Proceedings of the 2006 American Control Conference
Minneapolis, Minnesota, USA, June 14-16, 2006

Robust Output Feedback Stabilization of a Class of Nonminimum Phase Nonlinear Systems

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Abstract— **The paper presents an output feedback approach to globally and robustly stabilize a class of nonminimum phase nonlinear systems with bounded nonlinear uncertainties. A robust observer and the standard backstepping design procedure are integrated to synthesize output feedback robust controllers that achieve global regulation of system output with bounded internal states while being robust to bounded modeling uncertainties. Furthermore, in the absence of modeling uncertainties, asymptotic global stabilization is achieved. Simulation results are presented to illustrate the proposed robust output feedback controller scheme.**

I. Introduction

In the past two decades, a great deal of effort has been devoted to the development of global or semi-global analysis and design of nonlinear feedback controller. Recent results include the notion and properties of input-to-state stability [4], the methods of adaptive output feedback control [6], the approaches for nonlinear adaptive control with backstepping [3], the adaptive robust control [7], and nonlinear adaptive robust observer design [10]. The basic objective for these research efforts is to design a state or output feedback to deal with the structural and/or unstructural uncertainties, which is motivated by the fact that physical systems are subject to certain degrees of model and/or unmodelled uncertainties. Adaptive control and robust control have been developed successfully to deal with the parametric and unparametric uncertainties, respectively. It is noted that a class of nonlinear system with a special structure (lower triangle or upper triangle) has received much attention due to the powerful recursive backstepping design procedures. A good survey paper [2] reviewed the constructive nonlinear control development from the historical point of view and predicted the trend of the constructive procedure in the nonlinear control design.

Most of the above approaches are proposed for full state feedback only. In practice, however, some states may not be measurable during the control implementation due to sensor cost or availability. For example, it is very difficult to measure the flexible modes in a flexible structure[11]. In these applications, the output feedback control design becomes necessary but posts a more challenging control problem to solve since the state observers need to be developed and integrated in the control law design as well. It is shown [14] that the nonlinear system may be semiglobally stabilized by output feedback if the open-loop system is globally stabilizable with full state feedback and uniformly observable. Some high gain observers [13] are proposed to estimate the unmeasurable states of a class of nonlinear system transformable to an output feedback form with linear error dynamics. In addition, adaptive robust observer[10] is used in a wide range of applications such as model based virtual sensing, fault diagnosis and health monitoring.

The global output feedback methods for nonlinear systems [3] are only available for a class of minimum phase systems, while it is well known that the linear system can be globally stabilized via output feedback as long as it is detectable and stabilizable. The existing nonlinear output feedback approaches assume that the internal dynamics or zero dynamics of the nonlinear system is asymptotically stable. For the nonminimum phase system, feedback limitation has been well understood for linear systems with classical Bode integrals and the limits of the cheap control and for the nonlinear system with solution of Hamilton-Jacobian-Bellman equation [1]. The performance limitation for the nonminimum phase system is structural and can not be avoided without changing the input-output relationship.

A semiglobal stabilization tool for a class of nonminimum phase nonlinear system via output feedback [5] is proposed with the assumption that an auxiliary system with reduced dimension is globally stabilizable. The high gain observer proposed in [13] is used to estimate the unmeasured state and then the output feedback controller with the estimated states is used to recover the performance of the full state feedback [13]. The main idea is to adopt the separation principle of the linear system in the nonlinear controller design. Due to the interaction between the observer design and the dynamic output feedback design, the saturation level is introduced to curb the peaking phenomenon, which can only achieve the semiglobal stabilization for a class of nonminimum phase systems transformable to the output feedback form [13]. It is claimed [6] that nonminimum phase nonlinear systems transformable to the output feedback form can be globally stabilized via a dynamical output feedback controller with order equal to $n + \rho - 1$, with n and ρ being the system order and the relative degree, respectively.

In this paper, a globally robust output feedback stabilization for a class of single-input-single-output (SISO) nonminimum phase nonlinear system is proposed, which is based on the recently developed adaptive robust control (ARC) [7], adaptive robust observer design (ARO) [10], observer based adaptive robust control[10] and the input-to-state stability [4] [14]. The standard backstepping design procedure [3] can

+ This work is supported in part by National Science Foundation under the grant CMS-0220179.

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not be directly applied for the nonminimum phase system with unstable zero dynamics, since the backstepping design usually starts from the output to ensure that the output and its derivatives converge to zeros asymptotically. In the dynamical output feedback for the nonminimum phase system, a robust observer design approach through coordinate transformation [10] is employed to estimate the unmeasured states of the nonlinear system with uncertain nonlinearities. During the dynamical output feedback controller design, the unstable internal dynamics are stabilized first for a class of nonminimum phase system when the output is viewed as a virtual control input. The objective of controller design via backstepping is to ensure the error between the measured output and the virtual control input and its derivatives converge to zeros asymptotically. With the input-to-state stability, the control law will guarantee that the control input and the internal signals are bounded, so the system is globally stabilized even with the unstructural uncertainties.

The rest of paper details the dynamical output feedback design for a class of nonminimum phase nonlinear system. Section II introduces the class of nonlinear system to be addressed and gives some illustrations via the nonminimum phase linear system. Section III states the assumption for the robust observer design with unstructural uncertainties. The proposed dynamical output feedback controller is shown in section IV, simulation results are presented in section V and the conclusions are drawn in section VI.

II. Problem Formulation

We consider a class of single input single output(SISO) nonminimum phase nonlinear systems transformable to the following feedback form

$$\begin{aligned} \dot{\zeta} &= F(x_1)\zeta + G(x_1) + \Delta_0(\zeta, x, t) \\ \dot{x}_i &= x_{i+1}, 1 \le i \le r-1 \\ \dot{x}_r &= H_0(x_1)\zeta + \sum_{i=1}^{r} H_i(x_1)x_i + b(x_1)u + \Delta_r(\zeta, x, t) \\ y &= x_1 \end{aligned} \tag{1}$$

where $\zeta \in R^{n-r}$ and $x = [x_1, \cdots, x_r]^T \in R^r$ are states. $y \in R$ and $u \in R$ are the measured output and the control input respectively. The functions of $F(x_1), G(x_1), H_0(x_1), H_i(x_1), i = 1, \cdots, r$ are assumed to be known and sufficiently smooth with compatible dimensions. The unknown nonlinearity functions $\Delta_0(\zeta, x)$ and $\Delta_r(\zeta, x)$ represent the unstructural uncertainties. It is also assumed that $b(x_1) \neq 0, F(0) = 0$, $G(0) = 0$, $H_i(0) = 0$, $i = 0, 1, \cdots, r$.

***Remark** 1:* For nonminimum phase linear system with relative degree equal to r and transfer function give by

$$G(s) = \frac{b_0 + b_1 s + \cdots + b_{n-r}s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-r} + s^n} = \frac{n(s)}{d(s)} \tag{2}$$

in which $b_{n-r} \neq 0$, and $n(s)$ and $d(s)$ are coprime, it is well known that the system can be transformed into the normal form via coordination transformation

$$\begin{aligned} \dot{\zeta}_i &= \zeta_{i+1}, 1 \le i \le n-r-1 \\ \dot{\zeta}_{n-r} &= \frac{1}{b_{n-r}}(-b_0\zeta_1 - \cdots - b_{n-r-1}\zeta_{n-r}) + \frac{1}{b_{n-r}}x_1 \\ \dot{x}_i &= x_{i+1}, 1 \le i \le r-1 \\ \dot{x}_r &= \sum_{i=1}^{n-r} c_i\zeta_i + \sum_{i=1}^{r} d_i x_i + b_{n-r}u \\ y &= x_1 \end{aligned}$$

where c_i and d_i depend on a_i and b_i. It is noted that the above nonminimum phase linear system is a special case of the nonminimum phase nonlinear system in (1). ♢

The following assumptions are made to the above nonlinear systems:

***Assumption** 1: The unknown nonlinearity functions $\Delta_0(\zeta, x)$ and $\Delta_r(\zeta, x)$ are bounded by unknown constants, i.e.,*

$$\begin{aligned} |\Delta_0(\zeta, x, t)| &\le \delta_1 \\ |\Delta_r(\zeta, x, t)| &\le \delta_2 \end{aligned} \tag{3}$$

where δ_1 and δ_2 are unknown constants. ♢

***Assumption** 2: The internal dynamics of ζ in (1) can be stabilized by a state feedback control law $x_1 = \alpha_0(\zeta)$ with $\alpha_0(0) = 0$. Furthermore, the system*

$$\dot{\zeta} = F(\alpha_0(\zeta) + z_1)\zeta + G(\alpha_0(\zeta) + z_1) + \Delta_0 \tag{4}$$

where $z_1 = x_1 - \alpha_0(\zeta)$, is input-to-state practically stable (ISpS) [14] with respect to z_1, i.e., there exists a exp-ISpS positive definite Lyapunov function $V_0(\zeta)$ such that

$$\begin{aligned} \gamma_0(||\zeta||) \le V_0(\zeta) &\le \gamma_3(||\zeta||) \\ \dot{V}_0(\zeta) &\le -c_\zeta V_0(\zeta) + \gamma_1(|z_1|) + d_\zeta \end{aligned} \tag{5}$$

where γ_0, γ_1 and γ_3 are some class κ_∞ functions, $c_\zeta > 0$ and $d_\zeta \ge 0$. ♢

The objective is to design a dynamic output feedback controller for the above nonminimum phase nonlinear system with bounded control input and internal states such that the system is globally stable in spite of the uncertain nonlinearities $\Delta_0(\zeta, x, t)$ and $\Delta_r(\zeta, x, t)$.

III. Robust Observer Design

Since only the state x_1 is measurable, the robust observer is needed to design dynamic output feedback controller. Let us rearrange the system equation (1)

$$\begin{aligned} \dot{\eta} &= F_\eta(x_1)\eta + G_\eta(x_1, u) + \Delta_\eta \\ \dot{x}_1 &= C\eta \\ y &= x_1 \end{aligned} \tag{6}$$

where $\eta = [\zeta, x_2, \cdots, x_r]^T \in R^{n-1}$ is the unknown states of the system, $\Delta_\eta = [\Delta_0, 0, \cdots, \Delta_r]^T \in R^{n-1}$ is the unmodelled uncertainty, $C = [0, 1, 0, \cdots, 0] \in R^{n-1}$,

$G_\eta(x_1,u) = [G(x_1), 0, \cdots, H_1(x_1)x_1 + b(x_1)u]^T \in R^{n-1}$, and

$$F_\eta(x_1) = \begin{pmatrix} F(x_1) & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ H_0(x_1) & H_2(x_1) & H_3(x_1) & \dots & H_r(x_1) \end{pmatrix} \tag{7}$$

Based on the above information, the idea from the reduced order observer in the linear system analysis will be applied to design the robust observer for the unknown states. Following the procedure of the adaptive robust observer design[3][10], we first design the coordination transformation

$$\xi = \eta - \phi(x_1) \tag{8}$$

where $\phi(x_1) \in R^{n-1}$ is a continuous vector function of the measured state x_1 only and will be determined later. The ξ dynamics will be

$$\begin{aligned} \dot{\xi} &= (F_\eta - \phi^{'}(x_1)C)\xi + (F_\eta - \phi^{'}(x_1)C)\phi(x_1) \\ &\quad + G_\eta(x_1,u) + \Delta_\eta \\ &= A_\xi(x_1)\xi + A_\xi(x_1)\phi(x_1) + G_\eta(x_1,u) + \Delta_\eta \end{aligned} \tag{9}$$

in which the prime represents the derivative with respect to the measurement output x_1, and $A_\xi(x_1) = F_\eta(x_1) - \phi^{'}(x_1)C$.

It is noted From (7) and (8) that $A_\xi(x_1)$ has a special structure.

$$\begin{pmatrix} F(x_1) & -\phi_1^{'}(x_1) & 0 & \dots & 0 \\ 0 & -\phi_2^{'}(x_1) & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ H_0(x_1) & H_2(x_1) - \phi_r^{'}(x_1) & H_3(x_1) & \dots & H_r(x_1) \end{pmatrix} \tag{10}$$

***Assumption** 3: There exists a smooth vector function $\phi(x_1) \in R^{n-1}$ such that the matrix*

$$A_\xi(x_1) = F_\eta(x_1) - \phi^{'}(x_1)C \tag{11}$$

is exponentially stable,i.e, there exist two positive definite matrices P and Q such that

$$A_\xi^T P + PA_\xi + Q = 0 \tag{12}$$

***Remark** 2:* The above assumption is a nonlinear generalization of the detectability of the unknown states of ζ and $x_i, i = 2, \cdots, r$. For linear time-invariant system, $F_\eta(x_1)$ is a constant matrix. Since the pair (F_η, C) is assumed to be detectable, there exist constant vector $L \in R^{n-1}$ such that the eigenvalues of

$$A_\xi = F_\eta - LC$$

are in the open left-half plane. $\diamondsuit$

With the exponentially stable property of A_ξ, the robust nonlinear observer for (9) is chosen as

$$\dot{\hat{\xi}} = A_\xi(x_1)\hat{\xi} + A_\xi(x_1)\phi(x_1) + G_\eta(x_1,u) \tag{13}$$

Then the error dynamics of $\tilde{\xi} = \hat{\xi} - \xi$ is governed by

$$\dot{\tilde{\xi}} = A_\xi(x_1)\tilde{\xi} + \Delta_\eta \tag{14}$$

With the above robust observer design, the estimations and the estimation error of the unknown state η satisfy

$$\begin{aligned} \hat{\eta} &= \hat{\xi} + \phi(x_1) \\ \tilde{\xi} &= \hat{\xi} - \xi \\ \tilde{\eta} &= \hat{\eta} - \eta = \tilde{\xi} \end{aligned} \tag{15}$$

As seen from (14), the estimation error $\tilde{\xi} = \tilde{\eta}$ have two components.

$$\tilde{\xi} = \tilde{\eta} = \varepsilon_e(t) + \varepsilon_u(t) \tag{16}$$

The first and second term on the right hand side are determined by initial condition and the unmodelled uncertainty Δ_η, respectively. Since the unmodelled uncertainty Δ_η is bounded by some unknown constant, it guarantees that the error $\varepsilon_u(t)$ is bounded. The error $\varepsilon_e(t)$ is bounded but unknown, which depends on the unknown initial condition. So $\tilde{\xi}$ is bounded by unknown function, i.e.,

$$|\tilde{\xi}| = |\tilde{\eta}| \le \bar{\delta}(t) \tag{17}$$

in which $\bar{\delta}(t)$ is a bounded function but unknown.

In addition, from (6), (13) and (15) , the dynamics of the unknown states satisfy the following equation

$$\begin{aligned} \dot{\hat{\zeta}} &= F(x_1)\hat{\zeta} + G(x_1) - \phi_1^{'}\tilde{x}_2, \hat{\zeta} \in R^{n-r} \\ \dot{\hat{x}}_i &= \hat{x}_{i+1} - \phi_i^{'}\tilde{x}_2, 2 \le i \le r-1 \\ \dot{\hat{x}}_r &= H_0(x_1)\hat{\zeta} + \sum_{j=2}^{r} H_j(x_1)\hat{x}_j \\ &\quad + H_1(x_1)x_1 + b(x_1)u - \phi_r^{'}\tilde{x}_2 \end{aligned} \tag{18}$$

It is noted that the above dynamics can not be used directly to estimate the unknowns states. The observer in (13) is used to estimate the states instead.

IV. Robust Output Feedback Controller Design

In the following robust output feedback controller design, it first utilizes the input-to-state stability to stabilize the internal states when the output x_1 is viewed as the virtual control input α_0, and then synthesizes the control input u to guarantee that the difference $z_1 = x_1 - \alpha_0$ and its derivatives are stabilized.

A. Step 0

From (4) and the assumption (5), we know that there exist a smooth function $\alpha_0(\zeta)$ which can stabilize the internal dynamics ζ. When the estimation $\hat{\zeta}$ is used, ζ dynamics is governed by

$$\begin{aligned} \dot{\zeta} =& F(\alpha_0(\hat{\zeta}) + z_1)\zeta + G(\alpha_0(\hat{\zeta}) + z_1) + \Delta_0(\zeta, x, t) \\ =& F(\alpha_0(\zeta + \tilde{\zeta}) + z_1)\zeta + G(\alpha_0(\zeta + \tilde{\zeta}) + z_1) \\ &+ \Delta_0(\zeta, x, t) \end{aligned} \tag{19}$$

where $z_1 = x_1 - \alpha_0(\hat{\zeta})$. Viewing Assumption 2, it is natural to make the following assumption:

***Assumption** 4: The internal dynamics* (19) *is input-to-state practically stable(ISpS) [14] with respect to the inputs z_1 and $\tilde{\zeta}$, i.e, there exists a positive function $V_0(\zeta)$ such that*

$$\dot{V}_0(\zeta) \le -c_\zeta V_0(\zeta) + \gamma_1(|z_1|) + \gamma_2(||\tilde{\zeta}||) + d_\zeta \tag{20}$$

where the smooth functions of γ_1 and γ_2 belongs to the class of κ_∞, $c_\zeta > 0$ and $d_\zeta \geq 0$. $\diamondsuit$

The robust output feedback controller design using standard backstepping design will start from the estimation error z_1 rather than the internal state ζ. The second and the third term (20) will be taken into account through the input-to-state practical stability assumption.

B. Step 1

From (1) and (18), the error dynamics $z_1 = x_1 - \alpha_0(\hat{\zeta})$ can be written as

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 - \dot{\alpha}_0(\hat{\zeta}) \\ &= x_2 - \frac{\partial \alpha_0}{\partial \hat{\zeta}}(F(x_1)\hat{\zeta} + G(x_1) - \phi_1^{'}\tilde{x}_2) \end{aligned} \tag{21}$$

If the state x_2 is measurable, we would synthesize a virtual control law α_1 for x_2. Since x_2 is not available, we replace it by (15)

$$\dot{z}_1 = \hat{x}_2 - \tilde{x}_2 - \frac{\partial \alpha_0}{\partial \hat{\zeta}}(F(x_1)\hat{\zeta} + G(x_1) - \phi_1^{'}\tilde{x}_2) \tag{22}$$

A control law α_1 can be synthesized for $\hat{x}_2$ if $\hat{x}_2$ is viewed as the virtual control input. The control law has to be robust to the estimation error $\tilde{x}_2$. Following the robust controller design techniques [7][10], the control law is designed as

$$\begin{aligned} \alpha_1(\hat{\zeta}, x_1) &= \alpha_{1f} + \alpha_{1s} \\ \alpha_{1f} &= \frac{\partial \alpha_0}{\partial \hat{\zeta}}(F(x_1)\hat{\zeta} + G(x_1)) \\ \alpha_{1s} &= \alpha_{1s1} + \alpha_{1s2}, \alpha_{1s1} = -k_1 z_1 \end{aligned} \tag{23}$$

where $k_1 > 0$ and the robust term α_{1s2} is any smooth function to satisfy the following condition

$$\begin{aligned} &\text{i} \quad z_1(\alpha_{1s2} - (1 - \frac{\partial \alpha_0}{\partial \hat{\zeta}}\phi_1^{'})\tilde{x}_2) \leq \epsilon_{12}\bar{\delta}_2^2(t) \\ &\text{ii} \quad z_1\alpha_{1s2} \leq 0 \end{aligned} \tag{24}$$

where ϵ_{12} is a design parameters which can be arbitrarily small. It is noted that the right hand side of the first requirement arises from the fact that the bound of the estimation error $\tilde{x}_2$ is unknown. One smooth function satisfying the above robust condition is $-\frac{1}{4\epsilon_{12}}(1 - \frac{\partial \alpha_0}{\partial \hat{\zeta}}\phi_1^{'})^2 z_1$, with $\epsilon_{12} > 0$.

Define $z_2 = \hat{x}_2 - \alpha_1(\hat{\zeta}, x_1)$. Substituting the controller (23) into (22), the error dynamics of z_1 becomes

$$\dot{z}_1 = z_2 - k_1 z_1 + \alpha_{1s2} - (1 - \frac{\partial \alpha_0}{\partial \hat{\zeta}}\phi_1^{'})\tilde{x}_2 \tag{25}$$

Define a positive definite function $V_1 = \frac{1}{2}z_1^2$. From (25), its time derivative is

$$\begin{aligned} \dot{V}_1 &= z_1 z_2 - k_1 z_1^2 + z_1(\alpha_{1s2} - (1 + \frac{\partial \alpha_0}{\partial \hat{\zeta}}\phi_1^{'})\tilde{x}_2) \\ &\leq z_1 z_2 - k_1 z_1^2 + \epsilon_{12}\bar{\delta}_2^2 \end{aligned} \tag{26}$$

C. Step i

For the control law design in the intermediate step $2 \leq i < r$, the mathematical induction procedure will be used. The above procedure and the designed techniques will be employed to design the control through constructive procedure. Define $\bar{\hat{x}}_i = [\hat{\zeta}, \hat{x}_2, \cdots, \hat{x}_i]^T$. For step i, define the error $z_i = \hat{x}_i - \alpha_{i-1}$, in which α_{i-1} is derived through the recursive backstepping design procedure.

$$\begin{aligned} \alpha_i(\bar{\hat{x}}_i, x_1) &= \alpha_{if} + \alpha_{is} \\ \alpha_{if} &= \frac{\partial \alpha_{i-1}}{\partial \hat{\zeta}}(F(x_1)\hat{\zeta} + G(x_1)) \\ &\quad + \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}\hat{x}_{j+1} + \frac{\partial \alpha_{i-1}}{\partial x_1}\hat{x}_2 \\ \alpha_{is} &= \alpha_{is1} + \alpha_{is2}, \alpha_{is1} = -z_{i-1} - k_i z_i \end{aligned} \tag{27}$$

where $k_i > 0$.
Define

$$\tau_i = -\frac{\partial \alpha_{i-1}}{\partial \hat{\zeta}}\phi_1^{'} - \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}\phi_j^{'} + \phi_i^{'} - \frac{\partial \alpha_{i-1}}{\partial x_1} \tag{28}$$

For the robust term α_{is2} design, it should satisfy the same kind of condition (24).

$$\begin{aligned} &\text{i} \quad z_i(\alpha_{is2} - \tau_i\tilde{x}_2) \leq \epsilon_{i2}\bar{\delta}_2^2(t) \\ &\text{ii} \quad z_i\alpha_{is2} \leq 0 \end{aligned} \tag{29}$$

Then the error dynamics of z_i at the *i-th* step is

$$\dot{z}_i = z_{i+1} - k_i z_i + (\alpha_{is2} - \tau_i\tilde{x}_2) \tag{30}$$

Consider the augmented positive definite function $V_i = V_{i-1} + \frac{1}{2}z_i^2$. From (30) (26), its time derivative is

$$\begin{aligned} \dot{V}_i &= z_i z_{i+1} - k_i z_i^2 + \sum_{j=1}^{i} z_j(\alpha_{js2} - \tau_j\tilde{x}_2) \\ &\leq z_i z_{i+1} - k_i z_i^2 + \sum_{j=1}^{i} \epsilon_{j2}\bar{\delta}_2^2 \end{aligned} \tag{31}$$

D. Step r

This is the final control design step, in which the control input u will be synthesized such that the estimated state $\hat{x}_r$ tracks the virtual control input $\alpha_{r-1}(\bar{\hat{x}}_r, x_1)$. The error dynamics of $z_r = \hat{x}_r - \alpha_{r-1}$ can be obtained from (18)(27).

$$\begin{aligned} \dot{z}_r &= \dot{\hat{x}}_r - \dot{\alpha}_{r-1}(\hat{\zeta}, \hat{x}_1, \cdots, \hat{x}_{r-1}, x_1) \\ &= H_0(x_1)\hat{\zeta} + \sum_{j=2}^{r} H_j(x_1)\hat{x}_j + H_1(x_1)x_1 + b(x_1)u \\ &\quad - \phi_r^{'}\tilde{x}_2 - \dot{\alpha}_{r-1}(\hat{\zeta}, \hat{x}_1, \cdots, \hat{x}_{r-1}) \end{aligned} \tag{32}$$

Design the control law as

$$u = \frac{1}{b(x_1)}(\alpha_r - \alpha_{rf2}) \tag{33}$$

where α_r is given in (27) and α_{rf2} is a feedfoward compensation term.

$$\alpha_{rf2} = H_0(x_1)\hat{\zeta} + \sum_{j=1}^{r} H_j(x_1)\hat{x}_j + H_1(x_1)x_1$$

Theorem 1: With the above control law, it guarantee that **A.** The control input and all internal states are bounded. In addition, the control law also guarantees that

$$V_r(z) \leq \exp(-k_v t)V_r(0) + \frac{\bar{\epsilon}}{k_v}||\bar{\delta}_2||^2[1-\exp(-k_v t)] \quad (34)$$

where $k_v = 2\min(k_1, \cdots, k_r)$, and $\bar{\epsilon} = \sum_{i=1}^{r} \epsilon_{i2}$.
B. If the unstructural uncertainties of Δ_0 and Δ_r vanish after a finite time t_0, the errors $z_i, i = 1, \ldots, r$ asymptotically converge to zero for any controller gain k_i and ϵ_{i2} .

Proof: Noting that $z_{r+1} = 0$, from (33), we have

$$\begin{aligned} V_r &\leq \sum_{i=1}^{r}(-k_i z_i^2 + \epsilon_{i2}\bar{\delta}_2^2) \\ &\leq -\sum_{i=1}^{r} k_i z_i^2 + \sum_{i=1}^{r} \epsilon_{i2}||\bar{\delta}_2||^2 \end{aligned} \quad (35)$$

which proved the claim (34).

Now consider the situation that there has no unmodelled uncertainty. From the estimation error dynamics of $\tilde{\xi}$ in (14), we have

$$\dot{\tilde{\xi}} = A_\xi(x_1)\tilde{\xi} \quad (36)$$

It means that the error $\tilde{\xi}$ will converge to zero asymptotically. So does $\tilde{\zeta}, \tilde{x}_2, \cdots, \tilde{x}_r$.

From the robust condition (24) and (29), it is easy to say that $z_i(\alpha_{is2} - \tau_i\tilde{x}_2) \leq \epsilon_{i2}\varepsilon_{e2}^2(t), i = 1, \cdots, r$. Then noting the condition

$$V_r \leq \sum_{i=1}^{r}(-k_i z_i^2 + \epsilon_{i2}\varepsilon_{e2}^2(t)) \quad (37)$$

Let us define a new positive definite function V_ε as

$$V_\varepsilon = V_r + \gamma\tilde{\xi}^T P\tilde{\xi} \quad (38)$$

where $\gamma > \frac{1}{\lambda_{min}(Q)}(\bar{\epsilon} + \beta), \beta > 0$

From the error dynamics (14) and the time derivative without unmodelled uncertainties (37), the derivative is computed as

$$\begin{aligned} \dot{V}_\varepsilon &\leq \sum_{i=1}^{r}(-k_i z_i^2 + \epsilon_{i2}\varepsilon_{e2}^2(t)) - \gamma\lambda_{min}(Q)||\tilde{\xi}||^2 \\ &= -\sum_{i=1}^{r} k_i z_i^2 + \bar{\epsilon}\varepsilon_{e2}^2(t) - \gamma\lambda_{min}(Q)||\tilde{\xi}||^2 \\ &\leq -\sum_{i=1}^{r} k_i z_i^2 - \beta||\tilde{\xi}||^2 \end{aligned} \quad (39)$$

Thus, $\dot{V}_\varepsilon$ is negative positive. From the Lyapunov theory, z and $\tilde{\xi}$ are bounded and will converge to zero asymptotically as $t \to \infty$. With the condition in (20) and the input-to-state stability, we know that the internal state ζ will converges to zero. ■

V. Simulation

In order to illustrate the proposed robust output feedback controller, simulation result are obtained for the following example.

$$\begin{aligned} \dot{\zeta} &= \zeta + x_1 + \Delta_0(\zeta, x, t) \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 3\zeta + x_1^2 + 2x_1x_2 + u + \Delta_2(\zeta, x, t) \\ y &= x_1 \end{aligned} \quad (40)$$

where $\Delta_0(\zeta, x, t)$ and $\Delta_2(\zeta, x, t)$ are two unknown bounded disturbances and ζ and x_2 are unmeasurable states. In the simulation, we take $\Delta_0(\zeta, x, t) = 0.05sin(2t)$ and $\Delta_2(\zeta, x, t) = 0.1$. The control objective is to design globally robust output feedback controller to stabilize the nonminimum phase nonlinear system.

In the coordination transformation $\xi = \eta - \phi(x_1)$, we choose $\phi(x)$ as

$$\begin{aligned} \phi_1(x_1) &= 5x_1 + 2x_1^3 + \frac{10}{4}x_1^5 \\ \phi_2(x_1) &= 6x_1 + x_1^2 + x_1^3 + \frac{10}{4}x_1^5 \end{aligned} \quad (41)$$

Then the robust observer design can be applied. The resulting observer is

$$\dot{\hat{\xi}} = A_\xi(x_1)\hat{\xi} + A_\xi(x_1)\phi(x_1) + G_\zeta(x1) \quad (42)$$

where

$$A_\xi = \begin{pmatrix} 1 & -(5 + 6x_1^2 + \frac{50}{4}x_1^4) \\ 3 & -(6 + 3x_1^2 + \frac{50}{4}x_1^4)) \end{pmatrix} \quad (43)$$

is exponentially stable with P and Q given by

$$P = \begin{pmatrix} 100 & -100 \\ -100 & 200 \end{pmatrix}$$

$$Q = \begin{pmatrix} 400 & -600 + 300x_1^2 \\ -600 + 300x_1^2 & 1400 + 2500x_1^2 \end{pmatrix}$$

The estimation error dynamics of $\tilde{\xi}$ is governed by

$$\dot{\tilde{\xi}} = A_\xi(x_1)\tilde{\xi} + \Delta_\eta \quad (44)$$

Since A_ξ is exponentially stable and Δ_η is bounded above by an unknown constant, the error $\tilde{\xi} = \tilde{\eta}$ is bounded above by constant but unknown.

In the controller design, we first check the input-to-state stability for the internal dynamics of ζ. It is easy to find that the internal dynamics (40) satisfies the exp-ISpS condition with $\alpha_0(\hat{\zeta}) = -k_0\hat{\zeta}$ with $k_0 > 1 + \beta, \beta > 0$. Then following the robust output feedback controller design procedure in section IV, the resulting controller is given below:

A. Step 1

$$\begin{aligned} z_1 &= x_1 - \alpha_0\hat{\zeta} = x_1 + k_0\hat{\zeta} \\ \alpha_{1f} &= \frac{\partial\alpha_0}{\partial\hat{\zeta}}(F(x_1)\hat{\zeta} + G(x_1)) \\ \alpha_{1s1} &= -k_1 z_1 \\ \alpha_{1s2} &= -\frac{1}{4\epsilon_{12}}(1 - \frac{\partial\alpha_0}{\partial\hat{\zeta}}\phi_1')^2 z_1 \\ \alpha_1(\hat{\zeta}, x_1) &= \alpha_{1f} + \alpha_{1s1} + \alpha_{1s2} \end{aligned} \quad (45)$$

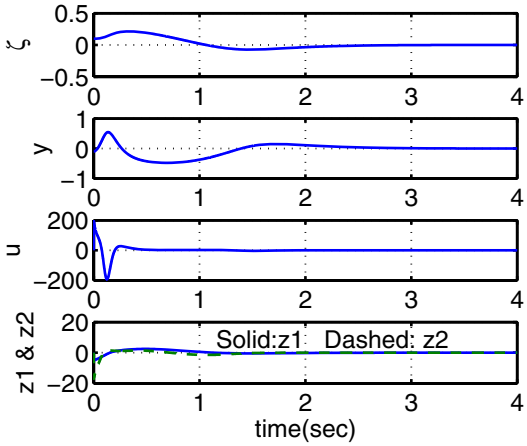

Fig. 1. The system response in the absence of the uncertain nonlinearities.

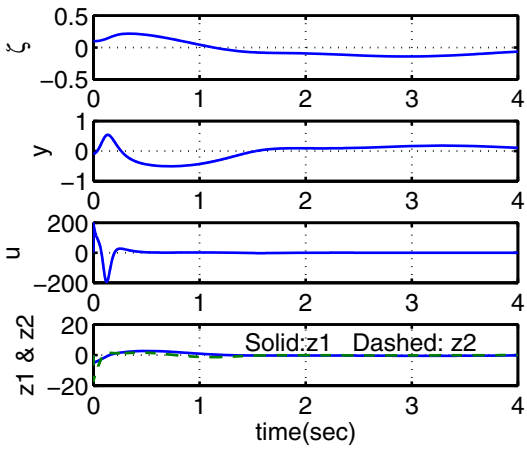

Fig. 2. The system response in the presence of the uncertain nonlinearities.

B. Step 2

$$
\begin{aligned}
z_2 &= \hat{x}_2 - \alpha_1(\hat{\zeta}, x_1) \\
\alpha_{2f} &= \frac{\partial \alpha_1}{\partial \hat{\zeta}}(F(x_1)\hat{\zeta} + G(x_1)) + \frac{\partial \alpha_1}{\partial x_1}\hat{x}_2 \\
\alpha_{2s1} &= -z_1 - k_2 z_2 \\
\alpha_{2s2} &= -\frac{1}{4\epsilon_{22}}(-\frac{\partial \alpha_1}{\partial \hat{\zeta}}\phi_1' + \phi_2' - \frac{\partial \alpha_1}{\partial x_1})^2 z_2 \\
\alpha_{2f2} &= 3\hat{\zeta} + x_1^2 + 2x_1\hat{x}_2 \\
\alpha_2(\hat{\zeta}, \hat{x}_2, x_1) &= \alpha_{2f} + \alpha_{2s1} + \alpha_{2s2} - \alpha_{2f2}
\end{aligned}
\tag{46}
$$

The initial condition and the parameters are given as $\zeta(0) = 0.1, x_1(0) = -0.1, x_2(0) = 0.1, k_0 = 10, k_1 = 1, \epsilon_{12} = 500, k_2 = 0.1$, and $\epsilon_{22} = 500$.

Figure 1 shows the response of internal state ζ, measured output y, control input u, and the error signal z for the closed-loop system in the absence of the nonlinear uncertainties. It indicates that the convergence of ζ, y to zero is very fast. Figure 2 shows the robustness performance of the controller when the system has the unmodelled uncertainties Δ_0 and Δ_2. The internal state ζ and output y do not converge, but it is robust with respect to the disturbances.

VI. Conclusion

This paper presents a robust dynamic output feedback controller to stabilize a class of nonlinear systems with unstable zero dynamics (e.e., nonminimum phase) globally and robustly. It has been shown that using the coordination transformation enables one to design a kind of reduced order robust observer to estimate the unmeasured states effectively. With the input-to-state stability and the standard backstepping design procedure, a robust controller has been constructed for a class of nonminimum phase nonlinear system.

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