INTRODUCTION TO NONLINEAR SYSTEMS

• Linear and Nonlinear Models
  – Linear analysis and design procedure
  – Nonlinear analysis and design procedure

• Unique Phenomena of Nonlinear Systems
  – Finite escape time
  – Multiple isolated equilibrium points
  – Limit cycles
  – Subharmonic, harmonic, or almost-periodic oscillations
  – Chaos (more complicated steady-state behavior other than the above)
  – Multiple models of behavior

• Examples
Nonlinear System Models

Finite Dimensional Systems only:
At any time $t$, status of the system is completely characterized by a finite number of independent variables $x_1(t), x_2(t), \ldots, x_n(t)$.

- State Space Component Form

\[
\begin{align*}
\dot{x}_1(t) &= f_1(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), w_1(t), \ldots, w_q(t)) \\
\dot{x}_2(t) &= f_2(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), w_1(t), \ldots, w_q(t)) \\
&\vdots \\
\dot{x}_n(t) &= f_n(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), w_1(t), \ldots, w_q(t)) \\
y_1(t) &= h_1(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), w_1(t), \ldots, w_q(t)) \\
&\vdots \\
y_p(t) &= h_p(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), w_1(t), \ldots, w_q(t))
\end{align*}
\]

- Compact Vector Form

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), u(t), w(t)) \\
y(t) &= h(t, x(t), u(t), w(t))
\end{align*}
\]

or in short notation

\[
\begin{align*}
\dot{x}(t) &= f(t, x, u, w) \\
y(t) &= h(t, x, u, w)
\end{align*}
\]

Graphically:

\[
\begin{align*}
\dot{x}(t) &= f(t, x, u, w) \\
y(t) &= h(t, x, u, w)
\end{align*}
\]
System Models

• **Time-Invariant (or Autonomous) Nonlinear Systems**
  State functions and output functions are independent of time

\[
\begin{align*}
\dot{x} &= f(x, u, w) \\
y &= h(x, u, w)
\end{align*}
\]

• **Linear Systems**
  State functions and output functions are linear functions of state and external input variables at any time

\[
\begin{align*}
\dot{x} &= A(t)x + B_u(t)u + B_w(t)w \\
y &= C(t)x + D_u(t)u + D_w(t)w
\end{align*}
\]

• **Linear Time-Invariant (LTI) Systems**

\[
\begin{align*}
\dot{x} &= Ax + Bu + B_ww \\
y &= Cx + Du + Dw
\end{align*}
\]
Nonlinear Models For Analysis

- Feedback Control Law
  - Static State-feedback
    \[
    \bar{u} = \bar{g}(t, \bar{x}(t))
    \]
  - Static output feedback
    \[
    \bar{u}(t) = \bar{g}_y(t, \bar{x}(t), \bar{y}(t))
    \]

- Closed-Loop Nonlinear System Models for Analysis
  \[
  \begin{align*}
  \dot{x} &= f(t, x, u, w) = f(t, x, \bar{g}(t, \bar{x}), \bar{u}) \\
  y &= h(t, x, u, w) = h(t, x, \bar{g}(t, \bar{x}), \bar{y})
  \end{align*}
  \]

- Unforced Closed-Loop System for Analysis
  When \( \bar{w}(t) = 0, \forall t \):
  \[
  \begin{align*}
  \bar{x} &= f_{cl}(t, \bar{x}, 0) = f_{cl}^u(t, \bar{x}) \\
  \bar{y} &= h_{cl}(t, \bar{x}, 0) = h_{cl}^u(t, \bar{x})
  \end{align*}
  \]
Linear Analysis and Design Procedure

- Linearization at a particular equilibrium point $(\bar{x}, \bar{u}, \bar{w}, \bar{y})$

\[
\begin{align*}
\bar{F}(t, \bar{x}, \bar{u}, \bar{w}) &= \bar{F}_x = 0 \\
\bar{y}(t, \bar{x}, \bar{u}, \bar{w}) &= \bar{y} = h(t, \bar{x}, \bar{u}, \bar{w})
\end{align*}
\]

\[
\text{Assume } || \bar{x}(t) || < 1, \quad || \bar{u}(t) || < 1, \quad || \bar{w}(t) || < 1
\]

\[
\bar{y}(t) = \bar{y}(0) + \int_0^t \bar{F}(s) \, ds + \frac{1}{2!} \int_0^t \int_0^s \bar{F}(r) \, dr \, ds + \cdots = \frac{1}{2!} \int_0^t \int_0^s \bar{F}(r) \, dr \, ds
\]

\[
\bar{y}(t) \approx \bar{y}(0) + \int_0^t \bar{F}(s) \, ds
\]

where

\[
\bar{A}(t) = \begin{bmatrix} \frac{\partial \bar{F}}{\partial \bar{x}} & \cdots & \frac{\partial \bar{F}}{\partial \bar{x}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \bar{F}}{\partial \bar{x}} & \cdots & \frac{\partial \bar{F}}{\partial \bar{x}} \end{bmatrix}, \quad \bar{B}(t) = \begin{bmatrix} \frac{\partial \bar{h}}{\partial \bar{x}} & \cdots & \frac{\partial \bar{h}}{\partial \bar{x}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \bar{h}}{\partial \bar{x}} & \cdots & \frac{\partial \bar{h}}{\partial \bar{x}} \end{bmatrix}, \quad \bar{C}(t) = \begin{bmatrix} \frac{\partial \bar{y}}{\partial \bar{x}} & \cdots & \frac{\partial \bar{y}}{\partial \bar{x}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \bar{y}}{\partial \bar{x}} & \cdots & \frac{\partial \bar{y}}{\partial \bar{x}} \end{bmatrix}
\]

Note

- Q: When are the neglected higher order terms important?
  1) Wide operating ranges which make the assumption of small perturbations invalid
  2) "hard" nonlinearities such as Coulomb friction

Intro Nonlinear Systems

Bin Yao

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For autonomous nonlinear systems, if $f$ and $g$ are not explicit functions of $t$, then, all the matrices in the linearized system are constant matrices, which leads to

**Properties of Linear Systems**

- **Model of Linear Time-Invariant (LTI) Systems**

\[
\begin{align*}
\dot{x} &= A x + B_u u + B_w w \\
\dot{y} &= C x + D_u u + D_w w
\end{align*}
\]

- **Equilibrium of unforced system:**

\[A x_e = x_e = 0 \Rightarrow x_e \text{ in null space of } A \Rightarrow \]

- **Asymptotic stability of unforced system:**

\[\text{If } \lambda_j(A) > 0, \text{ then } X(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \text{ which means } e^{At} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \text{ which means that all solutions will converge to 0 or the unique-equilibrium point.}

- **Forced Responses:**

1. **Satisfy the superposition principle.**

2. \[\text{Sufficient LTI: } X(t) = A X + B_u u \Rightarrow \text{ (input is sinusoidal)}\]
Properties of Nonlinear Systems

• Model of Time-Invariant Nonlinear Systems

\[
\begin{cases}
\dot{x} = f(x, u, w) \\
y = h(x, u, w)
\end{cases}
\]

- Equilibrium of unforced system: \((u=0, \dot{u}=0)\)

\[
\begin{cases}
\dot{x} = f(x, 0, 0) = f_u(x) \\
y = h(x, 0, 0) = h_u(x)
\end{cases}
\]

\[
\begin{cases}
\dot{x} = f_u(x_0) = 0 \\
y = h_u(x_0)
\end{cases}
\]

\[\Rightarrow \{ \text{no equilibrium points} \} \]

\[\Rightarrow \{ \text{multiple isolated equilibrium points} \} \]

\[\Rightarrow \{ \text{infinite number of} \} \]

- Stability of unforced system:

- Forced Responses:
Common Nonlinearities

- Memoryless Nonlinearities
  - Relay
  - Saturation
  - Dead zone
  - Quantization
Common Nonlinearities

- Memory and Hysteresis Nonlinearities

Hysteresis

Relay with hysteresis

Backlash
Example

- **Mass-Spring System with Friction**

  \[ m\ddot{y} = \sum F = F_r - F_f - F_c = F - ky - by - F_c(y) \]

  \( F_r = m\dot{u},\ u = \dot{F} \):

  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= \ddot{y} = -kx_1 - bx_2 - F_c(x_2) + u
  \end{align*}
  \]

- **Coulomb and linear viscous friction with Stribeck effect**

  \[ F_f = F_c + bv \]

  \[ F_c = \left[ f_c + (f_c - f_s) e^{-|\frac{v}{v_s}|} \right] \text{sign} (v) \]
Example (cts)

- Equilibrium Points (for unforced system, i.e., $u = 0$)

\[
\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= -kx_1 - b x_2 - \left[ f_e + (f_s - f_e) e^{-\frac{x_2}{\nu_s}} \right] \text{sign}(x_2)
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
\dot{x}_{2e} = \dot{x}_{1e} = 0 & \Rightarrow \quad x_{2e} = 0 \\
0 = \dot{x}_{2e} = -k x_{1e} - b x_{2e} - \left[ f_e + (f_s - f_e) e^{-\frac{x_{2e}}{\nu_s}} \right] \text{sign}(x_{2e})
\end{cases}
\]

\[
\dot{x}_{1e} \text{ is not defined} \quad \text{in the sense that } \quad x_{1e} \text{ can be any constant value between } \left[ -\frac{f_s}{k}, \frac{f_s}{k} \right]
\]

\[
\Rightarrow \quad \text{infinite number of equilibrium points at } x_e = \begin{bmatrix} x_{1e} \\ 0 \end{bmatrix} \text{ where } x_{1e} \in \left[ -\frac{f_s}{k}, \frac{f_s}{k} \right]
\]