Lyapunov Stability
– Stability of Equilibrium Points

1. Stability of Equilibrium Points - Definitions

System:

\[ \dot{x} = f(t, x), \quad x(t_o) = x_o \]  \hspace{1cm} (L.1)

or

\[ x(k+1) = f(k, x(k)), \quad x(k_o) = x_o, \]  \hspace{1cm} (L.2)

where \( x \in \mathbb{R}^n \), \( t \in \mathbb{R}_+ \), \( k \in \mathbb{Z}_+ \) and \( x_0 \) is the initial state

**Definition [Equilibrium State]**

An equilibrium state \( x_e \) is such that

- \( f(t, x_e) = 0 \), for all \( t \), for CT systems.
- \( f(t, x_e) = x_e \), for all \( k \), for DT systems.

Without loss of generality, we will assume that \( 0 \) is an equilibrium state. Notice that non-linear systems (and some linear systems) may have more than one equilibrium state.
The idea:

**Linear Systems:**
The closed-form solution is known \( \mathbf{x}(t) = e^{At} \mathbf{x}(0) \) for \( \dot{\mathbf{x}} = A\mathbf{x} \) and is determined by system matrix \( A \). Therefore, stability of the equilibrium point \( \mathbf{x} = 0 \) can be studied based on certain properties associated with the system matrix \( A \), namely, the eigenvalues of \( A \) or the poles of TFs.

**Nonlinear Systems:**
The closed-form solution is in general unknown and \( \mathbf{f}(t, \mathbf{x}) \) can have varieties of different forms, which prevents direct analysis of nonlinear systems via analytical solutions. As such, indirect means are sought to infer various properties of nonlinear systems. Noting that at any given time and state \( \mathbf{x}, \mathbf{f}(t, \mathbf{x}) \) can be calculated. Thus the evolving direction of the solution projected on the phase plane (i.e., the vector \( d\mathbf{x} \) in the phase plane) is known (since \( \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \)). Furthermore, for time-invariant nonlinear systems, such a direction only depends on the locations of states in the phase plane (as \( d\mathbf{x} = \mathbf{f}(\mathbf{x})dt \) in such a case). The graph of all the solutions projected on the phase plane does not depend on the time \( t \) and is unique, which can be graphically drawn for any second-order nonlinear systems -- the graphical tool of phase plane analysis introduced earlier. For higher-order or time-varying nonlinear systems, graphically displaying these directions in the phase plane is impossible. Instead, when looking at the distances of the
projected solutions to a fixed point in the phase plane representing a particular equilibrium or a constant solution, with the use of appropriate measure for distances (i.e, an energy-like positive definite (p.d.) scalar function of the states $V(x)$), it is still possible to use directions to judge if such distances $V(x(t))$ will continuously decrease or not with time. If the distances continuously decrease with time, then, the solution $x(t)$ will become closer and closer to the equilibrium point in the phase plane, indicating that the system is “stable” around the equilibrium point. The whole idea is conceived to mimic the physical phenomena that if we dissipate the energy of a system, (i.e., the p.d. energy function continuously decreases), then, the system will eventually come to rest and the solution converges to the equilibrium point. Lyapunov stability theory is a formalization of this type of energy based stability analysis.
**Definition [Ref.1]**

**[Stability and Uniform Stability in the sense of Lyapunov]**

The equilibrium state \( \mathbf{0} \) of (1) is (locally) **stable in the sense of Lyapunov** if for every \( \varepsilon > 0 \), there exists a \( \delta(\varepsilon, t_0) > 0 \) such that, if \( \|x(t_0)\| < \delta \) then \( \|x(t)\| < \varepsilon \) for all \( t > t_0 \) (respectively \( k_0 \) for DT).

In addition, if \( \delta \) can be chosen independent of \( t_0 \), i.e., \( \delta(\varepsilon) \), then, the origin is (locally) **uniformly stable**.
**Definition [Ref.1]**

**[Asymptotic Stability and Uniform Asymptotic Stability]**

The equilibrium state 0 of (1) is *(locally) asymptotically stable* if

1. It is stable in the sense of Lyapunov and
2. There exists a \( \delta'(t_o) \) such that,
   
   \[
   \text{If } \|x(t_o)\| < \delta', \text{ then, } x(t) \to 0 \text{ as } t \to \infty.
   \]

The equilibrium state 0 of (1) is *(locally) uniformly asymptotically stable* if

1. It is uniformly stable in the sense of Lyapunov and
2. There exists a \( \delta' \), independent of \( t_o \), such that, if \( \|x(t_o)\| < \delta' \), then, \( x(t) \to 0 \) as \( t \to \infty \), uniformly in \( t_o \); that is, for each \( \varepsilon > 0 \), there exists \( T=T(\varepsilon)>0 \), independent of \( t_o \), such that

\[
\|x(t)\| < \varepsilon, \quad \forall t \geq t_o + T(\varepsilon), \quad \forall \|x(t_o)\| < \delta'
\]
Note:

The definition of stability in the sense of Lyapunov is closely related to that of continuity of solutions. An equilibrium is stable if all solutions starting at nearby points stay nearby forever; otherwise, it is unstable, which includes solutions staying nearby eventually but away for some time during transient. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

Definition [Global asymptotic stability]
The equilibrium state 0 of (1) is globally asymptotically stable, if it is asymptotically stable for any $\delta' > 0$.

Definition [Exponential stability]
The equilibrium state 0 of (1) is exponentially stable, if it is stable in the sense of Lyapunov and there exists a $\delta' > 0$ and constants $M < \infty$ and $\alpha > 0$ such that

$$\|x(t)\| \leq e^{-\alpha(t-t_o)} M \|x_{\sigma}\|$$

for all $\|x(t_o)\| < \delta'$. $\alpha$ is called the rate of exponential convergence.
Example:

Stable and Asymptotic Stable but Not Uniformly Stable

Consider the system

$$\dot{x} = (8t \sin t - 2t)x$$

which has the closed-form solution

$$x = x(t_0) \exp \left[ \int_0^t (8\tau \sin \tau - 2\tau) \, d\tau \right]$$

$$= x(t_0) \exp \left[ -8t \cos t + 8\sin t - t^2 + 8t_0 \cos t_0 - 8\sin t_0 + t_0^2 \right]$$

For any $t_0$, the exponential term will eventually be dominated by $-t^2$. Thus, the exponential term is bounded for all $t > t_0$ by a constant $c(t_0)$ dependent on $t_0$. Hence,

$$x(t) \leq x(t_0)c(t_0), \quad \forall t > t_0$$

$$\Rightarrow \forall \epsilon > 0, \exists \delta(\epsilon, t_0) = \frac{\epsilon}{c(t_0)} \text{ s.t.}$$

$$\forall |x(t_0)| < \delta(\epsilon, t_0), \text{ we have, } |x(t)| < \epsilon, \forall t \geq t_0$$

$$\Rightarrow \text{ the origin is stable (in fact, asymptotically stable).}$$
However, the system is NOT uniformly stable since there does not exist a \( \delta(\varepsilon) > 0 \) such that \( |x(t_0)| < \delta(\varepsilon) \) implies \( |x(t)| < \varepsilon, \forall t \geq t_0 \). To prove this, use contradiction argument. Suppose that the statement is true, i.e., \( \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \) s.t. \( \forall t_0, |x(t_0)| < \delta(\varepsilon), \Rightarrow |x(t)| < \varepsilon \ \forall t \geq t_0 \). Let \( t_0 = 2n\pi \), \( t_1 = 2n\pi + \pi \). Then

\[
x(t_1) = x(t_0) \exp \left[ (4n+1)(6-\pi)\pi + 2\pi \right]
\]

Note that \( \exp \left[ (4n+1)(6-\pi)\pi + 2\pi \right] \to \infty \) as \( n \to \infty \). Then, for

\[
|x(t_0)| = \frac{1}{2} \delta(\varepsilon), \exists n \text{ s.t.}
\]

\[
\frac{|x(t_1)|}{|x(t_0)|} \geq \frac{2\varepsilon}{\delta} \quad \Rightarrow \quad |x(t_1)| \geq \frac{2\varepsilon}{\delta} |x(t_0)| = \varepsilon
\]

which contradicts with the assumption (i.e, \( \forall t_0, |x(t_0)| < \delta(\varepsilon), \Rightarrow |x(t)| < \varepsilon \ \forall t \geq t_0 \)).
Example:

**Uniformly Stable and Asymptotically Stable but Not Uniformly Asymptotically Stable**

Consider the first-order system given by

\[
\dot{x} = \frac{x}{1+t} \quad \Rightarrow \quad x(t) = x(t_0) \exp \left( \int_{t_0}^{t} \frac{-1}{1+\tau} d\tau \right) = x(t_0) \frac{1+t_0}{1+t}
\]

Thus, \( \forall t \geq t_0, \ |x(t)| \leq |x(t_0)| \) and the origin is uniformly stable and furthermore asymptotically stable. However, it is not uniformly asymptotically stable. To see this, \( \forall \delta' > 0 \), let \( |x(t_0)| = \frac{1}{2} \delta' \) and pick up a small \( \varepsilon \) such that \( 0 < \varepsilon < \frac{1}{2} \delta' \). Calculate the time that it takes for the solution to be in the \( \varepsilon \)-set:

\[
|x(t)| = |x(t_0)| \frac{1+t_0}{1+t} < \varepsilon \quad \Rightarrow \quad t > \frac{\delta'}{2\varepsilon} \left( 1+t_0 \right) - 1
\]

\[
\Rightarrow \quad t - t_0 > \left( \frac{\delta'}{2\varepsilon} - 1 \right) t_0 - 1 + \frac{\delta'}{2\varepsilon} \rightarrow \infty \quad \text{as} \quad t_0 \rightarrow \infty
\]

Thus, there does not exist a finite time \( T(\varepsilon) \), which is independent of \( t_0 \), such that \( \forall t \geq t_0 + T(\varepsilon) \Rightarrow |x(t)| < \varepsilon \). By definition, the origin is not uniformly asymptotically stable.
2. Lyapunov Stability Theorems for Autonomous Systems

When \( f \) in (1) does not depend on time \( t \) explicitly, i.e.,

\[
\dot{x} = f(x), \quad x(t_0) = x_o
\]  

for continuous time or

\[
x(k+1) = f(x(k)), \quad x(k_o) = x_o
\]  

for discrete time, then, the system becomes an autonomous system. The behavior of an autonomous system is invariant to shifts in the time origin. Thus, the solution \( x(t) \) depends on \( x_0 \) and \( t-t_0 \) only, and is independent of \( t_0 \). This leads to the following fact:

For autonomous system, uniform (asymptotic) stability is the same as (asymptotic) stability.
since $V(0)$ has the property that $V' = \frac{\partial V}{\partial x}$ for all solutions other than the origin. We have $V = \chi V = \chi V \cdot 0 = 0$ for $V > 0$.

$V \mapsto V(x, y)$ is strictly decreasing. Thus $\forall x, y \in \mathbb{R}$, $V(x, y) = V(0) - \int_0^x \frac{\partial V}{\partial x} \, dx - \int_0^y \frac{\partial V}{\partial y} \, dy$.

$V(x, y) = V(x, y) - V(x, 0)$ implies $V(x, y) = V(0) - \int_0^x \frac{\partial V}{\partial x} \, dx - \int_0^y \frac{\partial V}{\partial y} \, dy$.

To impose this problem, let us look at the square function of $r(x)$.

$V(0) = f(0) = \frac{x}{2}$, $V(x, y) = \frac{1}{2} x^2 + \frac{1}{2} y^2$. The value of $V(0) = \frac{x}{2}$, which exists everywhere.

\[ \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{x}{2}, \quad \frac{\partial V}{\partial y} = \frac{y}{2} \]

For $n = -2$, $-2x^2 + 2yx$.
2.2.
\[
\begin{align*}
    \dot{x}_1 &= x_2 - x_3^3 \\
    \dot{x}_2 &= -2x_1 - x_2^5 \\
\end{align*}
\]

Lyapunov.

If we still use the geometric distance function \( R(x) \) based
\[
    V_r(x) = r^2(x) = x_1^2 + x_2^2
\]

\[
\frac{dV_r(x(t))}{dt} = \frac{\partial V_r}{\partial x} \dot{x} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} \begin{bmatrix} x_2 - x_1^3 \\ -2x_1 - x_2^5 \end{bmatrix} \]

\[
= 2x_1x_2 - 2x_1^4 - 4x_2x_1 - 2x_2^6 \bigg|_{x(0)}
\]

could be positive & negative within any neighborhood around the origin.

\( \Rightarrow \) we don't know the sign of \( V_r(x) \), so still do not know if the origin is stable in the sense of Lyapunov or not.

Consider a nonlinear distance-like function:
\[
    V(x) = 2x_1^2 + x_2^2
\]

\( V(0) = 0 \iff x = 0 \)

\( V(x) > 0 \quad \forall x \neq 0 \)

& ordered by:
\[
    \Omega_r = \{ x : V(x) < r^2 \}
\]

\[
    \Omega_{r_0} = \{ x : V(x) \leq r_0^2 \}
\]

\( \Omega_{r_0} \subset \Omega_r \)
\[ V(t) = V(x(t)) : \quad \frac{dV(x)}{dt} = \left( \frac{2V}{dx} \right) x = \left[ \begin{array}{c} 4x_1, \ 2x_2 \end{array} \right] \cdot \left[ \begin{array}{c} x_1 - x_2^2 \\ -2x_1 - x_2 \end{array} \right] \mid_{x(t)} \]

\[ = 4x_1^2x_2 - 4x_1x_2^2 + 2x_2^2 - 2x_2^6 \mid_{x(t)} \]

Since \( \forall x \neq 0 \), \(-W(x)\) where \( W(x) = 4x_1^2 + 2x_2^6 \)

\( W(x) > 0 \) \( \Rightarrow \) For any solution, \( V(t) < 0 \)

\( \text{other than origin} \quad \Rightarrow \quad V(t) > V(t_0) \)

\( \Rightarrow \) the origin is stable in the sense of Lyapunov.
**Definition**  [Positive Definite (Semi-Definite) Function (PDF)]

A continuously differentiable function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) is called *positive definite* in a region \( U \subset \mathbb{R}^n \) containing the origin if

\[
\begin{align*}
& \text{a. } V(0)=0 \\
& \text{b. } V(x)>0, \ x \in U \text{ and } x \neq 0
\end{align*}
\]

A function is called *positive semi-definite* if Condition b is replaced by \( V(x) \geq 0 \).

**Note:**

i. For a p.d. function \( V(x) \) in \( \mathbb{R}^n \), the set \( S_c = \{ x \in \mathbb{R}^n : V(x) \leq c \} \) for any \( c > 0 \) is a closed set containing the origin. Let \( \Omega_c \) be the largest connected set in \( S_c \) that contains the origin. Then, \( \Omega_c \) is a closed set (\*but may not be bounded\*) and \( \Omega_{c_1} \subseteq \Omega_{c_2} \) for any \( c_2 > c_1 > 0 \). Furthermore, there exists a \( c_M > 0 \) such that \( \Omega_{c_M} \) is bounded and has a \( V(x) \) value of \( c_M \) on its boundary, i.e., \( V(x) = c_M \) \( \forall x \in \partial \Omega_{c_M} \). Consequently,

\( \Omega_{c_1} \subseteq \Omega_{c_2} \subseteq \Omega_{c_M}, \forall c_1 < c_2 \leq c_M. \)

For \( c_1 < c < c_2 \):

\( S_c = \{ x = -l_1, -l_2, -l_3, l_4, l_5, l_6 \} \)

\( S_c = \{ [-l_4, l_5], [-l_3, l_4], [l_5, l_6] \} \)

\( \partial S_c = \{ x = -l_1, -l_2, -l_3, l_4, l_5, l_6 \} \)

\( \Omega_{c_1} \subseteq \Omega_{c_2} \subseteq \Omega_{c_M} \).
ii. \( \forall r > 0, \) let \( c_{rl} = \min_{\|x\| = r} V(x) \) and \( c_{ru} = \max_{\|x\| = r} V(x) \) as \( \partial B_r = \{ x \in \mathbb{R}^n : \|x\| = r \} \) is a compact set and \( V(x) \) is continuous. Then, \( \forall 0 < c < c_{rl}, \) as \( \partial B_r \) separates \( \mathbb{R}^n \) into two disjoint spaces and \( \partial B_r \cap S_c = \emptyset, \Rightarrow \Omega_c \subset \bigcirc B_r. \)

iii. When \( V(x) \) is radially unbounded, \( c_M \) could be any positive value. Thus, \( V(x) \) can be used as a measure of distance globally, as graphically shown below (note that the boundedness of \( \Omega_c \) is essential to draw the graph as shown).
Example 1: 

\[ V(x) = \frac{x_1^2}{1 + x_1^4} + x_2^2 \]

For \( x = (0, 0) \), we have \( V(x) = 0 \). Since \( V(x) \) is positive definite, it is a Lyapunov function in 2-dimensional space. However, \( V(x) \) is not radially unbounded.

Example 2: 

\[ V(x) = x_1^4 + x_2^2 \]

For \( x = (1, 2) \), we have \( V(x) \) is positive definite. In both cases, \( V(x) \to \infty \) as \( |x| \to \infty \). 

With \( \mathcal{S}_{0.5} \), \( V(x) \) function as a measure of closeness to the origin.

For \( C = 0.2 \), 

\( \mathcal{S}_C = \{ \mathcal{S}_{0.2a}, \mathcal{S}_{0.2b}, \mathcal{S}_{0.2c} \} \)

\( \mathcal{S}_0 = \{ \mathcal{S}_{0.2a} \} \)

As \( C \) increases from 0 to 0.5, \( \mathcal{S}_C \) becomes larger but with a bounded boundary.

As \( C > 0.5 \), e.g., \( C = 0.6 \): \( \mathcal{S}_{0.6} \) with unbounded boundary.
Lyapunov Stability

Theorem L.1  [Theorem 4.1 of Ref1]  [Lyapunov Theorem]

For autonomous systems, let \( D \subset \mathbb{R}^n \) be a domain containing the equilibrium point of origin. If there exists a continuously differentiable positive definite function \( V: D \rightarrow \mathbb{R} \) such that

\[
\frac{dV(x)}{dt} = \nabla V(x) \cdot f(x) + \frac{\partial V}{\partial x} x \leq 0, \quad \forall x \in D, x \neq 0
\]

is negative semi-definite in \( D \), then, the equilibrium point \( 0 \) is stable. Moreover, if \( W(x) \) is positive definite, then, the equilibrium is asymptotically stable.

In addition, if \( D = \mathbb{R}^n \) and \( V \) is radially unbounded, i.e.,

\[
\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty
\]

then, the origin is globally asymptotically stable.

Note: In the following proof, choosing to work with \( B_r \subset B_\epsilon \cap D \cap \Omega_{c_M} \) instead of \( B_\epsilon \) is simply to make sure that all the assumptions used in the proof are valid -- \( V(x) \) is defined \( (B_r \subset D) \) and can be used as a measure of distance \( (B_r \subset \Omega_{c_M} \subset D) \), and \(-W(x)\) is negative semi-definite.
Proof of Theorem L.1:
As $V(x)$ is p.d., there exists a $c_M > 0$ such that $\Omega_{c_M}$ is bounded, $\Omega_{c_M} \subset D$, and $V(x) = c_M$, $\forall x \in \partial \Omega_{c_M}$, where $\Omega_{c_M}$ represents the largest connected set containing the origin with a $V(x)$ value less or equal to $c_M$.

Claim 1: $\forall 0 < c < c_M$, any solution starting from the set $\Omega_c$ will remain in $\Omega_c$, i.e., $\forall x_0 \in \Omega_c$, we have $x(t; x(0) = x_0) \in \Omega_c$, $\forall t \geq 0$.

Proof: For any trajectory starting within $\Omega_c$, $x(0) = x_0 \in \Omega_c$ or $V_0 = V(x(0)) \leq c$.

As $\dot{V} \leq 0$ in $\Omega_{c_M} \subset D$ and the fact that $\int_{t_0}^t \dot{V}(x(\tau)) \, d\tau = V(x(t)) - V(x(t_0))$,

$$V(x(t)) = V(x(0)) + \int_{t_0}^t \dot{V}(x(\tau)) \, d\tau \leq V(x(0)) \leq c, \quad \forall t \geq 0$$

as long as the solution $x(\tau) \in \Omega_{c_M}$, $\forall 0 \leq \tau \leq t$, which can be proved by noting that $x(0) \in \Omega_c \subset \overset{\circ}{\Omega}_{c_M}$.

#

1 Suppose not, then, there exists $\tau_1 \leq t$ such that $x(\tau_1) \notin \Omega_{c_M}$. As $x(0) \in \Omega_c \subset \Omega_{c_M}$, there exists $\tau_2 \leq \tau_1$ such that $x(\tau_2) \in \partial \Omega_{c_M}$ and $x(\tau) \in \Omega_{c_M}$, $\forall 0 \leq \tau \leq \tau_2$. Thus, $V(x(\tau_2)) \leq c$. Noting that $x(\tau_2)$ is also connected to $x(0) \in \Omega_c$, we have $x(\tau_2) \in \Omega_c \subset \overset{\circ}{\Omega}_{c_M}$, a contradiction with the assumption that $x(\tau_2) \in \partial \Omega_{c_M}$.
\( \forall \varepsilon > 0, \) choose \( r > 0 \) such that \( B_r = \{ x \in \mathbb{R}^n : \| x \| \leq r \} \subset B_\varepsilon \cap D \cap \Omega_{c_M} \), where \( B_\varepsilon = \{ x \in \mathbb{R}^n : \| x \| \leq \varepsilon \} \). The following is to show that there is a \( \delta > 0 \) such that any trajectory starting from the ball \( B_\delta = \{ x \in \mathbb{R}^n : \| x \| \leq \delta \} \) will remain in the ball \( B_r \), and thus in \( B_\varepsilon \), to prove the local stability of the origin.

As the surface of \( B_r \), i.e., \( \partial B_r = \{ x \in \mathbb{R}^n : \| x \| = r \} \), is a compact set and \( V(x) \) is continuous, \( V(x) \) will reach its maximum and minimum value on the set. So let \( c_{rl} \) be the minimum value, i.e., \( c_{rl} = \min_{\| x \| = r} V(x) \). Then, \( c_{rl} > 0 \) as \( \partial B_r \) does not contain the origin and \( V(x) \) is a positive definite function, and \( c_{rl} \leq c_M \) as \( B_r \subset \Omega_{c_M} \). It is thus clear from Claim 1 that for any \( c > 0 \) and \( c < c_{rl} \), the set \( \Omega_c \) is positive invariant, i.e., any trajectory starting from the set will remain within the set \( \Omega_c \). So any ball \( B_\delta \) within the set \( \Omega_c \) will serve our purpose as \( \Omega_c \subset B_r \). Such a \( B_\delta \) does exist as \( c_{\delta u} = \max_{\| x \| = \delta} V(x) > 0 \) could be arbitrarily small as \( \delta \to 0 \) due to the fact that \( V(x) \) is continuous and \( V(0) = 0 \). In fact, any \( \delta \) such that \( c_{\delta u} \leq c < c_{rl} \) will do as \( B_\delta \subset \Omega_{c_{\delta u}} \subset \Omega_c \). So every trajectory starting from \( B_\delta \) will remain in \( \Omega_{c_{\delta u}} \), and thus in \( B_r \) and \( B_\varepsilon \). By definition, \( 0 \) is locally stable in the sense of Lyapunov.
For asymptotic stability, we want to prove that

\[ \forall \varepsilon_a < \varepsilon, \exists T(\varepsilon_a) > 0, \text{ s.t. } \|x(t)\| \leq \varepsilon_a, \forall t > T + t_o \]

Proof:
\[ \dot{V} < 0 \implies V(x(t)) \text{ decreases all the time} \implies V(x(\infty)) \text{ exists as} V(x) \geq 0. \]
Say \( V(x(\infty)) = c \geq 0, \text{ i.e., } V(x(t)) \rightarrow V(x(\infty)) = c. \) We want to prove that \( c = 0 \) to complete the proof. For this purpose, let us use the contradiction argument. Namely, assume that \( c > 0 \) and show that this will lead to a contradiction:

\[ \therefore V(x(t)) \rightarrow c \text{ as } t \rightarrow \infty \text{ in decreasing fashion} \implies \forall \varepsilon_\beta > 0, \exists T > 0 \text{ s.t.} \]
\[ |V(x(t)) - c| < \varepsilon_\beta. \quad \forall t > T + t_o \implies c \leq V(x(t)) < c + \varepsilon_\beta \]

Consider the compact set
\[ H = \{ x \in \mathbb{R}^n : c \leq V(x) \leq c + \varepsilon_\beta \} \]

Since \(-\dot{V}(x(t)) = W(x)\) is continuous w.r.t. \( x \),
\[ \implies \gamma = \min_{x \in H} (W(x)) > 0 \iff \dot{V}(x(t)) \leq -\gamma, \quad \forall x(t) \in H \]
Let \( t_1 = T + t_o \):
\[ V(x(t)) = V(x(t_1)) + \int_{t_1}^{t} \dot{V}(x(\tau)) \ d\tau \]
\[ \leq V(x(t_1)) + \int_{t_1}^{t} (-\gamma) \ d\tau = V(x(t_1)) - \gamma (t - t_1) \]

which indicates that as \( t \rightarrow \infty \), \( V(x(t)) \rightarrow -\infty \), a contradiction!
Example: Pendulum Motion

State-Equation

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = g \sin x_1 - bx_2
\end{cases}
\]

\[ml^2 \ddot{\theta} = mgl \sin \theta - b \dot{\theta}\]

where \(x_1 = \theta, \ x_2 = \dot{\theta}, \ g = \frac{g}{l}, \ b = \frac{b}{ml^2}\)

Equilibrium Points:

\[
\begin{align*}
0 &= x_2 \\
0 &= g \sin x_1 - bx_2
\end{align*}
\]

\[
\begin{align*}
\dot{x}_2 &= 0 \\
\sin x_1 &= 0 \\
x_1 &= 0, \ x_2 = \pm m\pi, \ m = 1, 2, \ldots
\end{align*}
\]

In reality, there are only two equilibrium points: up position \((0, 0)^T\)

and down position \((\pi, 0)^T\)

Stability of the Equilibrium Point \((\pi, 0)^T\):

Define

\[
\begin{align*}
\bar{x}_1 &= x_1 - \pi \\
\bar{x}_2 &= x_2
\end{align*}
\]

\[
\Rightarrow \text{Equilibrium point } (\pi, 0)^T \text{ corresponds to } (0, 0)^T \text{ in } (\bar{x}_1, \bar{x}_2)^T \text{ plane.}\]
**New State-Equation:**

\[
\begin{align*}
\dot{x}_1 &= \bar{x}_2 \\
\dot{x}_2 &= -\bar{g} \sin x_1 - b\bar{x}_2,
\end{align*}
\]

\[
\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad f(\bar{x}) = \begin{bmatrix} \bar{x}_1 \\ -\bar{g} \sin \bar{x}_1 - b\bar{x}_2 \end{bmatrix}
\]

**Energy-function:**

\[
V(\bar{x}) = \int_{0}^{\bar{x}_1} \bar{g} \sin \bar{x}_1 d\bar{x}_1 + \frac{1}{2} \bar{x}_2^2 = \bar{g} (1 - \cos \bar{x}_1) + \frac{1}{2} \bar{x}_2^2
\]

Note that

\[V(0) = 0, \quad \text{and} \quad V(\bar{x}) > 0, \quad \forall \bar{x} \neq 0 \quad \& \quad \in D = \left\{ \bar{x} : \bar{x}_1 \in (-2\pi, 2\pi) \right\}
\]

\[\Rightarrow V(\bar{x}) \quad \text{is locally positive definite in} \quad D = \left\{ \bar{x} : \bar{x}_1 \in (-2\pi, 2\pi) \right\}.
\]

But \(V(\bar{x})\) can be used as a measure of distance-like closeness to the origin in state space only in

\[
\Omega_{c_M} = \left\{ \bar{x} : \bar{x}_1 \in (-\pi, \pi) \right\}
\]
\[ \dot{V}(\mathbf{x}(t)) = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) = \left( \frac{\partial V}{\partial x_1} \right) \dot{x}_1 + \left( \frac{\partial V}{\partial x_2} \right) \dot{x}_2 = g \sin x_1 \cdot \dot{x}_2 + \dot{x}_2 \left( -g \sin x_1 - b \dot{x}_2 \right) \]

\[ = -b \dot{x}_2^2 \]

Since \( W(\mathbf{x}) \) is positive semi-definite (why?), the equilibrium point \( \mathbf{x} = (0, 0)^T \) is locally stable by Lyapunov Theorem L1.

Q: Is the equilibrium point locally asymptotically stable?

Cannot conclude asymptotic stability by simply applying Theorem L1 as in the above analysis. The reason is that \( W(\mathbf{x}) \) is only positive semi-definite, and Lyapunov Theorem L1 needs \( W(\mathbf{x}) \) to be positive definite to conclude asymptotic stability. To obtain asymptotic stability, we have two options:
Option 1:

Try to find another Lyapunov function $V(x)$ such that $\dot{V}(\mathbf{x}(t))$ is negative definite as follows. Define a new function as

$$V_s(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{P} \mathbf{x} + \mathbf{g} \left( 1 - \cos \frac{\mathbf{x}_1}{2} \right)$$

$$= \frac{1}{4} b^2 \mathbf{x}_1^2 + \frac{1}{2} b \mathbf{x}_1 \mathbf{x}_2 + \frac{1}{2} \mathbf{x}_2^2 + \mathbf{g} \left( 1 - \cos \frac{\mathbf{x}_1}{2} \right)$$

where $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{g} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $b = 1$, $\mathbf{g} = 1$.

Note that $V_s(0) = 0$, and $V_s(\mathbf{x}) > 0$, $\forall \mathbf{x} \neq 0$ in $\mathbb{R}^2$, since $\mathbf{P} > 0$, $\Rightarrow V_s(\mathbf{x})$ is positive definite.

Furthermore, it is radially unbounded, so it can be used as a distance-like measure in state space globally as shown below (for simplicity, assume $b = 1$, $\mathbf{g} = 1$).
Lyapunov Stability

\[ \dot{V}_s(x(t)) = \frac{1}{2} \dddot{x}_1 x_2 + \frac{1}{2} x_2^2 + \frac{1}{2} (\dddot{x}_1 - \dddot{x}_2) - \dddot{x}_2^2 = -\frac{1}{2} x_2^2 - \frac{1}{2} x_1 \sin x_1 \]

which is negative definite over \( D_s = \{ \dddot{x}_1 : \dddot{x}_1 \in (-\pi, \pi) \} \subseteq D \) (why?). By Lyapunov Theorem L1, \( \dddot{x} = 0 \) is asymptotically stable.

Conservative Estimate of Region of Attraction:

\[ \Omega_{cR} = \{ \dddot{x} \in D_s : V_s(\dddot{x}) \leq c_R \} \]

where \( c_R < \inf_{\dddot{x} \in \partial D_s} \{ V_s(\dddot{x}) \} \) can be used as a measure of distance to the origin. \( \Omega_{cR} \approx R^2 \) for \( \tilde{g} = 1 \) & \( \tilde{b} = 1 \).

Option 2:

Use LaSalle’s Invariance Principle Theorem as introduced below.
Theorem L.2 [Ref1] [LaSalle’s Invariance Principle Theorem]

For autonomous systems, let $D \subset \mathbb{R}^n$ be a domain containing the equilibrium point of origin and $\Omega \subset D$ be a compact set that is positively invariant with respect to (L.4). Let $V: D \to \mathbb{R}$ be a continuously differentiable function (not necessarily positive definite) such that

$$
\dot{V}(x(t)) = \frac{\partial V}{\partial x} f(x) = -W(x) \leq 0 \quad \text{(L.8)}
$$

in $\Omega$. Let

$$
S = \{ x \in \Omega \mid W(x) = 0 \} \quad \text{(L.9)}
$$

and $M$ be the largest invariant set in $S$. Then, every solution starting in $\Omega$ approaches $M$ as $t \to \infty$.

A direct application of LaSalle’s Invariance Principle indicates that, in Theorem L.2, if $S$ contains no solution other than the origin, then, the origin is asymptotically stable with a region of attraction containing $\Omega$. 
Example: Pendulum Motion

\[ V(x) = \frac{1}{2} \ddot{x}^2 + g(1 - \cos(x)) \Rightarrow \dot{V}(\ddot{x}) = -\ddot{x}^2 \leq 0 \]

By Lyapunov theorem proof, we know any \( S_C = \{ x \in \mathbb{D} : V(x) \leq C < C_M \} \) is a positive invariant set. So choose \( S_C \) to be \( S_C, 1 < C < C_M \).

\[ S = \{ \dot{x}(t) = 0 \} = \{ x \in \mathbb{R} : \ddot{x} = 0 \} \]

To find \( M \), first note that \( M \subset S \Rightarrow \forall x \in M : \ddot{x} = 0 \]

Since \( M \) is an invariant set, \( \Rightarrow \forall x(t, x_0) \in M \)

\[ x_0 = \begin{bmatrix} \ddot{x}_0 \\ 0 \end{bmatrix} \& \ x(t, x_0) = \begin{bmatrix} \ddot{x}_0(t) \\ 0 \end{bmatrix}, \forall t \]

From first equation of the system dynamics:

\[ \ddot{x}_0(t) = 0 \Rightarrow \ddot{x}_0(t) = \text{constant} = \ddot{x}_{10} \]

From the second- equation:

\[ 0 = \ddot{x}_0(t) = -g \sin(\ddot{x}_0(t)) - b \cdot 0 \]

\[ \Rightarrow \sin(\ddot{x}_{10}) = 0 \Rightarrow \ddot{x}_{10} = 0 \text{ as } |\ddot{x}_{10}| < \pi \]

\( \Rightarrow M \) is nothing but the origin.
Lemma L.1 [Lemma 4.1 of Ref1]:

If a solution of (L.4) is bounded and belongs to D for \( t \geq 0 \), then, its positive limit set \( L^+ \) is a non-empty, compact, invariant set. Moreover, \( x(t) \) approaches \( L^+ \) as \( t \to \infty \).

Proof:

Since \( x(t) \) is bounded, by the Bolzano-Weierstrass theorem, it has a converging subsequence \( \{x(t_n), n = 1, 2, \ldots\} \) that \( x(t_n) \to y \) as \( n \to \infty \). By definition, \( y \in L^+ \). This shows that \( L^+ \) is non-empty.

To prove \( L^+ \) is compact, we need to show that \( L^+ \) is bounded and closed. \( \forall y \in L^+ \), by definition, \( \exists x(t_n) \) s.t. \( x(t_n) \to y \) as \( n \to \infty \). Since \( x(t) \) is bounded, i.e., \( \{\|x(t)\| \leq x_M, \forall t\} \) for some \( x_M \), then, \( \|y\| \leq x_M \Rightarrow L^+ \) is bounded.

To prove closeness of \( L^+ \), let us assume that if \( y \) is a limit of \( \{y_k \in L^+, k = 1, 2, \ldots\} \), i.e., \( y_k \to y \) as \( k \to \infty \). Since \( y_k \to y \) by assumption,

\[
\forall \varepsilon > 0, \exists N_1(\varepsilon) > 0, \text{ s.t. } \forall k > N_1, \|y_k - y\| < \frac{\varepsilon}{2}
\]
$\mathbf{x}(t_{11}) \quad \mathbf{x}(t_{12}) \quad \cdots \quad \mathbf{x}(t_{1n}) \quad \cdots \quad \rightarrow \quad \mathbf{y}_1$

$\mathbf{x}(t_{21}) \quad \mathbf{x}(t_{22}) \quad \cdots \quad \mathbf{x}(t_{2n}) \quad \cdots \quad \rightarrow \quad \mathbf{y}_2$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\mathbf{x}(t_{n1}) \quad \mathbf{x}(t_{n2}) \quad \cdots \quad \mathbf{x}(t_{nn}) \quad \cdots \quad \rightarrow \quad \mathbf{y}_n$

\[ \downarrow \]

$\mathbf{y}$

Let $\tau_1 = t_{11}$, and pick $\tau_2$ to be one of $\{t_{2k}\}$ s.t. $\tau_2 \geq t_{12}$ and $\|\mathbf{x}(\tau_2) - \mathbf{y}_2\| < \frac{1}{2}$.

Similarly, pick $\tau_3$ to be one of $\{t_{3k}, k \in I\}$ s.t. $\tau_3 \geq t_{13}$ and $\|\mathbf{x}(\tau_3) - \mathbf{y}_3\| < \frac{1}{3}$.

Let us now consider $\{\mathbf{x}(\tau_n), n = 1, 2, \ldots\}$. This sequence has the following properties. First, when $n \to \infty$, $\tau_n \to \infty$. Secondly, $\|\mathbf{x}(\tau_n) - \mathbf{y}_n\| < \frac{1}{n}$. Thus, $\forall \varepsilon > 0$, $\exists N_2(\varepsilon) > 0$, s.t., $\forall k \geq N_2$, $\|\mathbf{x}(\tau_k) - \mathbf{y}_k\| < \frac{\varepsilon}{2}$. Therefore, for any $n$ s.t. $n \geq N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$,

$$\|\mathbf{x}(\tau_n) - \mathbf{y}\| \leq \|\mathbf{x}(\tau_n) - \mathbf{y}_n\| + \|\mathbf{y}_n - \mathbf{y}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which shows that $\mathbf{x}(\tau_n) \to \mathbf{y}$ as $n \to \infty$. By definition, $\mathbf{y} \in L^+$. This shows that $L^+$ is closed.
To prove $L^+$ is invariant, let us assume that $y \in L^+$ and look at the solution $x(t; x(0) = y)$. We need to prove that $\forall t$, $x(t; x(0) = y) \in L^+$. Let $x(t; x(0) = x_o)$ be the original solution when defining $y \in L^+$. By definition, $\exists \{t_i\}$, s.t. as $i \to \infty$, $t_i \to \infty$ & $x(t_i; x(0) = x_o) \to y$. Consider the time sequence $\{\tau + t_i\}$ where $\tau$ is any fixed value.

$$x(\tau + t_i; x_o) = x(\tau; x(t_i; x_o))$$

Since $x(t_i; x_o) \to y$, by the continuous dependence of solutions on the initial conditions

$$\lim_{i \to \infty} x(\tau + t_i; x_o) = \lim_{i \to \infty} x(\tau; x(t_i; x_o)) = x(\tau; y)$$

which indicates that $x(\tau; y) \in L^+$. This shows that $L^+$ is invariant w.r.t. (L.4).
Proof of Theorem L.2:

\[ \dot{V}(x(t)) \leq 0 \implies V(x(t)) \text{ is non-increasing.} \]

V is differentiable on a compact set \( \Omega \implies V \text{ is lower bounded.} \)

\[ \implies V(x(t)) \to c \quad \text{as} \quad t \to \infty. \]

Consider \( L^+ \) for \( x(t; x_0) \). \( \forall p \in L^+ \), by definition, \( \exists t_n \) s.t.

\[ t_n \to \infty \quad \& \quad x(t_n; x_0) \to p \quad \text{as} \quad n \to \infty. \]

Thus,

\[ V(x(t_n; x_0)) \to c \quad \& \quad V(x(t_n; x_0)) \to V(p) \quad \text{as} \quad n \to \infty \]

So \( V(p) = c \). Now consider \( x(t; p) \). \( \forall t, x(t; p) \in L^+ \). Thus, \( \forall t, V(x(t; p)) = c. \)

\[ \implies \dot{V}(x(t; p)) = 0 \implies x(t; p) \in S \implies L^+ \subset M \]

By Lemma L.1, \( x(t; x_0) \to L^+ \subset M \quad \text{as} \quad t \to \infty. \)
2.2 Linear Time Invariant System

Theorem L.3

The following conditions are equivalent:

(a) *The equilibrium* \(0\) *of the nth order system*

\[
\dot{x} = Ax \tag{L.10}
\]

*is globally asymptotically stable (exponentially stable).*

(b) *All eigenvalues of* \(A\) *have negative real parts.*

(c) *For any positive definite symmetric matrix* \(Q\), *there exists a unique positive definite symmetric matrix* \(P\) *which is the solution of the following Lyapunov equation* \(^2\)

\[
P A + A^T P = -Q. \tag{L.11}
\]

Note: (c) indicates that the PDF function \(V(x) = x^T P x\) is a Lyapunov function for the system.

\(^2\)The MATLAB command for solving Lyapunov equation is “lyap” in continuous time and “dlyap” in discrete time.
Proof: We will demonstrate that (c) is a necessary and sufficient condition for (a) and (b).

Sufficiency:
Assume that given a positive definite symmetric matrix \( Q \) there exists a positive definite symmetric matrix \( P \) which satisfies (L.11). Define the PDF function \( V(x) = x^T P x \). Taking the time derivative of \( V \) along the trajectories of (L.10), we obtain

\[
\dot{V}(x(t)) = x^T \left\{ A^T P + PA \right\} x = -x^T Q x
\]  

(L.12)

Thus, \( V \) and \(-\dot{V}\) are both PDF, and the system is globally asymptotically stable. To prove exponential stability, we notice that

\[
x^T Q x \geq \lambda_{\min}(Q)x^T x = \lambda_{\min}(Q)\|x\|^2, \quad x^T P x \leq \lambda_{\max}(P)x^T x = \lambda_{\min}(P)\|x\|^2,
\]

where \( \lambda_{\min}(Q) \) and \( \lambda_{\max}(P) \) are respectively the minimum eigenvalue of \( Q \) and the maximum eigenvalue of \( P \), both of which are positive. Thus, defining \( \alpha = \lambda_{\min}(Q)/\lambda_{\max}(P) > 0 \), and using the short hand notation of \( V(t) = V(x(t)) \), we obtain, \( \forall x \neq 0 \),

\[
\frac{-\dot{V}(t)}{V(t)} \geq \frac{\lambda_{\min}(Q)\|x\|^2}{\lambda_{\max}(P)\|x\|^2} = \alpha
\]

Thus,
\[ \dot{V}(t) \leq -\alpha V(t). \]  

(L.13)

Integrating (L.13), we obtain

\[ V(t) \leq e^{-\alpha t} V(0) \]  

(L.14)

which implies that \( V \to 0 \) exponentially. Since

\[ V(x(t)) \geq \lambda_{\min}(P) \|x(t)\|^2, \]

where \( \lambda_{\min}(P) > 0 \) is the minimum eigenvalue of \( P \), \( \|x(t)\| \) must converge to zero exponentially as well.

Necessity:

We first define the unit ball:

\[ B_1 = \left\{ v \in \mathbb{R}^n : \|v\|^2 = v^T v = 1 \right\}, \]

and the vector induce 2 norm of a matrix \( M \in \mathbb{R}^{n \times n} : \)

\[ \|M\|_2 = \max_{v \in B_1} \|Mv\|_2 = \max_{v \in B_1} \left\{ \sqrt{v^T M^T M v} \right\} = \sqrt{\lambda_{\max}(M^T M)} = \sigma_{\max}(M) \]

Assume that the system given by (L.10) is asymptotically stable. Thus, all eigenvalues of \( A \) have negative real parts and, as a consequence, \( A \) is nonsingular and so is the solution matrix \( e^{At} \), for \( 0 \leq t < \infty \). Since \( Q \) is
positive definite, there exists a nonsingular matrix, $Q^{1/2}$, such that
$Q = (Q^{1/2})^T (Q^{1/2})$. Thus, the matrix $e^{At}Qe^{At}$ is positive definite for $0 \leq t < \infty$. Since all eigenvalues of $A$ have negative real parts, the matrix
\[ P = \int_0^\infty e^{At}Qe^{At} \, dt \tag{L.15} \]
exists, is unique and $\|P\|_2 < \infty$. (Remember that $e^{\lambda t}$, $t^m e^{\lambda t} \in L_1 \cap L_\infty$ for any $\lambda \in \mathbb{C}$ such that $\text{Re}(\lambda) < 0$ and any $0 < m < \infty$). Thus, $P$, as defined by (L.15), is positive definite. We now prove that $P$, as defined by (L.15), satisfies the Lyapunov equation (L.11).
\[ A^T P + PA = \int_0^\infty \left\{ A^T e^{At} Q e^{At} + e^{At} Q e^{At} A \right\} dt \]
\[ = \int_0^\infty d/dt \left\{ e^{At} Q e^{At} \right\} dt = \lim_{t \to \infty} \left\{ e^{At} Q e^{At} \right\} - Q = -Q, \]
since $\lim_{t \to \infty} \|e^{At}\|_2 = 0$. Q.E.D.
2.3 LTI Discrete Time Systems

Definition  [Change of $V(k, x)$ relative to a state trajectory]
Consider the system (L.2). The change of $\Delta V(k+1, x)$ relative to (L.2) is given by

$$\Delta V(k + 1, x) = V(k + 1, x(k + 1)) - V(k, x(k))$$

(L.16)

$$= V(k + 1, f(k, x(k))) - V(k, x(k)).$$

Theorem L.4  The following conditions are equivalent:

(a) **The equilibrium 0 of the nth order system**

$$x(k+1) = Ax(k)$$

(L.17)

*is globally asymptotically stable (exponentially stable).*

(b) **All eigenvalues of $A$ have magnitudes less than 1.**

(c) **For any positive definite symmetric matrix $Q$, there exist a unique positive definite symmetric matrix $P$ which is the solution of the following Discrete Time Lyapunov equation**

$$A^T PA - P = -Q.$$  

(L.18)

Proof:  The proof is very similar to the continuous time case and it’s left as an exercise.  

Q.E.D.
2.4 Lyapunov’s Indirect Method

**Theorem L.5 [Theorem 4.7 of Ref1]**

Consider the autonomous system (L.4) with the origin as an equilibrium point. If \( f: D \to \mathbb{R}^n \) is continuously differentiable and \( D \) is a neighborhood of the origin. Let

\[
A = \frac{\partial f(x)}{\partial x} \bigg|_{x=0} \quad (L.19)
\]

Then,

(a) The origin is locally asymptotically stable if \( A \) is asymptotically stable or all eigenvalues of \( A \) have negative real parts.

(b) The origin is unstable if one or more of the eigenvalues of \( A \) has positive real part.
Note:

(1). Both the Lyapunov’s indirect method (Theorem L.5) and the direct method (Theorem L.1) can be used to judge the local stability of an equilibrium point when the linearized system matrix $A$ is either asymptotically stable or unstable. However, the indirect method does not tell anything about the region of attraction\(^3\) (or domain of attraction) while the direct method gives at least some conservative estimate of the domain of attraction. For example, if conditions for asymptotic stability in Theorem 1 are satisfied and $\Omega_c=\{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is bounded and contained in D, then, every trajectory starting in $\Omega_c$ remains in $\Omega_c$ and approaches the origin as $t \to \infty$. Thus $\Omega_c$ is an estimate of the region of attraction.

(2). When some of the eigenvalues of $A$ have zero real parts and all the rest eigenvalues have negative real parts, local stability of the origin cannot be concluded from the above theorem. In such a case, local stability of the origin depends on higher-order nonlinear terms also. Advanced stability theorems such as Center Manifold Theorem may be used to judge local stability of the origin.

\(^3\) The region in which all trajectories converge to the equilibrium point as $t$ approaches $\infty$. 
Proof of Theorem L.5:

(a). \( f(0) = 0 : \quad f(x) = Ax + H.O.T. \quad \& \quad \lim_{\|x\| \to 0} \frac{g(x)}{\|x\|} = 0 \)

By assumption, \( \text{Re}\{\lambda_i(A)\} < 0, \forall i \). Thus, \( \forall Q > 0, \exists P > 0 \) s.t. \( PA + A^T P = -Q \).

Consider \( V(x) = x^T P x \quad \Rightarrow \quad V(x) \) is p.d. and
\[
\dot{V}(x(t)) = x^T P \dot{x} + x^T P x = x^T P[Ax + g(x)] + [(Ax)^T + g^T] P x \\
= x^T \{PA + A^T P\} x + 2x^T P g(x) = -x^T Q x + 2x^T P g(x)
\]

As \( \lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0 \quad \Rightarrow \quad \forall \gamma > 0, \exists r > 0, \text{ s.t. } \forall x \in B_r = \{x : \|x\| < r\}, \)
\( \|g(x)\| \leq \gamma \|x\| \). Thus, \( \forall x \in B_r : \)
\[
\dot{V}(x(t)) \leq -x^T Q x + 2\|x\|\|P\|\|g(x)\| \leq -\lambda_{\text{min}}(Q)\|x\|^2 + 2\|x\|\lambda_{\text{max}}(P) \gamma \|x\| \\
= -\left[\lambda_{\text{min}}(Q) - 2\gamma \lambda_{\text{max}}(P)\right]\|x\|^2
\]

Let \( \gamma < \frac{\lambda_{\text{min}}(Q)}{2 \lambda_{\text{max}}(P)} \). Then, \( k > 0 \) and \( \dot{V} \) is n.d. . By Lyapunov Theorem, \( 0 \) is asymptotically stable.
**Stability of Equilibrium Trajectories or Time-Varying Solutions**

Suppose \( \bar{y}(t) \) is the solution to the system

\[
\dot{y}(t) = g(t, y)
\]

e.i.

\[
\dot{\bar{y}}(t) = g(t, \bar{y}(t))
\]

We would like to study stability of the solution \( \bar{y}(t) \). The problem arises when we are interested in either the stability of a limit cycle of a system (correspondingly, \( \bar{y}(t) \) represents the limit cycle) or tracking a time-varying trajectory in a control problem (correspondingly, \( \bar{y}(t) \) represents the desired state trajectories). Define a new state-variable \( x \) as

\[
x(t) = y(t) - \bar{y}(t)
\]

Then

\[
x(t) = 0 \iff y(t) = \bar{y}(t)
\]

and

\[
\dot{x} = \dot{y}(t) - \dot{\bar{y}}(t) = g(t, y) - g(t, \bar{y}(t)) = g(t, x + \bar{y}(t)) - g(t, \bar{y}(t)) \triangleq f(t, x)
\]

Note that \( f(t, 0) = 0 \), and the origin \( x = 0 \) is an equilibrium point for the transformed system. Thus, the stability of the time-varying solution \( \bar{y}(t) \) is equivalent to the stability of the origin of the transformed system.
Note:
Even when the original system is autonomous, i.e., $\dot{y} = g(y)$, the transformed system is time-varying (or non-autonomous) in general since

$$f(t, x) = g(x + \bar{y}(t)) - g(\bar{y}(t))$$

is an explicit function of $t$ due to the appearance of explicit time function of $\bar{y}(t)$. 
Lyapunov Stability Theorems for Non-autonomous (or Time-Varying) Systems

Consider the non-autonomous system (L.1) where \( f: [0, \infty) \times D \rightarrow \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \) on \( [0, \infty) \times D \), and \( D \subset \mathbb{R}^n \) is a domain that contains the equilibrium point of origin \( x=0 \). Note that an equilibrium at origin of a non-autonomous system could be a translation of a non-zero time-varying solution of an autonomous system (e.g., trajectory tracking of an autonomous system). The solution of a non-autonomous system may depend on both the initial time \( t_0 \) and the time duration \( t-t_0 \). Consequently, Lyapunov function \( V(t, x) \) in general depends on \( t \) as well.

To characterize positive definiteness of such a time-varying function of the state \( x \), the following definitions are needed.

**Definition [Ref.1] [Class-K Function]**

A continuous function \( \alpha: [0,a) \rightarrow \mathbb{R}_+ \) is said to be a class-\( K \) function if,

(a) \( \alpha (0) = 0. \)

(b) \( \alpha \) is strictly increasing.

It is said to belong to class \( K_\infty \) if \( a=\infty \) and \( \alpha(r) \rightarrow \infty \) as \( r \rightarrow \infty \).
**Definition** [Ref.1] [Class-KL Function]

A continuous function $\beta: [0,a) \times [0,\infty) \rightarrow \mathbb{R}^+$ is said to belong to class-KL if, for each fixed $t$, the mapping $\beta(r,t)$ belongs to class $K$ with respect to $r$ and, for each fixed $r$, the mapping $\beta(r,t)$ is decreasing with respect to $t$ and $\beta(r,t) \rightarrow 0$ as $t \rightarrow \infty$.

**Definition** [Locally Positive Definite Function (LPDF)]

A continuous function $V: \mathbb{R}^+ \times D \rightarrow \mathbb{R}^+$ is said to be a Locally Positive Definite Function (LPDF) if there exists a class $K$ function $\alpha$ such that

(a) $V(t, x) \geq \alpha(\|x\|)$ for all $t \geq 0$ and for all $\|x\| \leq r$, for some $r > 0$.

(b) $V(t, 0) = 0$, $\forall t$
**Definition [Positive Definite Function (PDF)]**

A continuous function \( V: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \) is said to be a Positive Definite Function (PDF) if there exists a class \( K_\infty \) function \( \alpha \) such that

(a) \( V(t, x) \geq \alpha(\|x\|) \) for all \( t \geq 0 \) and \( x \in \mathbb{R}^n \).

(b) \( V(t, 0) = 0, \quad \forall t \)

\[ \]  

**Examples**

1) \( V(x_1, x_2) = x_1^2 + x_2^2 \) is a PDF and decrescent.

2) \( V(t, x_1, x_2) = (t + 1) (x_1^2 + x_2^2) \) is a PDF but not decrescent.

3) \( V(t, x_1, x_2) = e^{-t} (x_1^2 + x_2^2) \) is not a PDF.

4) \( V(x_1, x_2) = x_1^2 + \sin^2(x_2) \) is a LPDF and decrescent (but not PDF).
Lemma 4.2

Let \( \alpha_1 \) & \( \alpha_2 \) be class \( K \) functions on \([0, a]\), \( \alpha_3 \) & \( \alpha_4 \) be class \( K_\infty \) functions, and \( \beta \) be a class \( KL \) function. Denote the inverse of \( \alpha_i \) by \( \alpha_i^{-1} \). Then

(a) \( \alpha_1^{-1} \) is defined on \([0, \alpha_1(a)]\) and belongs to class \( K \)

(b) \( \alpha_3^{-1} \) is defined and belongs to class \( K_\infty \)

(c) \( \alpha_1 \cdot \alpha_2 \) belongs to class \( K \) where \( \alpha_1 \cdot \alpha_2 (\|x\|) = \alpha_1 (\alpha_2 (\|x\|)) \)

(d) \( \alpha_3 \cdot \alpha_4 \in K_\infty \)

(e) \( \sigma(r, s) = \alpha_1 (\beta(\alpha_2(r), s)) \) belongs to class \( KL \)
Lemma 4.3

Let $V(x)$ be a continuous positive definition function defined on a domain $D \subset \mathbb{R}^n$ that contains origin. Let $B_r \subset D$ for some $r > 0$. Then, there exists class $K$ functions $\alpha_1$ & $\alpha_2$, defined on $[0, r]$, such that

$$\alpha_1 (\|x\|) \leq V(x) \leq \alpha_2 (\|x\|)$$

$\forall x \in B_r$. If $D = \mathbb{R}^n$, then $\alpha_1$ & $\alpha_2$ will be defined in $[0, \infty]$ and the above inequality holds for all $x \in \mathbb{R}^n$. Furthermore, if $V(x)$ is radially unbounded, then, $\alpha_1$ & $\alpha_2$ can be chosen to belong to class $K \infty$. 
3.1 Continuous Time Systems

**Definition** [Derivative of \( V(t, x) \) relative to a state trajectory]

Consider the system (L.1). The derivative of \( V(t, x) \) relative to (L.1) is given by

\[
\dot{V}(t, x(t)) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x) \bigg|_{x=x(t)}
\]

(L.20)

where

\[
\frac{\partial V(t, x)}{\partial x} = \left[ \frac{\partial V(t, x)}{\partial x_1}, \frac{\partial V(t, x)}{\partial x_2}, \ldots, \frac{\partial V(t, x)}{\partial x_n} \right]
\]

**Theorem L.6** [Ref2]

*The equilibrium point \( 0 \) of (L.1) is locally stable in the sense of Lyapunov if there exists a LPDF \( V(t, x) \) such that*

\[
\dot{V}(t, x(t)) \leq 0
\]

*for all \( t \geq t_0 \) and all \( x \) such that \( \|x\| < r \) for some \( r > 0. \) \( \Delta \)
Proof of Theorem L.6:

\[ \dot{V} \leq 0 \quad \Rightarrow \quad V(t, x(t)) \leq V(t_0, x_0) = V_o, \quad \forall t \]

\( V \) is LPDF \( \Rightarrow \exists \alpha \in K \) s.t. \( \alpha(\|x(t)\|) \leq V(t, x(t)) \leq V_o \quad \Rightarrow \quad \|x(t)\| \leq \alpha^{-1}(V_o), \quad \forall t. \)

As \( V \) is continuous w.r.t. \( x \) & \( V(t, 0) = 0 \), we have, \( \forall \varepsilon > 0, \exists \delta(\varepsilon, t_0) > 0 \), s.t. \( \forall x_o \in B_\delta = \{x: \|x\| \leq \delta\} \),

\[ V_o = V(t_0, x_o) \leq \alpha(\varepsilon), \quad \Rightarrow \quad \|x(t)\| \leq \alpha^{-1}(\alpha(\varepsilon)) = \varepsilon \]

Thus, the origin 0 is stable in the sense of Lyapunov.
**Theorem L.7** [Ref2]  

The equilibrium point 0 of (L.1) is *locally uniformly stable* in the sense of Lyapunov if there exists a locally decrescent LPDF $V(t,x)$ such that

$$\dot{V}(t,x(t)) \leq 0$$

for all $t \geq 0$ and all $x$ such that $\|x\| < r$ for some $r > 0$.

**Proof of Theorem L.7:**

$V$ is decrescent $\Rightarrow$ $\exists \alpha_2 \in K$ s.t. $V(t,x) \leq \alpha_2 (\|x\|)$. Then, pick up $\delta(\varepsilon)$ s.t. $\alpha_2(\delta) \leq \alpha(\varepsilon)$. Then, $\forall x_o \in B_\delta$

$$\Rightarrow \quad V_o = V(t_o, x_o) \leq \alpha_2 (\|x_o\|) \leq \alpha_2(\delta) \leq \alpha(\varepsilon)$$

$$\Rightarrow \quad \alpha (\|x(t)\|) \leq V(t,x(t)) \leq V_o \leq \alpha(\varepsilon)$$

$$\Rightarrow \quad \|x(t)\| \leq \varepsilon, \quad \forall t$$
Theorem L.8 [Ref2]

The equilibrium point \(0\) of (L.1) is **locally uniformly asymptotically stable** if there exists a locally decrescent LPDF \(V(t, x)\) such that \(-\dot{V}(t, x)\) is a LPDF. \(\triangle\)

Theorem L.9 [Ref2]

The equilibrium point \(0\) of (L.1) is **globally uniformly asymptotically stable** if there exists a decrescent PDF \(V(t, x)\) such that \(-\dot{V}(t, x)\) is a PDF. \(\triangle\)

Theorem L.9b [Theorem 4.10 of Ref1] - for Exponential Stability

If there is a decrescent PDF \(V(t, x) : [0, \infty) \times D \to R\) such that

\[
\begin{align*}
    k_1 \|x\|^a &\leq V(t, x) \leq k_2 \|x\|^a \\
    \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -k_3 \|x\|^a
\end{align*}
\]

for some positive constants \(k_1, k_2, k_3\) and \(\lambda\), then, the **equilibrium point 0 of (L.1) is exponentially stable**. If the assumptions hold globally, then, the **equilibrium point 0 is globally exponentially sable.** \(\triangle\)

Proof: Using comparison lemma to show that

\[
V(t, x(t)) \leq V(t_0, x(t_0)) e^{-\lambda(t-t_0)}
\]

\(\triangle\)
In adaptive control problems, it is often the case that $\dot{V}(t, x)$ is only negative semi-definite, i.e., $\dot{V}(t, x) \leq 0$. If the system is autonomous, then, LaSalle’s Invariance Principle Theorem L.2 may be applied to obtain asymptotic tracking. For non-autonomous system, LaSalle’s Invariance Theorem L.2 cannot be applied. Instead, the following Barbalat’s lemma should be used.

**Lemma L.1 [Ref1] Barbalat’s Lemma**

Let $\phi(t)$ be a uniformly continuous real function of $t$ defined for $t \geq 0$.

Suppose that $\lim_{t \to \infty} \int_0^t \phi(\tau) d\tau$ exists and is finite. Then,

$$\phi(t) \to 0 \quad \text{as} \quad t \to \infty$$

Using Barbalat’s lemma, the following Lyapunov-like lemma can be obtained.

**Lemma L.2 [Ref2] “Lyapunov-Like Lemma”**

If a scalar function $V(t, x)$ satisfies the following conditions

- $V(t, x)$ is lower bounded
- $\dot{V}(t, x)$ is negative semi-definite.
- $\dot{V}(t, x)$ is uniformly continuous in time (A sufficient condition is that $\ddot{V}(t, x)$ is bounded)

then, $\dot{V}(t, x(t)) \to 0 \quad \text{as} \quad t \to \infty$
**Proof of Lemma L.1:**

Prove by contradiction, i.e., let us assume that \( \phi(t) \to 0 \) as \( t \to \infty \) first and try to prove that this will lead to a contradiction.

As \( \phi(t) \) is assumed to be uniformly continuous, then, \( \forall \varepsilon_\phi > 0, \exists \delta_\phi(\varepsilon_\phi) > 0 \) s.t. \( \forall t_1, t_2 \), as long as \( |t_1 - t_2| \leq \delta_\phi \), we have

\[
|\phi(t_1) - \phi(t_2)| < \varepsilon_\phi
\]

By assumption, if \( \phi(t) \to 0 \), then, \( \exists \varepsilon_t > 0 \), s.t. \( \forall T > 0, \exists t_3(T) > T \), s.t.

\[
|\phi(t_3)| > \varepsilon_t. \text{So let us choose } \varepsilon_\phi = \frac{1}{2} \varepsilon_t. \text{ Then, for this } \varepsilon_\phi, \exists \delta_\phi(\varepsilon_t) > 0 \text{ s.t.}
\]

\[
\forall t \in [t_3, t_3 + \delta_\phi) \text{ or } |t - t_3| \leq \delta_\phi, \text{ we have, } |\phi(t) - \phi(t_3)| < \varepsilon_\phi = \frac{1}{2} \varepsilon_t. \text{ Thus}
\]

\[
|\phi(t)| \geq |\phi(t_3)| - |\phi(t) - \phi(t_3)| > \varepsilon_t - \frac{1}{2} \varepsilon_t = \frac{1}{2} \varepsilon_t \quad \&
\]

\[
\text{sign}(\phi(t)) = \text{sign}(\phi(t_3))
\]

\[
\therefore \int_{t_3}^{t_3 + \delta_\phi} \phi(\tau)d\tau = \int_{t_3}^{t_3 + \delta_\phi} |\phi(\tau)|d\tau > \int_{t_3}^{t_3 + \delta_\phi} \frac{1}{2} \varepsilon_t d\tau = \frac{1}{2} \varepsilon_t \delta_\phi(\varepsilon_t)
\]
which contradicts the assumption that \( \int_{t_0}^{\infty} \phi(\tau) d\tau \) exists, as, if \( \lim_{t \to \infty} \int_{t_0}^{t} \phi(\tau) d\tau \) exists, then, \( \forall \epsilon > 0, \exists T_\epsilon > 0, \) s.t. \( \forall t > T_\epsilon, \forall \delta_t > 0, \left| \int_{t}^{t+\delta_t} \phi(\tau) d\tau \right| < \epsilon \). This shows that our assumption that \( \phi(t) \not\to 0 \) is not true \( \Rightarrow \phi(t) \to 0 \) as \( t \to \infty \).

**Proof of Lemma L.2:**

\[
\dot{V} \leq 0 \quad \Rightarrow \quad V(t) = V(0) + \int_{0}^{t} \dot{V} \, d\tau \quad \text{decreases as } t \to \infty
\]

\( V(t) \) is lower bounded

\( V(t) \) converges as \( t \to \infty \) or \( \lim_{t \to \infty} \int_{0}^{t} \dot{V} \, d\tau \) exists.

Since \( \dot{V} \) is uniformly continuous, by Barbalat’s lemma, \( \dot{V} \to 0 \) as \( t \to \infty \).
Lemma [Ref1] “Characterizing Lyapunov Stability via Class K Functions”

The equilibrium point 0 is

- **uniformly stable** iff there exists a class $K$ function $\alpha$ and a positive constant $c$, independent of $t_0$, such that
  \[ \|x(t)\| \leq \alpha\left(\|x(0)\|\right), \quad \forall t \geq t_0 \geq 0, \forall \|x(0)\| < c \]

- **uniformly asymptotically stable** iff there exist a class $KL$ function $\beta$ and a positive constant $c$, independent of $t_0$, such that
  \[ \|x(t)\| \leq \beta\left(\|x(0)\|, t - t_0\right), \quad \forall t \geq t_0 \geq 0, \forall \|x(0)\| < c \]

- **globally uniformly asymptotically stable** iff the above inequality is satisfied for all initial state $x(0)$. \[\Delta\]
3.2 Discrete Time Systems

Definition [Change of $V(k, x)$ relative to a state trajectory]
Consider the system (L.2). The change of $\Delta V(k+1, x)$ relative to (L.2) is given by

$$\Delta V(k+1, x) = V(k+1, x(k+1)) - V(k, x(k))$$

(L.21)

$$= V(k+1, f(k, x(k))) - V(k, x(k)).$$

Theorem L.10

The equilibrium point $0$ of (L.2) is locally stable in the sense of Lyapunov if there exists a LPDF $V(k,x)$ such that

$$\Delta V(k+1, x) \leq 0$$

for all $k \geq k_0$ and all $x$ such that $\|x\| < r$ for some $r > 0$.  

Theorem L.11

The equilibrium point $0$ of (L.2) is globally uniformly asymptotically if there exists a decrescent PDF $V(k,x)$ such that $\Delta V(k+1, x)$ is negative definite.
3.3 Linear Time-Varying Systems

The solution of the linear time-varying system described by
\[ \dot{x} = A(t)x \]  \hspace{1cm} (L.22)

is given by
\[ x(t) = \Phi(t,t_0)x(t_0) \]

where \( \Phi(t,t_0) \) is the state transition matrix.

**Theorem L.12** [Theorem 4.11 of Ref1]

The equilibrium point 0 of (L.22) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality
\[ \| \Phi(t,t_0) \| \leq ke^{-\gamma(t-t_0)}, \quad \forall t \geq t_0 \geq 0 \] \hspace{1cm} (L.23)

for some positive constant \( k \) and \( \gamma \).

Theorem L.12 shows that, for linear systems, uniform asymptotic stability of the origin is equivalent to exponential stability. Note that, for linear time-varying system, in general, uniform asymptotic stability cannot be characterized by the location of the eigenvalues of the matrix \( A \).
Example:
Stability of Linear-Time-Varying System cannot be Judged Based on Eigenvalues of $A(t)$

Consider the LTV system given by

$$\dot{x} = A(t)x, \quad A(t) = \begin{bmatrix}
-1 + 1.5 \cos^2 t & 1 - 1.5 \sin t \cos t \\
-1 - 1.5 \sin t \cos t & -1 + 1.5 \sin^2 t
\end{bmatrix}$$

Verify by yourself that

$$\lambda(A(t)) = -0.25 \pm 0.25 \sqrt{7}i, \quad \forall t$$

$$\Phi(t, 0) = \begin{bmatrix}
e^{0.5t} \cos t & e^{-t} \sin t \\
-e^{0.5t} \sin t & e^{-t} \cos t
\end{bmatrix}, \quad \left(i.e., \frac{d}{dt} \Phi(t, 0) = A(t)\Phi(t, 0)\right)$$

Thus, all eigenvalues of $A(t)$ are in LHP and, furthermore, are independent of $t$. Yet the system is unstable since some solutions become unbounded.
**Theorem L.13** [Theorem 4.12 of Ref1]

Suppose that the equilibrium point 0 of (L.22) is uniformly asymptotically stable, and \( A(t) \) is continuous and bounded. Let \( Q(t) \) be a continuous, bounded, symmetric positive definite matrix. Then, there is a continuously differentiable, bounded, symmetric positive definite matrix \( P(t) \) such that

\[
-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)
\]

(L.24)

Hence, \( V(t, x) = x^TP(t)x \) is a Lyapunov function for the system that satisfies the conditions of Theorem L.9.

\[\Delta\]

**Proof:**

Let

\[
P(t) = \int_t^\infty \Phi^T(\tau, t)Q(\tau)\Phi(\tau, t)\,d\tau
\]

(L.25)

It can be verified that \( P(t) \) given above satisfies (L.24). Details are omitted.
3.4 Linearization (Lyapunov's Indirect Method)

Theorem L.14 [Theorem 4.15 of Ref1]

Consider the non-autonomous system (L.1) with the origin as an equilibrium point. Suppose that \( f : [0, \infty) \times D \to \mathbb{R}^n \) is continuously differentiable, and the Jacobian matrix \( \left[ \frac{\partial f}{\partial x} \right] \) is bounded and Lipschitz on \( D \), uniformly in \( t \). Let

\[
A(t) = \left. \frac{\partial f(t, x)}{\partial x} \right|_{x=0} \quad (L.26)
\]

Then, the origin is an exponentially stable equilibrium point for the nonlinear system (L.1) if and only if it is an exponentially stable equilibrium point for the linearized linear system (L.22).
Q: Is there a function that satisfies conditions of Lyapunov Theorems?

A: Yes as seen from the following converse Lyapunov Theorems, which explicitly construct the needed Lyapunov functions. However, the construction assumes exact knowledge of the solutions, which is not helpful in practical search for an Lyapunov function as we do not know the solution in reality. However, knowing existence of such functions may be very useful in the analysis of nonlinear systems.

4. Converse Lyapunov Theorems

Theorem L.15 [Theorem 4.14 of Ref1] - Converse for Exponential Stability

Let \( x = 0 \) be an equilibrium point for the nonlinear system \( \dot{x} = f(t, x) \), where \( f : [0, \infty) \times D \to \mathbb{R}^n \) is continuously differentiable on \( D = B_r = \{ x \in \mathbb{R}^n : \|x\| < r \} \)

and the Jacobian matrix \( \frac{\partial f}{\partial x} \) is bounded on \( D \), uniformly in \( t \). Let \( K, \lambda \) and \( r_o \) be positive constants with \( r_o < \frac{r}{K} \). Let \( D_o = B_{r_o} = \{ x \in \mathbb{R}^n : \|x\| < r_o \} \).

Assume that the trajectories of the system satisfy

\[
\|x(t)\| \leq K \|x(t_o)\| e^{-\lambda(t-t_o)}, \quad \forall t, x(t_o) \in D_o
\]
Then, there is a function $V : [0, \infty) \times D_0 \rightarrow R$ that satisfies the inequalities:

$$c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\|$$

for some positive constants $c_1, c_2, c_3$ and $c_4$. Moreover, if $r = \infty$, and the origin is globally exponentially stable, then $V(t, x)$ is defined and satisfies the aforementioned inequalities on $R^n$. Furthermore, if the system is autonomous, $V$ can be chosen independent of $t$.

**Proof:** Show that

$$V(t, x) = \int_t^{t+\delta} \phi^T(\tau; t, x) \phi(\tau; t, x) d\tau$$

satisfies all the requirements, where $\phi(\tau; t, x)$ is the solution of the system at $\tau$ that starts at $(t, x)$, and $\delta$ is a positive constant that is large enough. #
For LTI:

\[ \dot{x} = Ax \implies \phi(\tau, t, x) = e^{A(\tau-t)x} \]

Then,

\[
V(t, x) = \int_t^{t+\delta} x^T e^{A^T(\tau-t)} e^{A(\tau-t)} x \, d\tau = x^T \left[ \int_t^{t+\delta} e^{A^T(\tau-t)} e^{A(\tau-t)} d\tau \right] x
\]

\[
P(t) = \int_0^{\delta} e^{A^T_s} e^{A_s} ds \to P_{\infty} \quad \text{as} \quad \delta \to \infty.
\]

which is the solution of Lyapunov equation for \( Q = I \).

**Remark:**

One application of the above converse Theorem is to show that the linearization Theorem L.14 also has a *only if* part, which is stated by Theorem 4.15 of textbook.
Theorem L.16 [Theorem 4.16 of Ref1]

- Converse Theorem for Uniformly Asymptotic Stability

Let $0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is continuously differentiable on $D = B_r$, and the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded on $D$, uniformly in $t$. Let $\beta$ be a class $KL$ function and $r_o$ be a positive constant that $\beta(r_o, 0) < r$. Assume that the trajectory of the system satisfies

$$\beta(||x(t_o)||, t - t_o), \quad \forall x(t_o) \in D_o = B_{r_o}$$

(an alternative way to state the uniformly asymptotic stability). Then, there is a continuously differentiable function $V : [0, \infty) \times D_o \to \mathbb{R}$ such that

$$\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(||x||)$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4(||x||)$$
where $\alpha_1, \alpha_2, \alpha_3,$ and $\alpha_4$ are class $K$ functions. If the system is autonomous, $V$ can be chosen independent of $t$.

Proof:

$$V(t, x) = \int_t^\infty G(\|\phi(\tau; t, x)\|) d\tau$$

where $G$ is a class $K$ function selected by using Massera’s Lemma (Lemma C.1).
5. Boundedness and Ultimate Boundedness

**Theorem L.17** [Theorem 4.18 of Ref1]:

Let \( D \subset R^n \) be a domain that contains the origin and \( V : [0, \infty) \times D \to R \) be a continuously differentiable function such that

\[
\alpha_1 (\|x\|) \leq V(t, x) \leq \alpha_2 (\|x\|)
\]

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall \|x\| \geq \mu > 0, t \geq 0, \text{ and } x \in D
\]

where \( \alpha_1 \) and \( \alpha_2 \) are class \( K \) functions and \( W_3(x) \) is a continuous positive definite function. Take \( r > 0 \) such that \( B_r \subset D \) and assume that

\[
\mu < \alpha_2^{-1} (\alpha_1(r))
\]

Then, there exists a class \( KL \) function \( \beta \) and for every initial state \( x(t_o) \) satisfying \( \|x(t_o)\| \leq \alpha_2^{-1} (\alpha_1(r)) \), there is a \( T(x(t_o), \mu) > 0 \) such that the solution satisfies

\[
\|x(t)\| \leq \beta (\|x(t_o)\|, t - t_o), \quad \forall t_o < t < t_o + T
\]

\[
\|x(t)\| \leq \alpha_1^{-1} (\alpha_2(\mu)), \quad \forall t > t_o + T
\]
**Intuitive Explanation of Theorem L.17 for Autonomous Systems**

\[ \Omega_\varepsilon = \{ \mathbf{x} : V(\mathbf{x}) \leq \varepsilon \} \]

\[ \Omega_c = \{ \mathbf{x} : V(\mathbf{x}) \leq c \} \]

\[ \varepsilon = \max_{\|\mathbf{x}\| = \mu} V(\mathbf{x}) = \min_{\|\mathbf{x}\| = \mu_\varepsilon} V(\mathbf{x}) \]

\[ \alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|) \]

If the assumptions of Theorem L.17 are true:

\[ \dot{V} = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \leq -W_3(\mathbf{x}), \quad \forall \|\mathbf{x}\| > \mu \]

which is also true for any \( \mathbf{x} \in \Omega_\varepsilon - \Omega_c \) and \( \dot{V} \leq -k \quad \Rightarrow \quad V(t) \leq V(t_o) - k(t - t_o) \).

Then the solution will eventually enter \( \Omega_\varepsilon \) after \( T = \frac{|V(t_o) - \varepsilon|}{k} \), which also indicates that \( \|\mathbf{x}(t)\| \leq \mu_\varepsilon \) after \( t - t_o \geq T \).

\[ \varepsilon = \max_{\|\mathbf{x}\| = \mu} V(\mathbf{x}) \leq \max_{\|\mathbf{x}\| = \mu_\varepsilon} \alpha_2(\|\mathbf{x}\|) = \alpha_2(\mu) \quad \Rightarrow \quad \alpha_1(\mu_\varepsilon) \leq \alpha_2(\mu) \quad \text{or} \]

\[ \varepsilon = \min_{\|\mathbf{x}\| = \mu_\varepsilon} V(\mathbf{x}) \geq \min_{\|\mathbf{x}\| = \mu} \alpha_1(\|\mathbf{x}\|) = \alpha_1(\mu_\varepsilon) \quad \mu_\varepsilon \leq \alpha_1^{-1}(\alpha_2(\mu)) \]
6. Input-to-State Stability

6.1 Linear Time-Invariant Systems:

\[ \dot{x} = Ax + Bu \quad \Rightarrow \quad x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau \]

For stable system,

\[ \Re \left[ \lambda_i(A) \right] < 0 \quad \Rightarrow \quad \left\| e^{At} \right\| \leq ke^{-\lambda t} \quad \text{for some } k \& \lambda > 0 \]

where \( 0 < \lambda \leq \min_i \{ -\Re \left[ \lambda_i(A) \right] \} \). Then,

\[
\left\| x(t) \right\| \leq \left\| e^{At} x_0 \right\| + \left\| \int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau \right\|
\]

\[
\leq \left\| e^{At} \right\| \left\| x_0 \right\| + \int_0^t \left\| e^{A(t-\tau)} \right\| \left\| B \right\| \left\| u(\tau) \right\| \, d\tau
\]

\[
\leq ke^{-\lambda t} \left\| x_0 \right\| + \int_0^t ke^{-\lambda(t-\tau)} \, d\tau \left\| B \right\| \left\| u \right\|_\infty
\]

\[
= ke^{-\lambda t} \left\| x_0 \right\| + \frac{k}{\lambda} [1 - e^{-\lambda t}] \left\| B \right\| \left\| u \right\|_\infty
\]

which means that bounded inputs will lead to bounded states.
6.2 Nonlinear Systems

**Definition:** [Input-to-State Stable (ISS)]

System \( \dot{x} = f(t, x, u) \) is said to be input-to-state stable if there exists a class \( KL \) function \( \beta \) and a class \( K \) function \( \gamma \) such that for any initial state \( x(t_o) \) and any bounded input \( u(t) \), the solution exists for all \( t \geq t_o \) and satisfies

\[
\|x(t)\| \leq \beta\left(\|x(t_o)\|, t - t_o\right) + \gamma\left(\sup_{t_o \leq \tau \leq t} \|u(\tau)\|\right)
\]

**Theorem L.18** [Theorem 4.19 of Ref1]:

Let \( V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function such that \( \forall (t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m : \)

\[
\alpha_1\left(\|x\|\right) \leq V(t, x) \leq \alpha_2\left(\|x\|\right)
\]

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W_3(x), \quad \forall \|x\| \geq \rho\left(\|u\|\right) > 0
\]

where \( \alpha_1, \alpha_2 \) are class \( K_\infty \) functions, \( \rho \) is a class \( K \) function, and \( W_3(x) \) is a continuous positive definite function on \( \mathbb{R}^n \). Then, the system \( \dot{x} = f(t, x, u) \) is ISS with \( \gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho \).
Proof of Theorem L.18:

For any bounded input, i.e., $\|u(t)\| \leq M$, for some $M$ and $\forall t$. Consider:

$$v(\tau) = \begin{cases} u(\tau), & \tau \leq t \\ 0, & \tau > t \end{cases}$$

Then, $x(\tau; t_o, x_o, v)$ is the same as $x(\tau; t_o, x_o, u)$, $\forall \tau \leq t$. Let

$$\mu = \rho\left\{ \sup_{t_o \leq \tau \leq t} \|v(\tau)\| \right\}$$

From Theorem L.17, the above theorem is true.

Note: [Lemma 4.6 of Ref1]

An immediate consequence of the converse Lyapunov theorem and the above theorem is that, if the unforced system, i.e., $\dot{x} = f(t, x, 0)$, has a globally exponentially stable equilibrium point at the origin and $f(t, x, u)$ is continuously differentiable and globally Lipschitz in $(x, u)$, uniformly in $t$, then, the system $\dot{x} = f(t, x, u)$ is input-to-state stable.

References: