2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

Experimental Design for Identification of Nonlinear Systems with Bounded Uncertainties

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Abstract— **This paper proposes an experimental design method for the identification of a class of nonlinear systems. The lumped uncertainty of the nonlinear system is assumed to be bounded by known bound. A closed-loop identification scheme is adopted for this system. Specifically, the problem of designing an optimal input that minimizes the worst-case identification error is converted to a constrained optimal trajectory planning problem. After this optimal desired trajectory is obtained, adaptive robust control (ARC) algorithm is utilized to design the control input such that the output of the system tracks the desired optimal trajectory as closely as possible. LSE is used to give an estimate of the unknown parameters based on the filtered input and output of the controlled plant after the input designed above is applied. Extensive experiments verify that the proposed identification method gives better results than the traditional open loop identification.**

Index Terms— **Least Squares Estimation; Set-Membership Identification; Adaptive Robust Control; Linear Motor**

I. INTRODUCTION

System identification, which plays an important role in control, vibration, fault detection, e.t.c., is a broad topic that has been studied extensively during the past several decades. Traditional study on system identification focused on how to estimate the unknown parameters of the system as accurately as possible given the measured input and output data. This type of research can be divided into two categories depending on the assumption on the unmodeled uncertainties. The first one assumes the unmodeled uncertainties to have certain statistical distribution. This assumption leads to a series of researches around the well-known Least Squares Estimation (LSE) method [8]. The second one assumes the unmodeled uncertainties to be bounded by known bounds, which is also called set-membership assumption. Based on this assumption, researchers developed Set-Membership Identification (SMI) method that has received increasing attentions these years due to the practicality of this assumption [4], [3], [9].

However, developing identification algorithms that best estimate the parameters of the systems using the measured input and output is only the last step for system identification. A lot of works need to be done before that. If the I-O data obtained is poor from identification point of view, then no algorithm would achieve accurate parameter estimation using that set of poor data. The research publications on input design problem have seen an increase in recent years. The input design problem, also called experimental design problem, focuses on what kind of input should be applied to the system to generate a good I-O data set for the subsequent identification. For linear systems with statistical assumption of unmodeled uncertainties, the way to find an optimal input is fairly simple, i.e., the input only has to be persistent exciting enough. However, for systems with set-membership assumption of unmodeled uncertainties, very few results are available. Some works such as [7], [2] assume that the system to be identified is a FIR linear filter. However, this assumption is restrictive. For input design of nonlinear systems with set-membership description of unmodeled uncertainties, no results are currently available. In fact, it is very difficult to quantify the worst-case identification error from the input side for general nonlinear systems.

On the other hand, most researches on input design until now follow an open-loop identification scheme, i.e., an optimal input is computed and directly applied to the system. However, open-loop identification has many drawbacks. For most physical systems, the input and states should satisfy certain constraints for safety considerations. Open-loop identification does not guarantee that these constraints can be satisfied. Moreover, if the original system is unstable, a stabilizing closed-loop controller needs be designed beforehand. Having realized these limitations of open-loop identification, the idea of closed-loop identification was proposed and has received many attentions recently [5], [6]. However, no effort has been done for the optimal input design problem in closed-loop identification mode.

After knowning the drawbacks of open-loop identification and the importance of optimal input selection, we propose a systematic procedure of the experimental design for the closed-loop identification of a class of nonlinear systems. Set-membership description of uncertainties is adopted. In order to minimize the worst-case estimation error bound in closed-loop identification mode, we convert the optimal input design problem to a constrained optimal trajectory planning problem. Specifically, an optimal desired trajectory that minimizes the worst-case identification error is obtained. Then, adaptive robust control (ARC) algorithm is utilized to design a control input such that the output of the system tracks the desired optimal trajectory as close as possible. LSE is used to give an estimate of the unknown parameters based on the filtered input and output of the controlled plant after the input designed above is applied. Experimental results verify that the proposed identification method gives more reasonable and reliable results than the traditional open loop identification.

The work is supported in part by the US National Science Foundation (Grant No. CMS-0600516)

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II. Second Order System

A. Problem Formulation

To illustrate our design philosophy, we consider a simple second order system:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \theta_{x2}\dot{x}_2 & = & u + \theta_\phi^T \phi(x) + \Delta(x,u,t), \end{array} \tag{1}$$

where $x = [x_1, x_2]^T \in R^n$ is the state of the system. u and y are the input and output, respectively.$\theta_\phi \in R^m$ is the vector of unknown constant parameters. θ_{x2} is an unknown parameter with $\theta_{x2} > 0$. $\phi \in R^{m-1}$ is the regressor vector which is a sufficiently smooth function with respect to x. $\Delta(x,u,t)$ represents the lumped unknown nonlinear functions such as disturbances and modeling errors.

Parameterizing (1) into the form of standard linear regression model, we have

$$u(t) = \varphi^T(y,\dot{y},\ddot{y})\theta - \Delta(y,\dot{y},u,t), \tag{2}$$

where $\varphi(y,\dot{y},\ddot{y}) = [\ddot{y},\ -\phi^T(y,\dot{y})]^T,\ \theta = [\theta_{x2},\ \theta_\phi^T]^T$.

Assumption 1: The uncertain parameters are bounded by known bounds. The lumped unmodeled uncertainties are bounded by known functions. In other words, θ and Δ satisfy

$$\begin{array}{rcl} \theta \in \Omega_\theta & \stackrel{\Delta}{=} & \{\theta:\ \theta_{min} \le \theta \le \theta_{max}\}, \\ \Delta(y,\dot{y},u,t) \in \Omega_\Delta & \stackrel{\Delta}{=} & \{\Delta:\ |\Delta(y,\dot{y},u,t)| \le \delta(y,\dot{y},u,t)\}. \end{array} \tag{3}$$

In the above, $\theta_{min} \in R^m$, $\theta_{max} \in R^m$ and $\delta(y,\dot{y},u,t) > 0$ are known.

Remark 1: Ω_θ is the apriori bounding set for θ known before the closed-loop system identification experiment is carried on. It can be obtained by some simple identification methods which may not be very accurate, by reasonable guess from the physical characteristics of the system or from the manual provided by the manufacturer. Ω_θ can be very conservative.

In addition, due to the physical limitations and safety requirements [5], the states and inputs of the system should satisfy the following constraints:

$$\begin{array}{rcl} x_{1min} & \le x_1 \le & x_{1max} \\ x_{2min} & \le x_2 \le & x_{2max} \\ u_{min} & \le u \le & u_{max} \end{array} \tag{4}$$

An identification algorithm is a mapping from $\{u(t), \varphi(t)\}$ to R^{m+1}. Denoting the estimated parameter as $\hat{\theta}$. Then, our objective is to design an optimal input $u_*(t)$, such that the worst-case estimation error $\tilde{\theta} = \hat{\theta} - \theta$ after applying this $u_*(t)$ to the system, measuring the output and performing the identification algorithm is as small as possible, while the input and output of the system does not violate the constraints (4).

B. Worst-Case Error Bound

For system (1), for each applied input $u(t)$ and disturbance $\Delta(t)$, the system will have a particular response $y(t)$. From Assumption 1, the unknown parameter θ must lie in the following set

$$\Theta_{\Delta u} = \left\{\hat{\theta} : |\varphi^T(t)\hat{\theta} - u(t)| \le \delta(t), \forall\ t \ge 0\right\} \tag{5}$$

The above feasible set $\Theta_{\Delta u}$ depends on $\Delta(t)$. Since $\Delta(t)$ is not known in advance and can take any value between $-\delta(t)$ and $\delta(t)$, we need to consider all the possible feasible sets $\Theta_{\Delta u}$ for a particular input $u(t)$. Let

$$\Theta_u = \bigcup_{\Delta(t) \in [-\delta(t),\ \delta(t)]} \Theta_{\Delta u}. \tag{6}$$

Noticing Assumption 1, $\Theta_u \bigcap \Omega_\theta$ is the worst-case feasible set that θ might lie in for all $\Delta(t) \in [-\delta(t),\ \delta(t)]$. The objective of input design is to try to find an optimal $u(t)$ that minimizes the size of the worst-case feasible set Θ_u, i.e., $|\Theta_u \bigcap \Omega_\theta|_{\mathcal{N}}$ where $|\bullet|_{\mathcal{N}}$ is certain measure of the set $\bullet$. For linear systems in impulse response form, directly solving $\min_{u(t)} |\Theta_u \bigcap \Omega_\theta|_{\mathcal{N}}$ is feasible because the assumed form of the system is simple [7], [2]. However, for nonlinear systems considered in this paper, although the optimizer $|\Theta_u \bigcap \Omega_\theta|_{\mathcal{N}}$ is a function of $u(t)$, it depends on $u(t)$ through complicated system dynamics and thus the dependence is too complicated to be derived explicitly. Thus finding a $u(t)$ that minimizes $|\Theta_u \bigcap \Omega_\theta|_{\mathcal{N}}$ is very difficult. Even if such an optimal input $u(t)$ can be found and is applied it to the system, there is no guarantee that the constraints (4) will be satisfied because the optimal input is obtained offline first and then applied to the system in an open-loop mode.

Thus, closed-loop identification has to be done [5], [6]. Here we are focusing on the input design problem for the closed-loop identification, which has not been studied before. In the closed-loop mode, $u(t)$ is obtained indirectly via a feedback loop. So we cannot directly optimize $|\Theta_u \bigcap \Omega_\theta|_{\mathcal{N}}$ with respect to $u(t)$. Keeping this in mind, we propose a new strategy that tries to obtain an optimal desired trajectory $y_*(t)$, and designs an input that makes the output $y(t)$ track $y_*(t)$ as accurately as possible.

Specifically, instead of optimizing the input $u(t)$, we let the optimized variable be $y(t)$. For each response $y(t)$, and a particular disturbance $\Delta(t)$, there will be a corresponding $u(t)$. Suppose that the corresponding $u(t)$ does not violate the input constraint (4), noting (2), we can rewrite the set $\Theta_{\Delta u}$ as a function of $\Delta(t)$ and $y(t)$ as follows (the notation is changed to $\Theta_{\Delta y}$):

$$\begin{array}{rcl} \Theta_{\Delta y} & = & \left\{\hat{\theta} : |\varphi^T(t)\hat{\theta} - u(t)| \le \delta(t), \forall\ t \ge 0\right\} \\ & = & \left\{\hat{\theta} : |\varphi^T(t)\hat{\theta} - [\varphi^T(t)\theta - \Delta(t)]| \le \delta(t), \forall\ t \ge 0\right\}. \end{array} \tag{7}$$

In the above, since the relationship between $y(t)$ and $\varphi(t)$ is known precisely, we just write $\varphi(t)$ instead of $\varphi(y(t))$. Let

$$\Theta_y = \bigcup_{\Delta(t) \in [-\delta(t),\ \delta(t)]} \Theta_{\Delta y}. \tag{8}$$

Then $\Theta_y \bigcap \Omega_\theta$ is the worst-case feasible set that θ might lie in for a particular $\varphi(t)$. We have the following theorem:

Theorem 1: Suppose that the identification process is to be done from time 0 to t_f. For each input $y(t)$, suppose that $P = \int_0^{t_f} \varphi(\tau)\varphi^T(\tau)d\tau$ is non-singular. Let

$$\Theta_{y*} \stackrel{\Delta}{=} \left\{\theta + 2P^{-1}\int_0^{t_f} \varphi(\tau)\epsilon(\tau)d\tau,\ \epsilon(\tau) \in [-\delta(\tau),\ \delta(\tau)]\right\}. \tag{9}$$

Then $\Theta_y \subset \Theta_{y*}$.

Proof: If $\bar{\theta} \in \Theta_y$. From (8), $\bar{\theta} \in \Theta_{\Delta y}$ for certain $\Delta(t)$. Then from (7), $\forall\ 0 \le t \le t_f$.

$$\varphi^T(t)\bar{\theta} - [\varphi^T(t)\theta - \Delta(t)] \in [-\delta(t),\ \delta(t)], \forall\ 0 \le t \le t_f.$$
$$\Downarrow$$
$$\int_0^{t_f} \varphi(\tau)[\varphi^T(\tau)(\bar{\theta} - \theta) + \Delta(\tau)]d\tau$$
$$\in \{\int_0^{t_f} \varphi(\tau)\epsilon(\tau)d\tau,\ \epsilon(\tau) \in [-\delta(\tau),\ \delta(\tau)]\}$$
$$\Downarrow$$
$$\bar{\theta} \in \left\{\theta + P^{-1}\int_0^{t_f} \varphi(\tau)[\epsilon(\tau) + \Delta(\tau)]d\tau,\ \epsilon(\tau) \in [-\delta(\tau),\ \delta(\tau)]\right\} \tag{10}$$

So, there must be a particular $\bar{\epsilon}(t) \in [-\delta(\tau),\ \delta(\tau)]$ such that $\bar{\theta} = \theta + P^{-1}\int_0^{t_f} \varphi(\tau)[\bar{\epsilon}(\tau) + \Delta(\tau)]d\tau$. Since $\bar{\epsilon}(\tau) + \Delta(\tau) \in [-2\delta(\tau),\ 2\delta(\tau)]$, it follows that $\bar{\theta} \in \Theta_{y*}$. Thus $\Theta_y \subset \Theta_{y*}$. ■

It should be noted that in general, $\Theta_y \neq \Theta_{y*}$. However, since the computation of Θ_y is extremely difficult, it is not advisable to compute the explicit expression of Θ_y.

It can be seen that the center of Θ_{y*} is θ, which is not known in advance. And each element in Θ_{y*} differs from its center θ by a vector $2P^{-1}\int_0^{t_f} \varphi(\tau)\epsilon(\tau)d\tau,\ \epsilon(\tau) \in [-\delta(\tau),\ \delta(\tau)]$. It means that for any identification algorithm mapping $u(t)$ and $\varphi(t)$ to Θ_{y*} after running the experiment, the estimation error is within the set $\{2P^{-1}\int_0^{t_f} \varphi(\tau)\epsilon(\tau)d\tau,\ \epsilon(\tau) \in [-\delta(\tau),\ \delta(\tau)]\}$. However, if we use LSE to estimate the unknown parameter, i.e.,

$$\hat{\theta} = P^{-1}\int_0^{t_f} \varphi(\tau)u(\tau)d\tau, \tag{11}$$

Then the estimation error $\tilde{\theta} = \hat{\theta} - \theta$ is equal to $P^{-1}\int_0^{t_f} \varphi(\tau)\Delta(\tau)d\tau$, which is half of the corresponding element in Θ_{y*}. Thus, it is obvious that **minimizing the norm of the feasible set Θ_{y*} is equivalent to minimizing the worst-case norm of the estimation error if LSE is used as estimation algorithm, i.e.,**

$$\min_{y(t)}\left\{\max_{\substack{\epsilon(\tau)\\ \in[-\delta(\tau),\ \delta(\tau)]}} [|P^{-1}\int_0^{t_f} \varphi(\tau)\epsilon(\tau)d\tau|_{\mathcal{N}}]\right\}, \tag{12}$$

By solving the above min-max problem with respect to $y(t)$, we can get an optimal trajectory $y_*(t)$ that minimizes the worst-case LSE identification error.

C. Choosing a Suitable Norm

1) H_∞ Bound Case:

Theorem 2:

$$\max|P^{-1}\textstyle\int_0^{t_f} \varphi(\tau)\epsilon(\tau)d\tau|_\infty = \left|\int_0^{t_f} \mathrm{abs}(P\varphi(\tau))\delta(\tau)d\tau\right|_\infty,$$
$$\forall\epsilon(\tau) \in [-\delta(\tau),\ \delta(\tau)] \tag{13}$$

where the operator abs($\bullet$) is defined as taking the absolute value of each element in the vector $\bullet$.

2) Individual Parameter Estimation Bound Case: Sometimes we only focus on one particular parameter of θ. In this case, only error bound for that parameter needs to be considered. We don't have to care about the estimation errors for other parameters. The following norm that measures the error bound for the i-th parameter of θ is proposed.

Definition 1:

$$|\bullet|_{\infty\ i} \overset{\Delta}{=} \text{the sum of elements in the i-th row of abs}(\bullet). \tag{14}$$

Theorem 3:

$$\max|P^{-1}\textstyle\int_0^{t_f} \varphi(\tau)\epsilon(\tau)d\tau|_{\infty i} = \left|\int_0^{t_f} \mathrm{abs}(P\varphi(\tau))\delta(\tau)d\tau\right|_{\infty i},$$
$$\forall\epsilon(\tau) \in [-\delta(\tau),\ \delta(\tau)] \tag{15}$$

D. Closed-Loop Identification

Incorporating the constraints (4), we propose a novel method to obtain the optimal trajectory that minimizes the worst-case estimation error by solving a constrained optimal control problem:

$$\min_{y(t),\ 0\le t\le t_f}(\left|\int_0^{t_f} \mathrm{abs}(P\varphi(\tau))\delta(\tau)d\tau\right|_\infty),$$
or
$$\min_{y(t),\ 0\le t\le t_f}(\left|\int_0^{t_f} \mathrm{abs}(P\varphi(\tau))\delta(\tau)d\tau\right|_{\infty i}),$$
subject to
$$\begin{gathered} y(t) \text{ is 3rd-order continuously differentiable,}\\ \dot{y}(t) = v(t),\\ x_{1min} \le y(t) \le x_{1max},\\ x_{2min} \le \dot{y}(t) \le x_{2max},\\ x_{3min} \le \ddot{y}(t) \le x_{3max},\\ x_{4min} \le \dddot{y}(t) \le x_{4max},\\ y(0) = x_{10},\ y(t_f) = x_{1t_f},\\ \dot{y}(0) = x_{20},\ \dot{y}(t_f) = x_{2t_f},\\ \ddot{y}(0) = x_{30},\ \ddot{y}(t_f) = x_{3t_f}, \end{gathered} \tag{16}$$

In the above, x_{3min} and x_{3max} are set as follows:

$$x_{3min} = \left[u_{min} + \max_{\substack{y_{min}\le x_1\le y_{max},\\ v_{min}\le x_2\le v_{max}}}(\phi(x)) \cdot \max_{\theta_\phi\in\Omega_\theta}(|\theta_\phi|_2)\right]/\theta_{x_2 max} \tag{17}$$

$$x_{3max} = \left[u_{max} - \max_{\substack{y_{min}\le x_1\le y_{max},\\ v_{min}\le x_2\le v_{max}}}(\phi(x)) \cdot \max_{\theta_\phi\in\Omega_\theta}(|\theta_\phi|_2)\right]/\theta_{x_2 max}. \tag{18}$$

It can be verified that, with these choices of x_{3min} and x_{3max}, the input constraint in (4) is satisfied, i.e., $u_{min} \le u \le u_{max}$. The constraint $x_{4min} \le \dddot{y}(t) \le x_{4max}$ is imposed to make sure that the input does not change too fast in order to reduce any possible unmodel high-frequency dynamics from the applied input u to the actual input to the system, e.g., amplifier dynamics. The initial conditions and final conditions on the states of the system specify how the system identification process should start and stop.

The optimal solution is denoted as $y_*(t)$. The corresponding optimal bound $\left|\int_0^{t_f} \text{abs}(P\varphi_*(\tau))\delta_*(\tau)d\tau\right|_{\mathscr{N}}$ is denoted as J_*. How to solve the above constrained optimal trajectory planning problem numerically can be found in [1].

E. Control Input Design

After the desired optimal response $y_*(t)$ is obtained, an optimal feedback control law should be designed such that the real output $y(t)$ of the system tracks the desired one $y_*(t)$ as accurately as possible. Various types of controllers with good tracking capabilities may apply. However, Adaptive Robust Control (ARC) algorithm will be used here due to its many advantages over other existing methods [11], [12].

For the system (1), define a switching-function-like quantity p as:

$$p \stackrel{\Delta}{=} \dot{e} + k_1 e = x_2 - x_{2eq}, \quad x_{2eq} \stackrel{\Delta}{=} \dot{y}_* - k_1 e, \tag{19}$$

where $e = x_1(t) - y_*(t)$ is the output tracking error and $k_1 > 0$ is a positive gain. If p is small or converges to zero, then the output tracking error e will be small or converge to zero since $G_p(s) = \frac{e(s)}{p(s)} = \frac{1}{s+k_1}$ is a stable transfer function. So the rest of the design is to make p as small as possible. Differentiating (19) and noting (1), one obtains

$$\theta_{x2}\dot{p} \quad = \quad u + \theta^T \varphi'(x) + \Delta(x,u,t), \tag{20}$$

where $\dot{x}_{2eq} \stackrel{\Delta}{=} \ddot{x}_d - k_1\dot{e}$ and $\varphi'^T = [\dot{x}_{2eq}, \phi^T]$. We propose the following ARC control law u_*:

$$u_* = u_a + u_s, \qquad u_a = -\varphi'^T \hat{\theta}_c, \tag{21}$$

where u_a is the adjustable model compensation needed for perfect tracking, and u_s is a robust control law to be synthesized later. $\hat{\theta}_c$ is obtained using the following discontinuous projection law:

$$\dot{\hat{\theta}}_c = Proj_{\hat{\theta}_c}(-\Gamma\phi p) \tag{22}$$

where $\Gamma > 0$ is a diagonal matrix. The projection mapping $\text{Proj}_{\hat{\theta}}(\bullet) = [\text{Proj}_{\hat{\theta}_{c1}}(\bullet_1), \cdots, \text{Proj}_{\hat{\theta}_{c4}}(\bullet_4)]^T$ is defined in element as

$$\text{Proj}_{\hat{\theta}_{ci}}(\bullet_i) = \begin{cases} 0 & \text{if } \hat{\theta}_{ci} = \theta_{imax} \text{ and } \bullet_i > 0 \\ 0 & \text{if } \hat{\theta}_{ci} = \theta_{imin} \text{ and } \bullet_i < 0 \\ \bullet_i & \text{otherwise} \end{cases} \tag{23}$$

Substituting (21) into (20), and then simplifying the resulting expression, one obtains

$$\theta_{x2}\dot{p} = u_s + \varphi'^T \tilde{\theta}_c + \Delta, \tag{24}$$

where $\tilde{\theta}_c = \hat{\theta}_c - \theta$. The robust control function u_s has the following structure:

$$u_s = u_{s1} + u_{s2}, \quad u_{s1} = -k_2 p, \tag{25}$$

where u_{s1} is a simple proportional feedback to stabilize the nominal system and u_{s2} is a smooth robust performance feedback term having the following properties[11], [12]:

$$\begin{array}{c} p\{u_{s2} - \varphi'^T \tilde{\theta}_c + \Delta\} \leq \varepsilon \\ pu_{s2} \leq 0 \end{array} \tag{26}$$

where ε is a design parameter that can be arbitrarily small. With the proposed control law, we have the following theorem:

Theorem 4: The ARC control law (21) guarantees that [13].

A. In general, all signals are bounded. Furthermore, the positive definite function $V_s = \frac{1}{2}\theta_{x2}p^2$ is bounded above by

$$V_s(t) \leq \exp(-\lambda t)V_s(0) + \frac{\varepsilon}{\lambda}[1 - \exp(-\lambda t)], \tag{27}$$

where $\lambda = 2k_2/\theta_{x2max}$.

B. If after a finite time t_N, there exist parametric uncertainties only (i.e., $\Delta = 0$, $\forall t \geq t_N$), then, in addition to results in A, zero final tracking error is also achieved, i.e, $e \longrightarrow 0$ and $p \longrightarrow 0$ as $t \longrightarrow \infty$.

The proof is similar to [13].

F. Post-Experiment Identification

After applying the optimal input $u_*(t)$ designed in the last subsection to the system (1) from time 0 to t_f, the response of the system $y(t)$ will be measured. Since $\varphi(t)$ involves $\ddot{y}$ which is not measurable, a first-order filter $H_f(s) = \frac{w}{s+w}$ is used so that only y and $\dot{y}$ are used in the identification. Specifically, let

$$u_f(t) = \varphi_f^T(y,\dot{y},t)\theta - \Delta_f(y,\dot{y},t), \tag{28}$$

where $u_f(t) = \mathscr{F}^{-1}\{H_f(s)\} * u(t)$, $\varphi_f^T(y,\dot{y},t) = [y_f^{(2)}, -\phi_f^T(y,\dot{y},t)]^T = [\mathscr{F}^{-1}\{sH_f(s)\} * \dot{y}, -\mathscr{F}^{-1}\{H_f(s)\} * \phi_f^T(y,\dot{y},t)]^T$, $\Delta_f(y,\dot{y},t) = \mathscr{F}^{-1}\{H_f(s)\} * \Delta(y,\dot{y},t)$. $\mathscr{F}^{-1}$ denotes the inverse Fourier transformation, and $*$ denotes the convolution operation. It is trivial to check that $|\Delta_f(x,t)| \leq \delta_f(y,\dot{y},t)$, where $\delta_f(y,\dot{y},t) = \mathscr{F}^{-1}\{H_f(s)\} * \delta(y,\dot{y},t)$ is the filtered bound of uncertainties. Thus, (28) still has a set-membership description of uncertainty.

With LSE used as identification algorithm, according to subsection B,

$$|\tilde{\theta}_{lse}|_{\mathscr{N}} \leq \left|\int_0^{t_f} \text{abs}(P_f\varphi_f(\tau))\delta_f(\tau)d\tau\right|_{\mathscr{N}} \stackrel{\Delta}{=} J_f. \tag{29}$$

It should be noted that $J_f \neq J_*$ because $\varphi(t) \neq \varphi_*(t)$. In order to make J_f close enough to J_*, $\varphi_f - \varphi_*$ has to be as small as possible. To achieve this, the bandwidth of the filter (w) should be properly selected. The first term of $\varphi_f - \varphi_*$ is

$$\begin{array}{rcl} y_f^{(2)} - \ddot{y} & = & \mathscr{F}^{-1}\{sH_f(s)\} * \dot{y} - \ddot{y}_* \\ & = & \mathscr{F}^{-1}\{sH_f(s)\} * \dot{\tilde{y}} + \mathscr{F}^{-1}\{H_f(s) - 1\} * \ddot{y}_* \\ & = & \mathscr{F}^{-1}\{\frac{w}{s+w}\} * \ddot{\tilde{y}} + \mathscr{F}^{-1}\{\frac{-s}{s+w}\} * \ddot{y}_*. \end{array} \tag{30}$$

Since $\dot{\tilde{y}}$ can be made arbitrarily small, $\ddot{\tilde{y}}$ is mainly composed of high frequency component. Thus, we see that if w is chosen small, the first term will shrink. But the second term will increase. This puts a tradeoff in choosing w, i.e., w can neither be chosen too large nor too small. However, if the level of $(\phi^T\theta_\phi + \Delta)/\theta_{x2}$ is much smaller than the level of $\ddot{y}$, then $\ddot{\tilde{y}}$ itself will be small. Then w can be chosen to be large.

The worst-case bound using LSE is thus $J_f = J_* + J_e$, where J_e is the additional worst-case bound level resulting from $\varphi_f - \varphi_*$.

III. EXPERIMENTAL RESULTS

In this section, we consider a real system as an example. In the Precision Mechatronics Laboratory at Zhejiang University, a two-axes commercial Anorad HERC-510-510-AA1-B-CC2 Gantry by Rockwell Automation has been set up. Both axes of the gantry are powered by Anorad LC-50-200 iron core linear motors and have a travel distance of 0.51 m. Linear encoders provide both axes a position measurement resolution of $0.5\mu m$. The entire system will be used as motion system hardware for our study. To implement the real-time control algorithm, the above system is connected to a dSPACE CLP1103 controller board.

The dynamics of 1-DOF linear motor systems can be represented by the following second order differential equation [13], [10]:

$$M\ddot{x} + B\dot{x} + F_c(\dot{x}) + \bar{d} + \Delta(x, \dot{x}, t) = u \tag{31}$$

where x represents the position of linear motor, with its velocity and acceleration denoted as $\dot{x}$ and $\ddot{x}$ respectively. M and B are the mass and viscous friction coefficient normalized with respect to the input gain, respectively. $F_c(\dot{x}) = A_f S_f(\dot{x})$ is the Coulomb friction term, where A_f represents the unknown normalized Coulomb friction coefficient and $S_f(\dot{x}) = \arctan(\beta\dot{x})$ is a known smooth function used to approximate the traditional discontinuous sign function $\mathrm{sgn}(\dot{x})$ for effective friction compensation in implementation. u is the control input voltage. d represents the constant portion of lumped disturbances, $\Delta(x, \dot{x}, t)$ represents the varying portion of lumped disturbances.

Due to physical limitations, the states and input of the system should satisfy the following constraints:

$$\begin{array}{rcl} -0.01m & \le x_1 \le & 0.4m \\ -2m/s & \le x_2 \le & 2m/s \\ -10V & \le u \le & 10V \end{array} \tag{32}$$

For this type of linear motor, the unmodeled uncertainties are mainly composed of dynamic friction and cogging force. Dynamic friction exist in low speed, and cogging force is position dependent and is varying in the full traveling range.

Denoting $\theta = [M,\ B,\ F_c,\ \bar{d}]^T$. After conducting some simple identification using square wave and reading the manual provided by the manufacturer, we can give a conservative guess of the range of θ, i.e., $\theta_{max} = [0.18,\ 0.5,\ 0.5,\ 0.5]^T$ and $\theta_{min} = [0.08,\ 0,\ 0,\ -0.5]$. The bound of the uncertain nonlinearity is characterized as follows with the effect of dynamic friction and cogging force being considered:

$$\begin{array}{rcl} |\Delta(x,\dot{x},t)| & \le & \delta(x,\dot{x},t), \\ \delta(x,\dot{x},t) & = & \left\{ \begin{array}{ll} 0.25 & |\dot{x}| \le 0.1 \\ 0.15 & |\dot{x}| > 0.1 \end{array} \right. . \end{array} \tag{33}$$

Now our objective is to estimate θ as accurately as possible, i.e., we want to shrink the conservative bound defined by θ_{max} and θ_{min} to our best.

The method proposed in section II will be applied here. Since what we care about are the values for M, B and F_c, the experiment will be done for three times. At each time, letting $|\bullet|_{\mathcal{N}} = |\bullet|_{\infty i},\ i = 1,2,3$, we want to estimate the i-th parameter of θ as accurately as possible. When solving the optimal trajectory planning problem in discrete-time domain at each time, we let $T = 0.01$ and try different $N_i - N_i = 20, 30, 40, 50, 60$. The N_i that achieves the smallest J_{*i} will be adopted. The initial state at $t = 0$ and final condition at $t = 2N_iT$ are set as $[0,\ 0,\ 0]$ and $[0,\ 0,\ 0]$, respectively. while at time $t = N_iT$ another constraint is added: the position at that instance should be at greater than 0.3 to guarantee sufficiently long traveling range. The above optimal desired trajectory obtained is from time 0 to $t_f = 2N_iT$. Concatenating this trajectory one by one, an optimal periodic desired trajectory is formed and used to design the ARC law.

The optimal desired trajectories $y_{*i}(t)$ at each time are plotted in Fig.1,2,3. The corresponding N_i and optimal J_{*i} are listed in Tab.I. Then, after conducting the closed-loop experiment for $i = 1,2,3$ with ARC control law applied, the control inputs and tracking errors at each time are plotted in Fig.4, 5 and 6, respectively. The bandwidth of the filter is selected as $w = 1000$. At each time, using LSE with filtered signals, we get the estimated values of M, B and F_c. They are listed in Tab.II together with the corresponding error bound.

It can seen that, with the proposed experimental design method, the feasible set for the parameters has been reduced substantially compared to the original Ω_θ. The reduction for M is most obvious. This is because the regressor term associated with M is the acceleration, and the maximal acceleration can be as high as $45m/s^2$. The estimation of M at high acceleration will significantly reduce the effect of unmodeled uncertainties. Thus, our algorithm in section II automatically computes that the desired trajectory needs high acceleration parts, as shown in Fig.1,2,3. In comparison, the error bound for B is reduced a little bit. This is because the maximal allowable speed $2m/s$ is much less than the maximal allowable acceleration $45m/s^2$ in magnitude. Thus, the reduction for the error bound of B is not so obvious. For F_c, because the associated regressor is $S_f(\dot{x})$ is always between -1 and 1, which is of very small magnitude, the error bound for F_c is thus large.

To compare the proposed closed-loop identification with the open loop identification, we draw the linear motor at some point in the middle of the traveling range, and apply two impulse sequences with magnitude of 0.3 and 0.5, respectively. The input and output are plotted in Fig.7,8. The filtered input and regressor are used for LSE. The estimated values and the corresponding error bound for each parameter is listed in Tab.III. As can be seen from the table, the error bound is very large for open-loop identification and does not reduce the pre-known bound Ω_θ at all. It is true that increasing the magnitude of input will make the acceleration and velocity vary at larger ranges and may thus reduce the error bounds. However, it's hard to manage it in open loop because the velocity or position can easily go out of bound for large input.

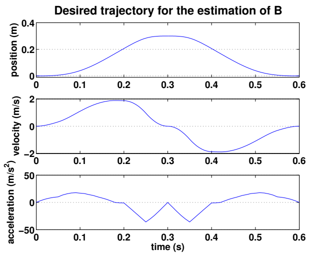

Fig. 1. Desired trajectory for the estimation of M

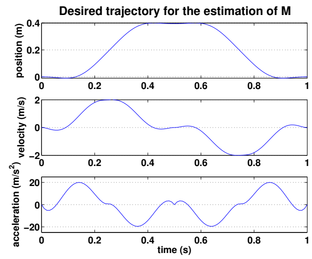

Fig. 2. Desired trajectory for the estimation of B

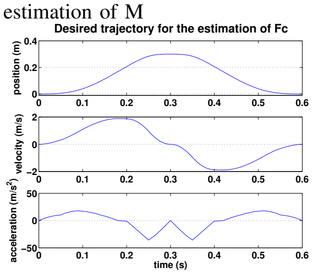

Fig. 3. Desired trajectory for the estimation of Fc

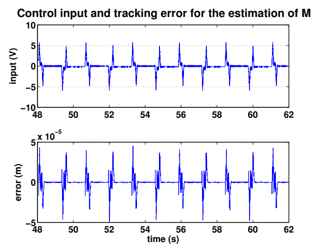

Fig. 4. Control input and tracking error for the estimation of M

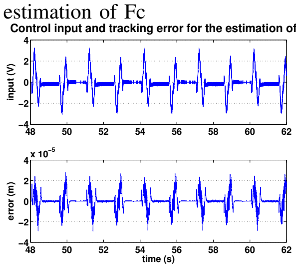

Fig. 5. Control input and tracking error for the estimation of B

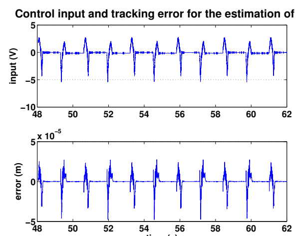

Fig. 6. Control input and tracking error for the estimation of Fc

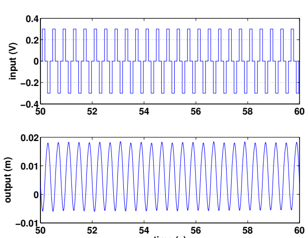

Fig. 7. Applied input and measured output for open-loop identification, with 0.3V magnitude of input

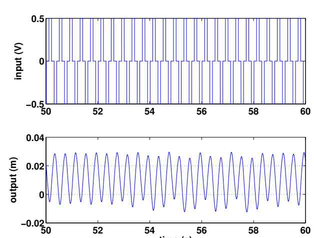

Fig. 8. Applied input and measured output for open-loop identification, with 0.5V magnitude of input

REFERENCES

[1] F. Borrelli, *Constrained optimal control of linear and hybrid systems*. Springer, 2003.

[2] M. Casini, A. Garulli, and A. Vicino, "On input design in l_∞ conditional set membership identification," *Automatica*, vol. 42, pp. 815–823, 2006.

[3] J. R. Deller, M. Nayeri, and S. F. Odeh, "Least-square identification with error bounds for real-time signalprocessing and control," *Proceedings of the IEEE*, vol. 81, no. 6, pp. 815–849, 1993.

[4] E. Fogel and Y.F.Huang, "On the value of information in system identification-bounded noise case," *Automatica*, vol. 18, no. 2, pp. 229–238, 1982.

TABLE I

N_i (NUMBER OF SAMPLING POINTS USED FOR ESTIMATION OF i-TH PARAMETER IN DISCRETE-TIME DOMAIN) AND J_{*i} (THE WORST-CASE ERROR BOUND FOR i-TH PARAMETER UNDER THE DESIRED TRAJECTORY

	N_i	J_{*i}
$i=1$	30	0.0055
$i=2$	50	0.1966
$i=3$	30	0.2598

TABLE II

THE ESTIMATED PARAMETERS AND THE CORRESPONDING ERROR BOUNDS

	$\hat{\theta}_i$	J_i
$i=1$ ($\hat{M}$)	0.12233	0.01820
$i=2$ ($\hat{B}$)	0.22593	0.24737
$i=3$ ($\hat{F}_c$)	0.19254	0.41782

[5] M. Gilson and P. V. den Hof, "Instrumental variable methods for closed-loop system identification," *Automatica*, vol. 41, no. 2, pp. 241–249, 2005.

[6] M. Gilson, H. Garnier, P. C. Young, and P. M. V. den Hof, "Refined instrumental variable methods for closed-loop system identification," in *15th IFAC Symposium on System Identification*, Saint-Malo, France, July 2009.

[7] B. Kacewicz and M. Milanese, "Optimality properties in finite sample l1 identification with bounded noise," *International Journal of Adaptive Control and Signal Processing*, vol. 9, no. 1, pp. 87–96, 1995.

[8] L. Ljung, *System Identification ł Theory For the User, 2nd Edition*. Upper Saddle River, N.J.: Prentice Hall, 1999.

[9] M. Milanese and C. Novara, "Set membership identification of nonlinear systems," *Automatica*, vol. 40, no. 6, pp. 957–975, 2004.

[10] L. Xu and B. Yao, "Adaptive robust precision motion control of linear motors with negligible electrical dynamics: theory and experiments," *IEEE/ASME Transactions on Mechatronics*, vol. 6, no. 4, pp. 444–452, 2001.

[11] B. Yao, "High performance adaptive robust control of nonlinear systems: a general framework and new schemes," in *Proceedings of the IEEE Conference on Decision and Control*, San Diego, 1997, pp. 2489–2494.

[12] B. Yao and M. Tomizuka, "Adaptive robust control of SISO nonlinear systems in a semi-strict feedback form," *Automatica*, vol. 33, no. 5, pp. 893–900, 1997, (Part of the paper appeared in Proceedings of 1995 American Control Conference, pp. 2500-2505, Seattle).

[13] B. Yao and L. Xu, "Adaptive robust control of linear motors for precision manufacturing," *International Journal of Mechatronics*, vol. 12, no. 4, pp. 595–616, 2002, part of the paper appeared in *The 14th IFAC World Congress, Vol. A* , pp25-30, Beijing, 1999.

TABLE III

THE ESTIMATED PARAMETERS AND THE CORRESPONDING ERROR BOUNDS FOR OPEN-LOOP IDENTIFICATION WITH $0.3V$ AND $0.5V$ INPUTS

input	$\hat{\theta}$	error bound of $\hat{\theta}$
$0.3V$	$[0.058\ 0.151\ 0.099\ 0.008]^T$	$[0.076\ 3.653\ 0.540\ 0.189]^T$
$0.5V$	$[0.083\ 0.119\ 0.109\ 0.013]^T$	$[0.056\ 2.507\ 0.510\ 0.182]^T$