INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING
Int. J. Adapt. Control Signal Process. (2010)
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/acs.1177

Adaptive robust control of a class of nonlinear systems in semi-strict feedback form with non-uniformly detectable unmeasured internal states

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SUMMARY

This paper proposes a novel control method for a special class of nonlinear systems in semi-strict feedback form. The main characteristic of this class of systems is that the unmeasured internal states are non-uniformly detectable, which means that no observer for these states can be designed to make the observation error exponentially converge to zero. In view of this, a projection-based adaptive robust control law is developed in this paper for this kind of system. This method uses a projection-type adaptation algorithm for the estimation of both the unknown parameters and the internal states. Robust feedback term is synthesized to make the system robust to uncertain nonlinearities and disturbances. Although the estimation error for both the unknown parameters and the internal states may not converge to zero, the tracking error of the closed-loop system is proved to converge to zero asymptotically if the system has only parametric uncertainties. Furthermore, it is theoretically proved that all the signals are bounded, and the control algorithm is robust to bounded disturbances and uncertain nonlinearities with guaranteed output tracking transient performance and steady-state accuracy in general. The class of system considered here has wide engineering applications, and a practical example—control of mechanical systems with dynamic friction—is used as a case study. Simulation results are obtained to demonstrate the applicability of the proposed control methodology. Copyright © 2010 John Wiley & Sons, Ltd.

Received 18 November 2008; Revised 25 February 2010; Accepted 7 March 2010

KEY WORDS: nonlinear control; dynamic uncertainties; adaptive robust control; dynamic friction

1. INTRODUCTION

The control of nonlinear systems with various kinds of uncertainties is receiving more and more attention these years. Typically, the uncertainties of the system fall into two categories: (i) repeatable and constant unknown uncertainties such as unknown physical parameters, (ii) non-repeatable uncertainties such as external disturbances and the imprecise modeling of some terms in the system dynamics. To deal with these uncertainties, two nonlinear control methods, the deterministic robust control (DRC) [1–3] and the adaptive control (AC) [4, 5], have been developed.

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Contract/grant sponsor: U.S. National Science Foundation; contract/grant number: CMS-0600516
Contract/grant sponsor: National Natural Science Foundation of China; contract/grant number: 50528505
Contract/grant sponsor: Ministry of Education of China

The deterministic robust controllers are able to guarantee transient performance and final tracking accuracy in the presence of various kinds of uncertainties. However, some problems like switching [1] or infinite-gain [3] feedback will happen, which are undesirable for industrial application. In contrast, the adaptive controllers [4, 5] are able to achieve asymptotic tracking in the presence of parametric uncertainties without using discontinuous or infinite-gain feedback. However, this approach may result in unstable closed-loop system in the presence of external disturbances. To remedy, a modification method called robust adaptive control (RAC) [4] has been developed to robust the system. But some trade offs have to be made, since the property of asymptotic tracking may be lost using this technique. In [6, 7], an adaptive robust control (ARC) algorithm has been proposed, which incorporates the design methods of DRC and AC effectively. The resulting ARC controllers have the advantages of both DRC and AC while overcoming their practical limitations. The proposed ARC algorithm has been successfully applied to various systems such as electro-mechanical systems [8, 9] and electro-hydraulic systems [10].

Besides parametric uncertainties and uncertain nonlinearities, some systems may be further subjected to dynamic uncertainties. This kind of system has exogenous dynamic systems whose states cannot be measured. The control of this kind of system has received more and more attention in the recent years not only because there are few previous results available, but also due to the many practical applications of this kind of system, e.g. dynamic friction model in [11, 12] and the control of eccentric rotor in [13, 14]. In [13], an adaptive controller was designed for a class of extended strict feedback nonlinear systems in which the unmeasured states enter the systems in a linear affine fashion. However, it is unclear how the approach can be made robust to uncertain nonlinearities and disturbances. In [15], Jiang and Praly proposed a modified RAC procedure [16] for a class of uncertain nonlinear systems subject to dynamic uncertainties satisfying certain conditions. However, since this method does not explicitly use the structural information of the original system, it does not have some desirable properties like asymptotic tracking in the presence of parametric uncertainties. In [17, 18], an observer-based ARC algorithm was proposed. Robustness and asymptotic tracking can both be achieved using this algorithm. However, the original system is assumed to be uniformly detectable. This assumption limits the application of this method because some systems, e.g. the mechanical systems with dynamic friction, does not satisfy the assumed detectability condition.

In this paper, we propose a novel ARC algorithm for the control of a class of nonlinear systems in semi-strict feedback form whose unmeasured internal states are bounded by known bounds but are not uniformly detectable. For this kind of system, no observer can be designed to make the observation error converge to zero. Instead, we design a projection-type adaptation algorithm to give the state estimation. It is theoretically proved that with the proposed control law, the closed-loop system is robust to nonlinear uncertainties and disturbances and has guaranteed output tracking transient performances and steady-state tracking accuracy. Furthermore, in the presence of parametric uncertainties only, asymptotic output tracking can be achieved. These two characteristics combine the good merits of DRC and AC while naturally overcoming their performance limitations. To illustrate the applicability of the proposed approach, we take a practical system—the control of linear motor systems with dynamic frictions—as a case study. This system satisfies all the assumptions made in the paper, mainly the unmeasured internal states are bounded but are not uniformly detectable. The simulation results show that our proposed control method achieves the claimed control performances for this kind of system with non-uniformly detectable internal states, demonstrating the applicability of the proposed method in practical applications.

2. PROBLEM FORMULATION

In this paper, we consider the following class of nonlinear systems:

$$\dot{\eta} = F_\eta(x)\theta + G_\eta(x)\eta + H_\eta(x) + \Delta_\eta(x, \eta, u, t),$$
$$\dot{x}_i = x_{i+1} + \theta^{\mathrm{T}}\varphi_{\theta i}(\bar{x}_i) + h_i(\bar{x}_i) + \Delta_i(x, \eta, u, t), \qquad 1 \leqslant i \leqslant l-1,$$

DOI: 10.1002/acs

$$\begin{aligned} \dot{x}_l &= u+\theta^{\mathrm{T}}\varphi_{\theta l}(x)+\varphi_{\eta l}^{\mathrm{T}}(x)\eta+h_l(x)+\Delta_l(x,\eta,u,t), \\ y &= x_1, \end{aligned} \tag{1}$$

where $x=[x_1,\ldots,x_l]^{\mathrm{T}}\in R^l$ is the vector of measurable states. $\bar{x}_i=[x_1,\ldots,x_i]^{\mathrm{T}}\in R^i$ is the vector of first i measurable states. u and y are the control input and output, respectively. $\eta\in R^m$ is the vector of unmeasured internal states. $\theta\in R^p$ is the vector of unknown constant parameters. $F_\eta\in R^{m\times p}$, $G_\eta\in R^{m\times m}$, $H_\eta\in R^m$, $\varphi_{\theta i}\in R^p$, $h_i\in R$ and $\varphi_{\eta l}\in R^m$ are matrices, vectors or scalars of known smooth functions. Δ_η and Δ_i represent the lumped unknown nonlinear functions such as disturbances and modeling errors.

Remark 1
The nonlinear systems considered in this paper are different from the system in [17] in that the internal states only appear in the dynamic equation directly related to input u here. But in [17], the internal states can appear from lth to nth dynamic equations. The reason we do this simplification is because our design method in this paper uses discontinuous projection algorithm to estimate the non-uniformly detectable internal states, and the discontinuous projection algorithm for the state estimation is neither continuously differentiable nor compensatory via large enough feedback gain as can be done for the discontinuous projection algorithm for the estimation of θ in [17]. Thus, in order to use discontinuous projection algorithm for the internal state estimation, we have to restrict the class of system we are dealing with such that the internal states appear only in the last dynamic equation, which is directly related to input u. Otherwise, the non-differentiability of the discontinuous projection algorithm will make it impossible to design virtual control laws for $x_{l+2},\ldots,x_n$.

It is, however, possible to use continuous or smooth projection algorithm instead of discontinuous one as in [19]. With projection algorithm that is continuous up to $n-l+1$th order, the class of systems that can be handled with the proposed method can be extended to the same one as in [17], i.e. the internal states can appear from lth to nth dynamic equation. Since the notations and deductions are very complicated in these cases, and the focus of this paper is on how to deal with non-uniformly detectable internal states, without loss of generality, only the class of systems in (1) are considered. Later in Remark 3, the continuous-projection laws for $\hat{\theta}$ and ζ_0 will be given and compared with the discontinuous one used in this paper in detail.

Now some practical assumptions are made as follows:

Assumption 1
The extents of parametric uncertainties are known. Also the uncertain nonlinearities are bounded by known functions. More precisely, parametric uncertainties and uncertain nonlinearities are assumed to satisfy

$$\begin{aligned} \theta\in\Omega_\theta &\triangleq \{\theta:\ \theta_{\min}\leqslant\theta\leqslant\theta_{\max}\}, \\ \Delta_i\in\Omega_{\Delta_i} &\triangleq \{\Delta_i:\ |\Delta_i(x,\eta,u,t)|\leqslant\delta_i(\bar{x}_i)\}. \end{aligned} \tag{2}$$

Assumption 2
The η dynamics is bounded-input–bounded-state stable with known bound. In other words, $\eta\in\Omega_\eta(t)$, where $\Omega_\eta(t)$ is a known time-varying convex set as a function of $x(\tau)$, $\tau\in[0,\ t]$, such that if $x(t)\in L_\infty^l[0\ \infty)$, then $\Omega_\eta(t)$ is uniformly bounded for all t.

Remark 2
This assumption is different from the uniform detectability assumption made in [17, 18]. In [17, 18], the pair $(\varphi_{\eta l}^{\mathrm{T}},\ G_\eta)$ is assumed to satisfy the uniform detectability condition, i.e. there exists an $\omega(x)=[\omega_1(x),\ldots,\omega_m(x)]^{\mathrm{T}}$,

DOI: 10.1002/acs

such that the unperturbed system $\dot{\varepsilon} = A(x)\varepsilon$ is exponentially stable, where $A(x) = G_\eta(x) - (\partial\omega/\partial x_l)\varphi_{\eta l}^{\mathrm{T}}$. But for some practical systems, e.g. mechanical systems with dynamic friction, this condition cannot be satisfied. In this paper, we will deal with the systems where $(\varphi_{\eta l}^{\mathrm{T}},\ G_\eta)$ may not be uniformly detectable, but the internal states are bounded by known time-varying bounds.

Assumption 3
There exists a positive-definite matrix $\Gamma_\eta \in R^{m\times m}$ such that $\Gamma_\eta^{-1} G_\eta(x) + G_\eta^{\mathrm{T}}(x)\Gamma_\eta^{-1} \leqslant 0,\ \forall\ x \in R^l$.

The following assumption is made on how the parametric uncertainties affect the dynamics of unmeasured internal states:

Assumption 4
Let $F_{\eta j}(x)$ be the jth column of $F_\eta(x)$. Then dynamic systems $\dot{\zeta}_j = F_{\eta j}(x) + G_\eta(x)\zeta_j$ $(1 \leqslant j \leqslant p)$ with the input x and states $[\zeta_1, \ldots, \zeta_p]$ are bounded-input–bounded-state stable in the sense that for every $x(t) \in L_\infty^l[0\ \infty)$, the solution $[\zeta_1, \ldots, \zeta_p]$ starting from any initial condition is bounded, i.e. $[\zeta_1(t), \ldots, \zeta_p(t)] \in L_\infty^{m\times p}[0\ \infty)$.

Let $y_d(t)$ be the desired motion trajectory, which is assumed to be known, bounded, with bounded derivatives up to lth order. The objective is to synthesize a bounded control input u such that output $y = x_1$ tracks $y_d(t)$ as closely as possible in spite of various model uncertainties and unmeasured states.

3. DISCONTINUOUS PROJECTION-BASED ARC BACKSTEPPING DESIGN

3.1. Parameter projection

Let $\hat{\theta}$ denote the estimate of θ and $\tilde{\theta}$ the estimation error, i.e. $\tilde{\theta} = \hat{\theta} - \theta$. A discontinuous projection-based ARC design will be constructed to solve the tracking control problem for (1). Specifically, under Assumption 1, the parameter estimate $\hat{\theta}$ is updated through a parameter adaptation law with the form

$$\dot{\hat{\theta}} = \mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta \tau_\theta), \tag{3}$$

where Γ_θ is a symmetric positive-definite (s.p.d.) diagonal adaptation rate matrix, τ_θ is an adaptation function to be synthesized later. $\mathrm{Proj}_{\hat{\theta}} = [\mathrm{Proj}_{\hat{\theta}_1}(\bullet_1), \ldots, \mathrm{Proj}_{\hat{\theta}_p}(\bullet_p)]^{\mathrm{T}}$ where each projection function is defined as

$$\mathrm{Proj}_{\hat{\theta}_i}(\bullet) = \begin{cases} 0 & \text{if } \hat{\theta}_i \geqslant \theta_{i\max} \quad \text{and} \quad \bullet > 0, \\ 0 & \text{if } \hat{\theta}_i \leqslant \theta_{i\min} \quad \text{and} \quad \bullet < 0, \\ \bullet & \text{otherwise.} \end{cases} \tag{4}$$

It can be shown that for any adaptation function τ_θ, the projection mapping guarantees

$$\begin{aligned} &\text{P1} \quad \hat{\theta} \in \Omega_\theta = \{\hat{\theta} :\ \theta_{\min} \leqslant \hat{\theta} \leqslant \theta_{\max}\} \\ &\text{P2} \quad \tilde{\theta}^{\mathrm{T}}(\Gamma_\theta^{-1}\mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta\tau_\theta) - \tau_\theta) \leqslant 0 \end{aligned} \tag{5}$$

DOI: 10.1002/acs

3.2. State estimation

The estimation of unmeasured states η forms the core part of this paper. Since the pair $(\varphi_{\eta l}^{\mathrm{T}}, G_\eta)$ is not assumed to be uniformly detectable, which is detailed in Remark 2, the observer-based approach such as those used in [17, 18] cannot be applied to this case any more. In [17, 18], using the detectability condition, the estimation error ε is proved to converge to zero exponentially; thus, the effect of ε will 'diminish' as seen in the proof of Theorem 1 of [17]. It is impossible, however, to make the estimation error converge to zero without the detectability condition. Instead, we will use a technique similar to parameter adaptation algorithm in ARC theory, i.e. to add an adaptation function to the state estimator and apply the projection algorithm. With this approach, although the estimation error may not converge to zero, the output tracking error will converge but in the presence of parametric uncertainties only, as will be proved later in this paper. Furthermore, the boundedness of the estimation signals is guaranteed with the projection algorithm, which will also be used later to synthesize the robust feedback term to guarantee transient performance and steady-state tracking accuracy in general.

Let $\zeta_j \in R^m$ $(0 \leqslant j \leqslant p)$ be the estimated variables with the estimation law as

$$
\begin{aligned}
\dot{\zeta}_0 &= \mathrm{Proj}_{\zeta_0}(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta),\\
&\triangleq \begin{cases} \left(I - \Gamma_\eta \dfrac{n_{\zeta_0} n_{\zeta_0}^{\mathrm{T}}}{n_{\zeta_0}^{\mathrm{T}} \Gamma_\eta n_{\zeta_0}} \right)(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta), \\ \text{if } \zeta_0 \in \partial\Omega_{\zeta_0} \quad \text{and} \quad n_{\zeta_0}^{\mathrm{T}}(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) > 0, \\ G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta \quad \text{otherwise}, \end{cases}\\
\dot{\zeta}_j &= G_\eta \zeta_j + F_{\eta j}, \ 1 \leqslant j \leqslant p,
\end{aligned} \tag{6}
$$

where $\Gamma_\eta \in R^{m\times m}$ is a positive-definite matrix satisfying Assumption 2. τ_η is any function to be synthesized later. Ω_{ζ_0} denotes the time-varying convex set that ζ_0 lies in (sometimes we drop the notation 't' for simplicity), $\partial\Omega_{\zeta_0}$ is its boundary. n_{ζ_0} represents the outward unit normal vector at $\zeta_0 \in \partial\Omega_{\zeta_0}$. Ω_{ζ_0} is derived as follows:

$$
\Omega_{\zeta_0}(t) = \left\{ a + b : a \in \Omega_\eta(t), \ |b| \leqslant \sup_{t>0} \left[\sum_{j=1}^{p} \max(|\theta_{\max j}|, |\theta_{\min j}|) |\zeta_j(t)| \right] \right\}, \tag{7}
$$

where $\sup_{t>0}(\bullet)$ function denotes the supremum of all $\bullet(t)$ from the beginning to the current time. Since $\Omega_\eta(t)$ is convex, it can be easily checked that Ω_{ζ_0} is also convex.

Now put ζ_j, $1 \leqslant j \leqslant p$, into a matrix $\zeta = [\zeta_1 \ \ldots \ \zeta_p]$. We have $\dot{\zeta} = G_\eta \zeta + F_\eta$. Defining the estimation error to be $\varepsilon = \zeta_0 + \zeta\theta - \eta$, then we have the following lemma:

Lemma 1

For any function τ_η,

(i) If $\zeta_0(0) \in \Omega_{\zeta_0}(0)$, then $\zeta_0(t) \in \Omega_{\zeta_0}(t)$.

(ii)

$$
\varepsilon^{\mathrm{T}} \Gamma_\eta^{-1} [\mathrm{Proj}_{\zeta_0}(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) - G_\eta \zeta_0 - H_\eta - \Gamma_\eta \tau_\eta] \leqslant 0. \tag{8}
$$

DOI: 10.1002/acs

Proof 1

At any time, if ζ_0 touches the bound, i.e. $\zeta_0 \in \partial\Omega_{\zeta_0}$, then according to (6),

$$n_{\zeta_0}^{\mathrm{T}} \mathrm{Proj}_{\zeta_0}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) = \begin{cases} n_{\zeta_0}^{\mathrm{T}} \left(I - \Gamma_\eta \dfrac{n_{\zeta_0} n_{\zeta_0}^{\mathrm{T}}}{n_{\zeta_0}^{\mathrm{T}} \Gamma_\eta n_{\zeta_0}} \right) (G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta), \\ \text{if } n_{\zeta_0}^{\mathrm{T}}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) > 0, \\ n_{\zeta_0}^{\mathrm{T}}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) \quad \text{otherwise,} \end{cases}$$
$$= \begin{cases} 0, \ \text{if } n_{\zeta_0}^{\mathrm{T}}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) > 0, \\ n_{\zeta_0}^{\mathrm{T}}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) \quad \text{otherwise} \end{cases}$$
$$\leqslant 0 \tag{9}$$

Thus, the derivative of ζ_0 always points inward or to the tangential direction of current Ω_{ζ_0} at the point ζ_0. From (7), $\Omega_{\zeta_0}(t)$ is monotonically expanding. Hence, we conclude that $\zeta_0(t) \in \Omega_{\zeta_0}(t)$ if $\zeta_0(0) \in \Omega_{\zeta_0}(0)$.

For (ii), we see that

Case 1: *If either* $\zeta_0 \in \partial\Omega_{\zeta_0}$ *or* $n_{\zeta_0}^{\mathrm{T}}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) > 0$ *is not true, then* $\mathrm{Proj}_{\zeta_0}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) = G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta$, (ii) *is obviously true*.

Case 2: *If* $\zeta_0 \in \partial\Omega_{\zeta_0}$ *and* $n_{\zeta_0}^{\mathrm{T}}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) > 0$, *then* ζ_0 *is on the boundary of* Ω_{ζ_0}. *From* (7), $\eta - \zeta\theta \in \Omega_{\zeta_0}$, *since* Ω_{ζ_0} *is convex*, $n_{\zeta_0}^{\mathrm{T}}\varepsilon = n_{\zeta_0}^{\mathrm{T}}(\zeta_0 - (\eta - \zeta\theta)) \geqslant 0$. *Then, a simple mathematical deduction leads to* (ii).

Lemma 1 is important. Although the proposed state estimator may not guarantee that the estimation error ε converges to zero, with Lemma 1, we still can construct a Lyapunov function different from those used in [17, 18] to prove asymptotic output tracking in the presence of parametric uncertainties as done later in the proof of part B of Theorem 1.

Remark 3

The parameter adaptation law and state estimation law proposed above are discontinuous, i.e. the differential equations (3) and (6) have discontinuous right-hand side with respect to $\hat{\theta}$ and ζ_0. As the only known sufficient condition for a differential equation described by $\dot{x} = f(t,x)$ to have the local existence and uniqueness of its solution is that the function $f(t,x)$ is locally Lipschitz in x (i.e. continuous in x at least), one may raise the issue of the existence and the uniqueness of the solutions of the proposed discontinuous projection-based ARC designs. Such a concern can be addressed in the following two distinct ways:

S1. The first way is simply to use the continuous or the smooth projection instead of the discontinuous projection to avoid any potential mathematical flaws, as done in the earlier publications on the proposed ARC; these include the continuous or smooth projections used for systems with matching [6] or extended matching uncertainties [19] and the smooth projections with backstepping designs for systems with unmatched uncertainties in [7, 20]. For example, if the continuous-projections are to be used, the parameter adaptation law (3) and the state estimation law (6) would become as

$$\begin{aligned} \dot{\hat{\theta}} &= \Gamma_\theta\tau_\theta + l_\theta(\hat{\theta}), \quad \hat{\theta}(0) \in \Omega_\theta, \\ \dot{\zeta}_0 &= G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta + l_{\zeta_0}(\zeta_0), \quad \zeta_0(0) \in \Omega_{\zeta_0}(0), \end{aligned} \tag{10}$$

DOI: 10.1002/acs

in which $l_\theta(\hat{\theta})$ and $l_{\zeta_0}(\zeta_0)$ represent the additional nonlinear damping terms to make the parameter adaptation and state estimation robust to unstructured uncertainties. As in [6], it can be shown that, in order to achieve the required robust performance results without losing the excellent steady-state performance result of asymptotic output tracking under parametric uncertainties, the damping terms $l_\theta(\hat{\theta})$ and $l_{\zeta_0}(\zeta_0)$ are only required to satisfy the following three conditions:

(i) $l_\theta(\hat{\theta})=0$ if $\hat{\theta}\in\Omega_\theta$ and $l_{\zeta_0}(\zeta_0)=0$ if $\zeta_0\in\Omega_{\zeta_0}$.
(ii) $\tilde{\theta}^{\mathrm{T}} l_\theta(\hat{\theta})\leqslant 0$ if $\hat{\theta}\notin\Omega_\theta$ and $\varepsilon^{\mathrm{T}} l_{\zeta_0}(\zeta_0)\leqslant 0$ if $\zeta_0\notin\Omega_{\zeta_0}$.
(iii) The nonlinear damping terms $l_\theta(\hat{\theta})$ and $l_{\zeta_0}(\zeta_0)$ should be chosen in such a way that the parameter estimate $\hat{\theta}$ and the state estimate ζ_0 belong to some known sets $\Omega_{\hat{\zeta}_0}(t)$ and $\Omega_{\hat{\theta}}(t)$; these known sets are required to be bounded when the measured state $x(t)$ is guaranteed to be bounded.

In other words, the damping terms should be chosen such that the parameter adaptation law satisfies the properties similar to (5), and the state estimation law satisfies the properties similar to those in Lemma 1. A specific example of the continuous damping terms in the parameter adaptation and state estimation laws satisfying the above conditions is given by

$$l_\theta(\hat{\theta}) = \begin{cases} 0 & \text{if } |\hat{\theta}-\hat{\theta}(0)|\leqslant\delta_\theta, \\ -\gamma_\theta \dfrac{|\hat{\theta}-\hat{\theta}(0)|-\delta_\theta}{\varepsilon_\theta}(\hat{\theta}-\hat{\theta}(0)) & \text{if } \delta_\theta\leqslant|\hat{\theta}-\hat{\theta}(0)|\leqslant\delta_\theta+\varepsilon_\theta, \\ -\gamma_\theta(\hat{\theta}-\hat{\theta}(0)) & \text{if } |\hat{\theta}-\hat{\theta}(0)|\geqslant\delta_\theta+\varepsilon_\theta, \end{cases} \tag{11}$$

$$l_{\zeta_0}(\zeta_0) = \begin{cases} 0 & \text{if } |\zeta_0-\zeta_0(0)|\leqslant\delta_{\zeta_0}(t), \\ -\gamma_{\zeta_0} \dfrac{|\zeta_0-\zeta_0(0)|-\delta_{\zeta_0}}{\varepsilon_{\zeta_0}}(\zeta_0-\zeta_0(0)) & \text{if } \delta_{\zeta_0}(t)\leqslant|\zeta_0-\zeta_0(0)|\leqslant\delta_{\zeta_0}(t)+\varepsilon_{\zeta_0}, \\ -\gamma_{\zeta_0}(\zeta_0-\zeta_0(0)) & \text{if } |\zeta_0-\zeta_0(0)|\geqslant\delta_{\zeta_0}(t)+\varepsilon_{\zeta_0}, \end{cases} \tag{12}$$

where δ_θ is the smallest positive number such that $\Omega_\theta\subset\{p\colon\ |p-\hat{\theta}(0)|\leqslant\delta_\theta\}$, $\delta_{\zeta_0}(t)$ is the smallest positive number such that $\Omega_{\zeta_0}(t)\subset\{p\colon |p-\zeta_0(0)|\leqslant\delta_{\zeta_0}(t)\}$, ε_θ and ε_{ζ_0} are positive numbers that represent the thickness of the boundary layer in which the nonlinear damping terms continuously change from 0 to $-\gamma_\theta(\hat{\theta}-\hat{\theta}(0))$ for $\hat{\theta}$ and from 0 to $-\gamma_{\zeta_0}(\zeta_0-\zeta_0(0))$ for ζ_0, respectively, and γ_θ and γ_{ζ_0} are some positive scalars representing the damping coefficients of the nonlinear damping terms at large. Noting the bounded-input–bounded-output assumption of the internal dynamics (Assumptions 2 and 4) and the definition of $\Omega_{\zeta_0}(t)$, we know that if $x(t)$ is bounded, $\Omega_{\zeta_0}(t)$ is contained in a bounded set for all $t>0$. Then, the same as in [6], it is easy to verify that these choices of $l_\theta(\hat{\theta})$ and $l_{\zeta_0}(\zeta_0)$ satisfy conditions (i), (ii), (iii).

In the cases of using the backstepping design procedure for systems of 'relative degree' of l, the projection laws may need to be lth order continuously differentiable. For these cases, one can use sufficiently smooth functions to replace the values of damping terms in the boundary layer $\delta_\theta\leqslant|\hat{\theta}-\hat{\theta}(0)|\leqslant\delta_\theta+\varepsilon_\theta$ and $\delta_{\zeta_0}(t)\leqslant|\zeta_0-\zeta_0(0)|\leqslant\delta_{\zeta_0}(t)+\varepsilon_{\zeta_0}$, just as done in a recent publication [21] on sufficiently smooth projection. In this sense, the continuous-projection laws (11) and (12) can be used as a typical example for all continuous or sufficiently smooth projections when comparing the designs using continuous or sufficiently smooth projections and the proposed discontinuous projection-based ARC designs. Hence in the following, only the essential differences

DOI: 10.1002/acs

between the design using the continuous-projections (11) and (12) and the proposed discontinuous projection-based ARC design are detailed, but the conclusions are valid for the use of smooth projections in general as well.

Combining the bounded state estimation design and the ARC design procedures in [6, 19], an ARC controller using the continuous-projections (11) and (12) instead of the discontinuous projections (3) and (6) can be easily worked out as well, which would have no formal mathematical flaws in terms of the existence and the uniqueness of the solutions.

S2. Though the continuous or the smooth projections as described above may be preferred by pure theoreticians, the resulting controller cannot achieve the same level of performance in implementation as the proposed discontinuous projection-based ARC design as detailed below:

(1) Nowadays almost all advanced nonlinear control laws have to be implemented approximately by a digital computer as there is no suitable hardware to truthfully implement complex nonlinear control laws in continuous time-domain. The essential treat of a digital computer lies in its ability to implement complex logic decisions in a straightforward way. In comparison, calculating values of complex nonlinear functions may need significant computation time, and, aside from the computation time issue, implementing a controller described by a set of differential equations having stiff nonlinearities in the right hand may have the issue of significant numerical approximation error problem. From this point of view, implementing the discontinuous projection-based controllers using a digital computer is rather straightforward and better conditioned in some sense than the continuous or smooth projection-based controllers. To see this, let us take a closer look at how the two classes of controllers are actually implemented by a digital computer. When the continuous-projections such as the ones in (11) and (12) are used in the parameter adaptation law and state estimation law in (10), the resulting controller would be described by a set of differential equations in which the right-hand side contains the continuous or smooth modification terms $l_\theta(\hat{\theta})$ and $l_{\zeta_0}(\zeta_0)$. Aside from the complexity of these modification terms in terms of real-time computation time needed[§], these modification terms tend to be very stiff during the transition periods when the parameter estimates are going out of their known ranges of Ω_θ, as the thicknesses of the boundary layers for the smoothing, ε_θ and ε_{ζ_0}, have to be very small for a better theoretically guaranteed control performance in general. The resulting controller is thus normally described by a set of differential equations having stiff nonlinearities in the right-hand side. It is well known that these classes of systems are hardly implemented well numerically by a digital computer. For example, when the Euler discretization method is used (normally done in actual implementation due to the much less online computation time needed), the parameter adaptation law (1) would be implemented as

$$\hat{\theta}((k+1)T) = \hat{\theta}(kT) + [\Gamma_\theta \tau_\theta + l_\theta(\hat{\theta})]|_{t=kT} T, \tag{13}$$

where T is the sampling period, $\hat{\theta}((k+1)T)$ and $\hat{\theta}(kT)$ are the values of parameter estimates at the sampling instances $(k+1)T$ and kT, respectively, and $\bullet|_{t=kT}$ represents the value of $\bullet$ at the sampling instance kT. As mentioned previously, when the parameter estimates start going out of the known bounded ranges of Ω_θ (i.e. wrong parameter adaptation in reality), the modification term $l_\theta(\hat{\theta})$ is stiff in the sense that $\partial l_{\hat{\theta}}/\partial\hat{\theta}$ is quite large due to the use of very small boundary layer thickness ε_θ for a better theoretically guaranteed control performance. Owing to the capacity limitation of hardware, the sampling period T in implementation cannot be chosen arbitrarily small. With this in mind, the moments when

[§]The sufficiently smooth projection in [21] is much more mathematically complex than the continuous modification in (11).

DOI: 10.1002/acs

these modification terms start acting, significant larger amount of numerical approximation errors of the parameter adaptation law (10) by the discretized version (13) in implementation could exist. This could easily lead to the undesirable chattering problem of parameter estimates at the boundary when the modification term $l_\theta(\hat{\theta})$ starts acting (i.e. when $|\hat{\theta}-\hat{\theta}(0)|=\delta_\theta$ in (11)). Furthermore, there is no guarantee that the resulting parameter estimates will actually lie within the pre-specified ranges that the original continuous or smooth modifications to the parameter adaptation law in continuous time-domain are supposed to achieve.

On the other hand, the digital implementation of the proposed discontinuous projection law is carried out by a combination of the usual approximation of differential equations having no stiff nonlinearities in the right-hand side and the logic operations that the discontinuous projection modification is supposed to achieve in continuous time-domain. As such it does not have the chattering problem of parameter estimates at the boundary and the resulting parameter estimates are kept within the pre-specified ranges of $\bar{\Omega}_\theta$ precisely. Specifically, noting that the real reason for the use of modifications to the parameter adaptation law is to keep the resulting parameter estimates within the known range of $\bar{\Omega}_\theta$ when the parameter estimates using the unmodified parameter adaptation law (i.e. $\dot{\hat{\theta}}=\Gamma_\theta\tau_\theta$) tends to go out of the known range of Ω_θ, the proposed discontinuous projection-based parameter adaptation law (10) is implemented digitally by

$$\hat{\theta}_u((k+1)T)=\hat{\theta}(kT)+[\Gamma_\theta\tau_\theta]|_{t=kT}T$$
$$\hat{\theta}_i((k+1)T)=\begin{cases}\theta_{i\max} & \text{if } \hat{\theta}_{iu}((k+1)T)>\theta_{i\max},\\ \theta_{i\min} & \text{if } \hat{\theta}_{iu}((k+1)T)<\theta_{i\min}, \quad i=1,\dots,p,\\ \hat{\theta}_{iu}((k+1)T) & \text{otherwise}\end{cases} \tag{14}$$

where $\hat{\theta}_u=[\hat{\theta}_{1u},\dots,\hat{\theta}_{pu}]^{\mathrm{T}}$. This implementation not only precisely guarantees that $\hat{\theta}\in\bar{\Omega}_\theta$, but also is free of the chattering problem of parameter estimates at the boundary when a stiff continuous or smooth modification term is used. In addition, much reduced online computation time is needed due to the avoid of the explicit calculation of the complex continuous or smooth modification terms.

(2) Aside from the more robust implementation of the proposed discontinuous projection-based adaptation law, the discontinuous projection-based backstepping ARC controller design is also better conditioned in implementation than the continuous or smooth projection-based backstepping ARC designs in [6, 7, 20] or the generalized σ-modification-based backstepping robust adaptive designs in [22]. Specifically, the backstepping ARC designs [7, 20] or the RAC design [22] are based on the brilliant tuning function-based adaptive backstepping design proposed in [4], which needs to incorporate the adaptation law in the design of control functions at each intermediate steps. As a result, the intermediate control functions involve the explicit calculation of the stiff smooth modification terms $l_\theta(\hat{\theta})$ and their higher derivatives, which will have exactly the same implementation problems as the smooth projection-based parameter adaptation law mentioned above. In contrast, the discontinuous projection-based ARC designs [23] such as the one detailed in this paper recognizes the implementation problems associated with the complex smooth modification terms and explicitly avoids the use of any direct cancellation of the parameter adaptation law in the design of intermediate control functions. Consequently, the resulting control law not only becomes simpler, but is also free from the possible numerical approximation errors associated with the stiff smooth modification terms and their higher derivatives.

(3) The existence and the uniqueness of the solutions of the proposed discontinuous projection-based ARC design in the implementation is not a significant issue either. As seen from the above explanations on

DOI: 10.1002/acs

the digital implementation of various control laws, there is no difficulty or ambiguity in the calculation of control output $u(kT)$ based on the proposed discontinuous projection-based control law at each sampling instances. With the zero-order hold (ZOH) circuity associated with the digital implementation, the actual input to the physical system to be controlled thus always exists and is uniquely determined by $u(t)=u(kT), \forall t \in [KT,(k+1)T)$. There should not be any issue on the existence and the uniqueness of the responses of the physical system to such a control input as well.

In summary, compared with the discontinuous projection-based adaptation laws, the continuous or smooth projection-based ones spend too much time on the practically not so meaningful smoothing process and get nothing out of it in the practical implementation except the mathematical completeness of the derivations in continuous time-domain. Instead of focusing on the practically not so meaningful problem of the existence and the uniqueness of the solutions, we believe that a control engineer should directly deal with the real issues in practical implementation and provide practically implementable solutions. In this regard, in this paper, we will keep using the discontinuous projection in the modification of parameter adaptation law and the state estimation law. It is also noted that the discontinuous projection-based ARC designs have been successfully implemented and tested in various applications [8, 10, 24–26].

3.3. *ARC controller design*

3.3.1. Step $1 \leqslant i \leqslant l-1$. To give initial values of the recursive function used in the backstepping design, we denote $\alpha_0(t)=y_{\mathrm{d}}(t)$.

At step i $(1 \leqslant i \leqslant l-1)$, let z_i be the error between the state x_i and the desired control signal α_{i-1}, then $z_i = x_i - \alpha_{i-1}$. Taking its derivative

$$\dot{z}_i = x_{i+1} + \theta^{\mathrm{T}} \varphi_{\theta i} + h_i + \Delta_i - \dot{\alpha}_{i-1}. \tag{15}$$

Noting that

$$\dot{\alpha}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \theta^{\mathrm{T}} \varphi_{\theta j} + h_j + \Delta_j) + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \alpha_{i-1}}{\partial t}, \tag{16}$$

we have

$$\begin{aligned} \dot{z}_i &= x_{i+1} + \theta^{\mathrm{T}} \varphi_{\theta i} + h_i + \Delta_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}[x_{j+1} + \theta^{\mathrm{T}} \varphi_{\theta j} + h_j + \Delta_j] - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{i-1}}{\partial t}, \\ &= x_{i+1} + \alpha_{ic} + \alpha_{iu}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \alpha_{ic} &= \hat{\theta}^{\mathrm{T}} \left(\varphi_{\theta i} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_{\theta j} \right) + h_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} h_j - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} - \frac{\partial \alpha_{i-1}}{\partial t}, \\ \alpha_{iu} &= -\tilde{\theta}^{\mathrm{T}} \left(\varphi_{\theta l} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_{\theta j} \right) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \Delta_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \Delta_j \end{aligned} \tag{18}$$

are computable and unknown parts, respectively.

DOI: 10.1002/acs

We construct a control function α_i for the virtual input x_{i+1} such that x_i tracks its desired control law α_{i-1} synthesized at step $i-1$.

$$\begin{aligned}
&\alpha_i(\bar{x}_i, \hat{\theta}, t) = \alpha_{ia} + \alpha_{is}, \\
&\quad \alpha_{ia} = -z_{i-1} - \alpha_{ic}, \quad \alpha_{is} = \alpha_{is1} + \alpha_{is2}, \quad \alpha_{is1} = -k_{is} z_i, \\
&\quad k_{is} \geqslant g_i + \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} C_{\theta i} \right|^2 + |C_{\phi i} \Gamma_\theta \phi_i|^2,
\end{aligned} \tag{19}$$

where g_i is a positive constant, $C_{\theta i}$ and $C_{\phi i}$ are positive constant diagonal matrices. Let $z_{i+1} = x_{i+1} - \alpha_i$ denote the input discrepancy. Substituting (19) into (17) leads to

$$\dot{z}_i + k_{Is} z_i = z_{i+1} - z_{i-1} + \alpha_{is2} - \tilde{\theta}^{\mathrm{T}} \phi_i + \tilde{\Delta}_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}, \tag{20}$$

where $\phi_i = \varphi_{\theta i} - \sum_{j=1}^{i-1} (\partial \alpha_{i-1}/\partial x_j) \varphi_{\theta j}$ and $\tilde{\Delta}_i = \Delta_i - \sum_{j=1}^{i-1} (\partial \alpha_{i-1}/\partial x_j) \Delta_j$ (let $\tilde{\Delta}_1 = \Delta_1$, $\phi_1 = \varphi_{\theta 1}$). Choosing $V_i = V_{i-1} + \frac{1}{2} z_i^2$, then its time derivative is

$$\dot{V}_i = z_i z_{i+1} + \sum_{j=1}^{i} \left[-k_{js} z_j^2 + z_j (\alpha_{js2} - \tilde{\theta}^{\mathrm{T}} \phi_j + \tilde{\Delta}_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} z_j \right]. \tag{21}$$

The ARC design can be applied to synthesize a robust control function α_{Is2} satisfying the following two conditions:

$$\begin{aligned}
&\text{(i)} \quad z_i (\alpha_{is2} - \tilde{\theta}^{\mathrm{T}} \phi_i + \tilde{\Delta}_i) \leqslant \varepsilon_i, \\
&\text{(ii)} \quad z_i \alpha_{is2} \leqslant 0,
\end{aligned} \tag{22}$$

where ε_i is a positive design parameter.

3.3.2. Step l. At the last step (step l), the derivative of $z_l = x_l - \alpha_{l-1}$ is

$$\begin{aligned}
\dot{z}_l &= u + \theta^{\mathrm{T}} \varphi_{\theta l} + \varphi_{\eta l}^{\mathrm{T}} \eta + h_l + \Delta_l - \dot{\alpha}_{l-1}, \\
&= u + \theta^{\mathrm{T}} \varphi_{\theta l} + \varphi_{\eta l}^{\mathrm{T}} (\zeta_0 + \zeta \theta - \varepsilon) + h_l + \Delta_l - \dot{\alpha}_{l-1}, \\
&= u + \theta^{\mathrm{T}} (\varphi_{\theta l} + \zeta^{\mathrm{T}} \varphi_{\eta l}) + \varphi_{\eta l}^{\mathrm{T}} \zeta_0 + h_l - \dot{\alpha}_{l-1} + \Delta_l - \varphi_{\eta l}^{\mathrm{T}} \varepsilon.
\end{aligned} \tag{23}$$

Noting that $\dot{\alpha}_{l-1} = \sum_{j=1}^{l-1} (\partial \alpha_{l-1}/\partial x_j)(x_{j+1} + \theta^{\mathrm{T}} \varphi_{\theta j} + h_j + \Delta_j) + (\partial \alpha_{l-1}/\partial \hat{\theta}) \dot{\hat{\theta}} + \partial \alpha_{l-1}/\partial t$, we have

$$\begin{aligned}
\dot{z}_l &= u + \theta^{\mathrm{T}} (\varphi_{\theta l} + \zeta^{\mathrm{T}} \varphi_{\eta l}) + \varphi_{\eta l}^{\mathrm{T}} \zeta_0 + h_l + \Delta_l - \varphi_{\eta l}^{\mathrm{T}} \varepsilon - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j} (x_{j+1} + \theta^{\mathrm{T}} \varphi_{\theta j} + h_j + \Delta_j) - \frac{\partial \alpha_{l-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{l-1}}{\partial t}, \\
&= u + \alpha_{lc} + \alpha_{lu},
\end{aligned} \tag{24}$$

DOI: 10.1002/acs

where

$$\alpha_{lc} = \hat{\theta}^{\mathrm{T}}\left(\varphi_{\theta l}+\zeta^{\mathrm{T}}\varphi_{\eta l}-\sum_{j=1}^{l-1}\frac{\partial\alpha_{l-1}}{\partial x_j}\varphi_{\theta j}\right)+\varphi_{\eta l}^{\mathrm{T}}\zeta_0+h_l-\sum_{j=1}^{l-1}\frac{\partial\alpha_{l-1}}{\partial x_j}h_j-\sum_{j=1}^{l-1}\frac{\partial\alpha_{l-1}}{\partial x_j}x_{j+1}-\frac{\partial\alpha_{l-1}}{\partial t},$$
$$\alpha_{lu} = -\tilde{\theta}^{\mathrm{T}}\left(\varphi_{\theta l}+\zeta^{\mathrm{T}}\varphi_{\eta l}-\sum_{j=1}^{l-1}\frac{\partial\alpha_{l-1}}{\partial x_j}\varphi_{\theta j}\right)-\frac{\partial\alpha_{l-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}-\varphi_{\eta l}^{\mathrm{T}}\varepsilon+\Delta_l-\sum_{j=1}^{l-1}\frac{\partial\alpha_{l-1}}{\partial x_j}\Delta_j. \tag{25}$$

We construct the control input u such that x_l tracks its desired ARC control law α_{l-1} synthesized at step $l-1$.

$$u(x,\zeta_0,\zeta,\hat{\theta},t) = \alpha_{la}+\alpha_{ls},$$
$$\alpha_{la} = -z_{l-1}-\alpha_{lc},\quad \alpha_{ls}=\alpha_{ls1}+\alpha_{ls2},\quad \alpha_{ls1}=-k_{ls}z_l,$$
$$k_{ls}\geqslant g_l+\left|\frac{\partial\alpha_{l-1}}{\partial\hat{\theta}}C_{\theta l}\right|^2+|C_{\phi l}\Gamma_\theta\phi_l|^2+c_\theta|\psi_l|^2, \tag{26}$$

where g_l and c_θ are positive constants, $\psi_l=\varphi_{\eta l}$, $C_{\theta l}$ and $C_{\phi l}$ are positive constant diagonal matrices to be specified later. Let $z_{l+1}=x_{l+1}-\alpha_l$ denote the input discrepancy. Substituting (26) into (24) leads to

$$\dot{z}_l+k_{ls}z_l=-z_{l-1}+\alpha_{ls2}-\tilde{\theta}^{\mathrm{T}}\phi_l-\psi_l^{\mathrm{T}}\varepsilon+\tilde{\Delta}_l-\frac{\partial\alpha_{l-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}, \tag{27}$$

where $\phi_l=\varphi_{\theta l}+\zeta^{\mathrm{T}}\varphi_{\eta l}-\sum_{j=1}^{l-1}(\partial\alpha_{l-1}/\partial x_j)\varphi_{\theta j}$ and $\tilde{\Delta}_l=\Delta_l-\sum_{j=1}^{l-1}(\partial\alpha_{l-1}/\partial x_j)\Delta_j$. Choosing $V_l=V_{l-1}+\frac{1}{2}z_l^2$, then its time derivative is

$$\dot{V}_l=\sum_{j=1}^{l}\left[-k_{js}z_j^2+z_j(\alpha_{js2}-\tilde{\theta}^{\mathrm{T}}\phi_j-\psi_j^{\mathrm{T}}\varepsilon+\tilde{\Delta}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}z_j\right], \tag{28}$$

where $\psi_j^{\mathrm{T}}=0,\ \forall j<l$. The ARC design can be applied to synthesize a robust control function α_{ls2} satisfying the following two conditions:

$$\text{(i)}\quad z_l(\alpha_{ls2}-\tilde{\theta}^{\mathrm{T}}\phi_l-\psi_l^{\mathrm{T}}\varepsilon+\tilde{\Delta}_l)\leqslant\varepsilon_l,$$
$$\text{(ii)}\quad z_l\alpha_{ls2}\leqslant 0. \tag{29}$$

Remark 4

One smooth example of α_{ls2} satisfying (29) can be found in the following way. Let h_l be any nth order continuous function satisfying

$$h_l\geqslant|\theta_M||\phi_l|+|\psi_l||\Omega_{\zeta_0}|+\tilde{\delta}_l, \tag{30}$$

where $\theta_M\triangleq\theta_{\max}-\theta_{\min}$ and $\tilde{\delta}_l\triangleq\sum_{j=1}^{l-1}|\partial\alpha_{l-1}/\partial x_j|\delta_j+\delta_l$. $|\Omega_{\zeta_0}|$ is the length of the set Ω_{ζ_0}, i.e. the maximum distance between any two points in Ω_{ζ_0}. Then α_{ls2} can be chosen as

$$\alpha_{ls2}=-\frac{1}{4\varepsilon_l}h_l^2z_l. \tag{31}$$

It is easy to verify that this choice of α_{ls2} satisfies (29).

DOI: 10.1002/acs

3.4. Main results

Theorem 1

Let the parameter estimates be updated by the adaptation law (3) in which τ_θ is chosen as

$$\tau_\theta = \sum_{j=1}^{l} \phi_j z_j \tag{32}$$

and τ_η is chosen as

$$\tau_\eta = \sum_{j=1}^{l} z_j \psi_j. \tag{33}$$

Let $c_{\theta ji}$ and $c_{\phi ki}$ be the ith diagonal elements of the diagonal matrices $C_{\theta j}$ and $C_{\phi k}$, respectively. If the controller parameters $C_{\theta j}$ and $C_{\phi k}$ are chosen such that $c^2_{\phi ki} \geqslant (n/4)\sum_{j=2}^{l} 1/c^2_{\theta ji}$, $\forall k, i$, then, the control law (26) guarantees that

A. In general, all signals are bounded. Furthermore, the positive-definite function V_l is bounded above by

$$V_l(t) \leqslant e^{-\lambda_l t} V_l(0) + \frac{\sum_{j=1}^{l} \varepsilon_j}{\lambda_l}(1 - e^{-\lambda_l t}), \tag{34}$$

where $\lambda_l = 2\min\{g_1, \ldots, g_l\}$.

B. If after a finite time t_0, there exist parametric uncertainties only (i.e. $\Delta_\eta = 0$ and $\Delta_i = 0$, $\forall t \geqslant t_0$), then, in addition to results in A, zero final output tracking error is also achieved, i.e $z_1 \longrightarrow 0$ and $t \longrightarrow \infty$.

Proof 2

For part A, from (19), (26) and (28), we have

$$\dot{V}_l \leqslant \sum_{j=1}^{l} \left\{ \left(-g_j - \left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 - |C_{\phi j} \Gamma_\theta \phi_j|^2 - c_\theta |\psi_j|^2 \right) z_j^2 + z_j(\alpha_{js2} - \tilde{\theta}^{\mathrm{T}} \phi_j - \psi_j^{\mathrm{T}} \varepsilon + \tilde{\Delta}_j) - z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right\}. \tag{35}$$

By completion of square

$$-\sum_{j=2}^{l} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \leqslant \left| \sum_{j=2}^{l} |z_j| \left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} C_{\theta j}^{-1} \dot{\hat{\theta}} \right| \right| \leqslant \sum_{j=2}^{l} \left(\left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 z_j^2 + \frac{1}{4} |C_{\theta j}^{-1} \dot{\hat{\theta}}|^2 \right). \tag{36}$$

Noting that $C_{\theta j}^{-1}$ and Γ_θ are diagonal matrices, from (3) and (4), we have

$$\begin{aligned}\sum_{j=2}^{l} |C_{\theta j}^{-1} \dot{\hat{\theta}}|^2 &= \sum_{j=2}^{l} |C_{\theta j}^{-1} \operatorname{Proj}_{\hat{\theta}}(\Gamma_\theta \tau)|^2 \leqslant \sum_{j=2}^{l} |C_{\theta j}^{-1} \Gamma_\theta \tau|^2, \\ &\leqslant \sum_{j=2}^{l} \left(\sum_{k=1}^{l} |C_{\theta j}^{-1} \Gamma_\theta \phi_k z_k| \right)^2 \leqslant l \sum_{j=2}^{l} \left(\sum_{k=1}^{l} |C_{\theta j}^{-1} \Gamma_\theta \phi_k|^2 z_k^2 \right).\end{aligned} \tag{37}$$

Thus, if $C_{\theta j}$ and $C_{\phi k}$ satisfy the conditions in the theorem, we have

$$\begin{aligned}-\sum_{j=2}^{l} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} &\leqslant \sum_{j=2}^{l} \left(\left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 z_j^2 + \frac{l}{4} \sum_{k=1}^{l} |C_{\theta j}^{-1} \Gamma_\theta \phi_k|^2 z_k^2 \right), \\ &\leqslant \sum_{j=2}^{l} \left| \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j} \right|^2 z_j^2 + \sum_{k=1}^{l} |C_{\phi k}^{-1} \Gamma_\theta \phi_k|^2 z_k^2.\end{aligned} \tag{38}$$

DOI: 10.1002/acs

From (35) and the properties of each α_{js2}, we have

$$\dot{V}_l \leqslant \sum_{j=1}^{l}(-g_j z_j^2+\varepsilon_j) \leqslant -\lambda_l V_l+\sum_{j=1}^{l}\varepsilon_j, \tag{39}$$

which leads to (34). The boundedness of z_j is thus proved. Using the standard arguments in the backstepping designs [4], it can be proved that all internal signals in the first $l-1$ steps are globally uniformly bounded. Furthermore, since $x_l = z_l+\alpha_{l-1}$, x_l is also bounded. Thus, $x=[x_1,\ldots,x_l]^{\mathrm{T}}$ are bounded. From the bounded-input–bounded-state Assumption 4, the projection-type estimation algorithm of ζ_0 and the bounded internal state Assumption 2, ζ_0, ζ and η are all bounded. Recursively using the fact that $x_i = z_i+\alpha_{i-1}$, it is obvious that α_i and x_i are bounded. Thus, the boundedness of u is apparent. This proves part A.

For part B, when $\Delta_\eta = 0$ and $\Delta_i = 0$, from (35) and (38), noting the condition (ii) of (29) and (22), we have

$$\dot{V}_l \leqslant \sum_{j=1}^{l}(-g_j z_j^2 - c_\theta|\psi_j|^2 z_j^2 - z_j\tilde{\theta}^{\mathrm{T}}\phi_j - z_j\psi_j^{\mathrm{T}}\varepsilon). \tag{40}$$

Define a new p.s.d function V_a as

$$V_a = V_l+\tfrac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma_\theta^{-1}\tilde{\theta}+\tfrac{1}{2}\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}\varepsilon. \tag{41}$$

Then

$$\begin{aligned}
\dot{V}_a &= \dot{V}_l+\tilde{\theta}^{\mathrm{T}}\Gamma_\theta^{-1}\dot{\tilde{\theta}}+\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}\dot{\varepsilon},\\
&\leqslant \sum_{j=1}^{l}(-g_j z_j^2 - c_\theta|\psi_j|^2 z_j^2 - z_j\tilde{\theta}^{\mathrm{T}}\phi_j - z_j\psi_j^{\mathrm{T}}\varepsilon)+\tilde{\theta}^{\mathrm{T}}\Gamma_\theta^{-1}\,\mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta\tau_\theta)+\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}(\dot{\zeta}_0+\dot{\zeta}\theta-\dot{\eta}),\\
&= \sum_{j=1}^{l}(-g_j z_j^2 - c_\theta|\psi_j|^2 z_j^2 - z_j\tilde{\theta}^{\mathrm{T}}\phi_j - z_j\psi_j^{\mathrm{T}}\varepsilon)+\tilde{\theta}^{\mathrm{T}}\Gamma_\theta^{-1}\,\mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta\tau_\theta)\\
&\quad +\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}[\mathrm{Proj}_{\zeta_0}(G_\eta\zeta_0+H_\eta+\Gamma_\eta\tau_\eta)+(G_\eta\zeta+F_\eta)\theta-F_\eta\theta-G_\eta\eta-H_\eta],\\
&= \sum_{j=1}^{l}(-g_j z_j^2 - c_\theta|\psi_j|^2 z_j^2)-\tilde{\theta}^{\mathrm{T}}\tau_\theta-\varepsilon^{\mathrm{T}}\tau_\eta+\tilde{\theta}^{\mathrm{T}}\Gamma_\theta^{-1}\mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta\tau_\theta)\\
&\quad +\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}[\mathrm{Proj}_{\zeta_0}(G_\eta\zeta_0+H_\eta+\Gamma_\eta\tau_\eta)+G_\eta(\varepsilon-\zeta_0)-H_\eta],\\
&= \sum_{j=1}^{l}(-g_j z_j^2 - c_\theta|\psi_j|^2 z_j^2)+\tilde{\theta}^{\mathrm{T}}(\Gamma^{-1}\,\mathrm{Proj}_{\hat{\theta}}(\Gamma\tau_\theta)-\tau_\theta)\\
&\quad +\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}[\mathrm{Proj}_{\zeta_0}(G_\eta\zeta_0+H_\eta+\Gamma_\eta\tau_\eta)-G_\eta\zeta_0-H_\eta-\Gamma_\eta\tau_\eta]+\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}G_\eta\varepsilon.
\end{aligned} \tag{42}$$

Since $\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}G_\eta\varepsilon = \frac{1}{2}\varepsilon^{\mathrm{T}}(\Gamma_\eta^{-1}G_\eta(x)+G_\eta^{\mathrm{T}}(x)\Gamma_\eta^{-1})\varepsilon$, from Assumption 2, $\varepsilon^{\mathrm{T}}\Gamma_\eta^{-1}G_\eta\varepsilon \leqslant 0$. Using (5) and (8), we have

$$\dot{V}_a \leqslant \sum_{j=1}^{l} -g_j z_j^2, \tag{43}$$

from which $z_j \in L_2[0, \infty)$. It is also easy to check that $\dot{z}_j$ is bounded. Hence, by the Barbalat's lemma, $z \to 0$ as $t \to 0$, which proves part B of Theorem 1.

4. PRACTICAL DESIGN EXAMPLE AND SIMULATION RESULTS

In order to see how the proposed algorithm can be applied, a practical design example is given here. The system in this practical example has bounded but non-uniformly detectable internal states, which is exactly the type of nonlinear system we considered in this paper.

4.1. Systems with dynamic friction

Nowadays, the control of mechanical systems with dynamic friction has become increasingly popular. A kind of friction model called the LuGre model [11] has seen wide application. Now we consider a linear motor-driven stage with dynamic friction existing between the contact surfaces. With the LuGre model proposed in [11, 12], the dynamic equation of the system can be written as

$$\dot{z} = \dot{x} - \frac{|\dot{x}|}{g(\dot{x})} z + \Delta_z, \quad g(\dot{x}) = \alpha_0 + \alpha_1 \mathrm{e}^{-(\dot{x}/v_s)^2}, \tag{44}$$

$$\begin{aligned} m\ddot{x} &= Ku - \sigma_0 z - \sigma_1 h(\dot{x})\dot{z} - \alpha_2 \dot{x} + \Delta_m, \\ &= Ku - \sigma_0 z - \sigma_1 h(\dot{x}) \left[\dot{x} - \frac{|\dot{x}|}{g(\dot{x})} z + \Delta_z \right] - \alpha_2 \dot{x} + \Delta_m, \end{aligned} \tag{45}$$

where m is the mass of the stage, u is the input voltage, K is the gain from voltage to the force applied to the stage, z represents the unmeasurable internal friction state, σ_0, σ_1, α_2 are unknown friction force parameters that can be physically explained as the stiffness, the damping coefficient of bristles and the viscous friction coefficient. x, $\dot{x}$ are the position and the velocity of linear motor, respectively. The function $g(\dot{x})$ is positive and it describes the Stribeck effect: $\sigma_0\alpha_0$ and $\sigma_0(\alpha_0+\alpha_1)$ represent the levels of the Coulomb friction and stiction force, respectively, and v_s is the Stribeck velocity. Δ_z and Δ_m represent the modeling errors of dynamic frictions and the disturbances of the stage, respectively. Let $y_d(t)$ be the desired motion trajectory, which is assumed to be known, bounded, with bounded derivatives up to the second order. We want to design a control law u, such that the output x can track $y_d(t)$ as close as possible, in spite of various uncertainties.

In (45), the dynamic friction $F_{dyn} = -\sigma_0 z - \sigma_1 h(\dot{x})[\dot{x} - (|\dot{x}|/g(\dot{x}))z + \Delta_z]$ is a complicated nonlinear function of three unknown variables: z, σ_0 and σ_1. As such, it is not possible to use an one-dimensional observer to estimate the internal state z directly as some sorts of adaptation laws are needed to estimate σ_0 and σ_1 online as well. Noting that the three unknown variables appear in F_{dyn} only through $\sigma_0 z$ and $\sigma_1 z$, we do not have to separate the estimations of z, σ_0 and σ_1 for dynamic friction compensation. Instead, we simply treat $\sigma_0 z$ and $\sigma_1 z$ as two different variables and estimate them directly through certain observers. Thus, denote $\eta = [\sigma_0 z/K \quad \sigma_1 z/K]^T$ as the unmeasured augmented internal states and $\Delta_x = (\sigma_1 h(\dot{x})\Delta_z + \Delta_m)/K$ as the lumped model uncertainty. Let $\theta = [\sigma_0/K \quad \sigma_1/K \quad \alpha_2/K \quad \bar{\Delta}_x]^T$ and $\theta_m = m/K$ be the unknown parameters with known bounds, in which $\bar{\Delta}_x$ represents the constant component of the lumped model uncertainty Δ_x. Denote $\tilde{\Delta}_x$ as the time-varying portion of Δ_x and define $x = [x_1 \quad x_2]^T = [x \quad \dot{x}]^T$. Then the system can be represented by

$$\begin{aligned} \dot{\eta} &= F_\eta \theta + G_\eta \eta + \Delta_\eta, \\ \dot{x}_1 &= x_2, \end{aligned}$$

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$$\begin{aligned} \theta_m \dot{x}_2 &= u + \theta^{\mathrm{T}} \varphi_\theta + \varphi_\eta^{\mathrm{T}} \eta + \Delta_2, \\ y &= x_1, \end{aligned} \tag{46}$$

where

$$F_\eta = \begin{bmatrix} x_2 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \end{bmatrix}, \tag{47}$$

$$G_\eta = \begin{bmatrix} -\dfrac{|x_2|}{g(x_2)} & 0 \\ 0 & -\dfrac{|x_2|}{g(x_2)} \end{bmatrix}, \tag{48}$$

$$\varphi_\theta = [0 \;\; -h(x_2)x_2 \;\; -x_2 \;\; 1]^{\mathrm{T}}, \tag{49}$$

$$\varphi_\eta = \left[-1 \;\; \frac{h(x_2)|x_2|}{g(x_2)}\right]^{\mathrm{T}}, \tag{50}$$

$$\Delta_\eta = [\sigma_0 \Delta_z \;\; \sigma_1 \Delta_z]^{\mathrm{T}}, \tag{51}$$

$$\Delta_2 = \tilde{\Delta}_x. \tag{52}$$

This system is of form (1), except that the unknown parameter θ_m appears in front of $\dot{x}_2$. For such a case, we only need to make slight modifications to the previously proposed algorithm, as shown later in this section. Now, we will show that this system satisfies all the assumptions made in Section 2 and the internal states are non-uniformly detectable.

Since the physical meanings of all unknown parameters in (46) are known, it is safe to assume that the unknown parameters, uncertain nonlinearities and disturbances are bounded by known bounds. Thus Assumption 1 is satisfied. It can also be seen that the pair $(\varphi_\eta^{\mathrm{T}}, G_\eta)$ is not uniformly detectable. Because at $x = [x_1 \;\; 0]^{\mathrm{T}}$, we have $G_\eta = \mathbf{0}^{2\times 2}$. Then for any $\omega(x) \in R^2$, the matrix

$$A(x) = G_\eta(x) - \frac{\partial \omega}{\partial x_2} \varphi_\eta^{\mathrm{T}} = \begin{bmatrix} \dfrac{\partial \omega_1}{\partial x_2} & 0 \\ \dfrac{\partial \omega_2}{\partial x_2} & 0 \end{bmatrix}$$

will always have zeros in the second column. Thus, techniques in [17, 18] do not apply here. However, since z represents the deflection of bristles between the contact surfaces, z is physically bounded by a fix bound [11]. Then $\eta = [\sigma_0 z/K \;\; \sigma_1 z/K]^{\mathrm{T}}$ is also bounded. Then it is easy to see that Assumption 2 is satisfied.

Since $G_\eta(x)$ is a diagonal negative semi-definite matrix for all x, for any diagonal matrix $\Gamma_\eta > 0$, we have $\Gamma_\eta^{-1} G_\eta(x) + G_\eta^{\mathrm{T}}(x) \Gamma_\eta^{-1} = 2\Gamma_\eta^{-1} G_\eta(x) \leqslant 0, \; \forall \, x \in R^2$. Hence Assumption 3 is satisfied. It can also be easily checked that Assumption 4 is satisfied. Since all assumptions required for the system are satisfied, we can use the technique proposed in this paper to design a control law.

DOI: 10.1002/acs

4.2. Control law design

Letting $\alpha_0 = y_d(t)$, and $z_1 = x_1 - \alpha_0$ be the tracking error, then, $\dot{z}_1 = \dot{x}_1 - \dot{\alpha}_0 = x_2 - \dot{\alpha}_0$. Selecting $\alpha_1 = -k_1 z_1 + \dot{\alpha}_0$ to be the desired x_2, and defining $z_2 = x_2 - \alpha_1$, then

$$\theta_m \dot{z}_2 = u + \theta^{\mathrm{T}}(\varphi_\theta + \zeta\varphi_\eta) + \varphi_\eta^{\mathrm{T}}\zeta_0 - \varphi_\eta^{\mathrm{T}}\varepsilon + \Delta_2 + k_1\theta_m\dot{z}_1 - \theta_m\ddot{\alpha}_0. \tag{53}$$

Then the state estimator and the parameter adaptation laws are chosen as

$$\begin{aligned} \dot{\zeta}_0 &= \mathrm{Proj}_{\zeta_0}(G_\eta\zeta_0 + \Gamma_\eta\varphi_\eta z_2), \quad \dot{\zeta} = G_\eta\zeta + F_\eta, \\ \dot{\hat{\theta}} &= \mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta(\varphi_\theta + \zeta^{\mathrm{T}}\varphi_\eta)z_2), \quad \dot{\hat{\theta}}_m = \mathrm{Proj}_{\hat{\theta}_m}[\gamma_{\theta_m}(k_1\dot{z}_1 - \ddot{\alpha}_0)z_2]. \end{aligned} \tag{54}$$

In this case, Ω_η is a square set, i.e. $\Omega_\eta = [\eta_{\min 1}, \eta_{\max 1}] \times [\eta_{\min 2}, \eta_{\max 2}]$. From the special structure of F_η and G_η, it is obvious that only ζ_{11} and ζ_{22} take effect. With this fact, $\Omega_{\zeta_0} = [\zeta_{0\min 1}, \zeta_{0\max 1}] \times [\zeta_{0\min 2}, \zeta_{0\max 2}]$, where

$$\begin{aligned} \zeta_{0\max i} &= \eta_{\max i} + \sup_{t>0}[\max(|\theta_{\max i}|, |\theta_{\min i}|)|\zeta_{ii}(t)|], \\ \zeta_{0\min i} &= \eta_{\min i} - \sup_{t>0}[\max(|\theta_{\max i}|, |\theta_{\min i}|)|\zeta_{ii}(t)|], \end{aligned} \tag{55}$$

for $i = 1, 2$. α_{2s1} is chosen as $\alpha_{2s1} = -k_2 z_2$. For α_{2s2}, we use the form given by (31): $\alpha_{2s2} = -(1/4\varepsilon)h^2 z_2$. h is chosen to be the right side of (30), which is a continuous function with respect to φ_θ, φ_η, ζ. The control law is thus given by

$$u = -k_2 z_2 + \alpha_{2s2} - \hat{\theta}^{\mathrm{T}}(\varphi_\theta + \zeta^{\mathrm{T}}\varphi_\eta) - \varphi_\eta^{\mathrm{T}}\zeta_0 - \hat{\theta}_m(k_1\dot{z}_1 + \ddot{\alpha}_0). \tag{56}$$

4.3. Simulation results

For simulation, we choose the system parameters to be the same as that used in [27], i.e. $\theta_m = 0.12$, $\theta = [7000\ \ 1176\ \ 0.166\ \ 0]^{\mathrm{T}}$, $g(x_2) = (0.1236 + 0.0861\,\mathrm{e}^{-|x_2/0.0022|})/7000$, $h(x_2) = 0.00013/(0.00013 + |x_2|)$. The internal state z is within ± 0.00005. In addition, we set the uncertain nonlinearity term $\Delta_m = 0.1\sin((2\pi/0.02)x_1) + 0.1\cos((2\pi/0.02)x_1) + 0.02\sin((4\pi/0.02)x_1) + 0.02\cos((4\pi/0.02)x_1)$ to simulate the effect of cogging forces on the linear motor, and $\Delta_\eta = n(t)|x_2|$ to represent the modeling error of the internal state dynamics where $n(t)$ is a uniformly distributed random number between -0.2 and 0.2.

The controller parameters are chosen as $\hat{\theta}_m(0) = 0.1$, $\hat{\theta}(0) = [6000\ \ 1100\ \ 0.2\ \ 0]^{\mathrm{T}}$, $\theta_{m\max} = 0.2$, $\theta_{m\min} = 0.08$, $\theta_{\max} = [10000\ \ 1500\ \ 0.5\ \ 0.5]^{\mathrm{T}}$, $\theta_{\max} = [4000\ \ 500\ \ 0\ \ -0.5]^{\mathrm{T}}$, $k_1 = 50$, $k_2 = 10$, $\varepsilon = 0.008$, $\gamma_{\theta_m} = 5 \times 10^8$, $\Gamma_\theta = \mathrm{diag}\{2.5 \times 10^{11}\ \ 2.5 \times 10^9\ \ 10\ \ 500\}$, $\Gamma_\eta = \mathrm{diag}\{1500\ \ \ 200\}$, $\eta_{\max} = [z_{\max}\cdot\theta_{1\max}\ \ z_{\max}\cdot\theta_{2\max}]^{\mathrm{T}} = [0.5\ \ 0.075]^{\mathrm{T}}$, $\eta_{\min} = [z_{\min}\cdot\theta_{1\max}\ \ z_{\min}\cdot\theta_{2\max}]^{\mathrm{T}} = [-0.5\ \ -0.075]^{\mathrm{T}}$, $\zeta_0(0) = [0\ \ 0]^{\mathrm{T}}$ and $\zeta(0) = \mathbf{0}^{2\times 4}$.

The design trajectory is chosen as a sinusoidal signal, with the amplitude of 0.002 and the frequency of 1 Hz. To see the transient performances, we set the initial value of the unknown internal state z to be 0.00003. The tracking error with disturbances added to the system is plotted in Figure 1 and the input is shown in Figure 2. As can be seen from the plots, the tracking error converges very fast after the first few cycles, showing a good transient performance and final tracking accuracy. Furthermore, the input signal is bounded, and the tracking error is less than 0.05% magnitude of the desired trajectory in spite of large disturbances and modeling errors, showing the robustness and good capability of disturbance rejection of the proposed ARC algorithm.

Then, we keep all parametric uncertainties but remove all the disturbances and unstructured modeling errors, and use the same trajectory, same initial internal state value and same controller parameters. The tracking error is

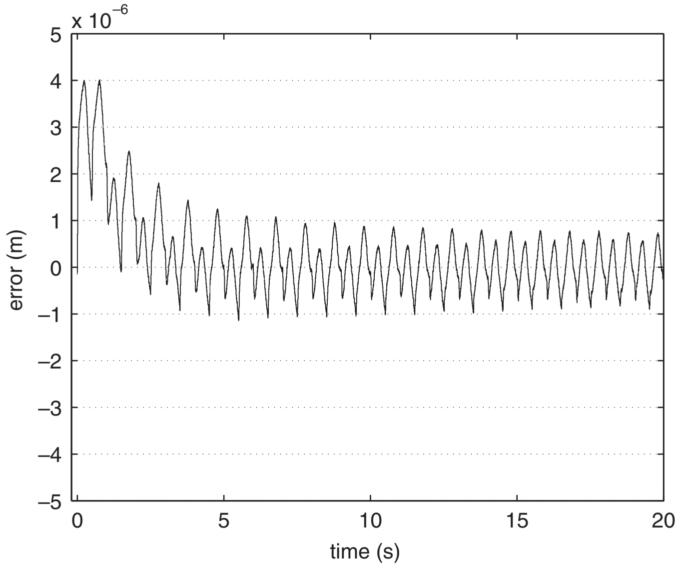

Figure 1. Tracking error, system with disturbances.

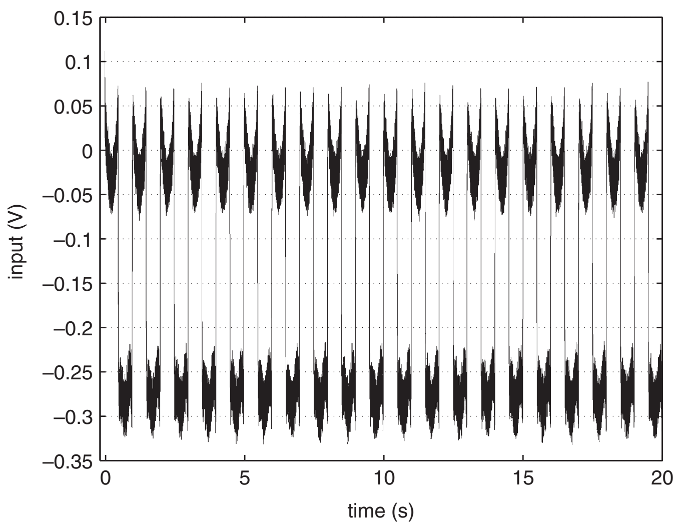

Figure 2. Input, system with disturbances.

plotted in Figure 3 and the input is shown in Figure 4. As can be seen from the plots, in the presence of parameters uncertainties only, asymptotic output tracking is achieved. All these results demonstrate the applicability of the proposed theoretical method in practical design cases.

DOI: 10.1002/acs

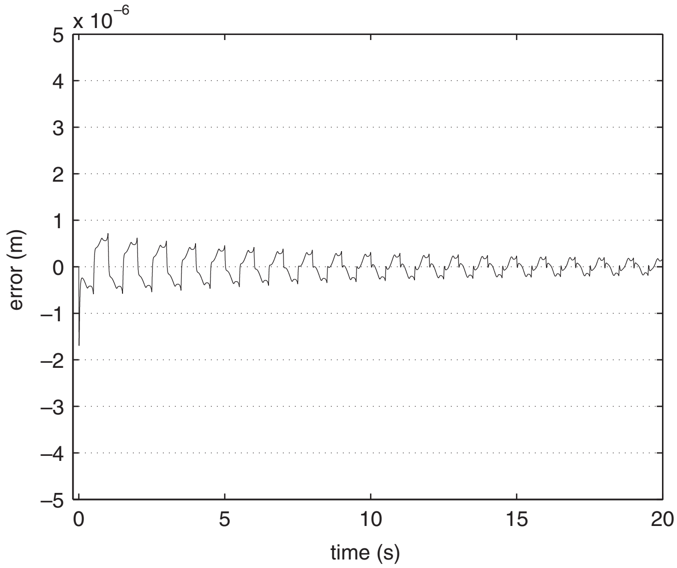

Figure 3. Tracking error, system without disturbances.

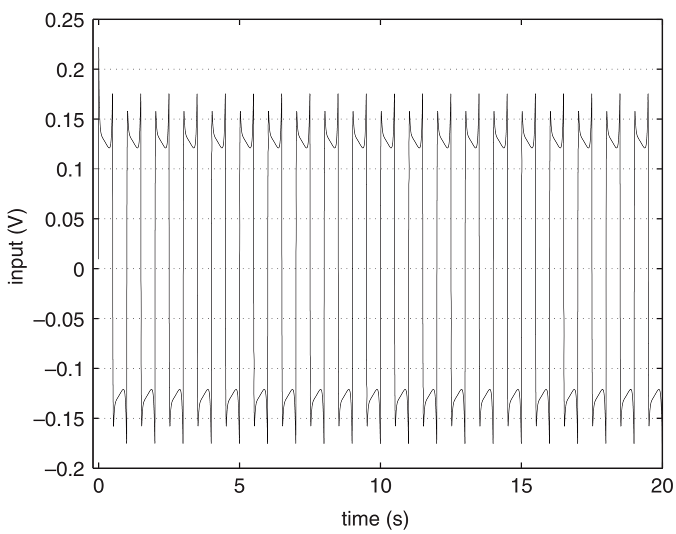

Figure 4. Input, system without disturbances.

5. CONCLUSION

In this paper, a discontinuous projection-based ARC algorithm has been designed for a class of nonlinear systems in semi-strict feedback form with bounded but non-uniformly detectable internal states. Specifically, discontinuous projection algorithm has been used to give the estimation of both the internal states and the unknown parameters.

DOI: 10.1002/acs

This algorithm has been theoretically proved to be robust to disturbances and uncertain nonlinearities with guaranteed transient performance while having asymptotic output tracking performance in the presence of parametric uncertainties only. The class of system considered in this paper has a lot of applications, and a practical example—control of mechanical systems with dynamic friction—is used for case study. The simulation results demonstrate the applicability of the proposed control methodology.

ACKNOWLEDGEMENTS

The work is supported in part by the U.S. National Science Foundation (Grant No. CMS-0600516) and in part by the National Natural Science Foundation of China (NSFC) under the Joint Research Fund for Overseas Chinese Young Scholars (Grant No. 50528505) and the Ministry of Education of China through a Chang Jiang Chair Professorship.

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DOI: 10.1002/acs