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Set-Membership Identification Based Adaptive Robust Control of Systems With Unknown Parameter Bounds

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Abstract— **In this paper, a hybrid control architecture is proposed for the adaptive robust control of a class of nonlinear systems with uncertain parameter variation ranges. Specifically, the standard set-membership description of uncertainty is adopted – the bounds of the structural approximation errors associated with the parametrized models are assumed to be known but the variation ranges of model parameters are not available or poorly known. To effectively control this class of systems, set-membership identification (SMI) is performed in discrete-time domain and a simple bound-shrinking algorithm is developed to obtain non-conservative real-time estimation of the regions where model parameters could actually be. The estimated parameter variation bounds are subsequently used to construct a continuous-time domain projection type parameter adaptation law with varying boundaries to achieve a controlled learning process. An adaptive robust control (ARC) algorithm is then synthesized to handle the effect of both parametric uncertainties and the model approximation error effectively. It is theoretically shown that in general the proposed approach achieves a guaranteed transient and steady-state output tracking performance. In addition, asymptotic output tracking can also be achieved when certain conditions hold.**

I. INTRODUCTION

Set-membership identification (SMI) has been studied back in 1980's [5]. It is a system identification technique assuming that the system to be identified has bounded noises rather than statistical noises in the least-square estimation (LSE) case. With this assumption, SMI algorithms compute a valid set for the unknown parameters to lie in. Since the bounded noise assumption is more realistic than the statistical noise assumption in a lot of practical applications, SMI soon gained popularity among researches in the field of system identification. Many different types of SMI algorithms have been developed, such as the the optimal bounding ellipse (OBE) algorithm [5], [4], and the optimal sequential parallelotope (OSP) algorithm [9]. Among them, the OBE algorithm is the most widely used one due to the easiness of implementing it online and the availability of convergence analysis [5], [1], [3].

In spite of the thriving theoretical researches in SMI, there have been very few studies on its use in control designs [7]. This might be due to the following several practical problems of using SMI for real-time controls. Firstly, convergence analysis of SMI is quite complicated and the resulting conditions for the convergence of parameter estimates to their true values are usually too stringent to be practically meaningful. For example, the convergence condition in [1] assumes that noises have to satisfy certain pre-specified bounds. Secondly, existing SMI algorithms are not very robust as they sometimes experience divergence [3], especially in presence of outliers. Though some remedies have been proposed to avoid divergence problem, the resulting SMI algorithms become quite complicated and their effects on the achievable control performance are even harder to analyze.

During the past decade, a novel control strategy, the adaptive robust control (ARC), has been developed for the control of uncertain nonlinear systems [12]. The approach incorporates the merits of deterministic robust control (DRC) and adaptive control (AC), which guarantees certain robust performance in presence of uncertainties while having the capability of asymptotic output tracking in absence of the unstructured uncertainties. These two advantages make it suitable to be applied to the real systems. Many good results have been obtained [10], [11]. However, there are still many interesting issues in ARC. For example, if the bounds of the parameters are unknown or too conservative, then how to design an ARC algorithm to effectively control the system and to estimate the parameters of the system more accurately is a very challenging research topic. [6] used fixed guesses for the bounds of the unknown parameters. However, if the guessed values of the bounds are fixed, then it must be conservative in order to contain the real value of the parameters. Using fixed large bounds may lead to a series of problems regarding the control performances and parameter estimations which will be detailed in this paper.

To this end, because SMI can give a good estimate of the parameter bound, it is natural to use SMI algorithm to shrink the bound in the ARC design. However, many things have to be taken care of to bypass the non-robustness problem of SMI. In this paper, a hybrid control architecture is proposed for the adaptive robust control of a class of nonlinear systems with uncertain parameter variation ranges. Specifically, the bounds of the structural approximation errors associated with the parametrized models are assumed to be known but the variation ranges of model parameters are not available or poorly known. To effectively control this class of systems, SMI is performed in discrete-time domain and a simple bound-shrinking algorithm is developed to obtain non-conservative real-time estimation of the regions where model parameters could actually be. The estimated parameter variation bounds are subsequently used to construct a continuous-time domain projection type parameter adaptation law with varying boundaries to achieve

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a controlled learning process. An adaptive robust control (ARC) algorithm is then synthesized to handle the effect of both parametric uncertainties and the model approximation error effectively. It is theoretically shown that the proposed scheme is robust in the sense that all the variables used to generate the input signal are bounded. In general the proposed approach achieves a guaranteed transient and steady-state output tracking performance. In addition, asymptotic output tracking can also be achieved when certain conditions hold.

II. Problem Formulation

The system to be considered in this paper has the following form

$$\theta_{xn} x^{(n)} = \phi^T(\bar{x},t)\theta_\phi + u(t) + \Delta(\bar{x},t), \tag{1}$$

where x is the system output, $\bar{x} = [\,x, \cdots, x^{(n-1)}]^T$ is the measurable state variable vector with $x^{(i)}$ denoting the i-th time derivative of the output x, $\phi(\bar{x},t) = [\phi_1(\bar{x},t), \cdots, \phi_m(\bar{x},t)]^T$ is the vector of known basis functions, $\theta_\phi = [\theta_{\phi 1}, \cdots, \theta_{\phi m}]^T$ is the vector of constant unknown parameter vector, $u(t)$ is the system input, $\Delta(\bar{x},t)$ represents the lumped disturbances.

The following practical assumption is made:

Assumption 1: Unmodeled Uncertainty term $\Delta(\bar{x},t)$ is bounded by a known function with known shape, i.e.,

$$|\Delta(\bar{x},t)| \leq \delta(\bar{x},t), \tag{2}$$

where $\delta(\bar{x},t)$ is a known positive-valued function.

The control objective is to synthesize a bounded control law u, such that the output x tracks its desired trajectory $x_d(t)$ which is assumed to be bounded and have bounded derivatives up to n-th order as quickly and accurately as possible.

III. Set-Membership Identification of Parameter Bounds

A. Computation of the Bounding Ellipse

In this section, optimal bounding ellipse (OBE) algorithm is used to obtain the bounds of parameters under Assumption 1. It should be noted that, since the SMI algorithm is completely separated from the ARC controller design, it is OK to use any other kind of SMI algorithm, as long as it can give a good estimate of parameters' bounds online.

Computation of the optimum bounding ellipse for the unknown parameters involves the regression model (1) with set membership description of uncertainty. Parameterizing (1) into the standard linear regression model form, we have

$$u(t) = \varphi^T(\bar{x}, x^{(n)}, t)\theta - \Delta(\bar{x},t), \tag{3}$$

where $\varphi(\bar{x}, x^{(n)}, t) = [x^{(n)},\ -\phi^T(\bar{x},t)]^T$, $\theta = [\theta_{xn},\ \theta_\phi^T]^T$. Since only $\bar{x}$ is measurable, $x^{(n)}$ is not, the regressor $\varphi(\bar{x}, x^{(n)}, t)$ cannot be directly obtained. To bypass this problem, a filter in the continuous-time domain is constructed. Specifically, a first-order filter $H_f(s) = \frac{w}{s+w}$ is used, so that

$$u_f(t) = \varphi_f^T(\bar{x},t)\theta - \Delta_f(\bar{x},t), \tag{4}$$

where $u_f(t) = \mathscr{F}^{-1}\{H_f(s)\} * u(t)$, $\varphi_f^T(\bar{x},t) = [x_f^{(n)}, -\phi_f^T(\bar{x},t)]^T = [\mathscr{F}^{-1}\{sH_f(s)\} * x^{(n-1)}, -\mathscr{F}^{-1}\{H_f(s)\} * \phi_f^T(\bar{x},t)]^T$, $\Delta_f(\bar{x},t) = \mathscr{F}^{-1}\{H_f(s)\} * \Delta(\bar{x},t)$. $\mathscr{F}^{-1}$ denotes the inverse Fourier transformation, and $*$ denotes the convolution operation. It is trivial to check that $|\Delta_f(\bar{x},t)| \leq \delta_f(\bar{x},t)$, where $\delta_f(\bar{x},t) = \mathscr{F}^{-1}\{H_f(s)\} * \delta(\bar{x},t)$ is the filtered bound of uncertainties. Thus, (6) still has a set-membership description of uncertainty.

In discrete-time domain, the OBE algorithm proposed in [5] is implemented. Suppose that the sampling interval is T. Then, in each sampling point, (6) becomes

$$u_f(kT) = \varphi_f^T(kT)\theta - \Delta_f(kT), \tag{5}$$

where $|\Delta_f(kT)| \leq \delta_f(kT)$. In order to treat all data points equally in the sense of uncertainty level, we normalize the above equation, i.e.,

$$u'_f(kT) = \varphi_f'^T(kT)\theta - \Delta'_f(kT), \tag{6}$$

where $u'_f(kT) = \frac{1}{\delta_f(kT)} \cdot u_f(kT)$, $\varphi_f'^T(kT) = \frac{1}{\delta_f(kT)} \cdot \varphi_f^T(kT)$, $\Delta'_f(kT) = \frac{1}{\delta_f(kT)} \cdot \Delta_f(kT)$. With this normalization, we have $|\Delta'_f(kT)| \leq 1$.

The bounding ellipse for the unknown parameter θ is

$$\begin{aligned} &\Theta(kT) \\ &= \left\{ \theta : \sum_1^k \omega(kT)(u'_f(kT) - \varphi_f'^T(kT)\theta)^2 \leq \sum_1^k \omega(kT) \right\} \\ &= \left\{ \theta : [\theta - \theta_c(kT)]^T P^{-1}(kT)[\theta - \theta_c(kT)] \leq 1; \theta \in \mathscr{R}^{m+1} \right\} \end{aligned} \tag{7}$$

where $\omega(kT) \geq 0$ is the weight for each data point. The computation of $\theta_c(kT)$ and $P(kT)$ is through a recursive process:

$$\begin{aligned} z(kT) \;=\; & 1 + \omega(kT) - \frac{\omega(kT)}{1+\omega(kT)\varphi_f'^T(kT)P((k-1)T)\varphi'_f(kT)} \\ & \cdot [u'_f(kT) - \varphi_f'^T(kT)\theta_c((k-1)T)]^2, \end{aligned} \tag{8}$$

$$Z(kT) = P((k-1)T) - \frac{\omega(kT)P((k-1)T)\varphi'_f(kT)\varphi_f'^T(kT)P((k-1)T)}{1+\omega(kT)\varphi_f'^T(kT)P((k-1)T)\varphi'_f(kT)}, \tag{9}$$

$$\begin{aligned} \theta_c(kT) \;=\; & \theta_c((k-1)T) \\ & + \omega(kT)Z(kT)\varphi'_f(kT)[u'_f(kT) - \varphi_f'^T(kT)\theta_c((k-1)T)] \end{aligned} \tag{10}$$

and

$$P(kT) = z(kT)Z(kT). \tag{11}$$

The initiations $P(0) = p_0 I$ and $\theta_c(0) = 0$ mean that the initial ellipse is the sphere of radius p_0 centered at the origin. p_0 can be chosen arbitrarily large.

The computation of $\omega(kT)$ can be based on any reasonable optimization criterion. Here, the minimal volume sequential algorithm [5] is utilized, which is to minimize the volume of the ellipse ($det(P(kT))$) at each step:

$$\epsilon(kT) = u'_f(kT) - \varphi_f'^T(kT)\theta_c((k-1)T), \tag{12}$$

$$G(kT) = \varphi_f'^T(kT)P((k-1)T)\varphi'_f(kT), \tag{13}$$

$$\alpha_1 = mG^2(kT), \tag{14}$$

$$\alpha_2 = G(kT)(2m+1-G(kT)+\epsilon^2(kT)), \tag{15}$$

$$\alpha_3 = (m+1)(1-\epsilon^2(kT)) - G(kT), \tag{16}$$

$$\omega(kT) = \begin{cases} 0 \begin{cases} \text{if } \alpha_2^2 - 4\alpha_1\alpha_3 < 0 \\ \text{or} -\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3} \le 0 \end{cases} \\ \frac{-\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3}}{2\alpha_1} \text{otherwise.} \end{cases} \quad (17)$$

B. Computation of H_∞ Bounds For Unknown Parameters

The bound information given by the above OBE algorithm is an ellipse. However, for control applications, we need to know the H_∞ bound of each unknown parameter, which involves the computation of the two limiting points of that ellipse in each dimension. Through a series of tedious deduction, one can find an iterative procedure to do this, which involves no computation of inverse matrix in each step.

Let us put the H_∞ bounds of the unknown parameters to a matrix β, where $\beta = [\beta_l \ \beta_u]$. $\beta_l = [\beta_{1l}, \cdots, \beta_{m+1l}]^T$ and $\beta_u = [\beta_{1u}, \cdots, \beta_{m+1u}]^T$ are the lower and upper bounds of parameters respective. Each β_{il} and β_{iu} is obtained as follows.

Let $Q(kT) = P(kT)^{-1}$, then according to [5],

$$Q(kT) = \frac{1}{z(kT)} \cdot (Q((k-1)T) + \omega(kT)\varphi'_f(kT)\varphi'^T_f(kT)). \quad (18)$$

Let $Q_{*i}(kT)$ denote the matrix obtained by erasing the i-th row and i-th column of Q, $P_{*i}(kT) = Q_{*i}(kT)^{-1}$. Let $\varphi'_{f*i}(kT)$ be the vector obtained by removing the i-th element of $\varphi'_f(kT)$. Then

$$P_{*i}(kT) = z_k\Big[P_{*i}((k-1)T) - \frac{\omega(kT)P_{*i}((k-1)T)\varphi'_{f*i}(kT)\varphi'^T_{f*i}(kT)P_{*i}((k-1)T)}{1+\omega(kT)\varphi'^T_{f*i}(kT)P_{*i}((k-1)T)\varphi'_{f*i}(kT)}\Big], \quad (19)$$

Let $\theta_{ci}(kT)$ denote the i-th element of $\theta_c(kT)$, $Q_{-i}(kT)$ be the i-the column of $Q(kT)$ with the i-th element of it being removed, $q_i(kT)$ be the element of $Q(kT)$ at the i-th row and i-th column. Let

$$a_{i1} = q_i(kT) - Q^T_{-i}(kT)P_{*i}(kT)Q_{-i}(kT), \quad (20)$$

$$a_{i2} = -2\theta_{ci}(kT)(q_i - Q^T_{-i}(kT)P_{*i}(kT)Q_{-i}(kT)), \quad (21)$$

$$a_{i3} = [q_i(kT) - Q^T_{-i}(kT)P_{*i}(kT)Q_{-i}(kT)]\theta_{ci}(kT)^2 - 1, \quad (22)$$

then,

$$\beta_{il} = \frac{-a_{i2} - \sqrt{a_{i2}^2 - 4a_{i1}a_{i3}}}{2a_{i1}}, \quad \beta_{iu} = \frac{-a_{i2} + \sqrt{a_{i2}^2 - 4a_{i1}a_{i3}}}{2a_{i1}}. \quad (23)$$

IV. Set-membership Identification Based Adaptive Robust Control

A. Discontinuous Projection With Varying Boundaries (DPVB)

Adaptive robust control (ARC) algorithm with fixed bounds of parameters has been developed in [12]. However, if the bounds of the parameters are continuously changing, the traditional discontinuous projection method [12] needs to be revised.

Let $\hat{\theta}$ denote the estimate of θ and $\tilde{\theta}$ the estimation error (i.e., $\tilde{\theta} = \hat{\theta} - \theta$), $\hat{\rho} = [\hat{\rho}_l \ \hat{\rho}_u]$ denote the estimates of the parameter bounds. $\hat{\rho}_l = [\hat{\rho}_{1l}, \cdots, \hat{\rho}_{m+1l}]^T$ and $\hat{\rho}_u = [\hat{\rho}_{1u}, \cdots, \hat{\rho}_{m+1u}]^T$. Each $\hat{\rho}_{il}$ and $\hat{\rho}_{iu}$ is differentiable. These estimates will be synthesized later. In order to simplify the notations, we sometimes drop '(t)', for example, $\hat{\theta}$ means $\hat{\theta}(t)$. The following discontinuous projection with varying boundaries (DPVB) based adaptation law is proposed:

$$\dot{\hat{\theta}} = \text{Proj}_{\hat{\theta}\hat{\rho}}(\Gamma\tau) \quad (24)$$

where $\Gamma > 0$ is a diagonal matrix and τ is an adaptation function to be synthesized later. The DPVB mapping $Proj_{\hat{\theta}\hat{\rho}}(\bullet) = [Proj_{\hat{\theta}_1\hat{\rho}_1}(\bullet_1), \cdots, Proj_{\hat{\theta}_{m+1}\hat{\rho}_{m+1}}(\bullet_{m+1})]^T$ is defined in element as

$$Proj_{\hat{\theta}_i\hat{\rho}}(\bullet_i) = \begin{cases} \dot{\hat{\rho}}_{iu} & \text{if } \hat{\theta}_i = \hat{\rho}_{iu} \text{ and } \bullet_i > \dot{\hat{\rho}}_{iu} \\ \dot{\hat{\rho}}_{il} & \text{if } \hat{\theta}_i = \hat{\rho}_{il} \text{ and } \bullet_i < \dot{\hat{\rho}}_{il} \\ \bullet_i & \text{otherwise} \end{cases} \quad (25)$$

The DPVB based adaptation law proposed above is an extension of traditional discontinuous projection based adaptation law where the parameter bounds are assumed to be fixed. It can be checked that, with the DPVB based adaptation law, if we set the initial parameter estimates to be within the initial bound estimates, i.e, $\hat{\rho}_l(0) \le \hat{\theta}(0) \le \hat{\rho}_u(0)$, then the following two good properties hold true.

P1:
$$\hat{\rho}_l \le \hat{\theta} \le \hat{\rho}_u, \quad (26)$$

P2: If the estimated bounds for the parameters always contain the true values of parameters, i.e., $\hat{\rho}_l \le \theta \le \hat{\rho}_u$, then
$$\tilde{\theta}^T[\Gamma^{-1}Proj_{\hat{\theta}\hat{\rho}}(\Gamma\tau) - \tau] \le 0. \quad (27)$$

It can be easily checked that $P1$ is satisfied, because when $\hat{\theta}_i$ touches the varying bound, its inward derivative is always less than or equal to the inward changing rate of the parameter bound. So $\hat{\theta}_i$ will be kept within the bound. Substituting (25) into (27), we will immediately see that $P2$ is also satisfied.

B. Bound Shrinking Algorithm

The DPVB proposed in the above subsection has effectively handled the case when the bounds of parameters are changing. However, in order to seamlessly combine the discrete-time-domain SMI algorithm with the continuous-time-domain ARC, we must set another criterion as to how to choose the continuously varying bound $\hat{\rho}$ based on β obtained by discrete-time SMI. This interconnection between continuous-time domain and discontinuous-time domain should be carefully dealt with. First, we have to notice the fact that the SMI algorithm we used may not be robust. In some extreme cases, especially in presence of the outliers, it is possible that the bounds can go to infinity, i.e., $det(P(kT)) \to \infty$ or $trace(P(kT)) \to \infty$ as $k \to \infty$. Then, some elements of β will also go to infinity. Thus, for the robustness consideration, β cannot be directly used as $\hat{\rho}$.

In view of this, we propose a simple bound-shrinking algorithm. Namely, after setting an initial bound estimates $\hat{\rho}(0)$, at each time, $\hat{\rho}_i$ is only updated when the bound β_i obtained by SMI is stricter than before. If the upper bound and lower bound of a particular parameter meets, simply stop the changing of that bound. This leads to monotonically decreasing upper bound $\hat{\rho}_{iu}$ and monotonically increasing lower bound $\hat{\rho}_{il}$. Thus, the parameter estimates will be at least within the initial bound $\hat{\rho}(0)$. And the boundedness of the parameter estimates follows. Specifically, $\forall t \in (kT,\ (k+1)T]$,

$$\hat{\rho}_{iu}(t) = \begin{cases} \hat{\rho}_{iu}(kT), & \text{if } \hat{\rho}_{iu}(kT) \leq \beta_{iu}(kT) \\ \hat{\rho}_{iu}(kT)(1 - f(\frac{t-kT}{T})) + f(\frac{t-kT}{T})\beta_{iu}(kT), & \text{if } \hat{\rho}_{iu}(kT) > \beta_{iu}(kT) \end{cases} \tag{28}$$

$$\hat{\rho}_{il}(t) = \begin{cases} \hat{\rho}_{il}(kT), & \text{if } \hat{\rho}_{il}(kT) \geq \beta_{il}(kT) \\ \hat{\rho}_{il}(kT)(1 - f(\frac{t-kT}{T})) + f(\frac{t-kT}{T})\beta_{il}(kT), & \text{if } \hat{\rho}_{il}(kT) < \beta_{il}(kT) \end{cases}, \tag{29}$$

where $f(x)$ is an arbitrary continuous function defined on $[0,\ 1]$, with the properties $f(0) = 0$ and $f(1) = 1$. Theoretically, in continuous-time domain, if we want $\hat{\rho}$ to be differentiable, as required for VBDP law, then $f(t)$ should be chosen such that $df/dt(0) = 0$ and $df/dt(1) = 0$. But in digital implementation, we can simply choose $f(x) = x$ since the derivative of $\hat{\rho}$ at point kT can be computed by backward difference.

C. Adaptive Robust Control Law Synthesis

Define a switching-function-like quantity s as:

$$s(t) \triangleq (\frac{d}{dt} + \lambda)^{n-1} e(t), \tag{30}$$

where $e = x_1 - x_d(t)$ is the output tracking error and $\lambda > 0$ is a positive constant. (31) can be rewritten as

$$s(t) = \bar{\lambda}^T \bar{e}(t), \tag{31}$$

where the i-th element of vector $\bar{\lambda}$ being given by $C_{n-1}^{i-1}\lambda^{n-i} = \frac{(n-1)!}{(n-i)!(i-1)!}\lambda^{n-i}$. The i-th element of vector $\bar{e}(t)$ is given by $\frac{d^{i-1}e(t)}{dt}$. Differentiating $s(t)$, we get

$$\begin{aligned} \theta_{xn}\dot{s} &= u(t) + \theta_{xn}\phi_{xn} + \phi^T\theta_\phi + \Delta, \\ &= u + \varphi_c^T\theta + \Delta \end{aligned} \tag{32}$$

where $\phi_{xn} = \bar{\lambda}_v^T \bar{x}_v - \bar{\lambda}^T \bar{x}_{dv}$ and $\varphi_c^T = [\phi_{xn},\ \phi^T]^T$. $\bar{\lambda}_v$ is the vector of first $n-1$ elements of $\bar{\lambda}$, $\bar{x}_v$ is the vector of first $n-1$ elements of $\bar{x}$, $\bar{x}_{dv} = [x_d^{(1)}, \cdots, x_d^{(n)}]^T$. We propose the following ARC control law:

$$u = u_a + \text{sign}(\theta_{xn})u_s,\ u_a = -\varphi_c^T\hat{\theta}, \tag{33}$$

where u_a is the adjustable model compensation needed for perfect tracking, and u_s is a robust control law to be synthesized later. Substituting (33) into (32), and then simplifying the resulting expression, one obtains

$$\theta_{xn}\dot{s} = \text{sign}(\theta_{xn})u_s - \varphi_c^T\tilde{\theta} + \Delta. \tag{34}$$

The robust control function u_s has the following structure:

$$u_s = u_{s1} + u_{s2}, \quad u_{s1} = -k_{s1}\cdot s, \quad u_{s2} = -\frac{h^2}{4\varepsilon}s, \tag{35}$$

where u_{s1} is a simple proportional feedback to stabilize the nominal system and u_{s2} is a robust performance feedback term. k_{s1} is the proportional feedback gain, h can be any i-th order continuous function with respect to $\bar{x}$ and t satisfying $h(\bar{x},t) \geq |\varphi_c(\bar{x},t)|[|\hat{\rho}_u(t) - \hat{\rho}_l(t)| + k_{s2}] + \delta(\bar{x},t)$, $\varepsilon > 0$ and $k_{s2} > 0$ are a design constants. With this choice of u_{s2}, it can be verified that

P3:

$$s u_{s2} \leq 0 \tag{36}$$

P4: If θ lies within the estimated bound $\hat{\rho}$, then

$$s[u_{s2} - \text{sign}(\theta_{xn})\cdot(\varphi_c^T\tilde{\theta} - \Delta)] \leq \varepsilon, \tag{37}$$

otherwise,

$$s[u_{s2} - \text{sign}(\theta_{xn})(\varphi_c^T\tilde{\theta} - \Delta)] \leq K_\varepsilon\varepsilon, \tag{38}$$

where $K_\varepsilon = \left(1 + \frac{\sqrt{\sum_{j=1}^{m+1}\max(|\theta_j - \hat{\rho}_{ju}(0)|, |\theta_j - \hat{\rho}_{jl}(0)|)^2}}{k_{s2}}\right)^2$.

Proof: P3 is obviously true. To prove P4, we see that if θ lies within the estimated bound $\hat{\rho}$, then,

$$\begin{aligned} & \varepsilon - s\cdot u_{s2} \\ = \; & \varepsilon + \tfrac{h^2}{4\varepsilon}s^2 \\ \geq \; & h\cdot|s| \\ \geq \; & \{|\varphi_c(\bar{x},t)|[|\hat{\rho}_u(t) - \hat{\rho}_l(t)| + k_{s2}] + \delta(\bar{x},t)\}\cdot|s| \\ \geq \; & s\cdot\text{sign}(\theta_{xn})\cdot(-\varphi_c^T\tilde{\theta} + \Delta). \end{aligned} \tag{39}$$

If θ does not lie within the estimated bound $\hat{\rho}$, then,

$$\begin{aligned} & K_\varepsilon\varepsilon - s\cdot u_{s2} \\ = \; & K_\varepsilon\varepsilon + \tfrac{h^2}{4\varepsilon}s^2 \\ \geq \; & \sqrt{K_\varepsilon}\cdot h\cdot|s| \\ \geq \; & \left(1 + \frac{\sqrt{\sum_{j=1}^{m+1}\max(|\theta_j - \hat{\rho}_{ju}(0)|, |\theta_j - \hat{\rho}_{jl}(0)|)^2}}{k_{s2}}\right)(|\varphi_c|[|\hat{\rho}_u - \hat{\rho}_l| + k_{s2}] + \delta)\cdot|s| \\ \geq \; & \left[|\varphi_c|\left(|\hat{\rho}_u - \hat{\rho}_l| + \sqrt{\sum_{j=1}^{m+1}\max(|\theta_j - \hat{\rho}_{ju}(0)|, |\theta_j - \hat{\rho}_{jl}(0)|)^2}\right) + \delta\right]\cdot|s| \\ \geq \; & \left[|\varphi_c|\left(|\hat{\theta} - \tfrac{\hat{\rho}_u + \hat{\rho}_l}{2}| + |\theta - \tfrac{\hat{\rho}_u + \hat{\rho}_l}{2}|\right) + \delta\right]\cdot|s| \\ \geq \; & s\cdot\text{sign}(\theta_{xn})\cdot(-\varphi_c^T\tilde{\theta} + \Delta). \end{aligned} \tag{40}$$

■

Theorem 1: If the adaptation function in (24) is chosen as

$$\tau = \text{sign}(\theta_{xn})\varphi_c s, \tag{41}$$

then the ARC control law (33) guarantees that.

A. In general, all signals in the continuous-time ARC loop are bounded. Furthermore, defining the positive definite function V_s as $V_s = \frac{1}{2}|\theta_{xn}|s^2$. If $\hat{\rho}_l(0) \leq \theta \leq \hat{\rho}_u(0)$ and $|\theta| \leq \sqrt{p_0}$, then

$$V_s(t) \leq \exp(-\lambda t)V_s(0) + \frac{\varepsilon}{\lambda}[1 - \exp(-\lambda t)], \tag{42}$$

where $\lambda = 2k_{s1}/\max(|\hat{\rho}_{1l}(0)|, |\hat{\rho}_{1u}(0)|)$. Otherwise,

$$V_s(t) \leq \exp(-\lambda t)V_s(0) + K_\varepsilon\frac{\varepsilon}{\lambda}[1 - \exp(-\lambda t)], \tag{43}$$

where $\lambda = 2k_{s1}/|\theta_{xn}|$.

B. If after a finite time t_0, there exist parametric uncertainties only (i.e., $\Delta(\bar{x},t) = 0,\ \forall t \geq t_0$), and $\hat{\rho}_l(0) \leq \theta \leq \hat{\rho}_u(0)$, $|\theta| \leq \sqrt{p_0}$, then, in addition to results in A, asymptotic output tracking error is also achieved, i.e, $e \longrightarrow 0$ and $s \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof: For part A, when $|\theta| \le \sqrt{p_0}$, then $[\theta - \theta_c(kT)]^T P^{-1}(kT)[\theta - \theta_c(kT)] \le 1$, $\forall k \in N$. Thus $\beta_l(kT) \le \theta \le \beta_u(kT)$. Since $\hat{\rho}_l(0) \le \theta \le \hat{\rho}_u(0)$, it follows that $\max(\max_{jT \le t}(\beta_l(jT)), \hat{\rho}_l(0)) \le \theta \le \min(\min_{jT \le t}(\beta_u(jT)), \hat{\rho}_u(0))$. By the bound shrinking algorithm proposed above,

$$\begin{aligned} \hat{\rho}_l(t) &\le \max(\max_{jT \le t}(\beta_l(jT)), \hat{\rho}_l(0)), \\ \hat{\rho}_u(t) &\ge \min(\min_{jT \le t}(\beta_u(jT)), \hat{\rho}_u(0)). \end{aligned} \quad (44)$$

Thus, $\hat{\rho}_l(t) \le \theta \le \hat{\rho}_u(t)$, $\forall t > 0$, i.e., θ lies within the estimated bound $\hat{\rho}$.

Take the derivative of $V_s(t)$, we get

$$\dot{V}_s = -k_{s1}s^2 + s[u_{s2} - \text{sign}(\theta_{xn}) \cdot (\varphi_c^T \tilde{\theta} - \Delta)]. \quad (45)$$

According to P4, if θ lies within the estimated bound $\hat{\rho}$, then

$$\dot{V}_s \le -k_{s1}s^2 + \varepsilon \le \frac{-2k_{s1}}{\max(|\hat{\rho}_{1l}(0)|, |\hat{\rho}_{1u}(0)|)} V_s + \varepsilon, \quad (46)$$

which leads to (42). If θ does not lie within the estimated bound $\hat{\rho}$, then

$$\dot{V}_s \le -k_{s1}s^2 + K_\varepsilon \varepsilon = \frac{-2k_{s1}}{|\theta_{xn}|} V_s + K_\varepsilon \varepsilon, \quad (47)$$

which leads to (43). Thus in both cases, s is bounded, so all the elements in $\bar{e}$ are also bounded because s and e are linked with a stable transfer function. Since the desired trajectory is assumed to be bounded and have bounded derivatives up to n-th order, all the states are also bounded because $\bar{x} = \bar{e} + \bar{x}_d$. By the VBDP projection law, $\hat{\theta}$ are bounded, and the control input u is thus bounded. This completes part A.

(It should be noted that we only proved that all the signals in continuous-time ARC loop is bounded. Although in the discrete-time domain the SMI algorithm may possibly blow up in some extreme cases, it will not affect the boundedness of the signals in the continuous-time ARC loop due to the use of the bound shrinking technique. Thus, our control algorithm is still safe and robust.)

For part B, it has been shown above that if $\hat{\rho}_l(0) \le \theta \le \hat{\rho}_u(0)$, $|\theta| \le \sqrt{p_0}$, then $\hat{\theta}$ always lie within the estimated bound. Then P2 is satisfied.

When, $\Delta = 0$ after a finite time t_0, choose Lyapunov function

$$V_a = \dot{V}_s + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (48)$$

and take its derivative, noting (24), (45), P2 and P3,

$$\begin{aligned} \dot{V}_a &= -k_{s1}s^2 + s[u_{s2} - \text{sign}(\theta_{xn})\varphi_c^T \tilde{\theta}] \\ &\quad + \tilde{\theta}^T \Gamma^{-1} \text{Proj}_{\hat{\theta}\hat{\rho}}(\Gamma \text{sign}(\theta_{xn})\varphi_c s) \\ &\le -k_{s1}s^2 \end{aligned} \quad (49)$$

Thus, $s \in L_2 \bigcap L_\infty$. It is clear that $\dot{s} \in L_\infty$ based on (32). So by applying Barbalat's lemma [8], $s \longrightarrow 0$ as $t \longrightarrow \infty$, so $e \longrightarrow 0$, which proves part B. ■

Remark 1: It can be seen from the above theorem that, $\hat{\rho}_l(0) \le \theta \le \hat{\rho}_u(0)$ and $|\theta| \le \sqrt{p_0}$ imply that $\hat{\theta}$ always lie within the estimated bound. If $\hat{\theta}$ always lie within the estimated bound, then the achieved theoretical performance is better than in the case where $\hat{\theta}$ may not lie within the estimated bound. In this sense, we need make $\hat{\rho}_l(0) \le \theta \le \hat{\rho}_u(0)$ and $|\theta| \le \sqrt{p_0}$ to the best of our ability. In other words, we need to i) set p_0 large enough such that the initial sphere encompasses the true value of θ; ii) set the initial bound $\hat{\rho}(0)$ large enough such that it contains the true value of θ. The first condition is not hard to satisfy due to the fact that the bound β obtained by SMI is not directly related to control signal in our hybrid control structure. Thus, p_0 can be set arbitrarily large so that the chance that θ is not contained in the initial sphere is very small. However, we can not follow the same procedure for ii). This is because $\hat{\rho}$ is directly used in ARC loop to generate the control signal. If $\hat{\theta}$ touches the preset large bound, then u will also be large. In reality, the control saturation will occur. Keeping this in mind, we should set $\hat{\rho}(0)$ to be as large as possible but also have to make sure that the control input is within the saturation limit. This leads to a choice of $\hat{\rho}(0)$ which is the largest value such that the control saturation does not occur. We can apply the method in [6] to choose $\hat{\rho}(0)$:

$$|\hat{\rho}_l(0)| = |\hat{\rho}_u(0)| = \frac{u_{max} - \epsilon_m}{\varphi_{cmax}}, \quad (50)$$

in which u_{max} is the control saturation limit, φ_{cmax} is the maximum of $|\varphi_c(\bar{x}_d, t)|$ for the known desired trajectory, ϵ_m is the margin left for robustness term.

D. Simulation

In this section, we consider a practical control problem. The dynamics of 1-DOF linear motor systems can be represented by the following equation [10]:

$$M\ddot{x} + B\dot{x} + F_c(\dot{x}) + F_r(x) + d = u \quad (51)$$

where x represents the position, with its velocity and acceleration denoted as $\dot{x}$ and $\ddot{x}$ respectively. M and B are the mass and viscous friction coefficient, respectively. $F_c(\dot{x})$ is the Coulomb friction term which is modeled by $S_f(\dot{x}) = A_f S_f(\dot{x})$, where A_f represents the unknown Coulomb friction coefficient and $S_f(\dot{x})$ is a known continuous or smooth function used to approximate the traditional discontinuous sign function $\text{sgn}(\dot{x})$ for effective friction compensation in implementation; one example is $\frac{\pi}{2}\arctan(\sigma\dot{x})$ with the value of σ being large enough to preserve the sgn function's characteristics when the speed is not too low. In (51), $F_r(x)$ represents the position dependent cogging force. u is the control input force. d represents the lumped effect of external disturbances and various types of modeling errors. The exact values of M, B, F_c, F_r and d are assumed to be unknown.

In industrial applications, the linear motor may have payloads of different weights. Thus, M varies a lot from case to case. Since the level of the friction depends on the payload. B and A_f also varies a lot. It is not advisable to give a fixed bounds for those parameters. However, it is well known that the cogging force F_r comes from the complicated electromagnetic interactions between coil and iron-core [2] and is bounded, independent of the payload. In this sense, we can

TABLE I

LIST OF PARAMETER VALUES

λ	30	k_{s1}	10
k_{s1}	2	$\hat{\theta}(0)$	$[0,\ 0,\ 0,\ 0]^T$
$\hat{\rho}_l(0)$	$[0,\ 0,\ 0,\ -0.5]^T$	$\hat{\rho}_u(0)$	$[0.2,\ 0.5,\ 0.5,\ -0.5]^T$
$\theta_c(0)$	$[1,\ 1,\ 1,\ 0]^T$	p_0	25
T	0.02	k_{max}	20
Γ	$[10,\ 50,\ 1000,\ 2000]^T$	δ	0.05

assume that it has known upper and lower bounds. In fact, most linear motor manufacturers provide the information of the upper bound and lower bound of the cogging force in the manual. So we can easily get its bound information.

If we also assume that d has known bound, then the system has exactly the form of (1), satisfying Assumption 1. $\bar{x} = [x,\ \dot{x}]^T$, $\theta_{xn} = M$ which is obviously positive, $\theta_\phi = [B,\ A_f,\ \bar{\bar{d}}]^T$, where $\bar{\bar{d}}$ is the nominal value of the cogging force and disturbances. $\Delta = -F_r(x) - d - \bar{\bar{d}}$ is the varying portion of cogging force and disturbances which is assumed to have known bound. $\phi = [-\dot{x},\ S_f(\dot{x}), -1]$.

The system parameters are selected as: $M = 0.1$, $B = 0.1$, $F_c = 0.1$, $u_{max} = 20$, and controller parameters are listed in Tab. I.

The desired trajectory is a sinusoidal signal with amplitude 0.5 and frequency $2Hz$. The tracking error, input, and $\hat{M}$, $\hat{B}$, $\hat{F}_c$ with their estimated upper and lower bounds are plotted in Fig 1 and Fig 2. As can be seen from the plots, the control signal is bounded, the tracking error converges to 0.1% of the amplitude of the desired trajectory. Furthermore, we can see that the SMI algorithm takes effect. The bounds shrink a lot compared to the initial values, especially for the $\hat{M}$. This is because the regressor for $\hat{M}$ is the acceleration, which can be very large sometimes. In contrast, the improvement for $\hat{F}_c$ is the smallest, since the regressor of it is $\frac{\pi}{2}\arctan(\sigma\dot{x})$, which is always less than one. But in general, all the bounds for three parameters show large improvement. For $\hat{M}$ and $\hat{F}_c$, the estimated values touch the bounds and are driven back by the proposed VBDP algorithm. This ensures that the parameter estimates do not go faraway from their true values.

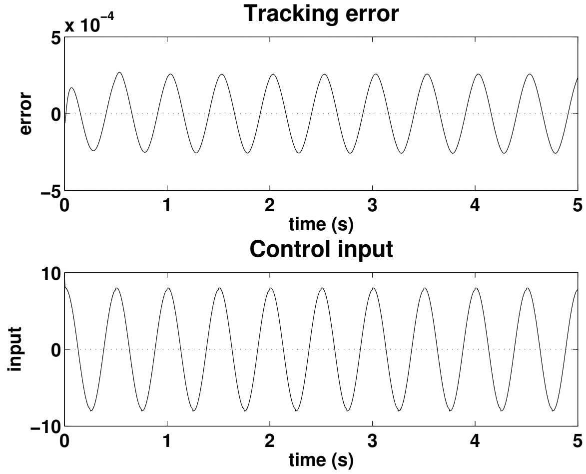

Fig. 1. Tracking error and input signal

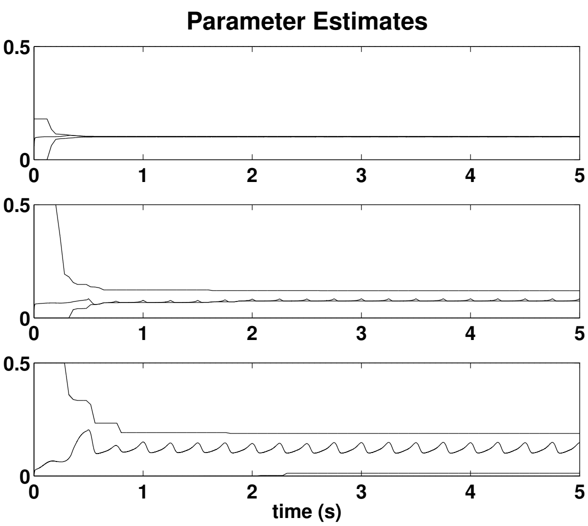

Fig. 2. $\hat{M}$, $\hat{B}$, $\hat{F}_c$ with their estimated upper and lower bounds

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