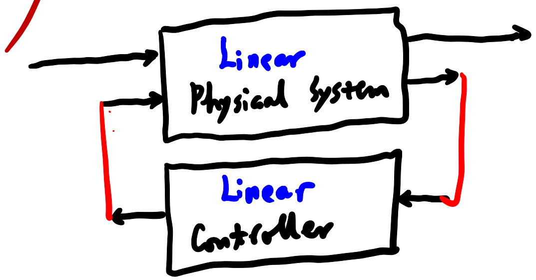
INTRODUCTION TO NONLINEAR SYSTEMS

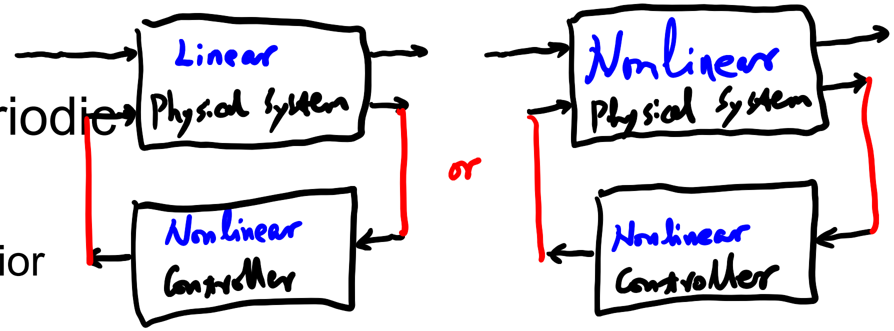
- Linear and Nonlinear Models
  - Linear analysis and design procedure
  - Nonlinear analysis and design procedure
- Unique Phenomena of Nonlinear Systems
  - Finite escape time
  - Multiple isolated equilibrium points
  - Limit cycles
  - Subharmonic, harmonic, or almost-periodic oscillations
  - Chaos (more complicated steady-state behavior other than the above)
  - Multiple models of behavior
- Examples

Traditionally:



Linear CL system

This Course:



Nonlinear CL system

Nonlinear System Models

Finite Dimensional Systems only:

At any time t, status of the system is completely characterized by a finite number of independent variables $x_1(t), x_2(t), \ldots, x_n(t)$

$\bar{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$, $\bar{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m$

$\bar{w}(t) = \begin{bmatrix} w_1(t) \\ \vdots \\ w_q(t) \end{bmatrix} \in \mathbb{R}^q$, $\bar{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in \mathbb{R}^n$

$\bar{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$, $\bar{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_p \end{bmatrix} \in \mathbb{R}^p$

- State Space Component Form

State equations

$$\begin{cases} \dot{x}_1(t) = f_1\left(t,\ x_1(t),\ldots,x_n(t), u_1(t),\ldots,u_m(t), w_1(t),\ldots,w_q(t)\right) \\ \dot{x}_2(t) = f_2\left(t,\ x_1(t),\ldots,x_n(t), u_1(t),\ldots,u_m(t), w_1(t),\ldots,w_q(t)\right) \\ \quad\vdots \\ \dot{x}_n(t) = f_n\left(t,\ x_1(t),\ldots,x_n(t), u_1(t),\ldots,u_m(t), w_1(t),\ldots,w_q(t)\right) \end{cases}$$

Output equations

$$\begin{cases} y_1(t) = h_1\left(t,\ x_1(t),\ldots,x_n(t), u_1(t),\ldots,u_m(t), w_1(t),\ldots,w_q(t)\right) \\ \quad\vdots \\ y_p(t) = h_p\left(t,\ x_1(t),\ldots,x_n(t), u_1(t),\ldots,u_m(t), w_1(t),\ldots,w_q(t)\right) \end{cases}$$

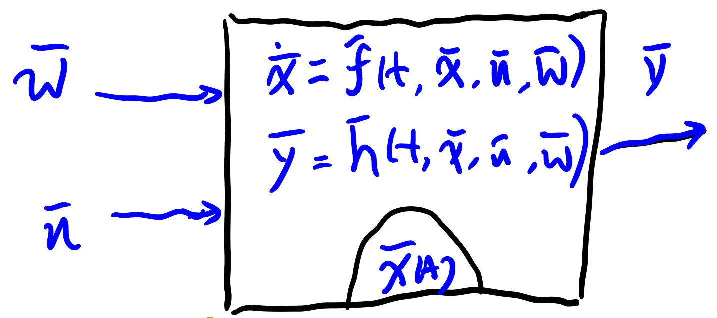
- Compact Vector Form

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)\right) \\ \mathbf{y}(t) = \mathbf{h}\left(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)\right) \end{cases}$$

or in short notation

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}\left(t, \mathbf{x}, \mathbf{u}, \mathbf{w}\right) \\ \mathbf{y} = \mathbf{h}\left(t, \mathbf{x}, \mathbf{u}, \mathbf{w}\right) \end{cases}$$

Graphically:



System Models

- Time-Invariant (or Autonomous) Nonlinear Systems

State functions and output functions are independent of time

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\ \mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \end{cases}$$

- Linear Systems

$\bar{h}$ $\bar{f} = \bar{A}(t)\bar{x}(t) + \bar{B}_u(t)\bar{u}(t) + \bar{B}_w(t)\bar{w}(t)$

State functions f and output functions are linear functions of state and external input variables at any time t

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}_u(t)\mathbf{u} + \mathbf{B}_w(t)\mathbf{w} \\ \mathbf{y} = \mathbf{C}(t)\mathbf{x} + \mathbf{D}_u(t)\mathbf{u} + \mathbf{D}_w(t)\mathbf{w} \end{cases}$$

$\bar{A}(t) \in R^{n\times n}$. $\bar{B}_u(t) \in R^{n\times m}$. $\bar{B}_w \in R^{n\times q}$

$C(t) \in R^{p\times m}$, $D_u(t) \in R^{p\times m}$, $\bar{D}_w \in R^{p\times q}$

- Linear Time-Invariant (LTI) Systems

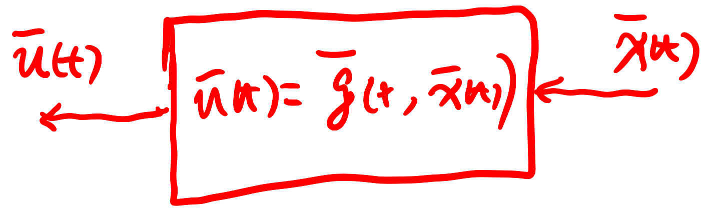
$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} + \mathbf{B}_w\mathbf{w} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}_u\mathbf{u} + \mathbf{D}_w\mathbf{w} \end{cases}$$

Nonlinear Models For Analysis

- Feedback Control Law
  - Static State-feedback

    $\bar u = \bar g(t, \bar x(t))$

    $$\bar g = \begin{cases} g_1(t, \bar x(t)) \\ \vdots \\ g_m(t, \bar x(t)) \end{cases}$$

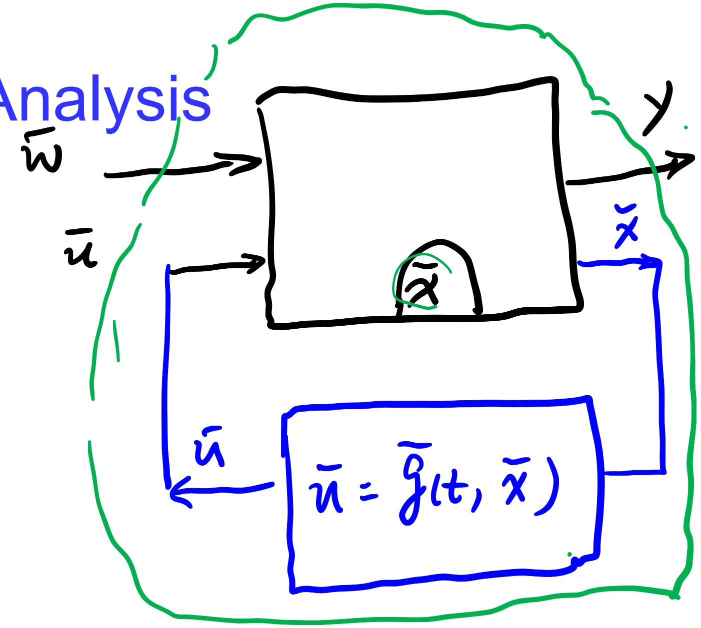


  - Static output feedback

    $$\bar u(t) = \bar g_y(t, \bar y(t)) = \underbrace{\bar g_y(t, \bar h(t, \bar x))}_{\bar g(t, \bar x(t))}$$

- Closed-Loop Nonlinear System Models for Analysis

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \mathbf{w}) = \underbrace{f(t, x, \bar g(t, \bar x), \bar w)}_{f_{cL}(\bar x, \bar w, t)} \\ \mathbf{y} = \mathbf{h}(t, \mathbf{x}, \mathbf{u}, \mathbf{w}) = \underbrace{h(t, \bar x, \bar g(t, \bar x), \bar w)}_{h_{cL}(t, \bar x, \bar w)} \end{cases}$$



- Unforced Closed-Loop System for Analysis

when $\bar w(t) = 0, \ \forall t$:

$$\dot{\bar x} = f_{cL}(t, \bar x, 0) = f^n_{cL}(t, \bar x)$$
$$\bar y = h_{cL}(t, \bar x, 0) = h^n_{cL}(t, \bar x)$$

Given $\dot{\bar{x}} = \bar{f}(t, \bar{x}, \bar{u}, \bar{w})$
$\bar{y} = \bar{h}(t, \bar{x}, \bar{u}, \bar{w})$

Linear Analysis and Design Procedure

- Linearization at a particular equilibrium point $(\bar{x}_e, \bar{u}_e, \bar{w}_e, \bar{y}_e)$

$$\begin{cases} \bar{f}(t, \bar{x}_e, \bar{u}_e, \bar{w}_e) = \dot{\bar{x}}_e = 0 \\ \bar{y}_e = h(t, \bar{x}_e, \bar{u}_e, \bar{w}_e) \end{cases} \Longrightarrow (\bar{x}_e, \bar{u}_e, \bar{w}_e, \bar{y}_e)$$

such that (s.t.) Taylor series expansions can be used for $\bar{f}(t, \bar{x}, \bar{u}, \bar{w})$ & $\bar{h}(t, \bar{x}, \bar{u}, \bar{w})$:

let
$$\bar{x}(t) = \bar{x}_e + \tilde{x}(t)$$
$$\bar{u}(t) = \bar{u}_e + \tilde{u}(t)$$
$$\bar{w}(t) = \bar{w}_e + \tilde{w}(t)$$
$$\bar{y}(t) = \bar{y}_e + \tilde{y}(t)$$

Assume
$$\|\tilde{x}(t)\| \ll 1, \quad \|\tilde{u}(t)\| \ll 1, \quad \|\tilde{w}(t)\| \ll 1$$

$$\dot{\tilde{x}}(t) = \dot{\bar{x}}(t) - \dot{\bar{x}}_e = \bar{f}(t, \bar{x}, \bar{u}, \bar{w}) - \bar{f}(t, \bar{x}_e, \bar{u}_e, \bar{w}_e)$$
$$= \left.\frac{\partial \bar{f}}{\partial \bar{x}}\right|_e \tilde{x}(t) + \left.\frac{\partial \bar{f}}{\partial \bar{u}}\right|_e \tilde{u}(t) + \left.\frac{\partial \bar{f}}{\partial \bar{w}}\right|_e \tilde{w}(t) + \text{H.O.T}$$
$$\approx \underbrace{\bar{A}(t)}\tilde{x}(t) + \underbrace{\bar{B}_u(t)}\tilde{u}(t) + \underbrace{\bar{B}_w(t)}\tilde{w}(t)$$

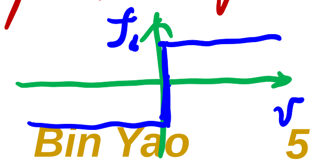
where
$$\bar{A}(t) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_e \in \mathbb{R}^{n \times n}, \quad \bar{B}_u(t) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_e \in \mathbb{R}^{n \times m}, \quad \bar{B}_w(t) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \cdots & \frac{\partial f_1}{\partial w_q} \\ \vdots & & \\ \frac{\partial f_n}{\partial w_1} & \cdots & \frac{\partial f_n}{\partial w_q} \end{bmatrix}\right|_e$$

Note

– Q: When are the neglected higher order terms important?

(1) Wide operating ranges which make the assumption of small perturbation invalid

(2) "hard" nonlinearities such as Coulomb friction



For autonomous nonlinear systems, $\mathcal{F}$ & h are not explicit functions of t, then, all the matrices in the linearized system are constant matrices, which leads to

Properties of Linear Systems

- Model of Linear Time-Invariant (LTI) Systems

$$\begin{cases} \dot{\tilde{x}} = \bar{A}\tilde{x} + \bar{B}_u\tilde{u} + \bar{B}_w\tilde{w} \\ \tilde{y} = \bar{C}\tilde{x} + \bar{D}_u\tilde{u} + D_w\tilde{w} \end{cases}$$

– **Equilibrium of unforced system:** $(\tilde{u}=0, \tilde{w}=0) \Rightarrow \begin{cases} \dot{\tilde{x}} = \bar{A}\tilde{x} \\ \tilde{y} = \bar{C}\tilde{x} \end{cases}$ then:

$\bar{A}\tilde{x}_e = \dot{\tilde{x}}_e = 0 \Rightarrow \tilde{x}_e \in$ Null space of $\bar{A}$ $\Rightarrow$

- If $\lambda_i(\bar{A}) \neq 0$, $\forall i$, $\bar{A}$ is nonsingular, i.e., $|\bar{A}| \neq 0$, only one solution exists: $\tilde{x}_e = 0$
- If $\lambda_i(A) = 0$, then, $\bar{A}$ is singular, then, there will be infinite number of solutions. Furthermore, they are not separable

– **Asymptotic stability of unforced system:**

$\tilde{x}(t) = e^{\bar{A}t}\tilde{x}(0)$. $\Rightarrow$ When $Re[\lambda_i(\bar{A})] < 0$, $\forall i$, then, $e^{\bar{A}t} \to 0$ as $t \to \infty$, which means that all solutions will converges to 0 or the unique equilibrium point

– **Forced Responses:**

(i) satisfy the superposition principle.

(ii). $\tilde{u} = \sin(\omega t) \longrightarrow \boxed{\begin{array}{c}\text{Stable LTI} \\ \dot{\tilde{x}}(t) = \bar{A}\tilde{x} + \bar{B}_u\hat{u}\end{array}} \longrightarrow y(t) \xrightarrow{t \to \infty} y_{ss}(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$

Properties of Nonlinear Systems

- Model of Time-Invariant Nonlinear Systems

$$\begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}, \bar{u}, \bar{w}) \\ \bar{y} = \bar{h}(\bar{x}, \bar{u}, \bar{w}) \end{cases}$$

  - **Equilibrium of unforced system:** $(\bar{u} = 0,\ \bar{w} = 0)$

$$\begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}, 0, 0) = \bar{f}_u(\bar{x}) \\ \bar{y} = \bar{h}(\bar{x}, 0, 0) = \bar{h}_u(\bar{x}) \end{cases}$$

$$\bar{f}_u(\bar{x}_e) = \dot{\bar{x}}_e = 0 \Rightarrow \bar{x}_e$$

$$\bar{y}_e = \bar{h}_u(\bar{x}_e)$$

$\Rightarrow$ no equilibrium points
multiple isolated equilibrium points.
infinite number of ..
⋮

  - **Stability of unforced system:**

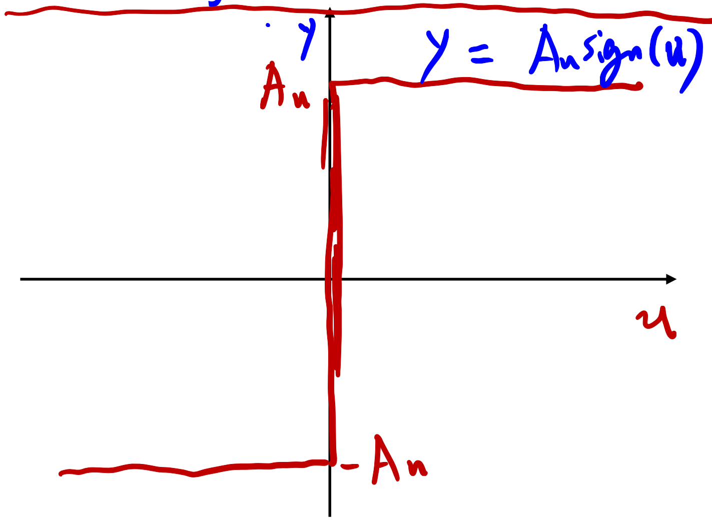
?

  - **Forced Responses:**

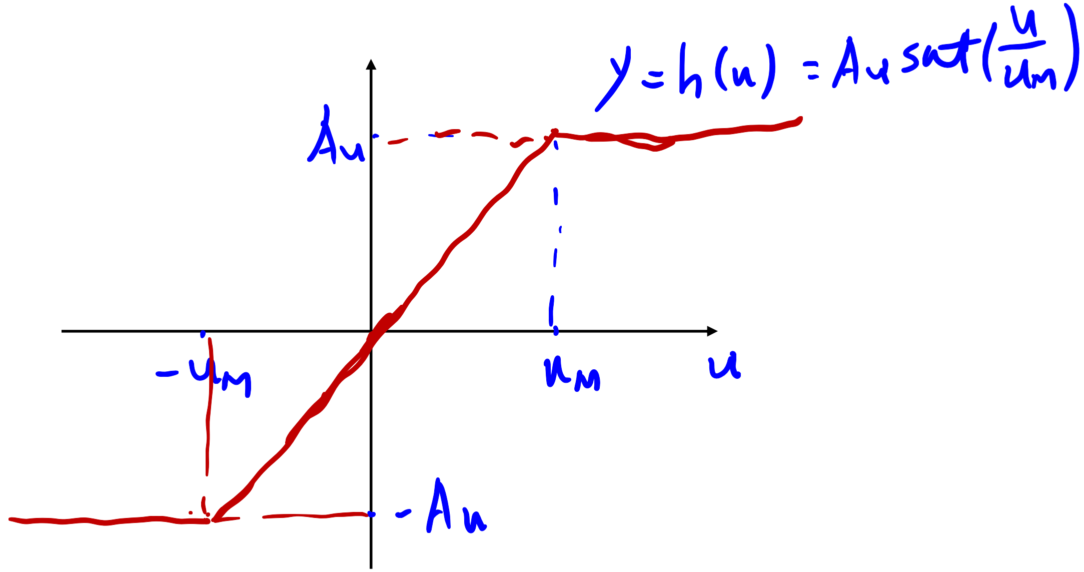
?

Common Nonlinearities

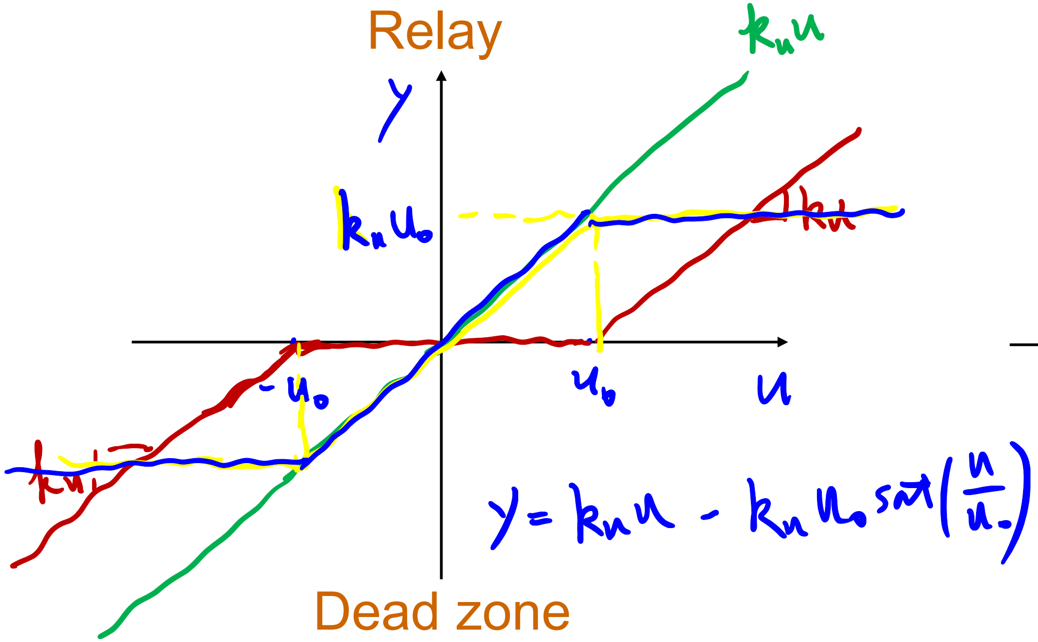
- Memoryless Nonlinearities



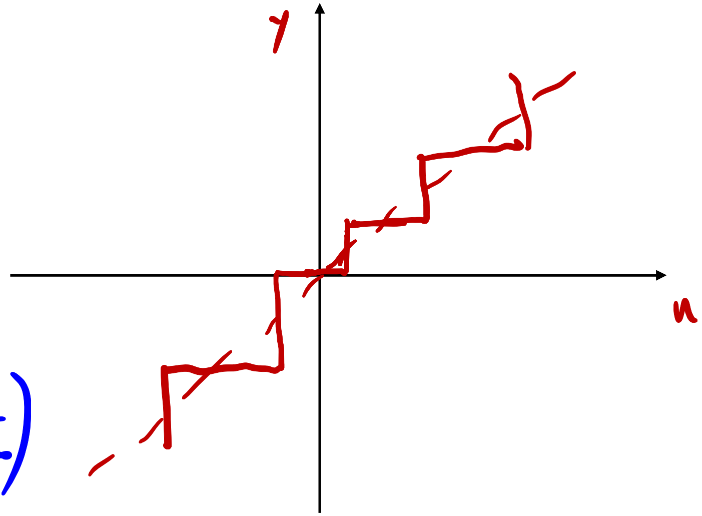
Relay



Saturation



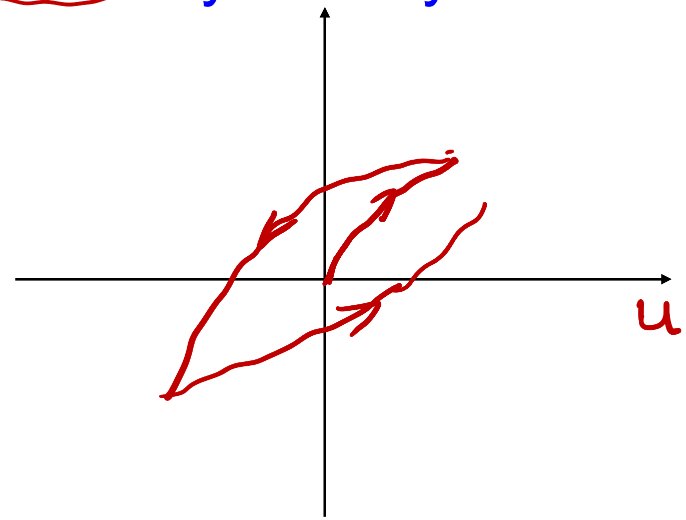
Dead zone



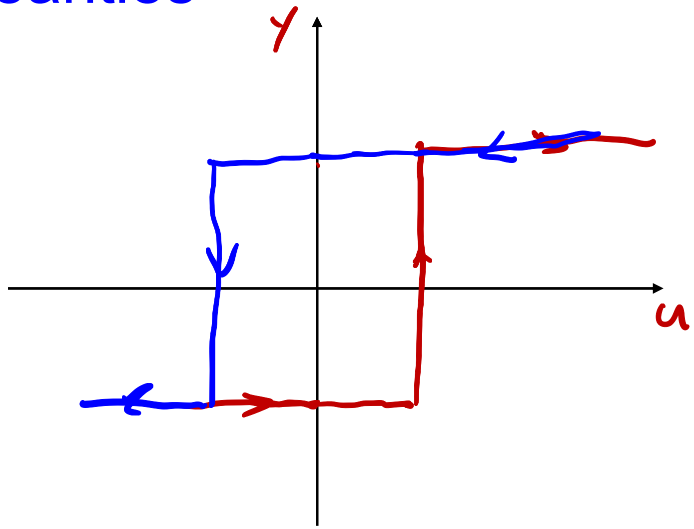
Quantization

Common Nonlinearities

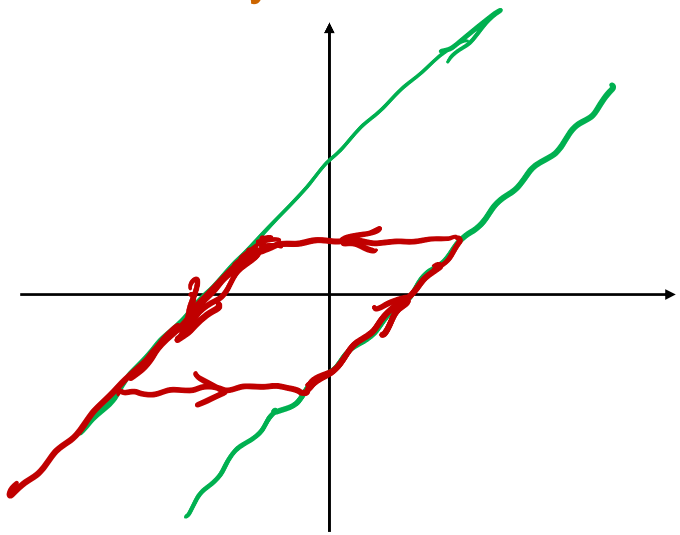
- Memory and Hysteresis Nonlinearities



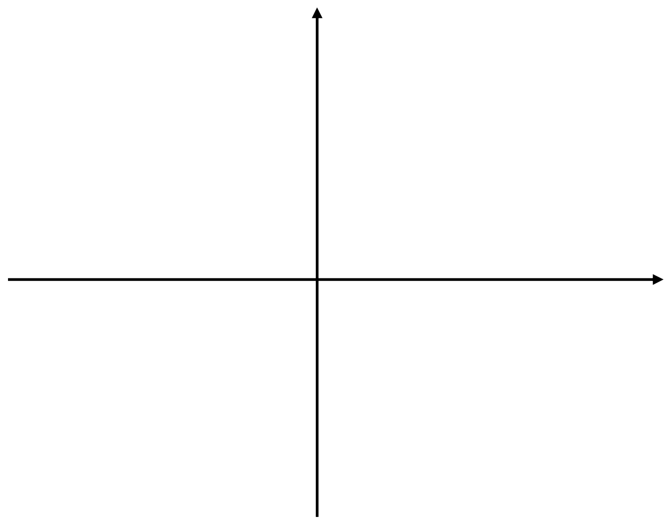
Hysteresis



Relay with hysteresis

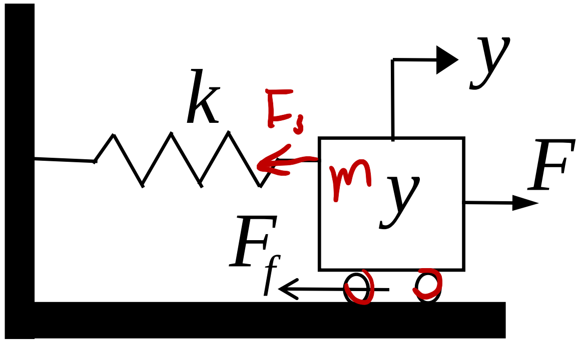


Backlash



Example

- Mass-Spring System with Friction



$$m\ddot{y} = \Sigma F = F - F_s - F_f = F - ky - b\dot{y} - F_c(\dot{y})$$

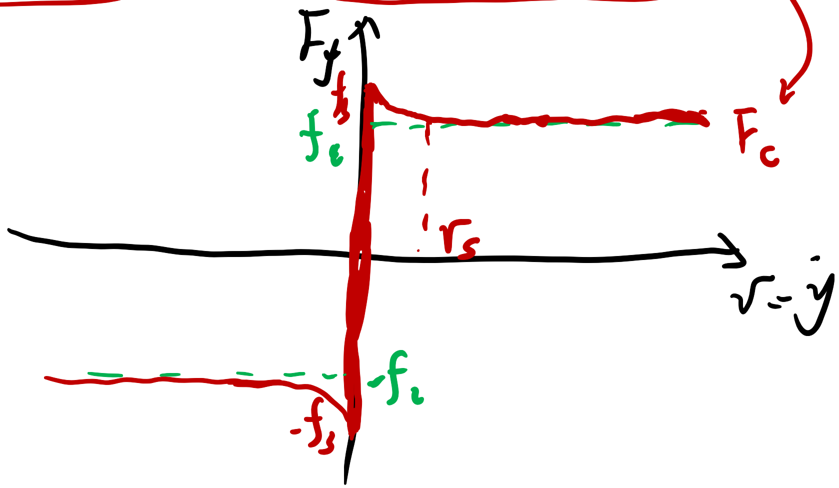
For $m=1$, $u=F$:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{y} = -kx_1 - bx_2 - F_c(x_2) + u \end{cases}$$

$x_1(t) = y(t)$

$x_2(t) = \dot{y}(t)$,

- Coulomb and linear viscous friction with Stribeck effect



$$F_f = F_c + bv$$

$$F_c = \left[f_c + (f_s - f_c)e^{-\left|\frac{v}{v_s}\right|^{\delta}}\right] \mathrm{sign}(v)$$

Example (cts)

- Equilibrium Points (for unforced system, i.e. $u=0$)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -kx_1 - bx_2 - \left[f_c + (f_s - f_c)e^{-\left|\frac{x_2}{v_s}\right|^\delta}\right] \mathrm{sign}(x_2) \end{cases}$$

$$\Rightarrow \begin{cases} x_{2e} = \dot{x}_{1e} = 0 \quad \Rightarrow \quad x_{2e} = 0 \\ 0 = \dot{x}_{2e} = -kx_{1e} - b\cancel{x_{2e}} - \underbrace{\left[f_c + (f_s - f_c)e^{-\left|\frac{x_{2e}}{v_s}\right|^\delta}\right]}_{f_c + f_s - f_c = f_s} \underbrace{\mathrm{sign}(x_{2e})} \end{cases}$$

sign(0) is not defined in the sense that it could be any value between -1 and 1.

$\Rightarrow$ x_{1e} can be any constant value between $\left[-\frac{f_s}{k} \quad \frac{f_s}{k}\right]$

$\Rightarrow$ infinite number of equilibrium points at $\bar{x}_e = \begin{bmatrix} x_{1e} \\ 0 \end{bmatrix}$ where $x_{1e} \in \left[-\frac{f_s}{k} \quad \frac{f_s}{k}\right]$