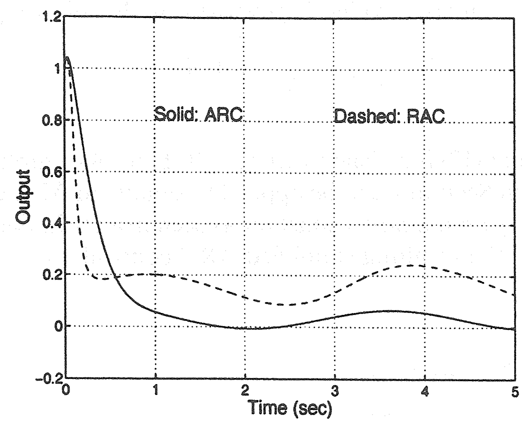

Figure 1. Set-point regulation in the presence of various model uncertainties.

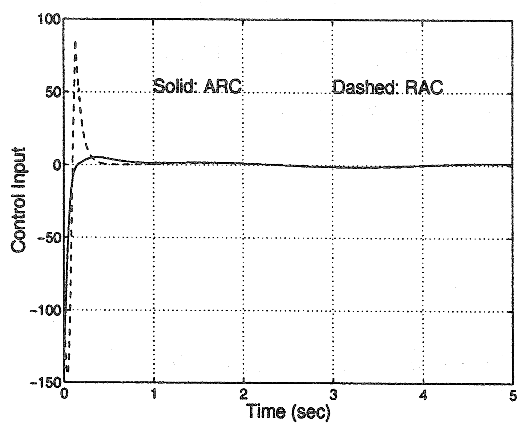

Figure 2. Control inputs in the presence of various model uncertainties.

[20] for the same initial conditions. The design parameters and initial conditions for ARC are given as follows:

$$
\begin{aligned}
&\eta(0) = x_1(0) = x_2(0) = 1, \quad \zeta_0(0) = \zeta_1(0) = 0, \quad \hat{\theta}(0) = 0.5 \\
&A = -10, \quad g_1 = g_2 = 10, \quad \Gamma = 1, \quad C_{\phi 1} = C_{\phi 2} = 0.25 \\
&C_{\theta 2} = 2, \quad \varepsilon_1 = 30, \quad \varepsilon_2 = 300
\end{aligned}
\tag{77}
$$

The plots in Figures 1 and 2 show that the ARC scheme achieves a better performance without using large control input. The parameter estimates of both controllers are given in Figure 3. It is seen that both estimates do not converge to the true value. The reason is that the persistent excitation condition is not satisfied.

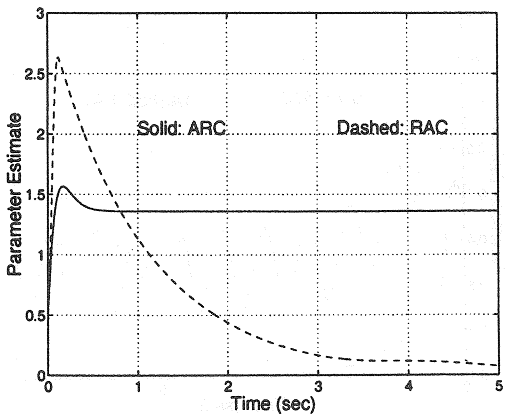

Figure 3. Parameter estimation of ARC and RAC.

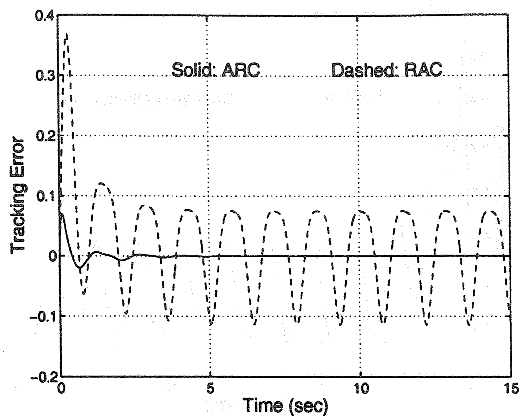

Figure 4. Tracking errors in the absence of uncertain nonlinearities.

Case 2: Trajectory tracking. To test the tracking capability of the proposed algorithm, a sinusoidal desired trajectory given by $x_d = 0.5(1 - \cos(1.4\pi t))$ is used. All design parameters and initial conditions remain unchanged except $\eta(0) = 0$ and $x_1(0) = x_2(0) = 0$. The following two conditions will be considered:

(i) *No disturbance*: To test the nominal performance of the two controllers, simulation is first run for the system without uncertain nonlinearities, i.e. $\bar{\Delta} = \Delta_1 = 0$ is used in the simulation. It is seen from Figure 4 that the ARC scheme has a much better transient performance and final tracking accuracy, which agrees with the theoretical results obtained in Theorem 1. Parameter estimation is given in Figure 5, and it shows that the parameter estimate of ARC converges to the true value. However, the parameter estimate of RAC does not converge because of the dynamic uncertainties. It also can be seen from Figure 6 that the state estimate $\hat{\eta}$ converges to the true value.

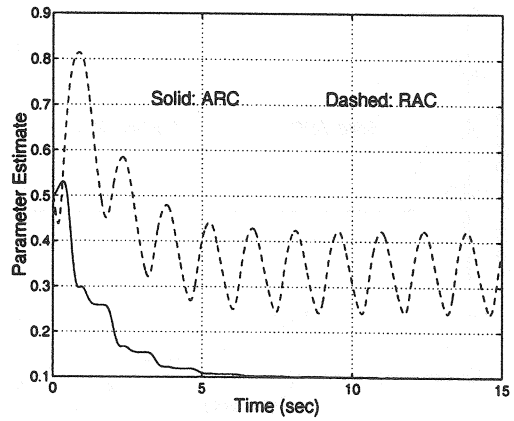

Figure 5. Parameter estimation in the absence of uncertain nonlinearities.

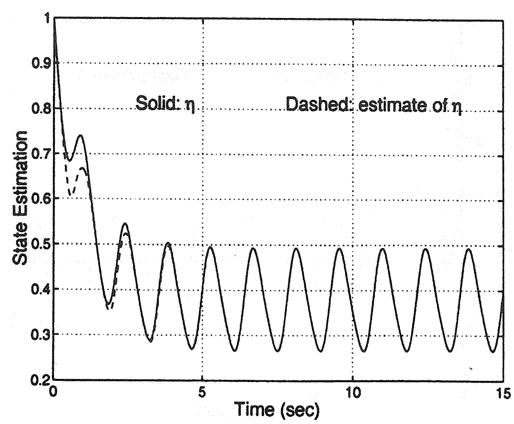

Figure 6. State estimation in the absence of uncertain nonlinearities.

(ii) *With disturbance*: To test the performance robustness of the controllers to uncertain nonlinearities, the assumed disturbances are used in the simulation, i.e. $\bar{\Delta} = 0.5$ and $\Delta_1(t) = 0.6\sin(2t)$. The tracking errors of both controllers are shown in Figure 7 with the control inputs shown in Figure 8. The plots in Figure 9 shows that the state estimate does not converge, but it is robust with respect to the disturbances.

6. CONCLUSIONS

In this paper, an observer-based adaptive robust control (ARC) scheme is presented for a class of nonlinear systems in an extended semi-strict feedback form, in which the unmeasured states enter the system equations in an affine fashion. The form allows for parametric uncertainties, uncertain

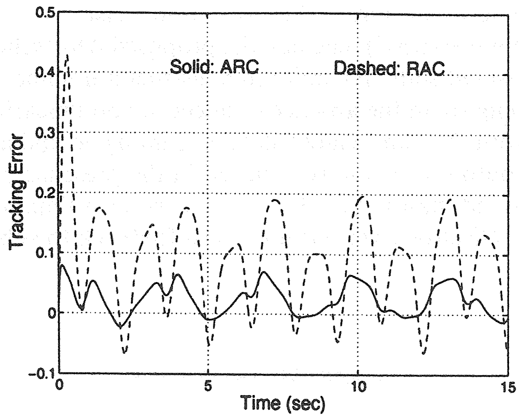

Figure 7. Tracking errors in the presence of various model uncertainties.

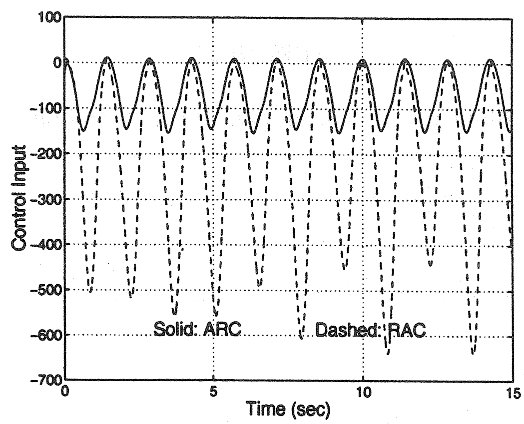

Figure 8. Control inputs in the presence of various model uncertainties.

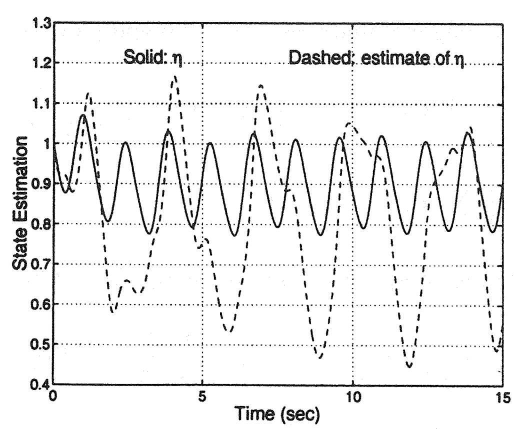

Figure 9. State estimation in the presence of various model uncertainties.

nonlinearities, and dynamic uncertainties due to the unmeasured states. In contrast to other existing robust adaptive control (RAC) schemes, the proposed ARC scheme utilizes the structural information of the unmeasured state dynamics to construct a nonlinear observer to recover the unmeasured states. By doing so, in the absence of uncertain nonlinearities, the effect of dynamic uncertainties associated with the unmeasured states is eliminated and an improved performance is obtained—asymptotic output tracking is achieved in the presence of both parametric uncertainties and the unmeasured states. In addition, the state estimation errors and uncertain nonlinearities are handled effectively via certain robust feedback to achieve a guaranteed robust performance.

ACKNOWLEDGEMENTS

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APPENDIX

Proof of Theorem 1. Noting $z_{n+1} = 0$, from (49), (30), (38) and (46)

$$\dot{V}_n \leqslant \sum_{j=1}^{n} \left\{ -\left(g_j + \left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 + |C_{\phi j}\Gamma\phi_j|^2 + c_\theta |\psi_j|^2 \right) z_j^2 + z_j(\alpha_{js2} - \tilde{\theta}^{\mathrm{T}}\phi_j - \psi_j^{\mathrm{T}}\varepsilon + \tilde{\Delta}_j) - z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right\} \tag{A1}$$

By completion of square

$$-\sum_{j=2}^{n} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \leqslant \sum_{j=2}^{n} |z_j| \left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} C_{\theta j}^{-1} \dot{\hat{\theta}} \right| \leqslant \sum_{j=2}^{n} \left(\left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 z_j^2 + \frac{1}{4} \left| C_{\theta j}^{-1} \dot{\hat{\theta}} \right|^2 \right) \tag{A2}$$

Noting that $C_{\theta j}^{-1}$ and Γ are diagonal matrices, from (25) and (26), we have

$$\sum_{j=2}^{n} |C_{\theta j}^{-1}\dot{\hat{\theta}}|^2 = \sum_{j=2}^{n} |C_{\theta j}^{-1} \operatorname{Proj}_{\hat{\theta}}(\Gamma\tau)|^2 \leqslant \sum_{j=2}^{n} |C_{\theta j}^{-1}\Gamma\tau|^2 \leqslant \sum_{j=2}^{n} \left(\sum_{k=1}^{n} |C_{\theta j}^{-1}\Gamma\phi_k z_k| \right)^2 \leqslant n \sum_{j=2}^{n} \left(\sum_{k=1}^{n} |C_{\theta j}^{-1}\Gamma\phi_k|^2 z_k^2 \right) \tag{A3}$$

Thus, if $C_{\theta j}$ and $C_{\phi k}$ satisfy the conditions in the theorem, from (A2) and (A3),

$$-\sum_{j=2}^{n} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \leqslant \sum_{j=2}^{n} \left(\left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 z_j^2 + \frac{n}{4} \sum_{k=1}^{n} |C_{\theta j}^{-1}\Gamma\phi_k|^2 z_k^2 \right) \leqslant \sum_{j=2}^{n} \left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 z_j^2 + \sum_{k=1}^{n} |C_{\phi k}\Gamma\phi_k|^2 z_k^2 \tag{A4}$$

From (A4) and (46)(i), (A1) becomes,

$$\dot{V}_n \leqslant \sum_{j=1}^{n} \{ -g_j z_j^2 + \epsilon_j + \epsilon_j \rho^2(t) \} \leqslant -\lambda_n V_n + \epsilon + \epsilon \rho^2(t) \tag{A5}$$

which leads to (54) and (55). The boundedness of $z_1, \dots, z_n$ is thus proved. Using the standard arguments in the backstepping designs [5, 12], it can be proved that all internal signals in the first $l-1$ steps are globally uniformly bounded. Furthermore, from $x_l = z_l + \alpha_{l-1}$, it follows that x_l is bounded. The boundedness of signals $x_1, \dots, x_l$, together with the bounded-input-bounded-state Assumption 2 for the η dynamics and Assumption 3 for the observer error dynamics, implies that η, ε, and r are bounded. Thus, all filter states ζ_0, ζ will be bounded. Thus, recursively using the fact that $x_i = z_i + \alpha_{i-1}$, it is easy to verify that all intermediate control functions α_i and states x_i are bounded. From (52) and (45), the boundedness of u is apparent. Theorem 1(A) is thus proved.

In the absence of uncertain nonlinearities, i.e. $\bar{\Delta} = 0$ and $\Delta_i = 0$, noting condition (ii) of (30), (38) and (46), from (A1) and (A4)

$$\dot{V}_n \leqslant \sum_{j=1}^{n} (-g_j z_j^2 - c_\theta |\psi_j|^2 z_j^2 - \tilde{\theta}^{\mathrm{T}} \phi_j z_j - z_j \psi_j^{\mathrm{T}} \varepsilon) \tag{A6}$$

Define a new p.s.d. function V_θ as

$$V_\theta = V_n + \tfrac{1}{2} \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta} \tag{A7}$$

Noting P2 of (27), from (A6), the derivative of V_θ satisfies

$$\begin{aligned} \dot{V}_\theta &\leqslant \sum_{j=1}^{n} \{ -g_j z_j^2 - c_\theta |\psi_j|^2 z_j^2 - z_j \psi_j^{\mathrm{T}} \varepsilon \} - \tilde{\theta}^{\mathrm{T}} \tau + \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \dot{\hat{\theta}} \\ &\leqslant \sum_{j=1}^{n} \{ -g_j z_j^2 - c_\theta |\psi_j|^2 z_j^2 + |z_j| |\psi_j| |\varepsilon| \} \\ &\leqslant -\sum_{j=1}^{n} g_j z_j^2 + \frac{n}{4c_\theta} |\varepsilon|^2 \end{aligned} \tag{A8}$$

By B of Assumption 3, $\varepsilon(t)$ exponentially converges to zero, and thus $\varepsilon(t) \in \mathscr{L}_2[0, \infty)$. From (A8), it is easy to prove that $z_j(t) \in \mathscr{L}_2[0, \infty)$. It is also easy to check that $\dot{\varepsilon}$ and $\dot{z}_j$ are bounded. Hence, by the Barbalat's lemma, $z \to 0$ as $t \to \infty$, which leads to Theorem (B).

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